VARIATIONAL INEQUALITY PROBLEMS

IN *H*-SPACES

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by

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CERTIFICATE

This is to certify that the thesis entitled "**VARIATIONAL INEQUALITY PROBLEMS IN *H*-SPACES**" which is being submitted by *Prasanta Kumar Das*, Ph.D. Student in Mathematics, Studentship Roll No. 50302001, National Institute of Technology, Rourkela - 769 008 (India), for the award of the Degree of Doctor of Philosophy in Mathematics from National Institute of Technology, is a record of bonafide research work done by him under my supervision. The results embodied in the thesis are new and have not been submitted to any other University or Institution for the award of any Degree or Diploma.

To the best of my knowledge Mr. Das bears a good moral character and is mentally and physically fit to get the degree.

(Akrur Behera)
(Supervisor)

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ABSTRACT

The concept of certain invex functions is introduced and applied in topological vector spaces and *H*-spaces to explore the properties of variational inequality and complementarity problems. The concept of *H*-differentiable manifold is introduced and the variational inequality problem is further studied in there. Also variational inequality and complementarity problem are discussed with the use of Lefschetz Fixed-Point Theorem.

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Chapter 0

INTRODUCTION

In many practical situations, both pure and applied mathematicians and applied scientists are often required to deal with nonlinear systems and their associated nonlinear equations. The theory of nonlinear analysis is of fundamental importance in the formulation and analysis of various classes of equations, which arise in physical, biological, social, engineering and technological sciences.

In the recent decades, there has been a great deal of development in the theory of optimization techniques. In this development, computer science has played a vital role for making it possible to implement such techniques for everyday use as well as stimulating new effort for finding solutions of much more complicated problems. The study of variational inequalities and complementarity problems is also a part of this development because optimization problems can often be reduced to the solution of variational inequalities and complementarity problems. It is important to remark that these subjects pertain to more than just optimization problems and therein lies much of their attractiveness.

Although variational inequalities and complementarity problems have much in common, historically there has been little direct contact among the researchers of these two fields. Despite some notable exceptions, it can be said that people who work in variational inequalities tend to be educated in the tradition of more classical applied mathematics, even if they use very modern tools such as computers. Models of physical problems, differential equations, and topological vector spaces are common elements of their works. On the other hand, people who work in complementarity problems, again with some exceptions, lie closer to other branches of mathematical sciences, such as, operations research and combinatorics. Their efforts are closely related to mathematical programming which is very often (though not always) motivated by management or economic problems and are mainly finite-dimensional.

In the past decades, variational inequalities have gained a great deal of importance both from theoretical and practical point of view. Variational inequalities are used in the study of calculus of variation and generally in the optimization problems. It is well known that variational inequality theory is a result of research works of Browder [6], Lions and Stampacchia [31], Mosco [39], Kinderlehrer and Stampacchia [28], etc., and it has interesting applications in the study of obstacles problems, confined plasmas, filtration phenomena, free boundary problems, plasticity and viscoplasticity phenomena, elasticity problem and stochastic optimal control problems.

The complementarity problem has also been considered by many mathematicians, as a large independent division of mathematical programming theory but Isac's opinion [23] is quite different. The complementarity problem represents as very deep, very interesting and very difficult mathematical problem. This problem is a very nice research domain because it has many interesting applications and deep connections with important area of nonlinear analysis [23].

The origin of the complementarity problem lies perhaps in the Kuhn-Turker Theorem for nonlinear programming (which gives the necessary conditions of optimality when certain conditions of differentiability are met, see [23] p.1) or perhaps in the old and neglected Du Val's paper [16]. In 1961, Dorn [15] proved that the minimum value of a certain quadratic programming problem of a positive definite matrix is zero. In fact Dorn's paper was the first step in treating the complementarity problema as an independent problem. In 1963, Dantzig and Cottle [14] generalized the Dorn's result to the case when all the principal minors of the matrix are positive. The result announced in 1963 by Dantzig and Cottle was generalized in 1964 and 1966 by Cottle [9,10] to a certain class of nonlinear functions. Also in 1965, Lemke [30] proposed the complementarity problem as a method for solving matrix games. Certainly, one of the first important papers on the complementarity problem is Ingleton's paper [22] which showed the importance of the complementarity problem in engineering applications. It seems Cottle, Habetler and Lemke proposed the term "complementarity" and the reason for this is explained in page 2 of [23]. After 1970, the theory of the complementarity problem has shown a strong and ascending development based on several important results obtained by Cottle [9-11], Eaves [18], Karamardian [24-26], Mangasarian [32,33], Saigal [41], More [36,37], Pang [40], etc.,.

In 1981, M. A. Hanson [21] introduced the historic notation named as "invex function" which abbreviates "invariant convex function" and is a generalization of convex function. Hanson's paper inspired a great deal of additional works in both variational and complementarity problems. The notion of invex function introduced by M. A. Hanson gave a wide scope to analyze the variational inequality problems, complimentarity problems and fixed-point theorems. In 1986, Ben-Israel and Mond [5] characterized invexity for both constrained and unconstrained problems and showed that invexity can be substituted for convexity in the saddle point problem and in the Slater constraint qualification. In 1986, Craven [13] used the nondifferentiable invex function to construct the symmetric duality problems. Basing on invex function several important results appear in the papers of Martin [34], Mititellu [35], Suneja-Aggarwal-Davar [42], Kaul-Kaur [27]. Specifically, Mititellu [35] studied the invex function on differentiable manifolds and Riemannian manifolds and developed a vector programming problem on a manifold. On the other hand, the fixed point problems lie closer to the topological vector spaces, homology groups and optimization and boundary value problems.

In this thesis we first deal with the generalization of the following known problems.

Let K be nonempty closed and convex subset of a reflexive real Banach space X with dual X^* and

$$T : K \to X^*$$

a nonlinear map. Let the value of $f \in X^*$ at $x \in X$ be denoted by $\langle f, x \rangle$.

(a) The *nonlinear variational inequality problem* (NVIP) is to find

$$x \in K \text{ such that } \langle T(x), y - x \rangle \geq 0 \quad \text{for all } y \in K.$$

The above problem has been generalized by Behera and Panda [2 - 4] using a function $\eta(-,-)$ introduced by Hanson [21]. The function $\eta(-,-)$ is quite general in nature and applicable to many cases of general interest. For the significance of the function $\eta(\text{-},\text{-})$, we refer to the papers of Hanson [21] and Ben-Israel and Mond [5]. In fact this function is a mapping

$$\eta : K \times K \to X$$

and it generalizes NVIP as follows :

(b) The *generalized nonlinear variational inequality problem* (GNVIP) is to find

$$x \in K \text{ such } \langle Tx, \eta(y, x)\rangle \geq 0 \quad \text{for all} \quad y \in K.$$

Our main aim in this thesis is to study GNVIP in certain spaces under moderate assumptions. Precisely speaking we bring NVIP and GNVIP under one umbrella. For this we concentrate our study on H-spaces and consider a variational-type inequality as follows :

Let X be a topological vector space (or H-space), (Y,P) be an ordered topological vector space equipped with closed convex pointed cone with $\text{int}P \neq \emptyset$ and $L(X,Y)$ be the set of continuous linear functionals from X to Y. Let the value of $f \in L(X,Y)$ at $x \in X$ be denoted by $\langle f, x\rangle$. Let K be a convex set in X, with $0 \in K$ and T: $K \to L(X,Y)$ be any map. The *nonlinear variational inequality problem* is to find

$$x_0 \in K \quad \text{such that} \quad \langle T(x_0), v - x_0\rangle \notin -\text{int}P \quad \text{for all} \quad v \in K.$$

The chapter description follows :

In Chapter 1, we recall some definitions and known results on variational inequality and complementarity problems on Banach spaces and Hilbert spaces. Some more definitions and results are included in the relevant chapters. This chapter serves as the base and background for the study of subsequent chapters and we shall keep on referring back to it as and when required.

In Chapter 2, we study the concept of invexity introduced by M.A. Hanson. This concept is broadly used in the theory of optimization. Many authors have studied different types of convex and invex functions in vector spaces with different assumptions. We introduce the concept of T-η-invex function in ordered topological vector spaces and prove some results using T-η-invexity property of some functions.

In Chapter 3, we study a vector complementarity problem in topological vector spaces. We extend Minty's lemma to invex sets and present an application of the same.

In Chapter 4, we establish an inequality in H-space and obtain the traditional variational and variational-type inequalities as particular cases of the newly obtained inequality. We also discuss the uniqueness of the solutions of the inequality with examples.

In Chapter 5, we recall Lefschetz number and Lefschetz fixed-point theorem and introduce the concept of H-differentiable manifolds. Using T-η-invex functions we study variational inequality problems in H-differentiable manifolds.

In Chapter 6, we study the behavior of continuous functions on manifolds with particular interest in finding the solutions of complementarily problems in the presence of fixed points or coincidences.

Chapter 1

PRE-REQUISITES

In this chapter we recall some definitions and known results on variational inequality and complementarity problems on Banach spaces and Hilbert spaces. Some more definitions and results are included in the relevant chapters. This chapter serves as the base and background for the study of subsequent chapters and we shall keep on referring back to it as and when required.

1.1 Some definitions in Banach spaces

Let X and Y be real Banach spaces, and let X^* be the dual of X. Denote the value of $f\in X^*$ at $x\in X$ by $\langle f, x\rangle$.

1.1.1 Definition [8] A sequence $\{x_n\}\subset X$ *converges* to x_0 (denoted $x_n\to x_0$) if

$$\lim_{n\to\infty} \| x_n - x_0\| = 0.$$

1.1.2 Definition [8] A mapping $T : X\to Y$ is said to be *continuous* at x_0 if for any sequence $\{x_n\}$ which converges to x_0, the sequence $\{T(x_\mathrm{n})\}$ converges to $T(x_0)$.

T is said to be *hemicontinuous* at x_0 if for any sequence $\{x_n\}$ converging to x_0 along a line, the sequence $\{T(x_n)\}$ weakly converges to $T(x_0)$, i.e., the map

$$\lambda \mapsto T(\lambda x + (1-\lambda)\, y)$$

of $[0,1]$ into Y is continuous for $x, y\in X$, when Y is endowed with its weak topology.

T is said to be *continuous* on finite dimensional subspaces if for every finite dimensional subspace S of X the mapping $T: S\to Y$ is weakly continuous.

1.1.3 Note [8] When $Y = X^*$, the convergence of $\{T(x_n)\}$ in the definitions of continuity and hemicontinuity on finite dimensional subspaces refer to convergence in the weak* topology.

1.2 Some results on nonlinear variational inequality problems.

Several authors have proved many fascinating results on variational inequality problem. We list some of them, which are used frequently in this thesis. The existence of the solution to the problem is studied by many authors such as, J.L. Lions and G. Stampacchia [31], R.W. Cottle, F. Giannessi and J.L. Lions [12] to name only a few. The most general and popular forms of inequality with very reasonable conditions are due to F. E. Browder [6], D. Kinderlehrer and G. Stampacchia [28].

The following known results on the existence of solutions of variational inequality problems can be found in the works of F.E. Browder [6], M. Chipot [8], G. Isac [23] and Behera and Panda [2,3]; the results are obtained as directly or as particular cases of some results proved in this thesis.

1.2.1 Theorem. ([8], Theorem 1.4, p.3) *Let K be a compact convex subset of a finite dimensional Banach space X with dual X^* and T a continuous mapping of K into X^*. Then there exists $x_0 \in K$ such that*

$$\langle Tx_0, y - x_0\rangle \geq 0 \qquad \cdots (1)$$

for all $y \in K$.

1.2.2 Theorem. ([8], Theorem 1.7, p.4) *Let X be a finite dimensional Banach space with dual X^*. Let K be an unbounded subset of X with $0 \in K$, and*

$$T : K \to X^*$$

a continuous coercive mapping. Then there exists $x_0 \in K$ such that

$$\langle Tx_0, y - x_0\rangle \geq 0 \qquad \cdots (2)$$

for all $y \in K$.

1.2.3 Theorem. ([6], Theorem 1, p.780; [8], Theorem 1, p.6) *Let* X *be a reflexive real Banach space with dual* X^*, K *a nonempty closed convex subset of* X, *with* $0 \in K$, *and*

$$T : K \to X^*$$

a monotone map, continuous on finite dimensional subspaces (*or at least hemicontinuous*). *If*

(a) K is bounded

or

(b) T *is coercive on* K,

then there exists $x_0 \in K$ *such that*

$$\langle Tx_0, y - x_0 \rangle \geq 0 \qquad \cdots (3)$$

for all $y \in K$. *Moreover if* T *is strictly monotone then the solution* x_0 *is unique*.

1.2.4 Theorem. ([23], Theorem 4.32, p.116) *Let* K *be a compact convex set in a Hausdorff topological vector space* X *and let* X^* *be the dual of* X. *Let*

$$T : K \to X^*$$

be a map such that

(a) *the map*

$$x \mapsto \langle Tx, y - x \rangle$$

is upper semicontinuous,

(b) *there exist a nonempty*, *compact and convex subset* $L \subset K$ *and* $u \in L$ *such that*

$$\langle Tx, u - x \rangle < 0 \text{ for every } x \in K - L.$$

Then there exists $x_0 \in L$ *such that*

$$\langle Tx_0, y - x_0 \rangle \geq 0 \qquad \cdots (4)$$

for all $y \in K$.

When K is compact and X is a Banach space then the above theorem takes the following form.

1.2.5 Theorem. *Let K be a nonempty compact convex set in a reflexive real Banach space X and let X^* be the dual of X. Let*

$$T : K \to X^*$$

be a map such that the map

$$x \mapsto \langle Tx, y - x\rangle$$

of K into $\mathbb{R}$ is upper semicontinuous for each $y \in K$. Then there exists $x_0 \in K$ such that

$$\langle Tx_0, y - x_0\rangle \geq 0 \qquad \cdots (5)$$

for all $y \in K$.

1.2.6 Theorem. ([23], Theorem 6.2.2, p.170) *Let K be a nonempty compact convex subset of a Hausdorff topological vector space X and let X^* be the dual of X. Let*

$$T : K \to X^*$$

and

$$g : K \to X$$

be two continuous maps such that

$$\langle Tx, x - g(x)\rangle \geq 0$$

for all $x \in K$. Then there exists $x_0 \in K$ such that

$$\langle Tx_0, y - g(x_0)\rangle \geq 0 \qquad \cdots (6)$$

for all $y \in K$.

The following results on the existence of the solution of the generalized nonlinear variational inequality occur in the works of Behera and Panda [2,3].

1.2.7 Theorem. ([2], Theorem 2.2, p.184) *Let K be a compact convex set in a reflexive real Banach space X, with $0 \in K$, and let X^* denote the dual of X. Let*

$$T : K \to X^*$$

and

$$\theta : K \times K \to X$$

be two continuous maps such that

(a) $\langle Ty, \theta(y, y)\rangle = 0$ *for all* $y \in K$,

(b) *for each fixed* $y \in K$, *the map*

$$\langle Ty, \theta(-, y)\rangle \; : \; K \to \mathbb{R}$$

is convex.

