FREE VIBRATION OF LAMINATED COMPOSITE CROSS-PLY SPHERICAL PANELS

A Thesis Submitted In Partial Fulfillment of
the Requirements for the degree of

Master of Technology

In

Civil Engineering

(Structural Engineering)

By

SUJITH RAJAN

DEPARTMENT OF CIVIL ENGINEERING
NATIONAL INSTITUTE OF TECHNOLOGY
ROURKELA, ORISSA

May 2009

FREE VIBRATION OF LAMINATED COMPOSITE CROSS-PLY SPHERICAL PANELS

A Thesis Submitted In Partial Fulfillment of the

Requirements for the degree of

Master of Technology

In

Civil Engineering

(Structural Engineering)

submitted by

SUJITH RAJAN

Roll No. 207CE207

under the guidance of

Prof. A. V Asha

DEPARTMENT OF CIVIL ENGINEERING
NATIONAL INSTITUTE OF TECHNOLOGY
ROURKELA, ORISSA

NATIONAL INSTITUTE OF TECHNOLOGY
ROURKELA – 769008, ORISSA
INDIA

CERTIFICATE

This is to certify that the thesis entitled, "**FREE VIBRATION OF LAMINATED COMPOSITE CROSS-PLY SPHERICAL PANELS"** submitted by Mr**. Sujith Rajan** in partial fulfillment of the requirement for the award of **Master of Technology** Degree in **Civil Engineering** with specialization in **Structural Engineering** at the National Institute of Technology, Rourkela (Deemed University) is an authentic work carried out by him under my supervision and guidance.

To the best of my knowledge, the matter embodied in the thesis has not been submitted to any other University/ Institute for the award of any degree or diploma

Date: May 30, 2009

Place: Rourkela

Prof. A. V. Asha
Dept of Civil Engineering
National Institute of Technology
Rourkela – 769008

ACKNOWLEDGEMENT

I express my gratitude and sincere thanks to **Prof. A. V. Asha,** for her guidance and constant encouragement and support during the course of my work in the last one year. I truly appreciate and value her esteemed guidance and encouragement from the beginning to the end of this thesis, her knowledge and company at the time of crisis remembered lifelong. My project could not have been done without the help of a lot of other people. So, it is my pleasure to have the opportunity to express my thanks to those who have ever helped me.

My sincere thanks to **Dr. S. K. Sarangi,** Director and **Prof M. Panda,** Head of the Civil Engineering Department, National Institute of Technology Rourkela, for his advice and providing necessary facility for my work. I am also very thankful towards Prof. S. K. Sahu, my faculty adviser and all faculty members of structural engineering, Prof. Asha patel, Prof. M. R. Barik, Prof. K. C. Biswal, for their help and encouragement during the project. I am also thankful to the all faculty of the Civil Department for their directly or indirectly help to my stay in NIT Rourkela. I also thank all my batch mates who have directly or indirectly helped me in my project work and in the completion of this report.

Finally yet importantly, I would like to thank my parents, who taught me the value of hard work by their own example . They rendered me enormous support during the whole tenure of my stay in NIT Rourkela.

Sujith Rajan

CONTENTS

Free vibration of laminated composite cross-ply spherical panels

ABSTRACT

The use of laminated composite curved panels is common in many engineering fields; the study of vibration problems arising in spherical shells/panels has become increasingly important. Free vibration frequencies and mode shapes are essential for the analysis of resonant response and flutter. Due to its significance in structural mechanics, many researchers have worked on the vibration characteristics of spherical shells/panels.

Composite structures have extensive use in aerospace, civil, marine and other engineering applications. Laminated composites are becoming key components in many of them. Their high performance places them at the top of the list of engineering materials needed for advanced design applications. This is because controlling the lamination angle and the stacking sequence can alter their structural properties leading to an optimal design. The higher specific modulus and specific strength of these composites means that the weight of certain components can be reduced. The increasingly wider application to other fields of engineering has necessitated the evolution of adequate analytical tools for the better understanding of the structural behavior and efficient utilization of the materials.

Composites have the specific advantage that their structural characteristics can be tailored to suit the design requirements. Hence they are finding increased use in primary and secondary structures in aerospace projects. The composites, like most structural materials, are fabricated

with appropriate quality control. The control works under finite limits due to practical and economic considerations. This results in variation in material properties, making them random.

In this thesis work, an analytical solution of frequency characteristics for the free vibration of laminated composite thin spherical panels has been obtained by using the first order shear deformation theory as well as a higher order shear deformation theory. Compared with classical theory and higher order theory, the "first order shear deformation theory" combines higher accuracy and lower calculation efforts. The higher order shear deformation theory is based on a displacement field in which the displacements of the middle surface are expanded as cubic functions of the thickness coordinate and the transverse displacement is assumed to be constant through the thickness. This displacement field leads to parabolic distribution of the transverse shear stresses and zero transverse normal strain and hence no shear correction factors are used.

The objective of this study is to examine the effect of various shell parameters on the frequency characteristics of laminated composite cross-ply thin spherical panels. For reasons of simplicity, the simply supported boundary conditions are assumed for the panels. The formulation is general. Frequency envelopes for different lamination schemes for the panels have been plotted. The results obtained by both the shear deformation theories have also been compared. From the results, the influence of radius to length ratio on natural frequency of the spherical shell is larger than that of length to thickness ratio. Also there is not much difference in the results by the first and higher order shear deformation theory.

LIST OF SYMBOLS

The principal symbols used in this thesis are presented for easy reference. A symbol is used for different meaning depending on the context and defined in the text as they occur.

English

Notation	Description
A_1 ,A_2	Lames Parameter or Surface Metrics
A_{ij} , B_{ij} , D_{ij} $E_{ij}, F_{ij,} H_{ij}$	Laminate Stiffnesses
$da_{1,} da_2$	Element of area of the cross section
C , $\overline{C}$	The matrix notation described in appendix-1
dS	The distance between points (α, β, z) and $(\alpha + d\alpha, \beta + d\beta, z + dz)$
dV	Volume of the shell element
dt	Time derivative
E_1 , E_2	Longitudinal and transverse elastic moduli respectively
G_{12} , G_{13} , G_{23}	In plane and Transverse shear moduli
h	Total thickness of the shell
h_k	Distance from the reference surface to the layer interface

K_{11}^2, K_{22}^2	The shear correction factors.
k1 , k2 , k6	Quantities whose expression are given in equation [11.b]
$k_1^0, k_2^0, k_6^0, k_1^2$ $k_2^2, k_6^2, k_4^1, k_5^1$	Quantities whose expression are given in equation [11.b]
L_1 , L_2 , L_3	Lames coefficient
N_i, M_i, P_i, Q_1 Q_2, K_1, K_2	Stress resultant and stress couples
$M, \overline{M}$	Matrix notation described in appendix
m	Axial half wave number
n	Circumferential wave number
$Q_{ij}^{(k)}$	Reduced stiffness matrix of the constituent layer
R	Radius of the reference surface of spherical shell
R_1 , R_2	Principal radii of curvature
T	Kinetic energy
U, V, W, Φ_1, Φ_2	Amplitude of displacement
U_s	Strain energy
$\overline{U}, \overline{V}, \overline{W}, \overline{\Phi}_1, \overline{\Phi}_2$	Non dimensionalised amplitude of displacement
a ,b	size of the spherical shell

u, v, w	Displacement component at the reference surface
$\bar{u}, \bar{v}, \bar{w}$	Non dimensionalised displacement component at any point in
$I_i, \bar{I}_i, \bar{I}_i'$	Inertia terms defined by equation 38.a.1
{X}	Column matrix of amplitude of vibration or eigenvector.
$\overline{\{X\}}$	Non dimensionalised form of column matrix

Greek

Notation	**Description**
α , β , z	Shell coordinates
ε_i	Strain components
$\varepsilon_1^{\ 0}, \varepsilon_2^{\ 0}, \varepsilon_4^{\ 0}$ $\varepsilon_5^{\ 0}, \varepsilon_6^{\ 0}$	Quantities whose expressions are given in equation 11.b and 36.b
λ_m, λ_n	Non dimensionalised axial and circumferential wave parameter
ρ	Material density
σ_i	Stress components
ϕ_1, ϕ_2	Rotation at z = 0 of normals to the mid-surface with respect to α and β axes
ω	Natural circular frequency parameter

LIST OF FIGURES

Figure No **Page No**

LIST OF TABLES

Chapter 1

Introduction

CHAPTER 1

INTRODUCTION

The ever-increasing use of thin-walled structural elements with relatively low flexural rigidity has necessitated the study of their dynamic behavior at large amplitudes. The shell-type structures are known as the most desired structural elements in modern construction engineering, aircraft construction, ship building, rocket construction, nuclear, aerospace as well as the petrochemical industries. The use of composite materials in many industries has increased greatly in recent years. This is largely due to the high strength-to-weight and stiffness-to-weight ratios and their ability to be tailored to meet design requirements of strength and stiffness.

The use of laminated composite materials in thin-walled structural applications has had a major impact on the entire design process for two-dimensional stress-bearing systems. Nowhere has this impact been greater than in the birthplace of the modern composite - the aerospace industry - where current design capabilities range from aero elastic tailoring to minimum weight structures. This advantageous state of affairs owes much to the vast amount of research and development that has been, and continues to be, expended in gaining a better understanding of the mechanical behaviour of composite materials under static and dynamic loading actions.

Laminated composite shells are increasingly being used in various engineering applications including aerospace, mechanical, marine and automotive engineering. With the increased awareness of, and sensitivity to, structural noise and vibration, research covering the vibration of composite shells has received considerable attention. Acoustic design and consideration of composite structures involve elastic wave propagation in composite materials and interaction of sound waves with the composite structures. During cruise flight of a modern aircraft, the vibration of the outer shell of the fuselage is transmitted to the interior and can produce a high noise level within the cabin. However, the acoustical properties of these light and stiff structures can often be less than desirable resulting in high aircraft interior noise levels. The development of lightweight structures, made of composite materials, has lowered the acoustic transmission loss of such structures and therefore further increased the acoustic transmission problem.

The spherical shell panels play an important role in modern engineering. Laminated spherical shells have been used extensively in various fields of modern engineering due to the distinct structural advantages they offer. One such field is aerospace engineering where structures are mostly assemblies of shell structures. Comprehensive understanding of the mechanical behavior of composite shells is vital to assure the integrity of these structures during their service life. Several studies have focused on predicting optimum laminate configurations for enhancing the load capacity of composite shells under various loading conditions such as pure axial compression, combined axial compression, torsion, transverse load etc.

The high performance of composite materials places them at the top of the list of engineering materials needed for advanced design applications. This is because controlling the lamination angle and the stacking sequence can alter their structural properties leading to an optimal design. The higher specific modulus and specific strength of these composites means that the weight of certain components can be reduced. The increasingly wider application to other fields of engineering has necessitated the evolution of adequate analytical tools for the better understanding of the structural behavior and efficient utilization of these materials.

In this thesis work, an analytical solution of the free vibration frequency characteristics of laminated composite cross-ply thin spherical panels is obtained by using the First order shear deformation theory and a Higher order shear deformation theory for laminated shells as proposed by Reddy and Liu for plates and shallow shells. The higher order shear deformation theory is based on a displacement field in which the displacements of the middle surface are expanded as cubic functions of the thickness coordinate and the transverse displacement is assumed to be constant through the thickness. This displacement field leads to parabolic distribution of the transverse shear stresses and zero transverse normal strain and hence no shear correction factors are used. The governing equations have been developed. These equations are then reduced to the equations of motion for spherical panel and the Navier solution has been obtained for cross-ply laminated composite spherical panels. The resulting equations are suitably nondimensionalised. The eigen value problem is then solved to obtain the free vibration frequencies.

This work also examines the effect of various shell parameters on the frequency characteristics of laminated composite cross-ply thin spherical panels. For reasons of simplicity, the simply supported boundary conditions are assumed for the panels. The formulation is general. Different boundary conditions, lamination schemes, order of shear deformation theories, and even forms of assumed solutions can be easily accommodated into the analysis. Frequency envelopes for different lamination schemes for the panels have been plotted. The results obtained by both the shear deformation theories have also been compared.