Then there exists $x_0 \in K$ *such that*

$$\langle Tx_0, \theta(y, x_0)\rangle \geq 0 \qquad \cdots (7)$$

for all $y \in K$.

1.2.8 Theorem. ([2], Theorem 2.1, p.184) *Let* K *be a closed convex set in a reflexive real Banach space* X, *with* $0 \in K$, *and let* X^* *be the dual of* X. *Let*

$$T \; : \; K \to X^*$$

and

$$\theta \; : \; K \times K \to X$$

be two continuous maps such that

(a) $\langle Ty, \theta(y, y)\rangle = 0$ *for all* $y \in K$,

(b) *for each fixed* $y \in K$, *the map*

$$\langle Ty, \theta(-, y)\rangle \; : \; K \to \mathbb{R}$$

is convex.

Then there exists $x_0 \in K$ *such that*

$$\langle Tx_0, \theta(y, x_0)\rangle \geq 0 \qquad \cdots (8)$$

for all $y \in K$, *under each of the following conditions* :

(A) *For at least one* $r > 0$, *there exists* $u \in \{x \in K : \|x\| < r\}$ *such that*

$$\langle Ty, \theta(u, y)\rangle \leq 0$$

for all $y \in \{x \in K : \|x\| = r\}$.

(B) *There exist a nonempty, compact and convex subset* C *of* K *and* $u \in C$ *such that*

$$\langle Ty, \theta(u, y)\rangle < 0$$

for every $y \in K - C$.

1.2.9 Theorem ([3], Theorem 2.2, p.347) *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Let*

$$T : K \to X^*,$$

$$\theta : K \times K \to X$$

and

$$\eta : K \times K \to \mathbb{R}$$

be three maps such that

(a) $\langle Tx, \theta(x, x) \rangle + \eta(x, x) \geq 0$ for each $x \in K$,

(b) *the map*

$$y \mapsto \langle Tx, \theta(y, x) \rangle + \eta(x, y)$$

of K into $\mathbb{R}$ is convex for each $x \in K$,

(c) *the map*

$$x \mapsto \langle Tx, \theta(y, x) \rangle + \eta(x, y)$$

of K into $\mathbb{R}$ is upper semicontinuous for each $y \in K$,

(d) *there exist a nonempty, compact and convex subset C of K and $u \in C$ such that*

$$\langle Tx, \theta(u, x) \rangle + \eta(x, u) < 0$$

for every $x \in K - C$.

Then there exists an $x_0 \in C$ such that

$$\langle Tx_0, \theta(y, x_0) \rangle + \eta(x_0, y) \geq 0 \qquad \cdots (9)$$

for all $y \in K$.

1.3 Ky Fan Theorem

In 1961 Ky Fan proved the generalized version Tychonoff's fixed-point theorem; this results involves the concept of *KKM*-map.

1.3.1 Definition [19] The set valued map

$$F : K \subset X \to 2^X$$

is said to be *a KKM-map if*

$$C_h(\{x_1, x_2, \ldots, x_n\}) \subset \bigcup \{ F(x_i) : i = 1, 2, \cdots, n\}$$

for each finite subset $\{x_1, x_2, \ldots, x_n\}$ of K where $C_h(\{x_1, x_2, \cdots, x_n\})$ denotes the convex hull of $\{x_1, x_2, \cdots, x_n\}$.

We state Ky Fan Theorem :

1.3.2 Theorem. ([19], Theorem 4.3.1, p116) *Let K be an arbitrary nonempty set in a Housdorff topological vector space X. Let the set valued mapping*

$$F : K \to 2^X$$

be a KKM map such that

(a) $F(x)$ *is closed for all* $x \in K$,

(b) $F(x)$ *is compact for at least one* $x \in K$.

Then $\bigcap \{F(x) : x \in K\} \neq \emptyset$.

Several researchers of the field have used the above theorem; in fact this theorem has been instrumental in proving many interesting results in variational inequality and complementarity problems.

1.3.3 Remarks. In 1988 Bardaro and Ceppitelli [1] have further generalized the Ky Fan theorem by considering the concepts of *H*-*KKM*-map, *H*-convexity, *H*-compactness, weakly *H*-convexity in *H*-spaces and thereby presented a *H*-*KKM* version of variational inequality problem in *H*-spaces. In 1990 Tarafdar presented the improved version of variational inequality problem in *H*-spaces. In this thesis further results on variational inequality problem in *H*-spaces are obtained by using the concepts of *H*-convexity and the concepts of T-η-invexity property of a function.

Chapter 2

η-INVEX SET AND T-η-INVEX FUNCTION IN ORDERED TOPOLOGICAL VECTOR SPACES

2.1 Introduction

The notion of invexity was introduced by M.A. Hanson [21] in 1981 as a generalization of the concept of convexity. Now this concept is broadly used in the theory of optimization. Many authors have studied different types of convex and invex functions in vector spaces with different assumptions. Suneja, Aggarwal and Davar [42] have studied K-convex functions in finite dimensional vector spaces. Mititelu [35] has studied the concept of η-invex functions in the differentiable manifolds.

2.2 η-invex set

We recall the definition of η-invex set

2.2.1 Definition. [21] Let X be a topological vector space and $K \subset X$ a nonempty subset of X. K is said to be an *η-invex set* if there exists a vector function

$$\eta : K \times K \to X$$

such that

$$y + t\eta(x, y) \in K$$

for all $x, y \in K$ and for all $t \in (0,1)$.

The example of η-invex set is given in Example 2.2.6, below. We make the following definitions for the vector function η for our need.

2.2.2 Definition. *Condition* C_0 : Let X be a topological vector space and K a nonempty subset of X. A vector function

$$\eta : K \times K \to X$$

is said to *satisfy condition* C_0 if the following hold :

$$\eta(x' + \eta(x, x'), x') + \eta(x', x' + \eta(x, x')) = 0,$$

$$\eta(x' + t\eta(x, x'), x') + t\eta(x, x') = 0$$

for all $x, x' \in K$ and for all $t \in (0,1)$.

2.2.3 Example. (Condition C_0) Let $X = \mathbb{R}$. Let $K = [0,\infty)$ be any nonempty convex subset of X. Define the vector function

$$\eta : K \times K \to X$$

by the rule

$$\eta(x, x') = x' - x$$

for all $x, x' \in K$. We have

$$\eta(x' + \eta(x, x'), x') + \eta(x', x' + \eta(x, x')) = x' - x' - \eta(x, x') + x' + \eta(x, x') - x'$$
$$= 0$$

and for all $t \in (0,1)$,

$$\eta(x' + t\eta(x, x'), x') = x' - x' - t\eta(x, x') = -t\eta(x, x').$$

Thus $\eta(x' + t\eta(x, x'), x') + t\eta(x, x') = 0$ for all $x, x' \in K$.

2.2.4 Theorem. *Let $K \subset X$ be a nonempty pointed subset of the topological vector space X. Let $p: X \to X$ be a linear projective map ($p^2 = p$). Let*

$$\eta : K \times K \to X$$

vector valued mapping defined by the rule

$$\eta(x, x') = p(x') - p(x).$$

Then η satisfies condition C_0.

Proof. For all $x, x' \in K$, we have

$$\eta(x' + \eta(x, x'), x') + \eta(x', x' + \eta(x, x'))$$
$$= p(x') - p(x' + \eta(x, x')) + p(x' + \eta(x, x')) - p(x')$$
$$= 0.$$

Since p is projective we have

$$\begin{aligned}\eta(x'+t\eta(x,x'),x') &= p(x')-p(x'+t\eta(x,x'))\\ &= p(x')-p(x')-p(t\eta(x,x'))\\ &= -tp(\eta(x,x'))\\ &= -tp(p(x')-p(x))\\ &= -t(p^2(x')-p^2(x))\\ &= -t(p(x')-p(x))\\ &= -t\eta(x,x'). \quad \square\end{aligned}$$

We use Ky Fan Theorem (Theorem 1.3.2) in our work to prove the following result.

2.2.5 Theorem. *Let X be a topological vector space* and *K be any nonempty η-invex subset of X. Let (Y,P) be an ordered topological vector space equipped with the closed convex pointed cone P with $intP\neq\emptyset$. Let $L(X,Y)$ be the set of continuous linear functionals from X to Y. Let*

$$T : K\to L(X,Y)$$

and

$$\eta : K\times K\to X$$

be continuous mappings. Assume that

(a) $\langle T(x),\eta(x,x)\rangle\notin -intP$ *for all* $x\in K$,

(b) *for each* $u\in K$, *the set*

$$B(u) = \{x\in K : \langle T(u),\eta(x,u)\rangle\in -intP\}$$

is an η-invex set,

(c) *the map*

$$u \mapsto \langle T(u),\eta(x,u)\rangle$$

of K into $L(X,Y)$ is continuous on the finite dimensional subspaces (or at least hemicontinuous),

(d) *for at least one* $x\in K$, *the set*

$$\{u\in K: \langle T(u),\eta(x,u)\rangle \notin -intP\}$$

is compact.

Then there exists $x_0\in K$ such that

$$\langle T(x_0),\eta(x,x_0)\rangle\notin -intP$$

for all $x\in K$.

Proof. For each $u \in K$, consider the set valued mapping

$$F : K \to 2^X$$

defined by the rule

$$F(x) = \{u \in K : \langle T(u), \eta(x, u) \rangle \notin -\text{int}P\}$$

for all $x \in K$.

We assert that $F(x)$ is closed for each $x \in K$. Let $\{u_n\}$ be a sequence in $F(x)$ such that $u_n \to u$. Since $u_n \in F(x)$ we have

$$\langle T(u_n), \eta(x, u_n) \rangle \notin -\text{int}P$$

for all $x \in K$, i.e.,

$$\langle T(u_n), \eta(x, u_n) \rangle \in (Y - (-\text{int}P))$$

for all $x \in K$. Since T and η are continuous, we have

$$\langle T(u_n), \eta(x, u_n) \rangle \to \langle T(u), \eta(x, u) \rangle.$$

Since $(Y - (-\text{int}P))$ is a closed set,

$$T(u), \eta(x, u \rangle \in (Y - (-\text{int}P))$$

for all $x \in K$, i.e.,

$$\langle T(u), \eta(x, u) \rangle \notin -\text{int}P \text{ for all } x \in K.$$

Thus $u \in F(x)$ and $F(x)$ is closed.

We claim that F is a KKM mapping. If not there exists a finite subset $\{x_1, x_2, \cdots, x_n\}$ of K such that

$$C_h(\{x_1, x_2, \cdots, x_n\}) \not\subset \bigcup \{F(x) : x \in \{x_1, x_2, \cdots, x_n\}\},$$

i.e,

$$C_h(\{x_1, x_2, \cdots, x_n\}) \not\subset F(x)$$

for any $x \in \{x_1, x_2, \cdots, x_n\}$. Let $w \in C_h(\{x_1, x_2, \cdots, x_n\})$ such that

$$w \notin \bigcup \{F(x) : x \in \{x_1, x_2, \cdots, x_n\}\},$$

i.e.,

$$w \notin F(x)$$

for any $x \in \{x_1, x_2, \cdots, x_n\}$. Note that $w \in K$. Thus

$$\langle T(w), \eta(x, w \rangle \in -\text{int}P$$

for all $x \in \{x_1, x_2, \cdots, x_n\}$; this shows that

$$x \in \{x_1, x_2, \cdots, x_n\} \subset C_h(\{x_1, x_2, \cdots, x_n\}) \subset B(w)$$

(since every convex set is a subset of invex set). Hence $w \in B(w)$, i.e.,

$$\langle T(w), \eta(w, w) \rangle \in -\text{int}P,$$

which contradicts (a). Thus F is a KKM mapping. Hence by Theorem 1.3.2,

$$\bigcap \{F(x) : x \in K\} \neq \emptyset,$$

i.e., there exists a $x_0 \in K$ such that

$$\langle T(x_0), \eta(x, x_0)\rangle \notin -\text{int}P$$

for all $x \in K$. □

We illustrate Theorem 2.2.5 by an example.

2.2.6 Example Let $X = \{si : s \in (-\infty,\infty)\}$, $K=\{si : s \in [0,\infty)\}$, $Y=\mathbb{R}$, $P=[0,\infty)$. Let

$$\eta : K \times K \to X$$

be defined by

$$\eta(u, v) = u - v.$$

Let

$$T : K \to L(X,Y)$$

be defined by

$$T(x) = -x$$

for all $x \in K$ and

$$\langle T(u), x\rangle = T(u)\cdot x$$

for all $u \in K$ and $x \in X$.

(a) For all $x \in K$, $\langle T(x), \eta(x, x)\rangle = x(x - x) =_P 0$.

(b) Let $a, b \in B(u)$. We show $b + t\eta(a, b) \in B(u)$. First we show that

$$t\langle T(u), \eta(a, u)\rangle + (1 - t)\langle T(u), \eta(b, u)\rangle - \langle T(u), \eta(b + t\eta(a, b), u)\rangle = 0$$

for all $a, b \in B(u)$ and for all $t \in (0,1)$:

$$t\langle T(u), \eta(a, u)\rangle + (1 - t) \langle T(u), \eta(b, u)\rangle - \langle T(u), \eta(b + t\eta(a, b), u)\rangle$$

$$= t(-u)(a - u) + (1 - t) (-u)(b - u) - (-u)\eta(b + t(a-b), u)$$

$$= (-u)(ta - tu + b - u - tb + tu) - (-u)(b + ta - tb - u)$$

$$= (-u)(ta - tu + b - u - tb + tu - b - ta + tb + u)$$

$$= (-u)0 = 0.$$

Hence for each $u \in K$,

$$t\langle T(u), \eta(a, u)\rangle + (1-t)\langle T(u), \eta(b, u)\rangle - \langle T(u), \eta(b + t\eta(a, b), u)\rangle = 0 \notin -\text{int}P$$

for all $a, b \in B(u)$.

As $a, b \in B(u)$, $\langle T(u), \eta(a, u)\rangle \in -\text{int}P$ and $\langle T(u), \eta(b, u)\rangle \in -\text{int}P$. Now for any $t \in (0,1)$,

$$\langle T(u), \eta(b + t\eta(a, b), u)\rangle = t\langle T(u), \eta(a, u)\rangle + (1 - t)\langle T(w), \eta(b, u)\rangle \in -\text{int}P$$

for all $a, b \in B(u)$. Hence $b + t\eta(a, b) \in B(u)$ for all $a, b \in B(u)$ and for all $t \in (0,1)$.

(c) It is obvious that the map $u \mapsto \langle T(u), \eta(x, u)\rangle$ is continuous.

(d) At $x = u$,

$$\langle T(u), \eta(x, u)\rangle = T(u)(x - u) = 0$$

for all $u \in K$ showing for at least one $x = 0 \in K$, the set

$$\{u \in K: \langle T(u), \eta(x, u)\rangle \notin -\text{int}P\}$$

is compact.