Chapter 2

Literature Review

CHAPTER 2

LITERATURE REVIEW

2.1 INTRODUCTION

The analysis of plate and shell structures has a long history starting with membrane theory and then the bending theories. Laminated composite plate analyses and shell analyses are mainly based on three theories:

(1) The classical laminated plate theory (CLPT),

(2) The first-order shear deformation theory (FSDT) and,

(3) The higher-order shear deformation theory (HSDT).

The effect of transverse shear deformation, which may be essential in some cases, is included in FSDT and HSDT, whereas it is neglected in CLPT due to the Kirchhoff hypothesis.

The classical laminate plate theory is based on the Kirchhoff hypothesis that straight lines normal to the undeformed midplane remain straight and normal to the deformed midplane and do not undergo stretching in the thickness direction. These assumptions imply the vanishing of the transverse shear and transverse normal strains. The classical laminate theory has been used in the stress analysis of composite plates. However, it is only accurate for thin composite laminates.

In FSDT, a first-order displacement field is assumed for transverse shear strain through the thickness. Appropriate shear correction factors are required in FSDT due to the assumption of constant transverse shear strain and shear stress through the plate thickness, which is contradictory to the zero shear stress condition on the bounding planes of the laminate and actual stress states through the layer thickness.

Higher-order polynomials are used to represent displacement components through the thickness of the laminates in HSDT, and the actual transverse strain/stress through the thickness and the zero stress conditions on the top and bottom of a general laminate can be represented. A more accurate approximation of the transverse shear effect can thus be obtained with no shear correction factors. However, complexities in formulation and large computational effort make it economically unattractive. The free vibration of plates has been largely studied using the first order shear deformation theory (FSDT).

2.2 REVIEW OF SPHERICAL SHELL

During the past three to four decades, there has been continuously increasing usage of laminated composite materials in structural applications. Often encountered among these applications are plate and shell structural components. Accompanying this increasing usage has been a growth in the literature of composite laminate structural analysis, particularly for plates and to a lesser extent for spherical shells. Equations have been thoroughly developed for the deformation analysis of laminated composite plates (Ambartsumyan, 1964, Ashton and Whitney, 1970), as well as for circular cylindrical shells.

Thin shells are widely used as structural elements. Studies of thin shells are extensive and many theories have been developed. The first to study the curved shell problem was Aron .In 1874, Aron derived the first set of equations for a cylindrical shell. Love was the first to provide a mathematical framework for a thin shell theory. Love’s mathematical framework, also known as Love's first approximation theory, consisted of four principal assumptions under which many thin shell theories were developed. These four assumptions, commonly known as the Kirchhoff-Love hypotheses form the background of many linear thin shell theories, which over the years have been modified and employed to varying degree. Reddy (1984)[24] has developed a higher order but simple shear deformation theory of laminated plates and shells . The developed theory is simple in the sense that it contains the same dependent unknowns as in the first order shear deformation theory . the u and v displacement are expanded as cubic functions of the thickness coordinates and the transverse displacement is assumed as constant.

K. P. Soldatos [28] (University of Ioannina, Greece- 1986) used Love's approximations to determine influence of thickness shear deformation on free vibrations of rectangular plates, cylindrical panels and cylinders of antisymmetric angle-ply construction. This paper presented the influence of thickness shear deformation and rotatory inertia on the free vibrations of antisymmetric angle-ply laminated circular cylindrical panels. Two kinds of thickness shear deformable shell theories were considered. In the first one, uniformly distributed thickness shear strains through the shell thickness were assumed and, therefore, thickness shear correction factors were used. In the second theory a parabolic variation of thickness shear strains and stresses with zero values at the inner and outer shell surfaces was assumed. The analysis is mainly based on Love's approximations but, for purposes of comparison, Donnell's shallow shell approximations were also considered.

A.V. Singh [1] **(**The University of Western Ontario, London-1996) used Rayleigh–Ritz method for the free vibration analysis of deep doubly curved sandwich panels. This paper presented the free vibration analysis of doubly curved open deep sandwich shells made of thin outer layers and a relatively thick core. The outer layers were assumed to be made of high strength and high density material. The core was a low strength low density material. The Rayleigh–Ritz method, in which the displacement fields were defined by Bezier surface patches, was used to obtain the natural frequencies. The formulation procedure was developed for the analysis of open panels circumscribed by four curvilinear edges. Numerical results were obtained for circular cylindrical and spherical sandwich panels with the boundary conditions given by one curved edge clamped and the remaining three edges free as well as two opposite edges clamped and the other two completely free.

A.Dasgupta and K.H . Huang [4] (University of Maryland-1997) used finite element method for a layer-wise analysis for free vibration of thick composite spherical panels. The authors had previously presented layer-wise model for modeling the vibrations of thick composite cylindrical shell. The layer-wise theory is needed to overcome the deficiencies of conventional shear deformable plate theories because the gradient of the deformation field are not necessarily continuous through the thickness, due to the discontinuity of material properties at layer interfaces. Fully three dimensional finite element models place prohibitive demands on computational resources, and are not economically feasible. In this paper a similar layer-wise laminated shell theory was developed for doubly curved thick composite panels subjected to different combination of three-dimensional boundary conditions. Piece-wise continuous, quadratic interpolation functions through the thickness, were combined with beam function expansions in the two in plane directions of the laminate, to model the dynamic behaviour of laminated spherical panels. This captured the discontinuities in the transverse shear and other strain distributions, from one layer to another.

Jose Simoes Moita ,Cristovao M. Mota Soares and Carlos A. Mota Soares [12] (Dep. Engenharia Mecanica, Portugal-1999) used a single layer higher order shear deformation theory for the buckling and dynamic behaviour of laminated composite structures using a discrete higher-order displacement model. .This model is based on an eight-node C^0 serendipity finite element with 10 degrees of freedom per node to contemplate general applications. The present model was tested on the evaluation of buckling loads and free vibrations of multilaminated plates and shells. The effects of different number of layers, lamination angles, material anisotropy, and length or radius to thickness ratios are studied.

Partha Bhattacharya , Hassan Suhail , Prasanta K. Sinha [22] (Indian Institute of Technology, Kharagpur, 2002) developed a shear deformable shell element based on Reissner's hypothesis for the analysis of smart laminated composite shells. The electric field was defined in the curvilinear co-ordinate system. The mathematics of arbitrary shell geometry was amenable to the flexible mapping characteristic of the finite element formulation developed in the present work. Lame's parameters and the radii of curvature are generated within the model. An IMSC based LQR control methodology was adopted for the active vibration control of laminated spherical shells with different fibre orientation and varying radius of curvature.

Latifa S K and P. K. Sinha [16] (Department of Aerospace Engineering, Kharagpur-2005) used Koiter's shell theory and Mindlin's hypotheses for Improved Finite Element Analysis of Multilayered, Doubly Curved Composite Shells. An improved finite element model for the bending and free vibration analysis of doubly curved, laminated composite shells having spherical and ellipsoidal shapes was presented. The present formulation was based on the stress resultant-type Koiter's shell theory and no restriction was imposed on the magnitude of curvature components to capture the deep and shallow shell cases. The twist curvature component was incorporated along with the normal curvatures to keep the strain equations complete. Transverse shear deformation was also considered according to Mindlin's hypotheses. The present five-DOF shell formulation was kept sufficiently general to capture both the bending- and membrane dominated problems. The accuracy and efficiency of the proposed finite element were illustrated by examples and were compared with those existing in the open literature. The comparison shows that the present analysis yielded accurate results with a relatively small number of elements.

Umut Topal [30] (Department of Civil Engineering, Turkey-2006) used first-order shear deformation theory for Mode-Frequency Analysis of Laminated Spherical Shell. The paper dealt with mode-frequency analysis of a simply-supported equal-sided sector of a laminated spherical shell. The problem was modelled using finite element package program ANSYS. The formulation was based on first-order shear deformation theory. Four elements were chosen along each edge of the sector. The reduced method of eigenvalue solution was chosen for the undamped mode-frequency analysis. The first five modes were extracted to obtain the fundamental frequency (first mode natural frequency). The numerical studies were conducted to determine the effects of width-to-thickness ratio (b/h), degree of orthotropy (E_1 / E_2), fiber orientations (θ) on the non-dimensional fundamental frequency.

H. Nguyen-Van, N. Mai-Duy and T. Tran-Cong [10] (University of Southern Queensland-2007) analyzed laminated plate/shell structures based on FSDT with a stabilized nodal-integrated quadrilateral element. This paper reported numerical analyses of free vibration of laminated composite plate/shell structures of various shapes, span-to-thickness ratios, boundary conditions and lay-up sequences. The method was based on a novel four-node quadrilateral element, namely MISQ20, within the framework of the first-order shear deformation theory (FSDT). The element was built by incorporating a strain smoothing method into the bilinear four-node quadrilateral finite element where the strain smoothing operation was based on mesh-free conforming nodal integration. The bending and membrane stiffness matrices were based on the boundaries of smoothing cells while the shear term was evaluated by 2×2 Gauss quadrature. Through several numerical examples, the capability, efficiency and simplicity of the element were demonstrated.

Ahmet Sinan Oktem, Reaz A. Chaudhuri, [2] (Technical University of Lisbon-2008) used Higher-order theory based boundary-discontinuous Fourier analysis of simply supported thick cross-ply doubly curved panels. A boundary-discontinuous double Fourier series based solution methodology was employed to solve the problem of a HSDT-based thick cross-ply doubly curved panel, characterized by a system of five highly coupled linear partial differential equations, with the SS1-type simply supported boundary condition prescribed at all four edges. For derivation of the complementary solution, the complementary boundary constraints were introduced through boundary discontinuities of some of the particular solution functions and their partial derivatives. Such discontinuities form a set of measure zero. The present solution methodology was based on the fact that the complementary boundary constraints, which are inequalities, played as important a role as the prescribed (admissible) boundary conditions, which were equalities. The effects of curvature, lamination, material property, thickness as well as their interactions were investigated in detail.

Ahmet Sinan Oktem and Reaz A. Chaudhuri **(**Department of Materials Science and Engineering, USA-2008) used higher order shear deformation theory (HSDT) for Sensitivity of the response of thick cross-ply doubly curved panels to edge clamping. . A solution methodology, based on a boundary-discontinuous generalized double Fourier series approach, was used to solve a system of five highly coupled linear partial differential equations, generated by the HSDT-based laminated shell analysis, with the C3-type clamped boundary condition prescribed at all four edges. The numerical accuracy of the solution was ascertained by studying the convergence characteristics of deflections and moments of a cross-ply spherical panel. The primary focus of the present study was to investigate the effect of edge clamping on the response of a thick laminated doubly-curved panel, while keeping the surface-parallel edge constraints unaltered.

2.3 OBJECTIVE AND SCOPE OF PRESENT INVESTIGATION

In this project, an analytical solution of frequency characteristics for the free vibration of laminated composite cross-ply spherical thin panels is presented. Both the first order shear deformation theory and a higher order shear deformation theory as proposed by Reddy and Liu were used in the formulation. Compared with classical theory and higher order theories, the first order shear deformation theory combines higher accuracy and lower calculation efforts.

2.3.1 Objectives

1. To investigate the effect of different layer configurations on the natural frequency of the laminated composite cross-ply thin spherical panels.

2. To compare the results obtained by the first order shear deformation theory and a higher order shear deformation theory as proposed by Reddy and Liu.

3. To plot the frequency envelopes for the different cross-ply spherical panels.

2.4.2 Present Work

The present study is carried out to determine the natural frequency of vibration of laminated cross-ply spherical panels that are simply supported .The formulation has been done using both the first order shear deformation theory and a higher order shear deformation theory as proposed by Reddy and Liu. The transverse displacement is assumed to be constant through the thickness. The thickness coordinate multiplied by the curvature is assumed to be small in comparison to unity and hence negligible.

The governing equations have been developed. These equations are then reduced to the equations of motion for spherical panel and the Navier solution has been obtained for cross-ply laminated composite spherical panels. The resulting equations are suitably nondimensionalised. The eigen value problem is then solved to obtain the free vibration frequencies.