All the conditions of the above Theorem 2.2.5 are satisfied. Hence there exists $x_0 = 0 \in K$ solving the problem

$$\langle T(x_0), \eta(x, x_0)\rangle \notin -\text{int}P$$

for all $x \in K$.

2.3 *T*-η-invex function

We introduce the concept of T-η-invexity in ordered topological vector spaces. Let X be a topological vector space and (Y,P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $\text{int}P \neq \emptyset$. Let $L(X,Y)$ be the set of continuous linear functionals from X to Y and

$$\eta : K \times K \to X$$

be a vector-valued function where K is any subset of X. Let

$$T : K \to L(X,Y)$$

be an operator.

2.3.1 Definition. A map $f : K \to Y$ is said to be T-η-*invex* in K if

$$f(x) - f(x') - \langle T(x'), \eta(x, x')\rangle \geq_P 0$$

(i.e., $f(x) - f(x') - \langle T(x'), \eta(x, x')\rangle \notin -\text{int}P$) for all $x, x' \in K$.

2.3.2 Remark. If $X = \mathbb{R}$, $Y = \mathbb{R}$, $K = (0,1)$, $T = \nabla f$ and $P = [0,\infty)$ and if f is differentiable then Definition 2.3.1 coincides with the definition of a differentiable invex function.

2.3.3 Definition. T is said to be *η-monotone* if there exists a vector function $\eta : K\times K\to X$ such that

$$\langle T(x'), \eta(x, x')\rangle + \langle T(x), \eta(x', x)\rangle \notin \text{int}P$$

for all $x, x' \in K$.

2.3.4 Example. Let $X = \mathbb{R}, K = \mathbb{R}_{+}, Y = \mathbb{R}^2, P = \mathbb{R}^2_{+}$. Let $f : K\to Y$ be defined by

$$f(u) = \begin{bmatrix} u^2 \\ 0 \end{bmatrix}$$

for all $u\in K$ and

$$T : K\to L(X,Y)$$

be defined by

$$T(u) = \begin{bmatrix} -2u \\ 0 \end{bmatrix}$$

for all $u\in K$ where

$$\langle T(u), x\rangle = T(u)\cdot x, \; u\in K, x\in X.$$

Define a vector function

$$\eta : K\times K\to X$$

by

$$\eta(u, v) = u + v$$

for all $u, v\in K$. Now for all $u, v\in K$, we have

$$f(u) - f(v) - \langle T(v), \eta(u, v)\rangle = \begin{bmatrix} u^2 \\ 0 \end{bmatrix} - \begin{bmatrix} v^2 \\ 0 \end{bmatrix} - \begin{bmatrix} -2v \\ 0 \end{bmatrix} [u + v]$$

$$= \begin{bmatrix} u^2 - v^2 + 2v(u+v) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} (u+v)^2 \\ 0 \end{bmatrix} \notin -intP$$

and

$$f(v) - f(u) - \langle T(u), \eta(v, u)\rangle = \begin{bmatrix} v^2 \\ 0 \end{bmatrix} - \begin{bmatrix} u^2 \\ 0 \end{bmatrix} - \begin{bmatrix} -2u \\ 0 \end{bmatrix} [v + u]$$

$$= \begin{bmatrix} v^2 - u^2 + 2u(v+u) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} (v+u)^2 \\ 0 \end{bmatrix} \notin -intP$$

showing f is T-η-invex in K. Next we have

$$\begin{aligned} \langle T(v), \eta(u, v)\rangle + \langle T(u), \eta(v, u)\rangle &= \begin{bmatrix} -2v \\ 0 \end{bmatrix} [u+v] + \begin{bmatrix} -2u \\ 0 \end{bmatrix} [v+u] \\ &= \begin{bmatrix} -2(u+v) \\ 0 \end{bmatrix} [u+v] \\ &= -\begin{bmatrix} 2(u+v) \\ 0 \end{bmatrix} [u+v] \\ &= -\begin{bmatrix} 2(u+v)^2 \\ 0 \end{bmatrix} \notin \text{int}P \end{aligned}$$

for all $u, v \in K$, showing that T is η-monotone in K.

2.3.5 Example. Let $X = \mathbb{R}$, $K = \mathbb{R}_+$, $Y = \mathbb{R}^2$, $P = \mathbb{R}^2_+$. Let

$$T : K \to L(X,Y)$$

be defined by

$$T(u) = \begin{bmatrix} -2u \\ -u \end{bmatrix}$$

for all $u \in K$ where

$$\langle T(u), x\rangle = T(u)\cdot x, \quad u \in K, x \in X.$$

Define a vector function

$$\eta : K \times K \to X$$

by

$$\eta(u, v) = u + v$$

for all $u, v \in K$. Now for all $u, v \in K$, we have

$$\begin{aligned} \langle T(v), \eta(u, v)\rangle + \langle T(u), \eta(v, u)\rangle &= \begin{bmatrix} -2v \\ -v \end{bmatrix} (u+v) + \begin{bmatrix} -2u \\ -u \end{bmatrix} (v+u) \\ &= \begin{bmatrix} -2 \\ -1 \end{bmatrix} (u+v)^2 \notin \text{int}P \end{aligned}$$

for all $u, v \in K$, showing that T is η- monotone in K.

2.3.6 Proposition. *Let X be a topological vector space and (Y,P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $\mathrm{int}P\neq\emptyset$. Let $L(X,Y)$ be the set of continuous linear functionals from X to Y and*

$$\eta : K\times K\to X$$

be a vector-valued function where K is any subset of X. Let

$$T : K \to L(X,Y)$$

be an operator. Let the function $f : K\to Y$ be T-η-invex in K. Then T is η-monotone.

Proof. Let f be T-η-invex in K. Then for all $x, x' \in K$, we have

$$f(x) - f(x') - \langle T(x'), \eta(x, x')\rangle \notin -\mathrm{int}P.$$

Interchanging x and x', we have

$$f(x') - f(x) - \langle T(x), \eta(x', x)\rangle \notin -\mathrm{int}P.$$

Adding the above we get

$$-\langle T(x'), \eta(x, x')\rangle - \langle T(x), \eta(x', x)\rangle \notin -\mathrm{int}P,$$

i.e.,

$$\langle T(x'), \eta(x, x')\rangle + \langle T(x), \eta(x', x)\rangle \notin \mathrm{int}P$$

for all $x, x' \in K$, showing T is η-monotone. □

The converse of Proposition 2.3.6 is not true, as shown in the following example.

2.3.7 Example. Let $X = Y = \mathbb{R}$ and $K = [-\pi/2, \pi/2]$, $P = [0,\infty]$, $T = \nabla f$ (derivative of f). Let $f : K\to\mathbb{R}$ be defined by $f(x) = \sin x$ for all $x\in K$ and

$$\eta : K \times K\to X$$

be defined by

$$\eta(x, x') = \cos x - \cos x'$$

for all $x, x' \in K$. We have

$$\langle T(x'), \eta(x, x')\rangle + \langle T(x), \eta(x', x)\rangle$$

$$= \nabla f(x')\,\eta(x, x') + \nabla f(x)\,\eta(x', x)$$

$$= \cos x' (\cos x - \cos x') + \cos x(\cos x' - \cos x)$$

$$= -(\cos x - \cos x')^2 \leq 0$$

for all $x, x' \in K$, showing that T is η-monotone. But at $x = -\pi/3$ and $x' = \pi/6$, we have

$$f(x) - f(x') - \langle T(x'), \eta(x, x') \rangle < 0,$$

showing f is not T-η-invex. □

Chapter 3

A COMPLEMENTARITY PROBLEM

3.1 Introduction

In this chapter we study a vector complementarity problem in topological vector spaces. One of the important results of variational inequality theory is Minty's Lemma, which has interesting applications in the study of obstacles problems, confined plasmas, filtration phenomena, free-boundary problems, plasticity and viscoplasticity phenomena, elasticity problems and stochastic optimal control problems. We extend Minty's lemma to invex sets and present an application of the same.

We use the notations of the following result.

3.1.1 Lemma. ([7], Lemma 2.1) *Let* (V, P) *be an ordered topological vector space with a closed, pointed and convex cone* P *with* $intP \neq \emptyset$. *Then, for all* $y, z \in V$, *we have*

(i) $y - z \in intP$ *and* $y \notin intP$ *imply* $z \notin intP$;

(ii) $y - z \in P$ *and* $y \notin intP$ *imply* $z \notin intP$;

(iii) $y - z \in -intP$ *and* $y \notin -intP$ *imply* $z \notin -intP$;

(iv) $y - z \in -P$ *and* $y \notin -intP$ *imply* $z \notin -intP$.

3.1.2 Remark. For simplicity, we use the following terminologies:

(a) $y \notin -\text{int}P$ if and only if $y \geq_P 0$;

(b) $y \in \text{int}P$ if and only if $y >_P 0$;

(c) $y \notin \text{int}P$ if and only if $y \leq_P 0$;

(d) $y \in -\text{int}P$ if and only if $y <_P 0$;

(e) $y - z \notin -\text{int}P$ if and only if $y - z \geq_P 0$ (i.e., $y \geq_P z$);

(f) $y - z \notin \text{int}P$ if and only if $y - z \leq_P 0$ (i.e., $y \leq_P z$);

(g) $y - z \notin (-\text{int}P \cup \text{int}P)$ if and only if $y - z =_P 0$, (i.e., $y =_P z$).

We also use the following terminologies as and when required:

(A) $y-z\notin -P$ and $z\notin -\text{int}P$ imply $y\notin -\text{int}P$;

(B) $y-z\notin -\text{int}P$ and $z\notin -\text{int}P$ imply $y\notin -\text{int}P$;

(C) $y-z\notin -P$ and $y\in -\text{int}P$ imply $z\in -\text{int}P$;

(D) $y-z\notin -\text{int}P$ and $y\in -\text{int}P$ imply $z\in -\text{int}P$;

(E) $y-z\in -\text{int}P$ and $z\in -\text{int}P$ imply $y\in -\text{int}P$;

(F) $y\notin -\text{int}P$ if and only if $-y\notin \text{int}P$;

(G) $y\notin -\text{int}P$ and $z\notin -\text{int}P$ imply $y+z\notin -\text{int}P$.

3.2 Vector complementarity problem

We state the generalized vector complementarity problem in a topological vector space as follows :

3.2.1 Definition. Let X be a topological vector space and (Y,P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $\text{int}P\neq\emptyset$. Let K be any subset of X, $L(X,Y)$ be the set of all continuous linear functionals from X to Y and

$$\eta : K\times K\to X$$

be a vector-valued function. Let

$$T : K\to L(X,Y)$$

be an arbitrary map.

The *Generalized Vector Variational Inequality Problem* (*GVVI*) and the *Generalized Vector Complementarity Problem* (*GVCP*) are defined as follows :

GVVI : Find $x_0\in K$ such that

$$\langle T(x_0), \eta(x, x_0)\rangle \notin -\text{int}P$$

for all $x\in K$,

GVCP : Find $x_0\in K$ such that

$$\langle T(x_0), \eta(x, x_0)\rangle \in Y-(-\text{int }P\cup \text{int}P)$$

for all $x\in K$.

We prove the following result concerning GVCP.

3.2.2 Theorem. *Let K be a nonempty compact cone in a topological vector space X and (Y,P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP \neq \varnothing$. Let $L(X,Y)$ be the set of all continuous linear functionals from X to Y and*

$$\eta : K \times K \to X$$

be a continuous vector-valued function. Let

$$T : K \to L(X,Y)$$

be an arbitrary continuous map and K be η-invex. Let the following conditions hold :

(a) $\langle T(x), \eta(x, x)\rangle =_P 0$ *for all $x \in K$,*

(b) *for each $u \in K$, the set*

$$B(u) = \{x \in K : \langle T(u), \eta(x, u)\rangle \in -intP\}$$

is an η-invex set,

(c) *η satisfies condition C_0.*

Then there exists $x_0 \in K$ such that

$$\langle T(x_0), \eta(x, x_0)\rangle \in Y - (-intP \cup intP)$$

for all $x \in K$.

Proof. By Theorem 2.2.5, there exists $x_0 \in K$ such that

$$\langle T(x_0), \eta(x, x_0)\rangle \notin -\text{int } P$$

i.e.,

$$\langle T(x_0), \eta(x, x_0)\rangle \geq_P 0$$

for all $x \in K$. Since K is η-invex cone,

$$x_0 + t\eta(x, x_0) \in K$$

for all $x \in K$. Replacing x by $x_0 + t\eta(x, x_0)$, we get

$$\langle T(x_0), \eta(x_0 + t\eta(x, x_0), x_0)\rangle \geq_P 0$$

for all $x \in K$. Thus

$$\begin{aligned} 0 \leq_P \langle T(x_0), \eta(x_0 + t\eta(x, x_0), x_0)\rangle &= \langle T(x_0), -t\eta(x, x_0)\rangle \text{ (by (c))} \\ &= -t\langle T(x_0), \eta(x, x_0)\rangle \end{aligned}$$

and hence

$$-t\langle T(x_0), \eta(x, x_0)\rangle \notin -\text{int}P$$

for all $x \in K$. Since $t > 0$ we have

$$\langle T(x_0), \eta(x, x_0)\rangle \notin \text{int}P.$$

Hence

$$\langle T(x_0), \eta(x, x_0)\rangle \notin -\mathrm{int}P \cup \mathrm{int}P$$

for all $x \in K$, showing

$$\langle T(x_0), \eta(x, x_0)\rangle \in Y - (-\mathrm{int}\, P \cup \mathrm{int}P)$$

for all $x \in K$. □

3.2.3 Theorem. *Let K be a nonempty compact η-invex cone in a topological vector space X and (Y,P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP \neq \emptyset$. Let $L(X,Y)$ be the set of all continuous linear functionals from X to Y and*

$$\eta : K \times K \to X$$

be a continuous vector-valued function. Let

$$T : K \to L(X,Y)$$

be an arbitrary continuous map. Let the following conditions hold :

(a) $\langle T(x), \eta(x, x)\rangle =_P 0$ *for all $x \in K$,*

(b) *for each $u \in K$, there exists an element $x^* \in K$ such that*

$$\langle T(u), \eta(x^*, u)\rangle \in intP$$

(c) *η satisfies condition C_0.*

Then there exists $x_0 \in K$ such that

$$\langle T(x_0), \eta(x, x_0)\rangle \in Y - (-intP \cup intP)$$

for all $x \in K$.

Proof. By assumption (b), for each $u \in K$, there exists an element $x^* \in K$ such that

$$\langle T(u), \eta(x^*, u)\rangle \in \mathrm{int}P$$

i.e.,

$$\langle T(u), \eta(x^*, u)\rangle >_P 0$$

for each $u \in K$. For each $u \in K$, define a map $B: K \to 2^K$ by the rule

$$B(u) = \{x \in K: \langle T(u), \eta(x, u)\rangle \in -\mathrm{int}P\}$$

We show that $B(u)$ is an η-invex set. Let $a,b \in B(u) \subset K$ be arbitrary. Then $\langle T(u), \eta(a, u)\rangle <_P 0$ and $\langle T(u), \eta(b, u)\rangle <_P 0$. Since K is η-invex, considering

$$a = u + s\eta(x^*, u), \quad 0 < s < 1/2$$

and

$$b = u + t\eta(x^*, u), \quad 0 < t < 1/2,$$

we have

$$
\begin{aligned}
\eta(a, b) &= \eta(u + s\eta(x^*, u), u + t\eta(x^*, u)) \\
&= \eta(u + s\eta(x^*, u), u + s\eta(x^*, u) + (t - s)\,\eta(x^*, u)) \\
&= -\eta(u + s\eta(x^*, u) + (t - s)\eta(x^*, u), u + s\eta(x^*, u)) \\
&\quad \text{(by condition } C_0\text{)} \\
&= -(-(t - s)\eta(x^*, u)) \quad \text{(by condition } C_0\text{)} \\
&= (t - s)\eta(x^*, u)
\end{aligned}
$$

and

$$
\begin{aligned}
b + t\eta(a, b) &= b + t(t - s)\eta(x^*, u) \\
&= u + t\eta(x^*, u) + t(t - s)\eta(x^*, u) \\
&= u + t(1 + t - s)\eta(x^*, u) \in K
\end{aligned}
$$

(since $0 < t(1 + t - s) < 1$ for all $0 < t, s < 1/2$). Let $t(1 + t - s) = \lambda$. Then

$$
\begin{aligned}
\langle T(u), \eta(b + t\eta(a, b), u)\rangle &= \langle T(u), \eta(u + \lambda\eta(x^*, u), u)\rangle \\
&= -\lambda\langle T(u), \eta(x^*, u)\rangle <_P 0
\end{aligned}
$$

giving $b + t\eta(a, b) \in B(u)$, i.e., for each $u \in K$, $B(u)$ is η-invex in K. Hence all the conditions of Theorem 2.2.5 are satisfied. Thus there exists $x_0 \in K$ such that

$$\langle T(x_0), \eta(x, x_0)\rangle \geq_P 0$$

for all $x \in K$. Since K is η-invex cone, replacing x by $x_0 + t\eta(x, x_0)$, we get

$$\langle T(x_0), \eta(x_0 + t\eta(x, x_0), x_0)\rangle \geq_P 0$$

for all $x \in K$. Thus

$$
\begin{aligned}
0 &\leq_P \langle T(x_0), \eta(x_0 + t\eta(x, x_0), x_0)\rangle \\
&= \langle T(x_0), -t\eta(x, x_0)\rangle \quad \text{(by (c))} \\
&= -t\langle T(x_0), \eta(x, x_0)\rangle
\end{aligned}
$$

and hence

$$-t\langle T(x_0), \eta(x, x_0)\rangle \notin -\text{int}P$$

for all $x \in K$. Since $t > 0$ we have

$$\langle T(x_0), \eta(x, x_0)\rangle \notin \text{int}P.$$

Hence

$$\langle T(x_0), \eta(x, x_0)\rangle \notin -\text{int}P \cup \text{int}P$$

for all $x \in K$ showing

$$\langle T(x_0), \eta(x, x_0)\rangle \in Y - (-\text{int}\, P \cup \text{int}P)$$

for all $x \in K$. □

3.3 Extension of Minty's Lemma to invex sets

We state Minty's lemma (Theorem, p.9; [8]).

3.3.1 Theorem *Let K be a nonempty, closed and convex subset of a reflexive real Banach space X and let X^* be the dual of X. Let*

$$T : K \to X^*$$

be a monotone operator which is continuous on finite dimensional subspaces (or at least hemicontinuous). Then the following are equivalent :

(a) $x_0 \in K, \ \langle Tx_0 , y - x_0 \rangle \geq 0$ *for all* $y \in K.$

(b) $x_0 \in K, \ \langle Ty , y - x_0 \rangle \geq 0$ *for all* $y \in K.$

In this section, following the traditional proof of the above result, we obtain a certain generalization of Minty's lemma.

3.3.2 Theorem *Let K be a nonempty compact η-invex cone in a topological vector space X and (Y,P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP \neq \varnothing$. Let $L(X,Y)$ be the set of all continuous linear functionals from X to Y and*

$$\eta : K \times K \to X$$

be a continuous vector-valued function. Let

$$T : K \to L(X,Y)$$

be an arbitrary continuous map. Let the following conditions hold :

(a) $\langle T(x), \eta(x, x)\rangle =_P 0$ *for all* $x \in K,$

(b) *the map*

$$\langle T(x), \eta(\text{-}, x)\rangle : K \to Y$$

satisfies

$$t\langle T(x), \eta(u, x)\rangle + (1 - t) \langle T(x), \eta(v, x)\rangle - \langle T(x), \eta(v + t\eta(u, v), x)\rangle) \notin -intP$$

for each fixed $x \in K$ *and for all* $t \in (0,1),\ u, v \in K.$

(c) T is *η-monotone in K*

Then the following are equivalent :

(A) $x_0 \in K$ *such that* $\langle T(x_0), \eta(x, x_0)\rangle \notin -intP$ *for all* $x \in K$

(B) $x_0 \in K$ *such that* $\langle T(x), \eta(x_0, x)\rangle \notin intP$ *for all* $x \in K$

Proof. Let

$$\langle T(x_0), \eta(x, x_0)\rangle \notin -\text{int}P$$

i.e.,

$$-\langle T(x_0), \eta(x, x_0)\rangle \notin \text{int}P$$

for all $x \in K$. Since T is η-monotone in K, we have

$$\langle T(x'), \eta(x, x')\rangle + \langle T(x), \eta(x', x)\rangle \notin \text{int}P$$

for all $x, x' \in K$. At $x' = x_0$, we get

$$\langle T(x_0), \eta(x, x_0)\rangle + \langle T(x), \eta(x_0, x)\rangle \notin \text{int}P$$

for all $x \in K$. The above gives

$$\langle T(x), \eta(x_0, x)\rangle \notin \text{int}P$$

for all $x \in K$. Conversely, let

$$\langle T(x), \eta(x_0, x)\rangle \notin \text{int}P$$

for all $x \in K$. Since K is η-invex, for any $t \in (0,1)$, $x_0 + t\eta(x, x_0) \in K$ for all $x \in K$. Taking $x_t = x_0 + t\eta(x, x_0)$ and using (b), we obtain

$$t\langle T(x_t), \eta(x, x_t)\rangle + (1 - t)\langle T(x_t), \eta(x_0, x_t)\rangle - \langle T(x_t), x_0 + t\eta(x, x_0), x_t)\rangle) \notin -\text{int}P$$

i.e.,

$$\langle T(x_t), x_0 + t\eta(x, x_0), x_t)\rangle) - t\langle T(x_t), \eta(x, x_t)\rangle - (1 - t)\langle T(x_t), \eta(x_0, x_t)\rangle \notin \text{int}P$$

for all $x \in K$. Thus

$$\langle T(x_t), \eta(x_t, x_t)\rangle) - t\langle T(x_t), \eta(x, x_t)\rangle - (1 - t)\langle T(x_t), \eta(x_0, x_t)\rangle \notin \text{int}P.$$

By (a) and assumption (B), we have

$$-t\langle T(x_t), \eta(x, x_t)\rangle \notin \text{int}P$$

i.e.,

$$\langle T(x_t), \eta(x, x_t)\rangle \notin -\text{int}P$$

for all $x \in K$. Since T and η are continuous, taking $t \to 0$, we have

$$\langle T(x_0), \eta(x, x_0)\rangle \notin -\text{int}P$$

for all $x \in K$. □

The following is an application of Theorem 3.3.2

3.3.3 Theorem *Let K be a nonempty compact η-invex cone in a topological vector space X and (Y,P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP \neq \emptyset$. Let $L(X,Y)$ be the set of all continuous linear functionals from X to Y and*

$$\eta : K \times K \to X$$

be a continuous vector-valued function. Let

$$T : K \to L(X,Y)$$

be an arbitrary continuous map. Let the following conditions hold :

(a) $\langle T(x), \eta(x, x)\rangle \notin -intP$ *for all* $x \in K$,

(b) *the map* $\langle T(x), \eta(-, x)\rangle : K \to Y$ *satisfies*

$$t\langle T(x), \eta(u, x)\rangle + (1-t)\langle T(x), \eta(v, x)\rangle - \langle T(x), \eta(v + t\eta(u, v), x)\rangle) \notin -intP$$

for each fixed $x \in K$ *and for all* $t \in (0,1)$, $u, v \in K$.

(c) *the map*

$$u \mapsto \langle T(u), \eta(x, u)\rangle$$

of K into $L(X,Y)$ is continuous on the finite dimensional subspaces (or at least hemicontinuous),

(d) *for at least one* $x \in K$, *the set*

$$\{u \in K: \langle T(u), \eta(x, u)\rangle \notin -intP\}$$

is compact,

(e) T is *η-monotone in K,*

(f) *η satisfies condition C_0* in K.

Then there exists $x_0 \in K$ such that

$$\langle T(x_0), \eta(x, x_0)\rangle \in Y - (-int\, P \bigcup intP)$$

for all $x \in K$.

Proof. By condition (b), the set

$$\{x \in K: \langle T(u), \eta(x, u)\rangle \in -\text{int}P\}$$

is η-invex for each fixed $u \in K$. By Theorem 2.2.5, there exists $x_0 \in K$ such that

$$\langle T(x_0), \eta(x, x_0)\rangle \notin -\text{int}P$$

for all $x \in K$. Since K is an η-invex cone, $x_0 + t\eta(x, x_0) \in K$ for any $t \in (0,1)$. By Theorem 3.3.2, we get

$$\langle T(x), \eta(x_0, x)\rangle \notin \text{int}P$$

for all $x \in K$. Replacing x by $x_0 + t\eta(x, x_0)$ in the above expression, we get

$$\langle T(x_0+t\eta(x,x_0)),\eta(x_0,x_0+t\eta(x,x_0))\rangle\notin \text{int}P$$

for all $x\in K$. By condition (g), we get

$$\langle T(x_0+t\eta(x,x_0)),t\eta(x,x_0)\rangle\notin \text{int}P$$

i.e.,

$$t\langle T(x_0+t\eta(x,x_0)),\eta(x,x_0)\rangle\notin \text{int}P$$

for all $x\in K$. Since $t>0$, we get

$$\langle T(x_0+t\eta(x,x_0)),\eta(x,x_0)\rangle\notin \text{int}P$$

for all $x\in K$. Since T and η are continuous, taking limit $t\to 0$ in the above expression, we get

$$\langle T(x_0),\eta(x,x_0)\rangle\notin \text{int}P$$

for all $x\in K$. Thus we get

$$\langle T(x_0),\eta(x,x_0)\rangle\notin -\text{int}P\bigcup \text{int}P$$

i.e.,

$$\langle T(x_0),\eta(x,x_0)\rangle\in Y-(-\text{int}\,P\bigcup \text{int}P)$$

for all $x\in K$. □

Chapter 4

VARIATIONAL INEQUALITY PROBLEMS IN *H*-SPACES

4.1 Introduction

In the recent past H-spaces have become an interesting area of research domain for studying variational-type inequality [1,43] because most of the pivotal concepts such as convex sets, weakly convex sets, *KKM*-maps in Banach spaces are respectively replaced by H-convex sets, H-weakly convex sets, H-*KKM*-maps in H-spaces. In this chapter we establish an inequality in H-spaces and obtain the traditional variational and variational-type inequalities as particular cases of the newly obtained inequality. We also discuss the uniqueness of the solutions of the inequality with examples.

Several generalizations of the celebrated Ky Fan minimax inequality [20] have already appeared.This study requires the use of *KKM*-Theorem. In [1] Bardaro and Ceppitelli have explained the necessity of generalizing the reformulation of the *KKM*-Theorem for generalizing minimax inequality for functions taking values in ordered vector spaces.

4.2 *H*-spaces

In this section we recall the definition of H-spaces and some known results there.

4.2.1 Definition. [1] Let X be a topological space and $\{\Gamma_A\}$ be a given family of nonempty contractible subsets of X, indexed by finite subsets of X. A pair $(X,\{\Gamma_A\})$ is said to be an *H-space* if $A\subset B$ implies $\Gamma_A \subset \Gamma_B$.

Let $(X,\{\Gamma_A\})$ be an H-space. A subset $D\subset X$ is said to be *H-convex* if $\Gamma_A \subset D$ for every finite subset $A\subset D$.

A subset $D \subset X$ is said to be *weakly H-convex* if $\Gamma_A \bigcap D$ is nonempty and contractible for every finite subset $A \subset D$. This is equivalent to saying the pair $(D, \{\Gamma_A \bigcap D\})$ is an H-space.

A subset $K \subset X$ is said to be *H-compact* if there exists a compact and weakly H-convex set $D \subset X$ such that $K \bigcup A \subset D$ for every finite subset $A \subset X$.

In a given H-space a multifunction $F : X \to 2^X$ is said to be *H-KKM* if $\Gamma_A \subset \bigcup \{F(x): x \in A\}$ for every finite subset $A \subset X$.

The following result has fundamental importance in H-spaces.

4.2.2 Theorem. ([1], Theorem 1, p.486) *Let* $(X, \{\Gamma_A\})$ *be an H-space and* F: $X \to 2^X$ *an H-KKM multifunction such that* :

(a) *For each* $x \in X$, $F(x)$ *is compactly closed, that is,* $B \bigcap F(x)$ *is closed in* B, *for every compact* $B \subset X$.

(b) *There is a compact set* $L \subset X$ *and an H-compact set* $K \subset X$, *such that for each weakly H-convex set* D *with* $K \subset D \subset X$, *we have*

$$\bigcap \{F(x) \bigcap D: x \in D\} \subset L.$$

Then $\bigcap \{F(x) : x \in X\} \neq \varnothing$.

4.3 Variational-type inequality in *H*-spaces.

In this section we present an application of Theorem 4.2.2. In fact we establish an inequality associated with the variational ineqaulity or variational-type inequality in H-spaces.

Let X be a topological vector space, (Y,P) be an ordered topological vector space equipped with closed convex pointed cone with $\text{int}P \neq \varnothing$ and $L(X,Y)$ be the set of continuous linear functionals from X to Y. Let the value of $f \in L(X,Y)$ at $x \in X$ be denoted by $\langle f, x \rangle$. Let K be a convex set in X, with $0 \in K$ and T : $K \to L(X,Y)$ be any map. The *nonlinear variational inequality problem* is to find

$$x_0 \in K \quad \text{such that} \quad \langle T(x_0), v - x_0 \rangle \notin -\text{int}P$$

for all $v \in K$.

4.3.1 Theorem. *Let* $(X,\{\Gamma_A\})$ *be an* H*-space. Assume that* X *is Hausdorff. Let* N *be a subset of* $X\times X$ *having the following properties* :

(a) *For each* $x\in X$, $(x,x)\in N$.

(b) *For each fixed* $y\in X$, *the set* $N(y) = \{x\in X : (x,y)\in N\}$ *is closed in* X.