Chapter 3

Theoretical Formulation

CHAPTER - 3

THEORETICAL FORMULATION

3.1 INTRODUCTION

The present study deals with the free vibration of thin laminated composite cross-ply spherical panels. The first order shear deformation theory and a higher order shear deformation theory as proposed by Reddy and Liu are used. The displacement components **u, v,** and **w** in the **α, β,** and **z** directions in a laminate element can be expressed in terms of the corresponding mid-plane displacement components **u°, v° ,w°,** and the rotations Φ_1 , Φ_2 of the mid-plane normal along **α** and **β** axes. The higher order shear deformation theory is based on a displacement field in which the displacements of the middle surface are expanded as cubic functions of the thickness coordinate and the transverse displacement is assumed to be constant through the thickness. This displacement field leads to parabolic distribution of the transverse shear stresses and zero transverse normal strain and hence no shear correction factors are used.

The governing equations including the effect of shear deformation are presented in orthogonal curvilinear co-ordinates for laminated composite shells. These equations are then reduced to the governing equations for free vibration of laminated composite cross-ply spherical shells. The equations are suitably non-dimensionalised. The Navier solution has been used and the generalized eigen value problem so obtained in matrix formulation is solved to obtain the eigen values which are the natural frequencies.

3.2 BASIC ASSUMPTIONS

A set of simplifying assumptions that provide a reasonable description of the behavior of thin elastic shells is used to derive the equilibrium equations that are consistent with the assumed displacement field.

1. No slippage takes place between the layers.
2. The effect of transverse normal stress on the gross response of the laminate is assumed to be negligible.
3. The line elements of the shell normal to the reference surface do not change their length after deformation.
4. The thickness coordinate of the shell is small compared to the principal radii of curvature (z/R_1, z/R_2 <<<1).
5. Normal to the reference surface of the shell before deformation remains straight, but not necessarily normal, after deformation (a relaxed Kirchhoff -Love hypothesis).

3.3 STRAIN DISPLACEMENT RELATIONS

Figure shows an element of a doubly curved shell. Here (α, β, z) denote the orthogonal curvilinear coordinates (shell coordinates) such that α and β curves are lines of curvature on the mid surface, z = 0, and z-curves are straight lines perpendicular to the surface, z = 0. For the doubly curved shells discussed here, the lines of principal curvature coincide with the co-ordinate lines. The values of the principal curvature of the middle surface are denoted by K_1 and K_2

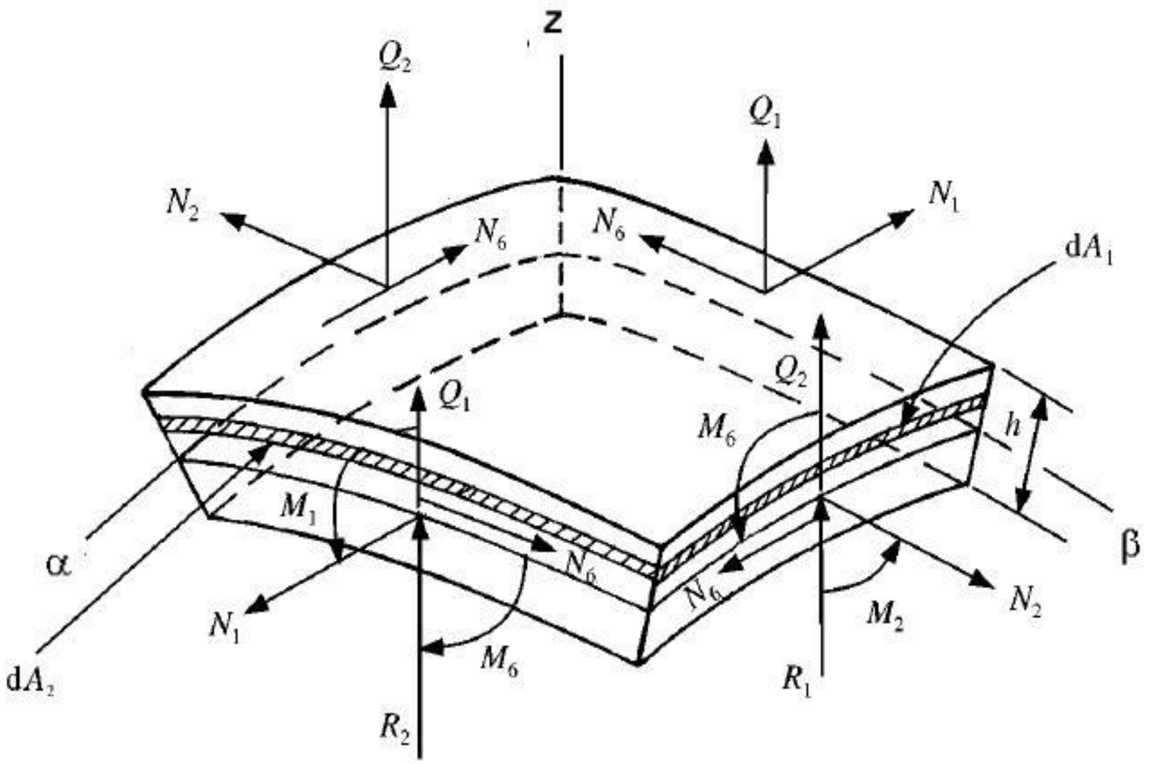

Figure 1: Geometry of laminated shell

The position vector of a point on the middle surface is denoted by r and the position of a point at distance, z, from the middle surface is denoted by R. The distance, ds, between points (α, β, z) and $(\alpha + d\alpha, \beta + d\beta, z + dz)$ is determined by

$$(ds)^2 = \overline{dr} \cdot \overline{dr} \tag{1}$$

$$\overline{dr} = \frac{\partial \bar{r}}{\partial \alpha} d\alpha + \frac{\partial \bar{r}}{\partial \beta} d\beta \tag{2}$$

The magnitude ds of $d\bar{r}$ is given in equation (2), the vectors $\frac{\partial \bar{r}}{\partial \alpha}$ and $\frac{\partial \bar{r}}{\partial \beta}$ are tangent to the and β coordinate lines. Then equation (1) can be modified as

$$(ds)^2 = \frac{\partial \bar{r}}{\partial \alpha}\frac{\partial \bar{r}}{\partial \alpha}(d\alpha)^2 + \frac{\partial \bar{r}}{\partial \beta}\frac{\partial \bar{r}}{\partial \beta}(d\beta)^2 + 2\frac{\partial \bar{r}}{\partial \alpha}\frac{\partial \bar{r}}{\partial \beta}d\alpha.d\beta \tag{3}$$

The following derivation is limited to orthogonal curvilinear coordinates which coincide with the lines of principal curvature of the neutral surface. The third term in equation (3) thus becomes

$$2\frac{\partial \bar{r}}{\partial \alpha}\frac{\partial \bar{r}}{\partial \beta}d\alpha.d\beta = 2\left|\frac{\partial \bar{r}}{\partial \alpha}\right|.\left|\frac{\partial \bar{r}}{\partial \beta}\right|\cos\frac{\pi}{2}d\alpha.d\beta = 0 \tag{4}$$

Where we define

$$\frac{\partial \bar{r}}{\partial \alpha}\frac{\partial \bar{r}}{\partial \alpha} = \left|\frac{\partial \bar{r}}{\partial \alpha}\right|^2 = A_1^{\ 2} \tag{5}$$

$$\frac{\partial \bar{r}}{\partial \beta}\frac{\partial \bar{r}}{\partial \beta} = \left|\frac{\partial \bar{r}}{\partial \beta}\right|^2 = A_2^{\ 2}$$

Now the equation (3) becomes

$$(ds)^2 = A_1^{\ 2}(d\alpha)^2 + A_2^{\ 2}(d\beta)^2 \tag{6}$$

This equation is called the fundamental form and A_1 and A_2 are the fundamental form parameters, Lame parameters, or surface metrics. The distance, dS, between points (α, β, z) and $(\alpha + d\alpha, \beta + d\beta, z + dz)$ is given by

$$(ds)^2 = d\overline{R}.d\overline{R}. = L_1{}^2(d\alpha)^2 + L_2{}^2(d\beta)^2 + L_3{}^2(dz)^2 \qquad (7)$$

In which $d\overline{R} = \frac{\partial \overline{R}}{\partial \alpha} d\alpha + \frac{\partial \overline{R}}{\partial \beta} d\beta + \frac{\partial \overline{R}}{\partial z} dz$ and L_1, L_2, and L_3 are the Lame's coefficients.

$$L_1 = A_1\left[1 + \frac{z}{R_1}\right]; L_2 = A_2\left[1 + \frac{z}{R_2}\right]; \quad L_3 = 1 \qquad (8)$$

The vectors $\frac{\partial \overline{R}}{\partial \alpha}$ and $\frac{\partial \overline{R}}{\partial \beta}$ are parallel to the vectors $\frac{\partial \overline{r}}{\partial \alpha}$ and $\frac{\partial \overline{r}}{\partial \beta}$.

From the figure the elements of area of the cross section are

$$\begin{aligned} da_1 &= L_1 d\alpha.dz = A_1\left[1 + \frac{z}{R_1}\right] d\alpha.dz \\ da_2 &= L_2 d\beta.dz = A_2\left[1 + \frac{z}{R_2}\right] d\beta.dz \end{aligned} \qquad (9)$$

The strain displacement equations of a shell are an approximation, within the assumptions made previously, of the strain displacement relations referred to orthogonal curvilinear coordinates. In addition, it is assumed that the transverse displacement, w, does not vary with z. According to the first order shear deformation theory, the displacement field is given by:

$$\begin{aligned} \bar{u} &= \left(1 + \frac{z}{R_1}\right) u + z\phi_1 \\ \bar{v} &= \left(1 + \frac{z}{R_2}\right) v + z\phi_2 \\ \bar{w} &= w \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (10)$$

Here $(\bar{u},\bar{v},\bar{w})$ = the displacement of a point (α, β, z) along the (α, β, z) coordinates; and (u, v, w) = the displacements of a point (α, β, 0). Now substituting equation (10) in the strain displacement relations referred to an orthogonal curvilinear coordinate system [Kraus-14] we get

$$\begin{aligned}
\varepsilon_1 &= \varepsilon_1{}^0 + zk_1 \\
\varepsilon_2 &= \varepsilon_2{}^0 + zk_2 \\
\varepsilon_4 &= \varepsilon_4{}^0 \\
\varepsilon_5 &= \varepsilon_5{}^0 \\
\varepsilon_6 &= \varepsilon_6{}^0 + zk_6
\end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (11.a)$$

Where,

$$\begin{aligned}
\varepsilon_1{}^0 &= \frac{1}{A_1}.\frac{\partial u}{\partial \alpha} + \frac{w}{R_1}; \\
\varepsilon_2{}^0 &= \frac{1}{A_2}.\frac{\partial v}{\partial \beta} + \frac{w}{R_2}; \\
\varepsilon_6{}^0 &= \frac{1}{A_1}.\frac{\partial v}{\partial \alpha} + \frac{1}{A_2}\frac{\partial u}{\partial \beta}; \\
\varepsilon_4{}^0 &= \phi_2 + \frac{1}{A_2}\frac{\partial w}{\partial \beta} - \frac{u}{R_1}; \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad 11.b \\
\varepsilon_5{}^0 &= \phi_1 + \frac{1}{A_1}\frac{\partial w}{\partial \beta} - \frac{v}{R_2}; \\
k_1 &= \frac{1}{A_1}.\frac{\partial \phi_1}{\partial \alpha}; \\
k_2 &= \frac{1}{A_2}.\frac{\partial \phi_2}{\partial \beta}; \\
k_6 &= \frac{1}{A_1}.\frac{\partial \phi_2}{\partial \alpha} + \frac{1}{A_2}\frac{\partial \phi_1}{\partial \beta} + \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{1}{A_1}\frac{\partial v}{\partial \alpha} - \frac{1}{A_2}\frac{\partial u}{\partial \beta}\right);
\end{aligned}$$

ϕ_1 and ϕ_2 are the rotations of the normals to the reference surface, z = 0, about the α and β coordinate axes, respectively. The displacement field in equation [10] can be used to derive the general theory of laminated shells.

3.5 STRESS-STRAIN RELATIONS

The stress-strain relation for the K$_{th}$ orthotropic layer takes the following form:

$$\begin{Bmatrix} \sigma_1{}^K \\ \sigma_2{}^K \\ \sigma_4{}^K \\ \sigma_5{}^K \\ \sigma_6{}^K \end{Bmatrix} = \begin{bmatrix} Q_{11}{}^K & Q_{12}{}^K & 0 & 0 & Q_{16}{}^K \\ Q_{12}{}^K & Q_{22}{}^K & 0 & 0 & Q_{26}{}^K \\ 0 & 0 & Q_{44}{}^K & Q_{45}{}^K & 0 \\ 0 & 0 & Q_{45}{}^K & Q_{55}{}^K & 0 \\ Q_{16}{}^K & Q_{26}{}^K & 0 & 0 & Q_{66}{}^K \end{bmatrix} \begin{Bmatrix} \varepsilon_1{}^K \\ \varepsilon_2{}^K \\ \varepsilon_4{}^K \\ \varepsilon_5{}^K \\ \varepsilon_6{}^K \end{Bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 12$$

For an orthotropic material, in which the principal axis direction coincides with the axis of the material direction,

$$Q_{16}{}^K = Q_{26}{}^K = Q_{45}{}^K = 0$$

Then,

$$\begin{Bmatrix} \sigma_1{}^K \\ \sigma_2{}^K \\ \sigma_4{}^K \\ \sigma_5{}^K \\ \sigma_6{}^K \end{Bmatrix} = \begin{bmatrix} Q_{11}{}^K & Q_{12}{}^K & 0 & 0 & 0 \\ Q_{12}{}^K & Q_{22}{}^K & 0 & 0 & 0 \\ 0 & 0 & Q_{44}{}^K & 0 & 0 \\ 0 & 0 & 0 & Q_{55}{}^K & 0 \\ 0 & 0 & 0 & 0 & Q_{66}{}^K \end{bmatrix} \begin{Bmatrix} \varepsilon_1{}^K \\ \varepsilon_2{}^K \\ \varepsilon_4{}^K \\ \varepsilon_5{}^K \\ \varepsilon_6{}^K \end{Bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 13$$

For generalized plane stress conditions, the above elastic moduli $Q_{ij}{}^k$ is related to the usual engineering constants as follows:

$$Q_{12} = \frac{\nu_{21}.E_{11}}{1-\nu_{12.}\nu_{21}} = \frac{\nu_{12}.E_{22}}{1-\nu_{12.}\nu_{21}};$$
$$Q_{66} = G_{12}; Q_{44} = G_{13}; Q_{55} = G_{23}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 14$$
$$\frac{E_1}{E_2} = \frac{\nu_{12}}{\nu_{21}}.$$

3.6 STRESS RESULTANTS AND STRESS COUPLES

Let N_1 be the tensile force, measured per unit length along β coordinate line, on a cross section perpendicular to α coordinate line. Then the total tensile force on the differential element in the α direction is

$$N_1.A_2 = \int_{-h/2}^{h/2} \sigma_1 da_2 dz \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

In which h = the thickness of the shell (z = -h/2 and z = h/2 denote the bottom and top surfaces of the shell) and da_2 is the area of cross section. Using equation (9), the remaining stress resultants per unit length are given by:

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_{12} \\ N_{21} \\ Q_1 \\ Q_2 \\ M_1 \\ M_2 \\ M_{12} \\ M_{21} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_1\left(1+\frac{z}{R_2}\right) \\ \sigma_2\left(1+\frac{z}{R_1}\right) \\ \sigma_6\left(1+\frac{z}{R_2}\right) \\ \sigma_5\left(1+\frac{z}{R_2}\right) \\ \sigma_4\left(1+\frac{z}{R_1}\right) \\ z\sigma_1\left(1+\frac{z}{R_2}\right) \\ z\sigma_2\left(1+\frac{z}{R_1}\right) \\ z\sigma_6\left(1+\frac{z}{R_2}\right) \\ z\sigma_6\left(1+\frac{z}{R_1}\right) \end{Bmatrix} dz \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad 16 \quad .$$

In contrast to the plate theory (where $1/R_1 = 0, 1/R_2 = 0$), the shear stress resultants, N_{12} and N_{21}, and the twisting moments, M_{12} and M_{21}, are, in general, not equal. For shallow shells the terms

z/R_1 and z/R_2 can be neglected in comparison with unity. Hence $N_{12} = N_{21} = N_6$ and $M_{12} = M_{21} = M_6$.

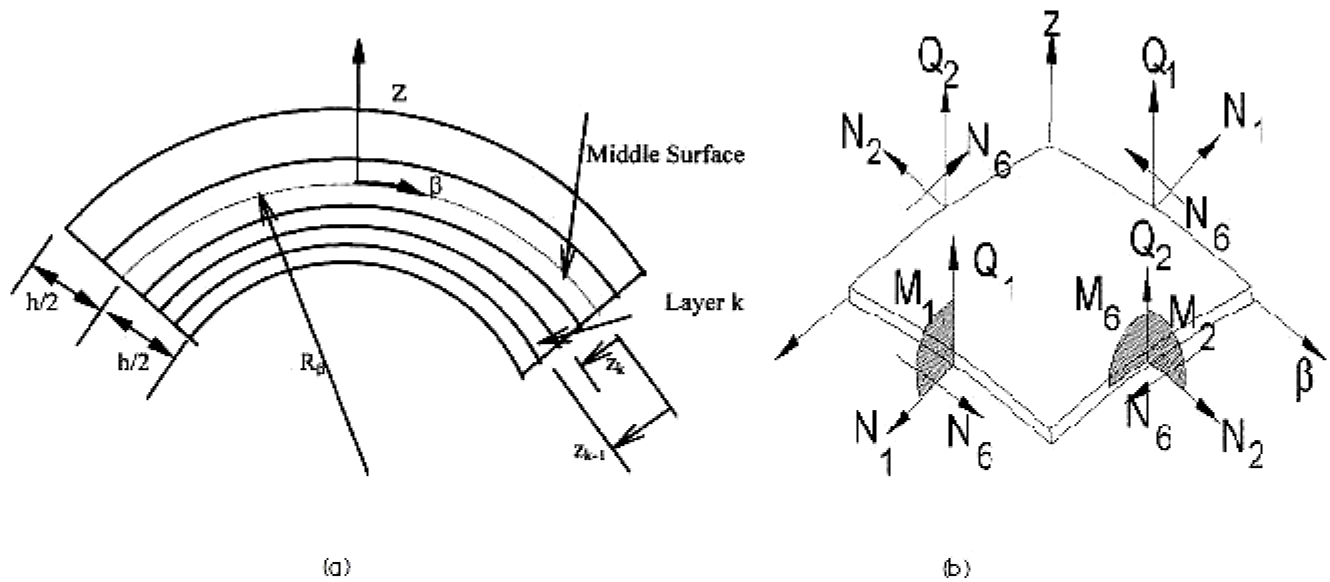

Figure 2: stress and moment resultants

The shell under consideration is composed of finite number of orthotropic layers of uniform thickness, as shown in Figure 2.(a). In view of assumption 1, the stress resultant in equation [16] can be expressed as

$$\left(N_{i,}M_i\right) = \sum_{k=1}^{n} \int_{h_{k-1}}^{hk} \sigma_i^{\,k}(1,z).dz \ldots\ldots\ldots\ldots\ldots\ldots(i = 1,2,6)$$

$$Q_i = \sum_{k=1}^{n} \int_{h_{k-1}}^{hk} \sigma_i^{\,k}.dz \ldots\ldots\ldots\ldots\ldots\ldots(i = 4,5) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (17)$$

In which n = the number of layers in the shell; h_k and h_{k-1} is the top and bottom z coordinates of the k^{th} lamina.

Substitution of equation [11] and [13] into equation [17] leads to the following expression for the stress resultants and stress couples

$$N_i = A_{ij}.\varepsilon_j^{\ 0} + B_{ij}.K_j^{\ 0};$$
$$M_i = B_{ij}\varepsilon_j^{\ 0} + D_{ij}K_j^{\ 0};$$
$$Q_1 = A_{45}.\varepsilon_4^{\ 0} + A_{55}\varepsilon_5^{\ 0};$$
$$Q_2 = A_{44}.\varepsilon_4^{\ 0} + A_{45}\varepsilon_5^{\ 0};$$

………………………….. (18)

Here A_{ij}, B_{ij} and D_{ij} denote the extensional, flexural-extensional coupling, and flexural stiffness. They may be defined as:

$$\left(A_{ij}, B_{ij}, D_{ij}\right) = \sum_{k=1}^{n} \int_{h_{k-i}}^{h_k} Q_{ij}^{\ k}(1, z, z^2).dz. \quad \text{………………} \quad (19)$$

For i, j = 1, 2, 4, 5, 6.

h_k and h_{k-1} are the distances measured as shown in figure -1

3.7 GOVERNING EQUATIONS

The governing differential equations, the strain energy due to loads, kinetic energy and formulations of the general dynamic problem are derived on the basis of Hamilton's principle.

3.7.1 GOVERNING DIFFERENTIAL EQUATIONS

The equation of motion is obtained by taking a differential element of the shell as shown in Figure 1. The figure shows an element with internal forces like membrane (N_1, N_2, and N_6), shearing forces (Q_1, and Q_2) and the moment resultants (M_1, M_2 and M_6).

3.7.1.1 Strain energy

The strain energy of a differential shell element can be written as,

$$U = \frac{1}{2}\int_{\alpha}\int_{\beta}\int_{z}\left[\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_6\varepsilon_6 + \sigma_4\varepsilon_4 + \sigma_5\varepsilon_5\right].dV \quad \ldots\ldots\ldots\ldots\ldots.. \quad (20)$$

dV = Volume of shell element

$$dV = A_1\left(1+\frac{z}{R_1}\right)A_2\left(1+\frac{z}{R_2}\right).d\alpha.d\beta.dz. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

The variation of strain energy U is given by

$$\delta U = \frac{1}{2}\int_{\alpha}\int_{\beta}\int_{z}\left[\sigma_1\delta\varepsilon_1 + \sigma_2\delta\varepsilon_2 + \sigma_6\delta\varepsilon_6 + \sigma_4\delta\varepsilon_4 + \sigma_5\delta\varepsilon_5\right].dV \quad \ldots\ldots\ldots\ldots\ldots \quad (22)$$

The equation [22] is independent of the material property. Substituting the variation of strain function and dV in equation (20),

$$U = \frac{1}{2}\int_{\alpha}\int_{\beta}\int_{z}[N_1\varepsilon_1{}^0 + M_1k_1 + N_2\varepsilon_2{}^0 + M_2k_2 + N_6w_1 + M_6\tau_1 + N_6w_2 + M_6\tau_2 + Q_1\left(1+\frac{z}{R_1}\right)\varepsilon_4 + Q_2\left(1+\frac{z}{R_2}\right)\varepsilon_5]A_1A_2d\alpha d\beta. \quad \ldots \quad (23)$$

The variation of strain energy is given as:

$$\delta U = \frac{1}{2}\int_{\alpha}\int_{\beta}\int_{z}[N_1\delta\varepsilon_1{}^0 + M_1\delta k_1 + N_2\delta\varepsilon_2{}^0 + M_2\delta k_2 + N_6\delta w_1 + M_6\delta\tau_1 + N_6\delta w_2 + M_6\delta\tau_2 + Q_1\left(1+\frac{z}{R_1}\right)\delta\varepsilon_4 + Q_2\left(1+\frac{z}{R_2}\right)\delta\varepsilon_5]A_1A_2d\alpha d\beta. \quad \ldots\ldots.. \quad (24)$$