(c) *For each* $x\in X$, *the set* $M(x) = \{y\in X : (x,y)\notin N\}$ *is* H*-convex.*

(d) *There exists a compact set* $L\subset X$ *and an* H*-compact set* $W\subset X$ *such that for each weakly* H*-convex set* D *with* $W\subset D\subset X$, *we have*

$$\bigcap\{(\{x\in X : (x,y)\in N\}\cap D) : y\in D\}\subset L.$$

Then there exists $x_0\in X$ *such that* $\{x_0\}\times X\subset N$.

Proof. Define a set valued map

$$F : X\to 2^X$$

by the rule

$$F(y) = \{x\in X : (x,y)\in N\}.$$

By (a), $F(y)\neq\varnothing$ for each $y\in X$. Since X is Hausdorff, by (b), $F(y)$ is compactly closed for each $y\in X$. We assert that F is an H-KKM map. Suppose to the contrary that F is not an H-KKM map.Then there exists a finite set $A\subset X$ such that

$$\Gamma_A\not\subset\bigcup\{F(y) : y\in A\}.$$

Thus there exists some $u\in\Gamma_A$ such that $(u,y)\notin N$ for all $y\in A$. Let

$$M(u) = \{y\in X : (u,y)\notin N\}.$$

By (c), $M(u)$ is H-convex. We observe that $A\subset M(u)$. By the H-convexity of $M(u)$, $\Gamma_A\subset M(u)$. Thus $u\in M(u)$, i.e., $(u,u)\notin N$, which is a contradiction to (a). Hence F is an H-KKM map.

By (d) there exists a compact set $L\subset X$ and an H-compact set $W\subset X$ such that for each weakly H-convex set D with $W\subset D\subset X$, we have

$$\bigcap\{(\{x\in X : (x,y)\in N\}\cap D) : y\in D\}\subset L,$$

i.e.,

$$\bigcap\{F(y)\cap D : y\in D\}\subset L.$$

Thus by Theorem 4.2.2,

$$\bigcap \{F(y) : y\in X\}\neq\emptyset.$$

Hence there exists $x_0 \in X$ such that $\{x_0\} \times X\subset N$. □

The following result is a slightly different version of Theorem 4.3.1.

4.3.2 Theorem. *Let* $(X,\{\Gamma_A\})$ *be an H-space and* (Y,P) *be an ordered topological vector space equipped with closed convex pointed cone with* $intP \neq \emptyset$. *Let* K *be a convex set in* X, *with* $0\in K$. *Assume that* X *is Hausdorff. Let*

$$f : K \times K\to Y$$

be a continuous map having the following properties :

(a) *For each* $x\in X$, $f(x, x) \notin -intP$.

(b) *For each fixed* $v\in K$, *the set* $\{x\in K : f(x, v) \notin -intP\}$ is *closed in* X.

(c) *For each* $x\in K$, *the set* $\{v \in K : f(x, v) \in -intP\}$ *is H-convex.*

(d) *There exists a compact set* $L \subset X$ *and an H-compact set* $W \subset X$, *such that for each weakly H-convex set* D *with* $W\subset D\subset X$, *we have*

$$\bigcap \{\{x\in K : f(x, v) \notin -intP\}\bigcap D : v\in D\} \subset L.$$

Then there exists $x_0\in K$ *such that* $f(x_0, v) \notin -intP$ *for all* $v\in K$.

Proof. Let

$$N = \{(x, v) : f(x, v) \notin -\text{int}P\}\subset K \times K.$$

By (a), N is nonempty since $(x, x) \in N$ for each $x\in K$. For each $v\in K$ consider the set

$$N(v) = \{x\in X : (x, v) \in N\} = \{x\in X : f(x, v) \notin -\text{int}P\}.$$

By (b), $N(v)$ is closed for each $v\in K$. By (c), for each $x\in K$, the set

$$M(x) = \{v\in K : (x, v) \notin N\} = \{v\in K : f(x, v)\in -\text{int}P\}$$

is H-convex . By (d), there exists a compact set $L \subset X$ and an H-compact set $W \subset X$ such that for each weakly H-convex set D with $W\subset D\subset X$, we have

$$\bigcap \{\{x\in K : f(x, v) \notin -\text{int}P\}\bigcap D : v\in D\}\subset L.$$

Thus all the conditions of Theorem 4.3.1, are satisfied and hence there exists $x_0 \in K$ such that $\{x_0\} \times K \subset N$, i.e., $(x_0, v) \in N$ for all $v \in K$. This means there exists $x_0 \in K$ such that $f(x_0, v) \notin -\text{int}P$ for all $v \in K$. □

4.3.3 Remarks. In Theorem 4.3.2 we take X to be a Hausdorff topological real vector space with dual X^*, $Y = \mathbb{R}$ and $P = [0,\infty)$. Clearly X is an H-space. Let K be a nonempty convex subset of X. Let

$$T : K \to X^*,$$

$$\eta : K \times K \to X,$$

$$\theta : K \times K \to \mathbb{R},$$

$$g : K \to X$$

be continuous functions satisfying some appropriate conditions as and when required. We consider the following cases :

Case I. If we define

$$f : K \times K \to \mathbb{R}$$

by the rule

$$f(x, y) = \langle Tx, y - x\rangle$$

then by Theorem 4.3.2, there exists $x_0 \in K$ such that

$$\langle Tx_0, y - x_0\rangle \geq 0$$

for all $y \in K$, which is the variational inequality as given in (1), (2), (3), (4) and (5) of Theorems 1.2.1, 1.2.2, 1.2.3, 1.2.4 and 1.2.5 respectively.

Case II. In Theorem 4.3.2 if we define

$$f : K \times K \to \mathbb{R}$$

by the rule

$$f(x, y) = \langle Tx, \eta(y, x)\rangle$$

then there exists $x_0 \in K$ such that

$$\langle Tx_0,\ \eta(y, x_0)\rangle \geq 0$$

for all $y \in K$, which is the variational-type inequality as given in (7) and (8) of Theorems 1.2.7 and 1.2.8 respectively.

Case III. In Theorem 4.3.2 if we define

$$f : K \times K \to \mathbb{R}$$

by the rule

$$f(x, y) = \langle Tx, y - g(x)\rangle$$

then there exists $x_0 \in K$ such that

$$\langle Tx_0, y - g(x_0)\rangle \geq 0$$

for all $y \in K$, which is the variational inequality as given in (6) of Theorem 1.2.6.

Case IV. In Theorem 4.3.2 if we define

$$f : K \times K \to \mathbb{R}$$

by the rule

$$f(x, y) = \langle Tx,\ \eta(y, x)\rangle + \theta(x, y),$$

then there exists $x_0 \in K$ such that

$$\langle Tx_0,\ \eta(y, x_0)\rangle + \theta(x_0, y) \geq 0$$

for all $y \in K$, which is the variational-type inequality as given in (9) of Theorem 1.2.9.

4.4 Uniqueness of solution

The following result characterizes the uniqueness of the solution of the inequality $f(x_0, v) \notin -\text{int}P$ obtained in Theorem 4.3.2.

4.4.1 Theorem. *Let* $(X, \{\Gamma_A\})$ *be an H-space and* (Y,P) *be an ordered topological vector space equipped with closed convex pointed cone with* $intP \neq \emptyset$. *Let* K *be a convex set in* X, *with* $0 \in K$. *Assume that* X *is Hausdorff. Let*

$$f : K \times K \to Y$$

be a continuous map such that

(a) $f(x, v) + f(v, x) \notin intP$ *for all* $x, v \in K$,

(b) $f(x, v) + f(v, x) =_P 0$ *implies* $x = v$.

Then if the problem $\begin{cases} \text{find} \quad x_0 \quad \text{such that} \\ f(x_0, v) \quad \text{for all} \quad v \in K \end{cases}$

is solvable, then it has a unique solution.

Proof. Let $x_1, x_2 \in K$, be such that,

$$f(x_1, v) \notin -\text{int}P \text{ and } f(x_2, v) \notin -\text{int}P$$

for all $v \in K$; putting $v = x_2$ in the former inequality and $v = x_1$ in the later inequality we see that

$$f(x_1, x_2) \notin -\text{int}P \text{ and } f(x_2, x_1) \notin -\text{int}P$$

and on adding we get

$$f(x_1, x_2) + f(x_2, x_1) \notin -\text{int}P.$$

This combined with inequality (a) gives,

$$f(x_1, x_2) + f(x_2, x_1) =_P 0.$$

Hence by (b), we have $x_1 = x_2$. □

The following examples illustrate Theorem 4.4.1. Example 4.4.2, given below, shows that fulfillment of conditions (a) and (b) does not guarantee the existence of the solution of the problem, stated in Theorem 4.4.1.

4.4.2 Example. Let $X = \mathbb{R}$ and define

$$f : X \times X \to \mathbb{R}$$

by

$$f(x, v) = -e^{-x} |x - v|.$$

Clearly

$$f(x, v) + f(v, x) = -(e^{-x} + e^{-v})|x - v| \leq 0.$$

Furthermore

$$f(x, v) + f(v, x) = 0$$

implies that $x = v$. It is clear that there is no $x_0 \in K$ satisfying

$$f(x_0, v) = -e^{-x_0}|x_0 - v| \geq 0$$

for all $v \in X$.

In the following example (Example 4.4.3) the function $f: X \times X \to \mathbb{R}$ satisfies conditions (a) and (b) of Theorem 4.4.1 and at the same time the problem stated in Theorem 4.4.1 has a unique solution.

4.4.3 Example. Let $X = [0, \infty)$ and define

$$f : X \times X \to \mathbb{R}$$

by

$$f(x, v) = -x^2|x - v|.$$

Clearly

$$f(x, v) + f(v, x) = -(x^2 + v^2)|x - v| \leq 0.$$

Furthermore

$$f(x, v) + f(v, x) = 0$$

implies that either

$$x^2 + v^2 = 0 \quad \text{or} \quad |x - v| = 0;$$

since x^2 and v^2 are nonnegative, when $x^2 + v^2 = 0$ we have $x = 0$ and $v = 0$ and when $|x - v| = 0$ we have certainly $x = v$. Thus conditions (a) and (b) of Theorem 4.4.1 hold. In this example we have a unique solution $x_0 = 0$ to the problem of Theorem 4.4.1, for $f(x_0, v) \geq 0$ for all $v \in X$ implies

$$-x_0^2|x_0 - v| \geq 0$$

for all $v \in X$; since $|x_0 - v| \neq 0$, the only solution is $x_0 = 0$.

4.5 The results

We explore some characteristics of generalized vector variational inequality problems in H-spaces.

4.5.1 Theorem. *Let* $(X,\{\Gamma_A\})$ *be an H-space and* (Y,P) *be an ordered topological vector space equipped with a closed convex pointed cone* P *such that* $intP \neq \emptyset$ *and* $K \subset X$. *Let* $L(X,Y)$ *be the set of all continuous linear functionals from* X *to* Y *and*

$$T : K \to L(X,Y)$$

be a mapping. Let

$$\eta : K \times K \to X$$

be a vector valued function and $f : K \to Y$ *be continuous. Assume that the following conditions hold* :

(a) $f(x) \notin -intP$ *for all* $x \in K$;

(b) $f : K \to Y$ *is* T-η-*invex in* K;

(c) *for each* $u \in K$ *the set* $B_u = \{x \in X : f(u) - f(x) \in -intP\}$ *is either H-convex or empty*.

Then

$$f(x') - \langle T(x'), \eta(x, x')\rangle \notin -intP$$

for all $x, x' \in K$.

Proof. Define a set valued mapping

$$F : K \to 2^X$$

by the rule

$$F(x) = \{x' \in X : f(x') - \langle T(x'),\ \eta(x, x')\rangle \notin -\text{int}P\}$$

for each $x \in K$. Clearly $F(x)$ is nonempty for each $x \in K$.

It is enough to prove that F is an H-KKM mapping. If not, then there exists a finite set $A \subset K$ such that

$$\Gamma_A \not\subseteq \bigcup \{F(x) : x \in A\}.$$

Let $z \in \Gamma_A$ such that

$$z \notin \bigcup \{F(x) : x \in A\}.$$

Thus

$$z \notin F(x), \text{ i.e., } f(z) - \langle T(z), \eta(x, z)\rangle \in -\text{int}P$$

for all $x \in A$. Since f is T-η-invex in X, at z, we have

$$f(x) - f(z) - \langle T(z), \eta(x, z)\rangle \notin -\text{int}P$$

for all $x \in A$. By (a), $f(z) \notin -\text{int}P$; so by Lemma 3.1.1(i) we have

$$f(x) - \langle T(z), \eta(x, z)\rangle \notin -\text{int}P$$

for all $x \in A$. Thus

$$f(x) - \langle T(z), \eta(x, z)\rangle \in \text{int}P \text{ i.e., } \langle T(z), \eta(x, z)\rangle - f(x) \in -\text{int}P$$

for all $x \in A$ and from the above we obtain

$$f(z) - f(x) \in -\text{int}P$$

for all $x \in A$. Hence $x \in B_z$ for all $x \in A$. Thus $A \subset B_z$ and by H-convexity of B_z, $\Gamma_A \subset B_z$. Since $z \in \Gamma_A$ we have

$$z \in B_z, \text{ i.e., } f(z) - f(z) \in -\text{int}P$$

giving $0 \in -\text{int}P$, which is a contradiction because by the pointed-ness condition of P,

$$0 \in -P \cap P$$

implies that

$$0 \notin -\text{int}P \cup \text{int}P.$$

Hence F is an H-KKM mapping. □

4.5.2 Theorem. *Let* $(X, \{\Gamma_A\})$ *be an H-space and* (Y, P) *be an ordered topological vector space equipped with a closed convex pointed cone* P *such that* $intP \neq \emptyset$. *Let* $L(X, Y)$ *be the set of all continuous linear functionals from* X *to* Y *and*

$$T : K \to L(X, Y)$$

be a mapping. Let

$$\eta : K \times K \to X$$

be a vector valued function and $f : K \to Y$ *be continuous. Assume that the following conditions hold* :

(a) $f(x) \notin -\text{int}P$ *for all* $x \in K$;

(b) $f : K \to Y$ *is* T-η-*invex in* K;

(c) *for each* $u \in K$, *the set* $B_u = \{x \in X : f(u) - f(x) \in -intP\}$ *is either H-convex or empty*;

(d) *the mapping* $v \mapsto f(v) - \langle T(v), \eta(x, v)\rangle$ *of* K *into* Y *is continuous*;

(e) *there exists a compact set* $L \subset X$ *and an H-compact* $C \subset X$, *such that*, *for each weakly H-convex set* D *with* $C \subset D \subset X$, *we have*

$$\bigcap \{x \in D : f(x') - \langle T(x'), \eta(x, x')\rangle \notin -intP\} \subset L.$$

Then there exists $x_0 \in K$ *such that*

$$f(x_0) - \langle T(x_0), \eta(x, x_0)\rangle \notin -intP$$

for all $x \in K$.

Proof. As in the proof of Theorem 4.5.1, for each $x \in K$ the set valued mapping

$$F(x) = \{x' \in X : f(x') - \langle T(x'), \eta(x, x')\rangle \notin -\text{int}P\}$$

is an *H-KKM* mapping. We show that

$$\bigcap \{F(x) \colon x \in X\} \neq \emptyset.$$

By Theorem 4.2.2 we need to show that F is closed. Let $\{z_n\} \subset F(x)$ be a sequence in $F(x)$ where $z_n \to z$; then we show that $z \in F(x)$. Since the map

$$z \mapsto f(z) - \langle T(z), \eta(x, z)\rangle$$

is continuous we have

$$f(z_n) - \langle T(z_n), \eta(x, z_n)\rangle \to f(z) - \langle T(z), \eta(x, z)\rangle.$$

But we have

$$f(z_n) - \langle T(z_n), \eta(x, z_n)\rangle \notin -\text{int}P.$$

Thus

$$f(z_n) - \langle T(z_n), \eta(x,z_n)\rangle \in Y-(-\text{int}P).$$

But $Y-(-\text{int}P)$ is closed. Therefore

$$f(z) - \langle T(z), \eta(x, z)\rangle \in Y-(-\text{int}P)$$

giving

$$f(z) - \langle T(z), \eta(x, z)\rangle \notin -\text{int}P.$$

Hence $z \in F(x)$. Thus there exists

$$x_0 \in \bigcap \{F(x) : x \in X\},$$

such that

$$f(x_0) - \langle T(x_0), \eta(x, x_0)\rangle \notin -\text{int}P$$

for all $x \in K$. □

The following result is a direct consequence of Theorem 4.5.2.

4.5.3 Theorem. *Let* $(X,\{\Gamma_A\})$ *be an H-space and* (Y,P) *be an ordered topological vector space equipped with a closed convex pointed cone P such that* $intP \neq \varnothing$. *Let* $L(X,Y)$ *be the set of all continuous linear functionals from* X *to* Y *and*

$$T : K \to L(X,Y)$$

be a mapping. Let

$$\eta : K \times K \to X$$

be a vector valued function and $f : K \to Y$ *be continuous. Assume that the following conditions hold* :

(a) $f(x) \notin -intP$ *for all* $x \in K$;

(b) $f : K \to Y$ *is* T-η*-invex in* K;

(c) *for each* $u \in K$, *the set* $B_u = \{x \in X : f(u) - f(x) \in -intP\}$ *is either H-convex or empty*;

(d) *the mapping* $v \mapsto f(v) - \langle T(v), \eta(x, v)\rangle$ *of* K *into* Y *is continuous*;

(e) *there exists a compact set* $L \subset X$ *and an H-compact* $C \subset X$, *such that*, *for each weakly H-convex set* D *with* $C \subset D \subset X$, *we have*

$$\bigcap \{x \in D : f(x') - \langle T(x'), \eta(x, x')\rangle \notin -intP\} \subset L;$$

(f) $\langle T(x'), \eta(x, x')\rangle \geq_P 0$ *for all* $x, x' \in K, x \neq x'$.

Then there exists $x_0 \in K$ *such that*

(i) $f(x) - \langle T(x), \eta(x_0, x)\rangle \notin -intP$ *for all* $x \in K$;

(ii) $\langle T(x_0), \eta(x, x_0)\rangle - \langle T(x), \eta(x_0, x)\rangle \notin -intP$ *for all* $x \in K$;

(iii) $f(x) - f(x_0) \notin -intP$ *for all* $x \in K$;

(iv) $\{f(x) - \langle T(x), \eta(x_0, x)\rangle\} - \{f(x_0) - \langle T(x_0), \eta(x, x_0)\rangle\} \notin -intP$ *for all* $x \in K$.

Proof. (i) Theorem 4.5.2, gives the existence of a $x_0 \in K$. Since T is η-monotone (Proposition 2.3.6)

$$\langle T(x_0), \eta(x, x_0)\rangle + \langle T(x), \eta(x_0, x)\rangle \leq_P 0$$

for all $x \in K$. By (f), at $x' = x_0$, we have

$$-\langle T(x), \eta(x_0, x)\rangle \geq_P \langle T(x_0), \eta(x, x_0)\rangle \geq_P 0$$

for all $x \in K$ and hence by (a)

$$f(x) - \langle T(x), \eta(x_0, x)\rangle \geq_P 0$$

for all $x \in K$, i.e.,

$$f(x) - \langle T(x), \eta(x_0, x)\rangle \notin -\mathrm{int}P$$

for all $x \in K$.

(ii) From (f), at $x' = x_0$, we have

$$\langle T(x_0),\ \eta(x, x_0)\rangle \geq_P 0$$

for all $x \in K$. As in the proof of (i)

$$-\langle T(x), \eta(x_0, x)\rangle \geq_P 0.$$

Thus

$$\langle T(x_0),\ \eta(x, x_0)\rangle - \langle T(x), \eta(x_0, x)\rangle \geq_P 0$$

i.e.,

$$\langle T(x_0), \eta(x, x_0)\rangle - \langle T(x), \eta(x_0, x)\rangle \notin -\mathrm{int}P$$

for all $x \in K$.

(iii) By T-η-invexity of f,

$$f(x) - f(x_0) - \langle T(x_0),\ \eta(x, x_0)\rangle \ \geq_P 0$$

i.e.,

$$f(x) - f(x_0) \geq_P \langle T(x_0), \eta(x, x_0)\rangle \geq_P 0$$

(by (f)) for all $x \in K$. Thus

$$f(x) - f(x_0) \notin -\mathrm{int}P$$

for all $x \in K$.

(iv) Addition of (ii) and (iii), gives

$$\{f(x) - \langle T(x), \eta(x_0, x)\rangle\} - \{f(x_0) - \langle T(x_0),\ \eta(x, x_0)\rangle\} \geq_P 0$$

i.e.,

$$\{f(x) - \langle T(x), \eta(x_0, x)\rangle\} - \{ f(x_0) - \langle T(x_0), \eta(x, x_0)\rangle\} \notin -\mathrm{int}P$$

for all $x \in K$. □

4.5.4 Theorem. *Let* $(X,\{\Gamma_A\})$ *be an H-space and* (Y,P) *be an ordered topological vector space equipped with a closed convex pointed cone* P *such that* $intP \neq \emptyset$. *Let* $L(X,Y)$ *be the set of all continuous linear functionals from* X *to* Y *and*

$$T : K \to L(X,Y)$$

be a mapping. Let

$$\eta : K \times K \to X$$

be a vector valued function and $f : K \to Y$ *be continuous. Assume that the following conditions hold* :

(a) $f(x) \notin -intP$ *for all* $x \in K$;

(b) $f : K \to Y$ *is* T-η*-invex in* K;

(c) *for each* $u \in K$ *the set* $B_u = \{x \in X : f(u) - f(x) \in -intP\}$ *is either H-convex or empty*;

(d) *the mapping* $v \mapsto f(v) - \langle T(v), \eta(x, v)\rangle$ *of* K *into* Y *is continuous*;

(e) *there exists a compact set* $L \subset X$ *and an H-compact* $C \subset X$, *such that, for each weakly H-convex set* D *with* $C \subset D \subset X$, *we have*

$$\bigcap \{x \in D : f(x') - \langle T(x'), \eta(x, x')\rangle \notin -intP\} \subset L;$$

(f) $\langle T(x'), \eta(x, x')\rangle \geq_P 0$ *for all* $x, x' \in K$, $x \neq x'$.

Then there exists $x_0 \in K$ *such that*

$$\{f(x) - \langle T(x), \eta(x_0, x)\rangle\} - \{f(x_0) - \langle T(x_0), \eta(x, x_0)\rangle\} \notin -intP$$

for all $x \in K$.

Proof. Theorem 4.5.2 gives the existence of $x_0 \in K$. Since f is T-η-invex in K

$$f(x) - f(x_0) - \langle T(x_0), \eta(x, x_0)\rangle \notin -\mathrm{int}P,$$

i.e.,

$$f(x) - f(x_0) - \langle T(x_0), \eta(x, x_0)\rangle \geq_P 0$$

for all $x \in K$. Thus

$$f(x) - \langle T(x), \eta(x_0, x)\rangle - f(x_0) \geq_P \langle T(x_0), \eta(x, x_0)\rangle - \langle T(x), \eta(x_0, x)\rangle$$

for all $x \in K$. Again since T is η-monotone, we have,

$$\langle T(x_0), \eta(x, x_0)\rangle + \langle T(x), \eta(x_0, x)\rangle \notin \mathrm{int}P,$$

i.e.,

$$\langle T(x_0), \eta(x, x_0)\rangle + \langle T(x), \eta(x_0, x)\rangle \leq_P 0$$

for all $x \in K$. Thus

$$-\langle T(x_0), \eta(x, x_0)\rangle - \langle T(x), \eta(x_0, x)\rangle \geq_P 0,$$

i.e.,

$$-\langle T(x), \eta(x_0, x)\rangle \geq_P \langle T(x_0), \eta(x, x_0)\rangle$$

for all $x \in K$. Since

$$T(x_0), \eta(x, x_0)\rangle \geq_P 0,$$

we have,

$$-\langle T(x), \eta(x_0, x)\rangle \geq_P -\langle T(x_0), \eta(x, x_0)\rangle$$

for all $x \in K$. From the above we conclude that

$$f(x) - \langle T(x), \eta(x_0, x)\rangle - f(x_0) \geq_P \langle T(x_0), \eta(x, x_0)\rangle - \langle T(x_0), \eta(x, x_0)\rangle,$$

i.e.,

$$f(x) - \langle T(x), \eta(x_0, x)\rangle - f(x_0) + \langle T(x_0), \eta(x, x_0)\rangle \geq_P \langle T(x_0), \eta(x, x_0)\rangle \geq_P 0.$$

Hence

$$\{f(x) - \langle T(x), \eta(x_0, x)\rangle\} - \{f(x_0) - \langle T(x_0), \eta(x, x_0)\rangle\} \geq_P 0$$

i.e.,

$$\{f(x) - \langle T(x), \eta(x_0, x)\rangle\} - \{f(x_0) - \langle T(x_0), \eta(x, x_0)\rangle\} \notin -\text{int}P$$

for all $x \in K$. □

4.5.5 Theorem. *Let $(X, \{\Gamma_A\})$ be an H-space and (Y, P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP \neq \varnothing$. Let $L(X,Y)$ be the set of all continuous linear functionals from X to Y and*

$$T : K \to L(X,Y)$$

be a mapping. Let

$$\eta : K \times K \to X$$

be a vector valued function and $f : K \to Y$ be continuous. Assume that the following conditions hold :

(a) *$f(x) \notin -intP$ for all $x \in K$;*

(b) *$f : K \to Y$ is T-η-invex in K;*

(c) *for each $u \in K$ the set $B_u = \{x \in X : f(u) - f(x) \in -intP\}$ is either H-convex or empty;*

(d) *the mapping* $v \mapsto f(v) - \langle T(v), \eta(x,v)\rangle$ *of* K *into* Y *is continuous*;

(e) *there exists a compact set* $L \subset X$ *and an H-compact* $C \subset X$, *such that, for each weakly H-convex set D with* $C \subset D \subset X$, *we have*

$$\bigcap \{x \in D : f(x') - \langle T(x'), \eta(x, x')\rangle \notin -intP\} \subset L;$$

(f) $\langle T(x'), \eta(x, x')\rangle \geq_P 0$ *for all* $x, x' \in K, x \neq x'$.

Then there exists $x_0 \in K$ *such that* $f(x) - f(x_0) \notin -intP$ *for all* $x \in K$.

Proof. Theorem 4.5.2 gives the existence of $x_0 \in K$. Since f is T-η-invex in K

$$f(x) - f(x_0) - \langle T(x_0), \eta(x, x_0)\rangle \notin -\mathrm{int}P$$

i.e.,

$$f(x) - f(x_0) - \langle T(x_0), \eta(x, x_0)\rangle \geq_P 0$$

for all $x \in K$. By (f), at $x' = x_0$, we have

$$\langle T(x_0), \eta(x, x_0)\rangle \geq_P 0.$$

Hence

$$f(x) - f(x_0) \geq_P 0, \text{ i.e., } f(x) - f(x_0) \notin -\mathrm{int}P$$

for all $x \in K$. □

4.5.6 Theorem. *Let* $(X, \{\Gamma_A\})$ *be an H-space and* (Y, P) *be an ordered topological vector space equipped with a closed convex pointed cone P such that* $intP \neq \emptyset$. *Let* $L(X,Y)$ *be the set of all continuous linear functionals from* X *to* Y *and*

$$T : K \to L(X,Y)$$

be a mapping. Let

$$\eta : K \times K \to X$$

be a vector valued function and $f : K \to Y$ *be continuous. Assume that the following conditions hold* :

(a) $f(x) \notin -intP$ *for all* $x \in K$;

(b) $f : K \to Y$ *is T-*η*-invex in* K;

(c) *for each* $u \in K$ *the set* $B_u = \{x \in X : f(u) - f(x) \in -intP\}$ *is either H-convex or empty*;

(d) *the mapping* $v \mapsto f(v) - \langle T(v), \eta(x, v)\rangle$ *of* K *into* Y *is continuous*;

(e) *there exists a compact set* $L \subset X$ *and an H-compact* $C \subset X$, *such that, for each weakly H-convex set D with* $C \subset D \subset X$, *we have*

$$\bigcap \{x \in D : f(x') - \langle T(x'), \eta(x, x')\rangle \notin -intP\} \subset L;$$

(f) $\langle T(x'), \eta(x, x')\rangle \geq_P 0$ *for all* $x, x' \in K, x \neq x'$;

(g) $\langle T(x'), \eta(x, x')\rangle - \langle T(x), \eta(x', x)\rangle \geq_P 0$ for all $x, x' \in K, x \neq x'$.

Then there exists $x_0 \in K$ *such that* $f(x) - \langle T(x), \eta(x_0, x)\rangle \notin -intP$ *for all* $x \in K$.