Substituting for $\varepsilon_1{}^0, \varepsilon_2{}^0, k_1, k_2$.....in equation (24) and integrating the resulting expression, the equation becomes,

$$\delta U=\int_{\alpha}\int_{\beta}\{-\frac{\partial(N_1A_2)}{\partial\alpha}\delta u+N_1\delta v\frac{\partial A_1}{\partial\beta}+\frac{N_1A_1A_2}{R_1}\delta w+M_1\delta\phi_2\frac{\partial A_1}{\partial\beta}-\frac{\partial(M_1A_2)}{\partial\alpha}\delta\phi_1+N_2\delta u\frac{\partial A_2}{\partial\alpha}$$

$$+N_2\frac{A_1A_2}{R_2}\delta w-\frac{\partial(N_2A_1)}{\partial\beta}\delta v+M_2\delta\phi_1\frac{\partial A_2}{\partial\alpha}-\frac{\partial(M_2A_1)}{\partial\beta}\delta\phi_2-N_6\delta u\frac{\partial A_1}{\partial\beta}-\frac{\partial(N_6A_2)}{\partial\alpha}\delta v-M_6\delta\phi_1\frac{\partial A_1}{\partial\beta}$$

$$-\frac{\partial(M_6A_2)}{\partial\alpha}\delta\phi_2-N_6\delta v\frac{\partial A_2}{\partial\alpha}-\frac{\partial(N_6A_1)}{\partial\beta}\delta u-M_6\delta\phi_2\frac{\partial A_2}{\partial\alpha}-\frac{\partial(M_6A_1)}{\partial\beta}\delta\phi_1+Q_1A_1A_2\left[\delta\phi_1-\frac{\delta u}{R_1}\right]-$$

$$\frac{\partial(Q_1A_2)}{\partial\alpha}\delta w+Q_2A_1A_2\left[\delta\phi_1-\frac{\delta u}{R_2}\right]-\frac{\partial(Q_2A_1)}{\partial\beta}\delta w\ \}d\alpha.d\beta \qquad +$$

$$\oint_{\beta}\left[N_1\delta u+N_6\delta v+Q_1\delta w+M_1\delta\phi_1+M_6\delta\phi_2\right]A_2d\beta$$

$$+\oint_{\alpha}\left[N_2\delta v+N_6\delta u+Q_2\delta w+M_2\delta\phi_2+M_6\delta\phi_1\right]A_1d\alpha \qquad \ldots\ldots\ldots\ldots.. \qquad (25)$$

3.7.1.2 Kinetic energy

If U be the displacement vector, the kinetic energy of the shell element is given by,

$$T=\frac{1}{2}\int_v\rho\,\overline{\dot{U}}.\overline{\dot{U}}\quad dV \qquad \ldots\ldots\ldots\ldots \qquad (26)$$

where ρ is the mass density and $\overline{\dot{U}}$ represents differentiation with respect to time.

$$\overline{U}=U_1\overline{n_1}+U_2\overline{n_2}+W\overline{n_1}$$

$$T=\frac{\rho h}{2}\int_{\alpha}\int_{\beta}\left\{\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2+\frac{h^2}{12}\left[\left(\frac{\partial\phi_1}{\partial t}\right)^2+\left(\frac{\partial\phi_2}{\partial t}\right)^2\right]\right\}A_1A_2.d\alpha.d\beta \ \ldots \qquad (27)$$

The variation of kinetic energy is given as

$$\delta T=\rho h\int_{\alpha}\int_{\beta}\left[\dot{u}\,\delta\dot{u}+\dot{v}\,\delta\dot{v}+\dot{w}\,\delta\dot{w}+\frac{h^2}{12}(\dot{\phi}_1\,\delta\dot{\phi}_1+\dot{\phi}_2\,\delta\dot{\phi}_2)\right]A_1A_2.d\alpha.d\beta \qquad \ldots\ldots\ldots \qquad (28)$$

This equation contains time derivatives of the variations, ie $\delta\dot{u}$ etc. To eliminate these terms, integrating equation (28) by parts (The variations of limits t = t_1 and t = t_2 must vanish) and neglecting the terms $\rho h^3 / 12\ddot{\phi}_1$ and $\rho h^3 / 12\ddot{\phi}_2$ which represent rotatory inertia, the above equation reduces to,

$$\int_{t1}^{t_2} \delta T.dt = -\rho h \int_{t1}^{t_2} \int_\alpha \int_\beta \left[\dot{u}\,\delta\dot{u} + \dot{v}\,\delta\dot{v} + \dot{w}\,\delta\dot{w} \right] A_1 A_2 .d\alpha.d\beta.dt \qquad \text{................} \qquad (29)$$

If the shell is subjected to both body and surface forces and if q_1,q_2 and q_n are the components of body and surface forces along the parametric lines, then the variation of work done by the external loads are,

$$\delta W_s = \int_\alpha \int_\beta \left(q_1 \delta u + q_2 \delta v - q_n \delta w \right) A_1 .A_2 .d\alpha.d\beta$$

$$\delta W_{e1} = \oint_\beta \left(\overline{N_1} \delta u + \overline{N_6} \delta v + \overline{Q_1} \delta w + \overline{M_1} \delta \phi_1 + \overline{M_6} \delta \phi_2 \right) A_2 .d\beta \qquad \text{...............} \qquad (30)$$

$$\delta W_{e2} = \oint_\alpha \left(\overline{N_6} \delta u + \overline{N_2} \delta v + \overline{Q_2} \delta w + \overline{M_6} \delta \phi_1 + \overline{M_2} \delta \phi_2 \right) A_1 .d\alpha$$

3.7.2 **HAMILTON'S PRINCIPLE**

The equations of equilibrium are derived by applying the dynamic version of the principle of virtual work that is the Hamilton's Principle.

It states that among the set of all admissible configurations of system, the actual motion makes the quantity $\int_{t1}^{t_2} L.dt$ stationary, provided the configuration is known at the limits t = t_1 and t = t_2.

Mathematically this means $\delta \int_{t1}^{t_2} L.dt$

Here, L is called Lagrangian and is equal to L= T - (U – V) ………. (31)

Where, T = Kinetic energy, U= Strain energy, V= potential of all applied loads (= 0, because of free vibration), δ = Mathematical operation called variation. It is analogous to partial differentiation. It is clear from equation [31] that the Lagrangian consists of kinetic, strain energy and potential of applied loads.

3.7.3 EQUATIONS OF EQUILIBRIUM

By applying the dynamic version of the principle of virtual work (Hamilton's Principle), integrating the displacement gradients by parts in the resulting equation and setting the coefficient of δ u, δ v, δ w, $\delta\phi_1, \delta\phi_2$ to zero separately, the following equations of equilibrium are obtained

$$\frac{\partial(N_1 A_2)}{\partial\alpha} + \frac{\partial(N_6 A_1)}{\partial\beta} + N_6\frac{\partial A_1}{\partial\beta} - N_2\frac{\partial A_2}{\partial\alpha} + A_1 A_2\left(\frac{Q_1}{R_1}\right) = A_1 A_2 \rho h \ddot{u}$$

$$\frac{\partial(N_6 A_2)}{\partial\alpha} + \frac{\partial(N_2 A_1)}{\partial\beta} + N_6\frac{\partial A_2}{\partial\alpha} - N_1\frac{\partial A_1}{\partial\beta} + A_1 A_2\left(\frac{Q_2}{R_2}\right) = A_1 A_2 \rho h \ddot{v}$$

$$\frac{\partial(Q_1 A_2)}{\partial\alpha} + \frac{\partial(Q_2 A_1)}{\partial\beta} - A_1 A_{2(\ )}\left(\frac{N_1}{R_1} + \frac{N_2}{R_2}\right) = A_1 A_2 \rho h \ddot{w} \qquad \ldots\ldots\ldots.. \qquad (32)$$

$$\frac{\partial(M_1 A_2)}{\partial\alpha} + \frac{\partial(M_6 A_1)}{\partial\beta} + M_6\frac{\partial A_1}{\partial\beta} - M_2\frac{\partial A_2}{\partial\alpha} - Q_1 A_1 A_2 = 0$$

$$\frac{\partial(M_6 A_2)}{\partial\alpha} + \frac{\partial(M_2 A_1)}{\partial\beta} + M_6\frac{\partial A_2}{\partial\alpha} - M_1\frac{\partial A_1}{\partial\beta} - Q_2 A_1 A_2 = 0$$

3.7.3.1 SPHERICAL SHELLS

The governing equations derived in orthogonal curvilinear coordinates in the previous section for general shell element is reduced for spherical shell. The equation of motion is represented in terms of displacements.

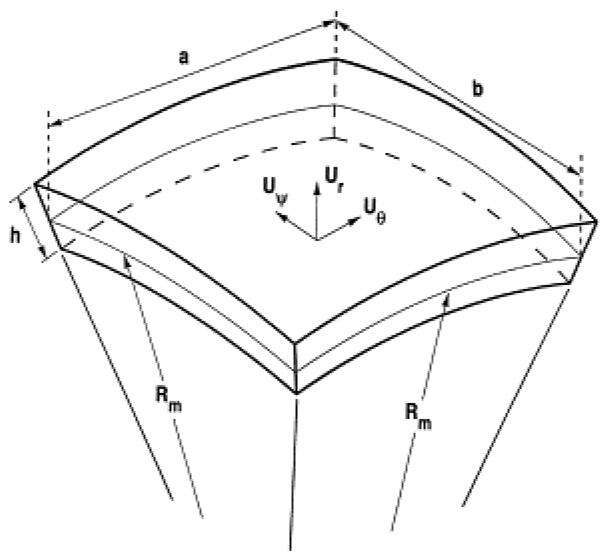

Figure 3: Geometry and co-ordinate system

3.7.3.2 Equations of equilibrium for laminated composite spherical shell

For the spherical shell configuration shown in figure 3, the co-ordinates are given. The Lame parameters $A_1 = A_2 = 1$(for thin shells) and the principal curvatures $K_1 = K_2 = 1/R$, where 'R' is the radius of the mid-surface of the spherical shell.

The equations of motion in terms of the stress resultants and stress couples obtained from Equations (32) become,

$$\frac{\partial N_1}{\partial \alpha}+\frac{\partial N_6}{\partial \beta}+\frac{Q_1}{R}-\rho h \ddot{u}=0$$

$$\frac{\partial N_6}{\partial \alpha}+\frac{\partial N_2}{\partial \beta}+\frac{Q_2}{R}-\rho h \ddot{v}=0$$

$$\frac{\partial Q_1}{\partial \alpha}+\frac{\partial Q_2}{\partial \beta}-\frac{1}{R}(N_1+N_2)-\rho h \ddot{w}=0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (33)$$

$$\frac{\partial M_1}{\partial \alpha}+\frac{\partial M_6}{\partial \beta}-Q_1=0$$

$$\frac{\partial M_6}{\partial \alpha}+\frac{\partial M_2}{\partial \beta}-Q_2=0$$

The strain displacement relations (11.b) are substituted in the equations for the stress resultants and stress couples given in equation (18). Since the solution for the equations of motion is done by using the Navier solution, therefore such a solution exist only for specially antisymmetric cross ply laminates for which the following laminate stiffness are zero.