Proof. Theorem 4.5.2 gives the existence of $x_0 \in K$. By condition (g), at $x' = x_0$,

$$\langle T(x_0), \eta(x, x_0)\rangle - \langle T(x), \eta(x_0, x)\rangle \notin -\text{int}P$$

i.e.,

$$\langle T(x_0), \eta(x, x_0)\rangle - \langle T(x), \eta(x_0, x)\rangle \geq_P 0$$

for all $x \in K$. Since f is T-η-invex in K, we have

$$f(x) - f(x_0) - \langle T(x_0), \eta(x, x_0)\rangle \geq_P 0$$

for all $x \in K$. Adding the above two inequalities, we get

$$f(x) - f(x_0) - \langle T(x), \eta(x_0, x)\rangle \geq_P 0$$

i.e.,

$$f(x) - \langle T(x), \eta(x_0, x)\rangle \geq_P f(x_0) \geq_P 0$$

for all $x \in K$. Hence

$$f(x) - \langle T(x), \eta(x_0, x)\rangle \notin -\text{int}P$$

for all $x \in K$. □

4.5.7 Theorem. *Let* $(X, \{\Gamma_A\})$ *be an H-space and* (Y, P) *be an ordered topological vector space equipped with a closed convex pointed cone P such that* $intP \neq \emptyset$. *Let* $L(X,Y)$ *be the set of all continuous linear functionals from* X *to* Y *and*

$$T : K \to L(X,Y)$$

be a mapping. Let

$$\eta : K \times K \to X$$

be a vector valued function and $f : K \to Y$ *be continuous. Assume that the following conditions hold* :

(a) $f(x) \notin -intP$ *for all* $x \in K$;

(b) $f : K \to Y$ *is* T-η-*invex in* K;

(c) *for each* $u \in K$ *the set* $B_u = \{x \in X : f(u) - f(x) \in -intP\}$ *is either H-convex or empty*;

(d) *the mapping* $v \mapsto f(v) - \langle T(v), \eta(x, v)\rangle$ *of* K *into* Y *is continuous*;

(e) *there exists a compact set* $L \subset X$ *and an* H-*compact* $C \subset X$, *such that, for each weakly* H-*convex set* D *with* $C \subset D \subset X$, *we have*

$$\bigcap \{x \in D : f(x') - \langle T(x'), \eta(x, x'\rangle \notin -intP\} \subset L;$$

(f) $\langle T(x'), \eta(x, x')\rangle \geq_P 0$ *for all* $x, x' \in K, x \neq x'$.

Then there exists $x_0 \in K$ *such that*

(A) $\langle T(x_0), \eta(x, x_0)\rangle - \langle T(x), \eta(x_0, x)\rangle \notin -intP$ *for all* $x \in K$.

(B) $\{f(x) - \langle T(x), \eta(x_0, x)\rangle\} - \{f(x_0) - \langle T(x_0), \eta(x, x_0)\rangle\} \notin -intP$ *for all* $x \in K$.

Proof. Theorem 4.5.5 gives the existence of $x_0 \in K$ with

$$f(x) - f(x_0) \notin -\text{int}P, \text{ i.e., } f(x) - f(x_0) \geq_P 0$$

for all $x \in K$.

(A) Since f is T-η-invex in K

$$f(x_0) - f(x) - \langle T(x), \eta(x_0, x)\rangle \notin -\text{int}P$$

i.e.,

$$f(x_0) - f(x) - \langle T(x), \eta(x_0, x)\rangle \geq_P 0$$

for all $x \in K$; adding this with

$$f(x) - f(x_0) \geq_P 0,$$

we get

$$-\langle T(x), \eta(x_0, x)\rangle \geq_P 0.$$

By condition (f) at $x' = x_0$, we have

$$\langle T(x_0), \eta(x, x_0)\rangle \geq_P 0.$$

Hence

$$\langle T(x_0), \eta(x, x_0)\rangle - \langle T(x), \eta(x_0, x)\rangle \notin -\text{int}P$$

for all $x \in K$.

(B) Addition of (A) with $f(x) - f(x_0) \geq_P 0$ gives

$$f(x) - f(x_0) + \langle T(x_0), \eta(x, x_0)\rangle - \langle T(x), \eta(x_0, x)\rangle \geq_P 0$$

i.e.,

$$\{f(x) - \langle T(x), \eta(x_0, x)\rangle\} - \{f(x_0) - \langle T(x_0), \eta(x, x_0)\rangle\} \geq_P 0$$

for all $x \in K$. Hence

$$\{f(x) - \langle T(x), \eta(x_0, x)\rangle\} - \{f(x_0) - \langle T(x_0), \eta(x, x_0)\rangle\} \notin -\text{int}P$$

for all $x \in K$. □

Chapter 5

VARIATIONAL INEQUALITY PROBLEM IN *H*-DIFFERENTIABLE MANIFOLDS

5.1 Introduction

In this chapter we recall Lefschetz number and Lefschetz Fixed-Point Theorem and introduce the concept of H-differentiable manifolds. Using T-η-invex functions we study variational inequality problems in H-differentiable manifolds. In order to make the thesis self-contained, in the next section, we recall the necessary terminologies with respect to Lefschetz number from [44].

5.2 Lefschetz number

In 1923 Lefschetz published the first version of his fixed point formula. Let M be a closed manifold and $f : M \to M$ be a map. Then for each k there is the induced homomorphism on homology with rational coefficients

$$f_k : H_k(M;\mathbb{Q}) \to H_k(M;\mathbb{Q}).$$

For each k a basis is chosen for the finite dimensional rational vector space $H_k(M;\mathbb{Q})$ and f_k is written as a matrix with respect to this basis. Denote by $tr(f_k)$ the trace of the matrix. The *Lefschetz number* of f is defined by

$$L(f) = \sum_{k=0}^{\infty}(-1)^k\, tr(f_k).$$

The Lefschetz number $L(f)$ of f is independent of the choices involved and hence is a well defined, rational-valued function of f and it depends only on the homotopy class of f [44].

In general, if $f, g : M_1 \to M_2$ are maps between closed oriented n-manifolds, a *coincidence* of f and g is a point $x \in M_1$ such that $f(x) = g(x)$. Geometrically, if $G(f)$ and $G(g)$ are the graphs of the respective functions in $M_1 \times M_2$, their points of intersection correspond to the coincidences [44].

Consider the diagram

$$\begin{array}{ccc} H_m(M_1;\mathbb{Q}) & \xrightarrow{f_*} & H_m(M_1;\mathbb{Q}) \\ \cong\uparrow\mu & & \cong\uparrow\nu \\ H^{n-m}(M_1;\mathbb{Q}) & \underset{g^*}{\leftarrow} & H^{n-m}(M_2;\mathbb{Q}) \end{array}$$

where the vertical homomorphisms are Poincaré duality isomorphisms. The homomorphism

$$\Theta_m: H_m(M_1;\mathbb{Q}) \to H_m(M_1;\mathbb{Q})$$

is defined by $\Theta_m = \mu g^* \nu^{-1} f_*$. Then the coincidence number of f and g is given by

$$L(f, g) = \sum_{k=0}^{n} (-1)^k \, tr(\Theta_m).$$

$L(f, g)$ is the intersection number of $G(f)$ and $G(g)$; hence if $L(f, g) \neq 0$, then f and g have a coincidence. Note that if $M_1 = M_2$ and g is the identity then $L(f, g) = L(f)$ [44].

Let M_1 and M_2 be closed, connected, oriented n-manifolds with fundamental classes $z_i \in H_n^*(M_i)$ and corresponding Thom classes

$$U_i \in H_0^n(M_i \times M_i, M_i \times M_i - \Delta(M_i)), \; i = 1, 2.$$

Suppose that W is an open set in M_1 and $f, g : W \to M_2$ are the maps for which the coincidence set $C = \{x \in W : f(x) = g(x)\}$ is a compact subset of W. By normality of M_1 there exists an open set V in M_1 with $C \subseteq V \subseteq \overline{V} \subseteq W$. The *coincidence index* of the pair (f, g) on W is defined to be the integer $I_{f,g}^W$ given by the image of the fundamental class of z_1 under the composition

$$H_n(M_1) \to H_n(M_1, M_1 - V) \underset{\cong}{\xrightarrow{excision}} H_n(W, W - V) \xrightarrow{(f,g)_*}$$

$$H_n(M_2 \times M_2, M_2 \times M_2 - \Delta(M_2)) \cong \mathbb{Z}.$$

where the map

$$(f, g) : W \to M_2 \times M_2$$

is given by

$$(f, g)(x) = (f(x), g(x)),$$

and the identification

$H_n(M_2 \times M_2, M_2 \times M_2 - \Delta(M_2)) \cong \mathbb{Z}$

is given by sending a class α into the integer $\langle U_2, \alpha\rangle$. If $M_1 = M_2$ (denoted by M) and g = identity on the open set W, the coincidence index $I^W_{f,id}$ is denoted by I^W_f and called the *fixed-point index* of f on W [44].

5.2.1 Lemma ([44], Lemma 6.7, p.180) *If* $I^W_f \neq 0$, *then* f *has a fixed point in* W.

5.2.2 Definition [44]] Let M be a closed oriented n-manifold. Let $f, g : M \rightarrow M$ be mappings from M to itself. Then the coincidence index of the pair (f, g) is said to be *fixed-point index* of f if g is identity mapping. The fixed-point index of f is denoted by I_f.

Lefschetz studied the fixed point theorem as follows.

5.2.3 Theorem ([44], Lefschetz Fixed-Point Theorem, Theorem 6.16, p.188) If $f : M \rightarrow M$ is a map of a closed oriented n-manifold to itself, then $I_f = L(f)$. Thus if $L(f) \neq 0$, then f has a fixed point.

5.3 The results

In this section, we study the concept of T-η-invex functions and its application in generalized vector variational inequality problems (in short, *GVVI*) over the manifolds.

Let X and Y be differentiable manifolds with tangent bundles τX and τY respectively. Let K a closed convex subset in the manifold X and P be a closed, convex, pointed ordered cone in Y with int$P \neq \emptyset$. Let $\eta : K \times K \rightarrow \tau X$ be an application and $T : K \rightarrow L(\tau X, \tau Y)$ be the linear application.

The *Generalized Vector Variational Inequality Problem* $(GVVI)_X$ on the X can be formulated as follows:

$(GVVI)_X$: *Find* $x_0 \in K$ *such that* $\langle T(x_0), \eta(x, x_0)\rangle \notin -intP$ *for all* $x \in K$.

5.3.1 Definition. [35] *Let* $\varphi : X \to \mathbb{R}^n$ be a differential vector function. The *differential of* φ *at* $u \in X$, denoted,

$$d\varphi_u\colon \tau(X, u) \to \tau(\mathbb{R}^n, \varphi(u)) \equiv \mathbb{R}^n,$$

is defined by the rule $d\varphi_u(v) = d\varphi(u)(v)$ for all $v \in \tau(X, u)$.

5.3.2 Definition. [35] A differential vector function $\varphi : X \to \mathbb{R}^n$ is said to be *invex* at $u \in X$ with respect to η (shortly, φ is η-*invex*) if there exists an application

$$\eta : K \times K \to \tau X$$

such that

$$\varphi(x) - \varphi(u) \geq d\varphi_u(\eta(x, u))$$

for all $x \in K$.

5.3.3 Definition. An H-space is called an *H-differentiable manifold* if it is also a differentiable manifold.

5.3.4 Example. $\mathbb{R}$ is an H-differentiable manifold.

In this section we prove some results in an H-differentiable manifold $(X, \{\Gamma_A\})$.

5.3.5 Theorem. *Let* $(X, \{\Gamma_A\})$ *be an H-differentiable manifold with the tangent bundle* τX *and* K *a nonempty closed convex subset of* X. *Let* $\varphi : X \to \mathbb{R}^n$ *be a linear application. Let* P *be a closed, convex and ordered pointed cone in* $\mathbb{R}^n$ *with* $intP \neq \emptyset$. $L(\tau X, \mathbb{R}^n)$ *denote the set of continuous linear maps from* τX to $\tau\mathbb{R}^n \equiv \mathbb{R}^n$. *Let*

$$T\colon K \to L(\tau X, \mathbb{R}^n)$$

and

$$\eta : K \times K \to \tau X$$

be an application such that $\eta(x, u) \in \tau(X, u)$ *for all* $x, u \in K$. *Suppose that*

(a) $-\varphi$ *is* T-η-*invex in* K,

(b) *for each* $u \in K$,

$$U(u) = \{x \in X\colon \varphi(x) - \varphi(u) \in -intP\}$$

is either H-convex or empty,

(c) *the application*

$$u \mapsto \langle T(u), \eta(x, u) \rangle$$

of K *into* $\mathbb{R}^n$ *is continuous* (*or at least hemicontinuous*) *for all* $x \in K$,

(d) *there exists a compact set* $L \subset X$ *and an* H*-compact set* $C \subset X$ *such that for each weakly* H*-convex set* D *with* $C \subset D \subset X$, *we have*

$$\bigcap \{\langle T(u), \eta(x, u) \rangle \notin -intP : x \in D\} \subset L.$$

Then $(GVVI)_K$ *is solvable*, *i.e.*, *there exists* $x_0 \in K$ *such that*

$$\langle T(x_0), \eta(x, x_0) \rangle \notin -intP$$

for all $x \in K$.

Proof. Let $F : K \to 2^X$ be a set valued application defined by the rule

$$F(x) = \{u \in K : \langle T(u), \eta(x, u) \rangle \notin -\mathrm{int}P\}$$

for all $x \in X$. We prove that

$$\bigcap \{F(x) : x \in K\} \neq \emptyset.$$

It is enough to show that F is an H-KKM mapping on the H-differentiable manifold X. If F is not an H-KKM then there exists a finite subset $A \subset X$ such that

$$\Gamma_A \not\subset \bigcup \{F(x) : x \in A\}.$$

Let there exists $w \in \Gamma_A$ such that

$$w \notin \bigcup \{F(x) : x \in A\}.$$

This implies that

$$w \notin F(x), \text{ i.e., } \langle T(w), \eta(x, w) \rangle \in -\mathrm{int}P$$

for all $x \in A$. Since $(-\varphi)$ is T-η-invex in K,

$$-\varphi(x) + \varphi(u) + \langle T(u), \eta(x, u) \rangle \notin -\mathrm{int}P;$$

equivalently

$$\varphi(x) - \varphi(u) - \langle T(u), \eta(x, u) \rangle \notin \mathrm{int}P$$

for all $x, u \in K$. At $w \in K$, we have

$$\varphi(x) - \varphi(w) - \langle T(w), \eta(x, w) \rangle \notin \mathrm{int}P$$

for all $x \in K$. Hence

$$\varphi(x) - \varphi(w) - \langle T(w), \eta(x, w) \rangle \notin \mathrm{int}P$$

for all $x \in A$ (since A is a nonempty finite subsets of X). Thus we get,

$$\varphi(x) - \varphi(w) \in -\mathrm{int}P$$

for all $x \in A$ and by assumption (b) we get $x \in U(w)$. Hence $A \subset B(w)$. By the H-convexity of $U(w)$, we get $\Gamma_A \subset U(w)$ for every finite subset $A \subset U(w)$. Since $w \in \Gamma_A$, we have $w \in U(w) \subset X$. Hence

$$0 = \varphi(w) - \varphi(w) \in -\text{int}P$$

which is a contradiction since $0 \notin -\text{int}P$. Hence F is an H-KKM map.

Next we prove that F is closed. Let $\{y_n\}$ be a sequence in $F(x)$ such that $y_n \to y$. We need to show that $y \in F(x)$. Since the application

$$y_n \mapsto \langle T(y_n), \eta(x, y_n)\rangle$$

is continuous, $y_n \to y$ gives

$$\langle T(y_n), \eta(x, y_n)\rangle \to \langle T(y), \eta(x, y)\rangle.$$

Also $y_n \in F(x)$ gives

$$\langle T(y_n), \eta(x, y_n)\rangle \notin -\text{int}P$$

for all $x \in K$. Since $\mathbb{R}^n - (-\text{int}P)$ is a closed set,

$$\langle T(y_n), \eta(x, y_n)\rangle \in \mathbb{R}^n - (-\text{int}P)$$

for all $x \in K$, showing

$$\langle T(y), \eta(x, y)\rangle \notin -\text{int}P$$

for all $x \in \mathbb{R}^n$. Hence $y \in F(x)$. By Theorem 4.2.2, we get

$$\bigcap \{F(x) : x \in K\} \neq \varnothing.$$

Hence there exists an $x_0 \in K$, such that

$$\langle T(x_0), \eta(x, x_0)\rangle \notin -\text{int}P$$

for all $x \in K$. □

5.3.6 Theorem. *Let* $(X, \{\Gamma_A\})$ *and* $(Y, \{\Gamma_B\})$ *be H-differentiable manifolds with the tangent bundles* τX *and* τY *respectively and* K *a nonempty closed convex subset of* X*. Let* P *be a closed, convex and pointed ordered cone* Y *in such that* $\text{int}P \neq \varnothing$*. Let* $L(\tau X, \tau Y)$ *denotes the set of continuous linear maps from* τX *to* τY *and*

$$\eta : K \times K \to \tau X$$

an application. Let $\varphi : X \to Y$ *be a linear application and for each* $u \in K$,

$$e_u = \tau\varphi_u : \tau X \to \tau Y$$

be the corresponding bundle map defined by

$$e_u(v) \in \tau(Y, \varphi(u))^-$$

for all $v \in \tau(X, u)$*, where*

$$\tau(Y,\varphi(u))^- = \{w\in\tau(Y,\varphi(u)) : w\notin intP\}.$$

Let

$$T : K\to L(\tau X,\tau Y)$$

be defined by the rule

$$\langle T(u),v\rangle = (d\varphi_u - e_u)(v)$$

for all $v\in\tau(X,u)$. *Let the following conditions hold* :

(a) $-\varphi$ *is differentiable-*η*-invex in* K,

(b) *for each* $u\in K$, *the set*

$$U(u) = \{x\in X : \varphi(x) - \varphi(u)\in -intP\}$$

is either H*-convex or empty*,

(c) *the application*

$$u\mapsto\langle d\varphi_u - e_u,\eta(x, u)\rangle$$

of K *into* τY *is continuous* (*or at least hemicontinuous*) *for all* $x\in X$,

(d) *there exists a compact set* $L\subset X$ *and an* H*-compact set* $V\subset X$ *such that for each weakly* H*-compact set* D *with* $V\subset D\subset X$ *and for each* $u\in K$, *we have*

$$\bigcap\{\langle d\varphi_u - e_u,\ \eta(x, u)\rangle\notin -intP : x\in D\}\subset L.$$

Then $(GVVI)_K$ *is solvable*, *i.e.*, *there exists* $x_0\in K$ *such that*

$$\langle d\varphi_{x_0} - e_{x_0},\eta(x, x_0)\rangle\notin -intP$$

for all $x\in K$.

Proof. First we show that $-\varphi$ is T-η-invex in K. Since $-\varphi : X\to Y$ is differentiable η-invex in K, we have

$$(-\varphi)(x) - (-\varphi)(u) - d(-\varphi)_u(\eta(x, u))\notin -\mathrm{int}P$$

for all $x, u\in K$, i.e.,

$$(-\varphi)(x) - (-\varphi)(u) + d\varphi_u(\eta(x, u))\notin -\mathrm{int}P$$

and this implies that

$$-\varphi(x) + \varphi(u) + d\varphi_u(\eta(x, u))\notin -\mathrm{int}P.$$

By definition of e_u,

$$e_u(\eta(x, u))\in\tau(Y,\varphi(u))^-$$

implies

$$e_u(\eta(x, u))\notin\mathrm{int}P,$$

i.e.,

$$-e_u(\eta(x, u)) \notin -\mathrm{int}P.$$

Hence

$$-\varphi(x) + \varphi(u) + d\varphi_u(\eta(x, u)) - e_u(\eta(x, u)) \notin -\mathrm{int}P,$$

i.e.,

$$-\varphi(x) + \varphi(u) + \langle T(u),(\eta(x, u)\rangle \notin -\mathrm{int}P$$

for all $x, u \in K$. Thus $-\varphi$ is T-η-invex in K.

Construct a set valued mapping $G : K \rightarrow 2^X$ by the rule

$$G(x) = \{z \in K : (d\varphi_z - e_z)(\eta(x, z)) \notin -\mathrm{int}\, P\}.$$

We show that G is an H-KKM mapping. Suppose to the contrary that G is not an H-KKM application. Then there exists a finite subset $A \subset K$ such that

$$\Gamma_A \not\subset \bigcup \{G(x) : x \in A\}.$$

Let $w \in \Gamma_A$ be such that

$$w \notin \bigcup \{G(x) : x \in A\}.$$

So $w \notin G(x)$ for all $x \in A$, i.e.,

$$(d\varphi_w - e_w)(\eta(x, w)) \in -\mathrm{int}P$$

for all $x \in A$. Since $-\varphi$ is T-η-invex in K, we have

$$-\varphi(x) + \varphi(u) + \langle T(u), \eta(x, u)\rangle \notin -\mathrm{int}P$$

for all $x, u \in K$, equivalently

$$\varphi(x) - \varphi(u) - \langle T(u), \eta(x, u)\rangle \notin \mathrm{int}P$$

for all $x, u \in K$ (by Lemma 3.1.2(F)). At the point w in K, we get

$$\varphi(x) - \varphi(w) - \langle T(w), \eta(x, w\rangle \notin \mathrm{int}P$$

for all $x \in K$ (since Γ_A is a nonempty contractible subset of K). Thus

$$\varphi(x) - \varphi(w) - \langle T(w), \eta(x, w)\rangle \notin \mathrm{int}P$$

for all $x \in A$ (since A is a nonempty finite subsets of K), i.e.,

$$\varphi(x) - \varphi(w) - (d\varphi_w - e_w)(\eta(x, w)) \notin \mathrm{int}P$$

for all $x \in A$. Therefore

$$\varphi(x) - \varphi(w) \in -\mathrm{int}P$$

for all $x \in A$, giving $x \in B(w)$. Hence $A \subset B(w)$. By the H-convexity of $B(w)$, we get $\Gamma_A \subset B(w)$ for every finite subset $A \subset B(w)$. Since $w \in \Gamma_A$, we have $w \in B(w) \subset X$. Hence

$$0 = \varphi(w) - \varphi(w) \in -\mathrm{int}P$$

gives a contradiction that, $0 \notin -\mathrm{int}P$. Hence G is an H-KKM map.

Next we prove that G is closed. Let $\{z_n\}$ be a sequence in $G(x)$ such that $z_n \to z$, then we show that $z \in G(x)$. Again since the application

$$z_n \mapsto (d\varphi_{z_n} - e_{z_n})\eta(x, z_n)$$

is continuous, we have

$$(d\varphi_{z_n} - e_{z_n})\eta(x, z_n) \to (d\varphi_z - e_z)\eta(x, z)$$

as $z_n \to z$. Since $z_n \in G(x)$, we have

$$(d\varphi_{z_n} - e_{z_n})\eta(x, z_n) \notin -\mathrm{int}P$$

for all $x \in K$. But $(Y - (-\mathrm{int}P))$ is a closed set; therefore

$$(d\varphi_z - e_z)\eta(x, z) \in Y - (-\mathrm{int}P)$$

for all $x \in X$ implying that

$$(d\varphi_z - e_z)\eta(x, z) \notin -\mathrm{int}P$$

for all $x \in X$. Hence $z \in G(x)$. Thus G is closed.

By Theorem 4.2.2**,** we get $\cap\{G(x) : x \in X\} \neq \emptyset$. Hence there exists an $x_0 \in K$ such that

$$(d\varphi_{x_0} - e_{x_0})\eta(x, x_0) \notin -\mathrm{int}P$$

for all $x \in K$, i.e., x_0 solves the problem. □

Chapter 6

COMPLEMENTARITY PROBLEM IN RIEMANNIAN *n*-MANIFOLD

6.1 Introduction

In this chapter we are interested in studying the behavior of continuous functions on manifolds with particular interest in finding the solutions of complementarily problems in the presence of fixed points or coincidences. Though the complementarily problem is a classical problem, the techniques involved in this section may prove enlightening to take a brief look at some development of the problems.

6.2 The problem

Let X be a closed, convex and oriented Riemannian n-manifold, modeled on the Hilbert space H with Riemannian metric g. It is well known that the tangent bundle $\tau(X)$ can be identified with the cotangent bundle $\tau^*(X)$ by the Riemannian metric, because H^*, the dual of H, can be identified with H [29].

If $v, w \in \tau(X, x)$, then we write $g_x(v, w) = \langle v, w\rangle_x$. Let

$$F : X \to H$$

be an operator.The Complementarity problem is :

Find $y_0 \in X$ *such that*

$$F(y_0) \in \tau^*(X)$$

and

$$g_{y_0}(F(y_0), y_0) = \langle F(y_0), y_0\rangle_{y_0} = 0.$$

6.3 The results

We prove the following results.

6.3.1 Theorem. *Let X be a closed, convex and oriented Riemannian n-manifold, modeled on the Hilbert space H with Riemannian metric g and $f : X \to X$ be a map with Lefschetz number $L(f)$. Let $F : X \to H$ be an operator. Then there exists a unique $y_0 \in X$ such that*

$$F(y_0) \in \tau^*(X)$$

and

$$g_{y_0}(F(y_0), y_0) = \langle F(y_0), y_0 \rangle_{y_0} = 0.$$

Proof. Since X is nonempty, closed and convex endowed with the Riemannian metric g, for every $y \in X$, there is an unique $x \in X$ closest to $y - F(y)$. Thus

$$\|x - y + F(y)\| \le \|z - y + F(y)\|$$

i.e.,

$$\langle x, z - x\rangle_x \ge \langle y - F(y), z - x\rangle_x$$

for all $z \in X$. Let $f: X \to X$ be defined by

$$f(y) = y - F(y) + x$$

for every $y \in X$ where x is the unique element corresponding to y. Now for every $y \in X$,

$$(1_X - f)(y) = 1_X(y) - f(y) = F(y) - x$$

and at the unique $x \in X$, we have,

$$(1_X - f)(x) = F(x) - x = (F - 1_X)(x),$$

i.e.,

$$1_X - f = F - 1_X$$

at the unique $x \in X$. Define

$$G : X \times I \to X$$

by the rule

$$G(y, t) = \begin{cases} (1_X - f)((1-2t)y + 2tx) & 0 \le t \le \frac{1}{2} \\ (F - 1_X)((2t-1)y + 2(1-t)x) & \frac{1}{2} \le t \le 1. \end{cases}$$

$$G(y, 0) = (1_X - f)(y),$$

$$G(y,1) = (F - 1_X)(y)$$

for each $y \in X$. At $t = 1/2$,

$$G(y,\frac{1}{2}) = (1_X - f)(x) = (F - 1_X)(x).$$

G is continuous by Pasting Lemma. Thus

$$G : (1_X - f) \simeq (F - 1_X).$$

Thus, the coincidence index set of f is given by

$$I_f = (1_X - f)* 0_X = (F - 1_X)* 0_X \text{ and } I_f \neq 0.$$

Since $f : X \to X$ is a mapping with $L(f) = I_f \neq 0$, by Theorem 5.2.3, f has a fixed point. Let the fixed point be y_0 in X, i.e., $f(y_0) = y_0$. Let x_0 be the unique element that corresponds y_0. Thus we have

$$\langle x_0, z - x_0 \rangle_{x_0} \geq \langle y_0 - F(y_0), z - x_0 \rangle_{x_0}$$

for all $z \in X$, giving

$$\langle x_0, z - x_0 \rangle_{x_0} \geq \langle f(y_0) - x_0, z - x_0 \rangle_{x_0}$$

i.e.,

$$\langle 2x_0 - y_0, z - x_0 \rangle_{x_0} \geq 0$$

for all $z \in X$. At $y = y_0$, we get

$$f(y_0) = y_0 - F(y_0) + x_0,$$

i.e., $x_0 = F(y_0)$. We show that $F(y_0) \in \tau^*(X)$. By definition of coincidence index set, we have

$$I_f = (1_X - f)* 0_X = (F - 1_X) * 0_X = I_F,$$

which means f and F has same fixed point in X, i.e.,

$$f(y_0) = y_0 = F(y_0).$$

So, we get $x_0 = y_0$. Putting $x_0 = y_0$ in

$$\langle 2F(y_0) - y_0, z - x_0 \rangle_{y_0} \geq 0,$$

we get

$$\langle F(y_0), z - y_0 \rangle_{y_0} \geq 0$$

for all $z \in X$, i.e., $F(y_0) \in \tau^*(X)$. Again putting $z = 0$ and $z = 2y_0$ in

$$\langle F(y_0), z - y_0 \rangle_{y_0} \geq 0,$$

respectively we get

$$\langle F(y_0), y_0 \rangle_{y_0} \leq 0 \text{ and } \langle F(y_0), y_0 \rangle_{y_0} \geq 0.$$

Hence $\langle F(y_0), y_0 \rangle_{y_0} = 0$. □

6.3.2 Theorem. *Let* $f : S^n \to S^n$, $n \geq 1$ *be a map of degree* $m \neq (-1)^{n+1}$. *Let* $T : S^n \to S^n$ *be any operator. Then there exists an* $y_0 \in S^n$ *such that*

$$\langle T(y_0), z - y_0 \rangle \geq 0$$

for all $z \in S^n$.

Proof: Since $S^n \subset R^{n+1}$ is closed, for each $y \in S^n$, there exists a unique $x \in S^n$ closest to $y - T(y)$, i.e.,

$$\|x - y + T(y)\| \leq \|z - y + T(y)\|$$

for all $z \in S^n$. Thus

$$\langle x, z - x \rangle \geq \langle y - T(y), z - x \rangle$$

i.e.,

$$\langle x - y + T(y), z - x \rangle \geq 0$$

for all $z \in S^n$. Define $f : S^n \to S^n$ by the rule $f(y) = x$. This is well defined since x is unique in S^n corresponding to each $y \in S^n$. Obviously f is an homeomorphism. Replacing x by $f(y)$ in the inequality

$$\langle x - y + T(y), z - x \rangle \geq 0$$

we get

$$\langle f(y) - y + T(y), z - f(y) \rangle \geq 0$$

for all $z \in S^n$. Since S^n is a closed n-manifold, and $f : S^n \to S^n$, $n \geq 1$ is a map of degree $m \neq (-1)^{n+1}$, f has a fixed point and is homotopic to the identity map $i: S^n \to S^n$, i.e., there is an element $y_0 \in S^n$ such that $f(y_0) = y_0$. Taking $y = y_0$ in the inequality

$$\langle f(y) - y + T(y), z - f(y) \rangle \geq 0,$$

we get

$$\langle T(y_0), z - y_0 \rangle \geq 0$$

for all $z \in S^n$. □

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