$$A_{16}=A_{26}=B_{16}=B_{26}=D_{16}=D_{26}=A_{45}=0$$

The expression for the stress resultants and stress couples so obtained are then substituted into the equations of motion (33). The equation of motion in terms of the displacements hence reduces to

$$A_{11}.\frac{\partial^2 u}{\partial \alpha^2}+A_{66}.\frac{\partial^2 u}{\partial \beta^2}+[A_{12}+A_{66}]\frac{\partial^2 v}{\partial \alpha \partial \beta}+\left[\frac{1}{R}[A_{11}+A_{12}+A_{55}]\right].\frac{\partial w}{\partial \alpha}+B_{11}\frac{\partial^2 \phi_1}{\partial \alpha^2}+B_{66}\frac{\partial^2 \phi_1}{\partial \beta^2}+$$
$$[B_{12}+B_{66}]\frac{\partial^2 \phi_2}{\partial \alpha \partial \beta}+A_{55}\frac{\phi_1}{R}-A_{55}\frac{u}{R^2}-\rho h\ddot{u}=0.$$

$$[A_{21}+A_{66}]\frac{\partial^2 u}{\partial\alpha\partial\beta}+A_{66}\frac{\partial^2 v}{\partial\alpha^2}+A_{22}\frac{\partial^2 v}{\partial\beta^2}+\left[\frac{1}{R}[A_{21}+A_{22}+A_{44}]\right]\frac{\partial w}{\partial\beta}+[B_{21}+B_{66}]\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}+B_{66}\frac{\partial^2\phi_2}{\partial\alpha^2}+B_{22}$$

$$\frac{\partial^2\phi_2}{\partial\beta^2}+A_{44}\frac{\phi_2}{R}-A_{44}\frac{v}{R^2}-\rho h\ddot{v}=0$$

$$\left[-\frac{1}{R}[A_{11}+A_{21}+A_{55}]\right]\frac{\partial u}{\partial\alpha}+\left[-\frac{1}{R}[A_{12}+A_{22}+A_{44}]\right]\frac{\partial v}{\partial\beta}+\left[A_{55}\frac{\partial^2 w}{\partial\alpha^2}+A_{44}\frac{\partial^2 w}{\partial\beta^2}\right]+\left[A_{55}-\frac{1}{R}[B_{11}+B_{21}]\right]$$

$$\frac{\partial\phi_1}{\partial\alpha}+\left[A_{44}-\frac{1}{R}[B_{12}+B_{22}]\right]\frac{\partial\phi_2}{\partial\beta}+\left[-\frac{1}{R}[A_{11}+A_{12}+A_{21}+A_{22}]\right]\frac{w}{R}-\rho h\ddot{w}=0$$

$$B_{11}\frac{\partial^2 u}{\partial\alpha^2}+B_{66}\frac{\partial^2 u}{\partial\beta^2}+[B_{12}+B_{66}]\frac{\partial^2 v}{\partial\alpha\partial\beta}+\left[\frac{1}{R}[B_{11}+B_{12}]-A_{55}\right]\frac{\partial w}{\partial\alpha}+D_{11}\frac{\partial^2\phi_1}{\partial\alpha^2}+D_{66}\frac{\partial^2\phi_1}{\partial\beta^2}+[D_{12}+D_{66}]$$

$$\frac{\partial^2\phi_2}{\partial\alpha\partial\beta}-A_{55}\phi_1+A_{55}\frac{u}{R}=0$$

$$[B_{66}+B_{21}]\frac{\partial^2 u}{\partial\alpha\partial\beta}+B_{66}\frac{\partial^2 v}{\partial\alpha^2}+B_{22}\frac{\partial^2 v}{\partial\beta^2}+\left[\frac{1}{R}[B_{21}+B_{22}]-A_{44}\right]\frac{\partial w}{\partial\beta}+D_{66}\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}+D_{21}\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}+D_{66}\frac{\partial^2\phi_2}{\partial\alpha^2}+$$

$$D_{22}\frac{\partial^2\phi_2}{\partial\beta^2}-A_{44}\phi_2+A_{44}\frac{v}{R}=0$$

…………………... (34)

3.8 HIGHER ORDER SHEAR DEFORMATION THEORY

The geometry of the layered shell is shown in figure 1 and 2(a). α,β and z are the orthogonal curvilinear coordinate such that α and β are the principle lines of curvature on the reference surface,which is the mid surface for the present analysis. The value of the principle curvature of the middle surface are denoted by $K_1 and\ K_2$.

The line element is given by

$$ds^2 = A_1^2(1+zK_1)^2 d\alpha^2 + A_2^2(1+zK_2)^2 d\beta^2 + dz^2 \qquad \text{……………………} \quad 35$$

Where A_j and K_j are the lame parameters and principle curvature respectively and are the functions of α,β only. The spherical panel under consideration is composed of '*n*' orthotropic layers of uniform thickness. The displacement field relations as proposed by Reddy and Liu [24] are

$$\begin{aligned}\bar{u}(\alpha,\beta,z,t) &= (1+K_1 z)u + z\phi_1 + z^2\varphi_1 + z^3\theta_1\\ \bar{v}(\alpha,\beta,z,t) &= (1+K_2 z)v + z\phi_2 + z^2\varphi_2 + z^3\theta_2\\ \bar{w}(\alpha,\beta,z,t) &= w\end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots \quad 35.a$$

Where t is the time , $(\bar{u},\bar{v},\bar{w})$ are the displacement along the (α,β,z)coordinates (u,v,w) are the displacements of a point on the middle surface and ϕ_1 and ϕ_2 are the rotation at z=0of normals to the midsurface with respect to α and β axes,respectively.The displacement field in equation (35.a) are so chosen that the transverse shear strains will be quadratic functions of the thickness coordinates, z and the transverse normal strain will be zero.

The functions φ_i and θ_i are determined using the coordination that the transverse shear stresses σ_4 and σ_5 vanish on the top and bottom surfaces of the shell;

$$\begin{aligned}\sigma_4(\alpha,\beta,\pm\frac{h}{2},t) &= 0\\ \sigma_5(\alpha,\beta,\pm\frac{h}{2},t) &= 0\end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad 35.b$$

For shells laminated of orthotropic layers, the conditions (35.b) are equivalent to the requirement that the corresponding strain be zero on these surfaces. The transverse shear strains of ashell with two principle radii of curvature are given by

$$\begin{aligned}\varepsilon_4 &= \frac{\partial \bar{u}}{\partial z} + \frac{1}{A_1}\frac{\partial w}{\partial \alpha} - K_1\bar{u}\\ \varepsilon_5 &= \frac{\partial \bar{v}}{\partial z} + \frac{1}{A_2}\frac{\partial \bar{w}}{\partial \beta} - K_2\bar{v}\end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad 35.c$$

Substituting for $\bar{u},\bar{v},\bar{w}$ from equation 35.a in the above equation and neglecting the term multiplied by $K_1 z$ and $K_2 z$. We have,

$$\varepsilon_4 = \phi_1 + 2z\varphi_1 + 3z^2\theta_1 + \frac{1}{A_1}\frac{\partial w}{\partial \alpha}$$

$$\varepsilon_5 = \phi_2 + 2z\varphi_2 + 3z^2\theta_2 + \frac{1}{A_2}\frac{\partial w}{\partial \beta} \qquad \text{………………. 35.d}$$

Setting $\varepsilon_4(\alpha,\beta,\pm\frac{h}{2},t)$ and $\varepsilon_5(\alpha,\beta,\pm\frac{h}{2},t)$ to zero,

$$\varphi_1 = \varphi_2 = 0$$

$$\theta_1 = -\frac{4}{3h^2}\left(\phi_1 + \frac{1}{A_1}\frac{\partial w}{\partial \alpha}\right) \quad \textit{and} \qquad \text{……………….35.e}$$

$$\theta_2 = -\frac{4}{3h^2}\left(\phi_2 + \frac{1}{A_2}\frac{\partial w}{\partial \beta}\right)$$

Substituting equation 35.e into equation 35.a

$$\bar{u} = \left(1 + \frac{z}{R_1}\right)u + z\phi_1 + z^3\left(\frac{4}{3h^2}\right)\left[-\phi_1 - \frac{1}{A_1}\left(\frac{\partial w}{\partial \alpha}\right)\right]$$

$$\bar{v} = \left(1 + \frac{z}{R_2}\right)v + z\phi_2 + z^3\left(\frac{4}{3h^2}\right)\left[-\phi_2 - \frac{1}{A_2}\left(\frac{\partial w}{\partial \beta}\right)\right] \qquad \text{………… 35.f}$$

$$\bar{w} = w$$

Now substituting equation [35.f] in strain displacement relations referred to an orthogonal curvilinear coordinate system, we get

$$\varepsilon_1 = \varepsilon_1{}^0 + z\left(k_1{}^0 + z^2 k_1{}^2\right)$$

$$\varepsilon_2 = \varepsilon_2{}^0 + z\left(k_2{}^0 + z^2 k_2{}^2\right)$$

$$\varepsilon_4 = \varepsilon_4{}^0 + z^2 k_4{}^1$$

$$\varepsilon_5 = \varepsilon_5{}^0 + z^2 k_5{}^1$$

$$\varepsilon_6 = \varepsilon_6{}^0 + z\left(k_6{}^0 + z^2 k_6{}^2\right) \qquad \text{…………………… (36.a)}$$

Where,

$$\varepsilon_1{}^0 = \frac{1}{A_1}.\frac{\partial u}{\partial \alpha} + \frac{w}{R_1};$$

$$\varepsilon_2{}^0 = \frac{1}{A_2}.\frac{\partial v}{\partial \beta} + \frac{w}{R_2};$$

$$\varepsilon_6{}^0 = \frac{1}{A_1}.\frac{\partial v}{\partial \alpha} + \frac{1}{A_2}\frac{\partial u}{\partial \beta};$$

$$\varepsilon_4{}^0 = \phi_2 + \frac{1}{A_2}\frac{\partial w}{\partial \beta};$$

$$\varepsilon_5{}^0 = \phi_1 + \frac{1}{A_1}\frac{\partial w}{\partial \beta};$$

$$k_1{}^0 = \frac{1}{A_1}.\frac{\partial \phi_1}{\partial \alpha};$$

$$k_2{}^0 = \frac{1}{A_2}.\frac{\partial \phi_2}{\partial \beta};$$

$$k_6{}^0 = \frac{1}{A_1}.\frac{\partial \phi_2}{\partial \alpha} + \frac{1}{A_2}\frac{\partial \phi_1}{\partial \beta} + \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{1}{A_1}\frac{\partial v}{\partial \alpha} - \frac{1}{A_2}\frac{\partial u}{\partial \beta}\right);$$

$$k_4{}^1 = -\left(\frac{4}{h^2}\right)\left(\phi_2 + \frac{\partial w}{\partial \beta}\right)$$

$$k_5{}^1 = -\left(\frac{4}{h^2}\right)\left(\phi_1 + \frac{\partial w}{\partial \alpha}\right)$$

$$k_1{}^2 = -\left(\frac{4}{3h^2}\right)\left(\frac{\partial \phi_1}{\partial \alpha} + \frac{\partial^2 w}{\partial \alpha^2}\right) \qquad \text{……………………} \qquad (36.b)$$

$$k_2{}^2 = -\left(\frac{4}{3h^2}\right)\left(\frac{\partial \phi_2}{\partial \beta} + \frac{\partial^2 w}{\partial \beta^2}\right)$$

$$k_6{}^2 = -\left(\frac{4}{3h^2}\right)\left(\frac{\partial \phi_2}{\partial \alpha} + \frac{\partial \phi_1}{\partial \beta} + 2\frac{\partial^2 w}{\partial \alpha \partial \beta}\right)$$

Where ϕ_1 and ϕ_2 are the rotations of the normals to the reference surface, z = 0, about α and β coordinate axes, respectively. The stress-strain relations are as given in equation (13).

3.8.1 STRESS RESULTANTS AND STRESS COUPLES

Stress resultants and stress couples are defined by the integrals of the stresses over each lamina thickness and summation of the expressions over the thickness consisting of 'n' laminas

$$\left(N_{i,}M_{i},P_{i}\right)=\sum_{k=1}^{n}\int_{h_{k-1}}^{hk}\sigma_{i}^{\ k}\left(1,z,z^{3}\right)dz..................\left(i=1,2,6\right)$$

$$\left(Q_{1},K_{1}\right)=\sum_{k=1}^{n}\int_{h_{k-1}}^{hk}\sigma_{4}^{\ k}\left(1,z^{2}\right)dz$$

$$\left(Q_{2},K_{2}\right)=\sum_{k=1}^{n}\int_{h_{k-1}}^{hk}\sigma_{5}^{\ k}\left(1,z^{2}\right)dz \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (37)$$

In which n = the number of layers in the shell; h_k and h_{k-1} [fig 2(a)] is the top and bottom z coordinates

of the kth lamina.

Substituting of equation [36.a] and [13] into equation [37] leads to the following expression for the stress resultants and stress couples

$$\begin{aligned}
N_{i} &= A_{ij}.\varepsilon_{j}^{\ 0}+B_{ij}.k_{j}^{\ 0}+E_{ij}k_{j}^{\ 2};\\
M_{i} &= B_{ij}\varepsilon_{j}^{\ 0}+D_{ij}k_{j}^{\ 0}+F_{ij}k_{j}^{\ 2};\\
P_{i} &= E_{ij}\varepsilon_{j}^{\ 0}+F_{ij}k_{j}^{\ 0}+H_{ij}k_{j}^{\ 2}; \qquad \left(i,j\right)=1,2,6\\
Q_{1} &= A_{5j}.\varepsilon_{j}^{\ 0}+D_{5j}k_{j}^{\ 1};\\
Q_{2} &= A_{4j}.\varepsilon_{j}^{\ 0}+D_{4j}k_{j}^{\ 1};\\
K_{1} &= D_{4j}\varepsilon_{j}^{\ 0}+F_{4j}k_{j}^{\ 1}; \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots...... \qquad (37.a)\\
K_{2} &= D_{5j}\varepsilon_{j}^{\ 0}+F_{5j}k_{j}^{\ 1}; \qquad \left(j=4,5\right)
\end{aligned}$$

where *Aij*, *Bij* , etc. are the laminate stiffnesses expressed as

$$\left(A_{ij},B_{ij},D_{ij},E_{ij},F_{ij},H_{ij}\right)=\sum_{k=1}^{n}\int_{h_{k-i}}^{h_{k}}Q_{ij}^{\ k}(1,z,z^{2},z^{3},z^{4},z^{5}).dz \quad ....37.b$$

3.8.2 Equations of equilibrium for the laminated composite spherical shell

For the spherical shell configuration shown in figure 3, the co-ordinates are given. The Lame parameters $A_1 = A_2 = 1$ and the principal curvatures $K_1 = K_2 = 1/R$,where 'R' is the radius of the mid-surface of the spherical shell.

By applying the dynamic version of the principle of virtual work (Hamilton's Principle), integrating the displacement gradients by parts in the resulting equation and setting the coefficient of δ u, δ v, δ w, $\delta\phi_1, \delta\phi_2$ to zero separately, the equations of equilibrium become:

$$\frac{\partial N_1}{\partial \alpha} + \frac{\partial N_6}{\partial \beta} = I_1 \ddot{u} + \bar{I}_2 \ddot{\phi}_1 - \left(\frac{4}{3h^2}\right) I_4 \frac{\partial \ddot{w}}{\partial \alpha}$$

$$\frac{\partial N_6}{\partial \alpha} + \frac{\partial N_2}{\partial \beta} = I_1 \ddot{v} + \bar{I}_2 \ddot{\phi}_2 - \left(\frac{4}{3h^2}\right) I_4 \frac{\partial \ddot{w}}{\partial \beta}$$

$$\frac{\partial Q_1}{\partial \alpha} + \frac{\partial Q_2}{\partial \beta} - \frac{4}{h^2}\left(\frac{\partial K_1}{\partial \alpha} + \frac{\partial K_2}{\partial \beta}\right) + \frac{4}{3h^2}\left(\frac{\partial^2 P_1}{\partial \alpha^2} + 2\frac{\partial^2 P_6}{\partial \alpha \partial \beta} + \frac{\partial^2 P_2}{\partial \beta^2}\right) - \frac{N_1}{R_1} - \frac{N_2}{R_2} = I_1 \ddot{w} - \left(\frac{4}{3h^2}\right) I_7 \left(\frac{\partial^2 \ddot{w}}{\partial \alpha^2} + \frac{\partial^2 \ddot{w}}{\partial \beta^2}\right)$$

$$+\left(\frac{4}{3h^2}\right) I_4 \left(\frac{\partial \ddot{u}}{\partial \alpha} + \frac{\partial \ddot{v}}{\partial \beta}\right) + \left(\frac{4}{3h^2}\right) \bar{I}_5 \left(\frac{\partial \ddot{\phi}_1}{\partial \alpha} + \frac{\partial \ddot{\phi}_2}{\partial \beta}\right)$$

$$\frac{\partial M_1}{\partial \alpha} + \frac{\partial M_6}{\partial \beta} - Q_1 + \frac{4}{h^2} K_1 - \left(\frac{4}{3h^2}\right)\left(\frac{\partial P_1}{\partial \alpha} + \frac{\partial P_6}{\partial \beta}\right) = \bar{I}_2 \ddot{u} + \bar{I}_3 \ddot{\phi}_1 - \left(\frac{4}{3h^2}\right) I_5 \frac{\partial \ddot{w}}{\partial \alpha} \quad \text{....................38}$$

$$\frac{\partial M_6}{\partial \alpha} + \frac{\partial M_2}{\partial \beta} - Q_2 + \frac{4}{h^2} K_2 - \left(\frac{4}{3h^2}\right)\left(\frac{\partial P_6}{\partial \alpha} + \frac{\partial P_2}{\partial \beta}\right) = \bar{I}_2 \ddot{v} + \bar{I}_3 \ddot{\phi}_2 - \left(\frac{4}{3h^2}\right) \bar{I}_5 \frac{\partial \ddot{w}}{\partial \beta}$$

The inertias $\bar{I}_i$, i=1, 2, 3, 4, 5 are defined by the equations,

$$\bar{I}_2 = I_2 - \left(\frac{4}{3h^2}\right) I_4$$

$$\bar{I}_5 = I_5 - \left(\frac{4}{3h^2}\right) I_7 \quad \text{………… (38.a)}$$

$$\bar{I}_3 = I_3 - \left(\frac{8}{3h^2}\right) I_5 + \left(\frac{16}{9h^4}\right) I_7$$

Where,

$$\left(I_1, I_2, I_3, I_4, I_5, I_7\right) = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \rho^k \left(1, z, z^2, z^3, z^4, z^6\right) dz \qquad \text{……………(38.a.1)}$$

Where ‘ ρ^k ’ is the density of the material of the kth layer.

The strain displacement relations (36.b) are substituted in the equations for the stress resultants and stress couples given in equation (37.a). Since the solution for the equations of motion is done by using the Navier solution, therefore such a solution exists only for specially antisymmetric cross ply laminate for which the following laminate stiffnesses are zero.

$$A_{16} = A_{26} = B_{16} = B_{26} = D_{16} = D_{26} = A_{45} = 0$$

The expression for the stress resultants and stress couples so obtained are then substituted into the equation of motion (38). The equation of motion in terms of the displacements hence reduces to

$$A_{11}\frac{\partial^2 u}{\partial \alpha^2} + A_{66}\frac{\partial^2 u}{\partial \beta^2} + [A_{12} + A_{66}]\frac{\partial^2 v}{\partial \alpha \partial \beta} + \frac{1}{R}[A_{11} + A_{12}]\frac{\partial w}{\partial \alpha} - \frac{4}{3h^2}E_{11}\frac{\partial^3 w}{\partial \alpha^3} - \frac{4}{3h^2}[E_{12} + 2E_{66}]\frac{\partial^3 w}{\partial \alpha \partial \beta^2}$$
$$+\left[B_{11} - \frac{4}{3h^2}E_{11}\right]\frac{\partial^2 \phi_1}{\partial \alpha^2} + \left[B_{66} - \frac{4}{3h^2}E_{66}\right]\frac{\partial^2 \phi_1}{\partial \beta^2} + \left[B_{12} - \frac{4}{3h^2}E_{12}\right]\frac{\partial^2 \phi_2}{\partial \alpha \partial \beta} + \left[B_{66} - \frac{4}{3h^2}E_{66}\right]\frac{\partial^2 \phi_2}{\partial \alpha \partial \beta} =$$
$$I_1\ddot{u} + \bar{I}_2\ddot{\phi}_1 - \left(\frac{4}{3h^2}\right)I_4\frac{\partial w}{\partial \alpha}$$

$$[A_{21} + A_{66}]\frac{\partial^2 u}{\partial \alpha \partial \beta} + A_{66}\frac{\partial^2 v}{\partial \alpha^2} + A_{22}\frac{\partial^2 v}{\partial \beta^2} + \frac{1}{R}[A_{21} + A_{22}]\frac{\partial w}{\partial \beta} - \frac{4}{3h^2}E_{22}\frac{\partial^3 w}{\partial \beta^3} - [2E_{66} + E_{21}]\frac{\partial^3 w}{\partial \alpha^2 \partial \beta}$$
$$+\left[B_{66} - \frac{4}{3h^2}E_{66}\right]\frac{\partial^2 \phi_1}{\partial \alpha \partial \beta} + B_{21}\frac{\partial^2 \phi_1}{\partial \alpha \partial \beta} - \frac{4}{3h^2}E_{21}\frac{\partial^2 \phi_1}{\partial \alpha \partial \beta} + \left[B_{66} - \frac{4}{3h^2}E_{66}\right]\frac{\partial^2 \phi_2}{\partial \alpha^2} + \left[B_{22} - \frac{4}{3h^2}E_{22}\right]\frac{\partial^2 \phi_2}{\partial \beta^2}$$
$$= I_1\ddot{v} + \bar{I}_2\ddot{\phi}_2 - \left(\frac{4}{3h^2}\right)I_4\frac{\partial \ddot{w}}{\partial \beta}$$

$$
\begin{aligned}
&A_{55}\left[\frac{\partial \phi_1}{\partial \alpha}+\frac{\partial^2 w}{\partial \alpha^2}\right]-D_{55}\left[\frac{4}{h^2}\left[\frac{\partial \phi_1}{\partial \alpha}+\frac{\partial^2 w}{\partial \alpha^2}\right]\right]+A_{44}\left[\frac{\partial \phi_2}{\partial \beta}+\frac{\partial^2 w}{\partial \beta^2}\right]-D_{44}\left[\frac{4}{h^2}\left[\frac{\partial \phi_2}{\partial \beta}+\frac{\partial^2 w}{\partial \beta^2}\right]\right]-\frac{4}{h^2}\bigg\{D_{55}\bigg[\frac{\partial \phi_1}{\partial \alpha}+\\
&\frac{\partial^2 w}{\partial \alpha^2}\bigg]-F_{55}\left[\frac{4}{h^2}\left[\frac{\partial \phi_1}{\partial \alpha}+\frac{\partial^2 w}{\partial \alpha^2}\right]\right]+D_{44}\left[\frac{\partial \phi_2}{\partial \beta}+\frac{\partial^2 w}{\partial \beta^2}\right]-F_{44}\left[\frac{4}{h^2}\left[\frac{\partial \phi_2}{\partial \beta}+\frac{\partial^2 w}{\partial \beta^2}\right]\right]\bigg\}+\frac{4}{3h^2}\bigg\{E_{11}\left[\frac{\partial^3 u}{\partial \alpha^3}+\frac{1}{R}\frac{\partial^2 w}{\partial \alpha^2}\right]\\
&+E_{12}\left[\frac{\partial^3 v}{\partial \alpha^2\partial \beta}+\frac{1}{R}\frac{\partial^2 w}{\partial \alpha^2}\right]+F_{11}\frac{\partial^3 \phi_1}{\partial \alpha^3}+F_{12}\frac{\partial^3 \phi_2}{\partial \alpha^2\partial \beta}-H_{11}\left[\frac{4}{3h^2}\left[\frac{\partial^3 \phi_1}{\partial \alpha^3}+\frac{\partial^4 w}{\partial \alpha^4}\right]\right]-H_{12}\left[\frac{4}{3h^2}\left[\frac{\partial^3 \phi_2}{\partial \alpha^2\partial \beta}+\frac{\partial^4 w}{\partial \alpha^2\partial \beta^2}\right]\right]\\
&+2E_{66}\left[\frac{\partial^3 u}{\partial \alpha\partial \beta^2}+\frac{\partial^3 v}{\partial \alpha^2\partial \beta}\right]+2F_{66}\left[\frac{\partial^3 \phi_2}{\partial \alpha^2\partial \beta}+\frac{\partial^3 \phi_1}{\partial \alpha\partial \beta^2}\right]-2H_{66}\left[\frac{4}{3h^2}\left[\frac{\partial^3 \phi_2}{\partial \alpha^2\partial \beta}+\frac{\partial^3 \phi_1}{\partial \alpha\partial \beta^2}+2\frac{\partial^4 w}{\partial \alpha^2\partial \beta^2}\right]\right]+E_{21}\\
&\left[\frac{\partial^3 u}{\partial \alpha\partial \beta^2}+\frac{1}{R}\frac{\partial^2 w}{\partial \beta^2}\right]+E_{22}\left[\frac{\partial^3 v}{\partial \beta^3}+\frac{1}{R}\frac{\partial^2 w}{\partial \beta^2}\right]+F_{21}\frac{\partial^3 \phi_1}{\partial \alpha\partial \beta^2}+F_{22}\frac{\partial^3 \phi_2}{\partial \beta^3}-H_{21}\left[\frac{4}{3h^2}\left[\frac{\partial^3 \phi_1}{\partial \alpha\partial \beta^2}+\frac{\partial^4 w}{\partial \alpha^2\partial \beta^2}\right]\right]-H_{22}\\
&\left[\frac{4}{3h^2}\left[\frac{\partial^3 \phi_2}{\partial \beta^3}+\frac{\partial^4 w}{\partial \beta^4}\right]\right]\bigg\}-\frac{1}{R}\bigg\{A_{11}\left[\frac{\partial u}{\partial \alpha}+\frac{w}{R}\right]+A_{12}\left[\frac{\partial v}{\partial \beta}+\frac{w}{R}\right]+B_{11}\frac{\partial \phi_1}{\partial \alpha}+B_{12}\frac{\partial \phi_2}{\partial \beta}-E_{11}\left[\frac{4}{3h^2}\left[\frac{\partial \phi_1}{\partial \alpha}+\frac{\partial^2 w}{\partial \alpha^2}\right]\right]\\
&-E_{12}\left[\frac{4}{3h^2}\left[\frac{\partial \phi_2}{\partial \beta}+\frac{\partial^2 w}{\partial \beta^2}\right]\right]\bigg\}-\frac{1}{R}\bigg\{A_{21}\left[\frac{\partial u}{\partial \alpha}+\frac{w}{R}\right]+A_{22}\left[\frac{\partial v}{\partial \beta}+\frac{w}{R}\right]+B_{21}\frac{\partial \phi_1}{\partial \alpha}+B_{22}\frac{\partial \phi_2}{\partial \beta}-E_{21}\left[\frac{4}{3h^2}\left[\frac{\partial \phi_1}{\partial \alpha}+\frac{\partial^2 w}{\partial \alpha^2}\right]\right]\\
&-E_{22}\left[\frac{4}{3h^2}\left[\frac{\partial \phi_2}{\partial \beta}+\frac{\partial^2 w}{\partial \beta^2}\right]\right]\bigg\}=I_1\ddot{w}-\left[\frac{4}{3h^2}\right]^2 I_7\left[\frac{\partial^2 \ddot{w}}{\partial \alpha^2}+\frac{\partial^2 \ddot{w}}{\partial \beta^2}\right]+\frac{4}{3h^2}I_4\left[\frac{\partial \ddot{u}}{\partial \alpha}+\frac{\partial \ddot{v}}{\partial \beta}\right]+\frac{4}{3h^2}\bar{I}_5\left[\frac{\partial \ddot{\phi}_1}{\partial \alpha}+\frac{\partial \ddot{\phi}_2}{\partial \beta}\right]
\end{aligned}
$$

$$
\begin{aligned}
&B_{11}\left[\frac{\partial^2 u}{\partial \alpha^2}+\frac{1}{R}\frac{\partial w}{\partial \alpha}\right]+B_{12}\left[\frac{\partial^2 v}{\partial \alpha\partial \beta}+\frac{1}{R}\frac{\partial w}{\partial \alpha}\right]+D_{11}\frac{\partial^2 \phi_1}{\partial \alpha^2}+D_{12}\frac{\partial^2 \phi_2}{\partial \alpha\partial \beta}-F_{11}\left[\frac{4}{3h^2}\left[\frac{\partial^2 \phi_1}{\partial \alpha^2}+\frac{\partial^3 w}{\partial \alpha^3}\right]\right]-F_{12}\\
&\left[\frac{4}{3h^2}\left[\frac{\partial^2 \phi_2}{\partial \alpha\partial \beta}+\frac{\partial^3 w}{\partial \alpha\partial \beta^2}\right]\right]+B_{66}\left[\frac{\partial^2 u}{\partial \beta^2}+\frac{\partial^2 v}{\partial \alpha\partial \beta}\right]+D_{66}\left[\frac{\partial^2 \phi_2}{\partial \alpha\partial \beta}+\frac{\partial^2 \phi_1}{\partial \beta^2}\right]-F_{66}\left[\frac{4}{3h^2}\left[\frac{\partial^2 \phi_2}{\partial \alpha\partial \beta}+\frac{\partial^2 \phi_1}{\partial \beta^2}+\frac{2\partial^3 w}{\partial \alpha\partial \beta^2}\right]\right]\\
&-A_{55}\left[\phi_1+\frac{\partial w}{\partial \alpha}\right]+D_{55}\frac{4}{h^2}\left[\phi_1+\frac{\partial w}{\partial \alpha}\right]+\frac{4}{h^2}\left[D_{55}\left[\phi_1+\frac{\partial w}{\partial \alpha}\right]-F_{55}\frac{4}{h^2}\left[\phi_1+\frac{\partial w}{\partial \alpha}\right]\right]-\frac{4}{3h^2}\bigg\{E_{11}\left[\frac{\partial^2 u}{\partial \alpha^2}+\frac{1}{R}\frac{\partial w}{\partial \alpha}\right]+\\
&E_{12}\left[\frac{\partial^2 v}{\partial \alpha\partial \beta}+\frac{1}{R}\frac{\partial w}{\partial \alpha}\right]+F_{11}\frac{\partial^2 \phi_1}{\partial \alpha^2}+F_{12}\frac{\partial^2 \phi_2}{\partial \alpha\partial \beta}-H_{11}\left[\frac{4}{3h^2}\left[\frac{\partial^2 \phi_1}{\partial \alpha^2}+\frac{\partial^3 w}{\partial \alpha^3}\right]\right]-H_{12}\left[\frac{4}{3h^2}\left[\frac{\partial^2 \phi_2}{\partial \alpha\partial \beta}+\frac{\partial^3 w}{\partial \alpha\partial \beta^2}\right]\right]+\\
&E_{66}\left[\frac{\partial^2 u}{\partial \beta^2}+\frac{\partial^2 v}{\partial \alpha\partial \beta}\right]+F_{66}\left[\frac{\partial^2 \phi_2}{\partial \alpha\partial \beta}+\frac{\partial^2 \phi_1}{\partial \beta^2}\right]-H_{66}\frac{4}{3h^2}\left[\frac{\partial^2 \phi_2}{\partial \alpha\partial \beta}+\frac{\partial^2 \phi_1}{\partial \beta^2}+\frac{2\partial^3 w}{\partial \alpha\partial \beta^2}\right]\bigg\}=\bar{I}_2\ddot{u}+\bar{I}_3\ddot{\phi}_1-\left[\frac{4}{3h^2}\right]\bar{I}_5\frac{\partial \ddot{w}}{\partial \alpha}
\end{aligned}
$$

$$B_{66}\left[\frac{\partial^2 u}{\partial\alpha\partial\beta}+\frac{\partial^2 v}{\partial\alpha^2}\right]+D_{66}\left[\frac{\partial^2\phi_2}{\partial\alpha^2}+\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}\right]-F_{66}\frac{4}{3h^2}\left[\frac{\partial^2\phi_2}{\partial\alpha^2}+\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}+2\frac{\partial^3 w}{\partial\alpha^2\partial\beta}\right]+B_{21}\left[\frac{\partial^2 u}{\partial\alpha\partial\beta}+\frac{1}{R}\frac{\partial w}{\partial\beta}\right]$$

$$+B_{22}\left[\frac{\partial^2 v}{\partial\beta^2}+\frac{1}{R}\frac{\partial w}{\partial\beta}\right]+D_{21}\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}+D_{22}\frac{\partial^2\phi_2}{\partial\beta^2}-F_{21}\frac{4}{3h^2}\left[\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}+\frac{\partial^3 w}{\partial\alpha^2\partial\beta}\right]-F_{22}\frac{4}{3h^2}\left[\frac{\partial^2\phi_2}{\partial\beta^2}+\frac{\partial^3 w}{\partial\beta^3}\right]-A_{44}$$

$$\left[\phi_2+\frac{\partial w}{\partial\beta}\right]+D_{44}\frac{4}{h^2}\left[\phi_2+\frac{\partial w}{\partial\beta}\right]+\frac{4}{h^2}\left[D_{44}\left[\phi_2+\frac{\partial w}{\partial\beta}\right]-F_{44}\frac{4}{h^2}\left[\phi_2+\frac{\partial w}{\partial\beta}\right]\right]-\frac{4}{3h^2}\left\{E_{66}\left[\frac{\partial^2 u}{\partial\alpha\partial\beta}+\frac{\partial^2 v}{\partial\alpha^2}\right]+F_{66}\right.$$

$$\left[\frac{\partial^2\phi_2}{\partial\alpha^2}+\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}\right]+H_{66}\left[\frac{\partial^2\phi_2}{\partial\alpha^2}+\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}+2\frac{\partial^3 w}{\partial\alpha^2\partial\beta}\right]+E_{21}\left[\frac{\partial^2 u}{\partial\alpha\partial\beta}+\frac{1}{R}\frac{\partial w}{\partial\beta}\right]+E_{22}\left[\frac{\partial^2 v}{\partial\beta^2}+\frac{1}{R}\frac{\partial w}{\partial\beta}\right]+F_{21}\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}$$

$$\left.+F_{22}\frac{\partial^2\phi_2}{\partial\beta^2}-H_{21}\frac{4}{3h^2}\left[\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}+2\frac{\partial^3 w}{\partial\alpha^2\partial\beta}\right]-H_{22}\frac{4}{3h^2}\left[\frac{\partial^2\phi_2}{\partial\beta^2}+\frac{\partial^3 w}{\partial\beta^3}\right]\right\}=\bar{I}_2\ddot{v}+\bar{I}_3\ddot{\phi}_2-\left[\frac{4}{3h^2}\right]\bar{I}_5\frac{\partial\ddot{w}}{\partial\beta} \qquad (39)$$

3.9 BOUNDARY CONDITIONS

Up to now, the analysis has been general without reference to the boundary conditions.Considering the line integrals while integrating by parts the displacement gradient in the Hamilton principle ,the boundary conditions at an edge α =constant and β =constant are obtained .For reasons of simplicity, only simply supported boundary condition are considered along all edges for the shell. The boundary conditions for the simply supported spherical shell are obtained as given below

$$N_1 = 0 , v = 0 , w = 0,$$

Following the Navier solution procedure, the following solution form which satisfies
the boundary conditions in equations is assumed:

$$
\begin{aligned}
u &= U\cos\lambda_m\alpha.\sin\lambda_n\beta \quad e^{i\omega t}\\
v &= V\sin\lambda_m\alpha.\cos\lambda_n\beta \quad e^{i\omega t}\\
w &= W\sin\lambda_m\alpha.\sin\lambda_n\beta \quad e^{i\omega t}\\
\phi_1 &= \Phi_1\cos\lambda_m\alpha.\sin\lambda_n\beta \quad e^{i\omega t}\\
\phi_2 &= \Phi_2\sin\lambda_m\alpha.\cos\lambda_n\beta \quad e^{i\omega t}
\end{aligned}
\qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (40)
$$

where , $\lambda_m = \frac{m\pi}{a}$, $\lambda_n = \frac{n\pi}{b}$ and U, V, W, Φ_1 and Φ_2 are the maximum amplitudes, m and n are known as the axial half wave number and circumferential wave number respectively. This implies during vibration, the shell generators are assumed to subdivide into m half waves and the circumference subdivide into 2n half waves.

Introducing the expressions [40] into the governing equations of motion in terms of displacements [34] and [39], the following equation in matrix form is obtained, which is a general eigen value problem.

$$
[C]\{X\} = \omega^2[M]\{X\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (41)
$$

Where,

ω^2 is the eigenvalue

{X} is a column matrix of amplitude of vibration or eigenvector.

[C] and [M] are 5 x 5 matrices.

The coefficients of the matrices are described in Appendix 1

For convenience, the elements of the above matrices are suitably non-dimensionalised as follows

$$
\begin{aligned}
&U = \overline{U}h \\
&V = \overline{V}h \\
&W = \overline{W}h \\
&\Phi_1 = \overline{\Phi}_1 \\
&\Phi_2 = \overline{\Phi}_2 \\
&A_{ij} = \overline{A}_{ij}Q_2h \qquad E_{ij} = \overline{E}_{ij}Q_2h^4 \\
&B_{ij} = \overline{B}_{ij}Q_2h^2 \qquad F_{ij} = \overline{F}_{ij}Q_2h^5 \\
&D_{ij} = \overline{D}_{ij}Q_2h^3 \qquad H_{ij} = \overline{H}_{ij}Q_2h^7
\end{aligned}
\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (42)
$$

(- (bar) on top indicates non-dimensionalised quantities)

And

$$
\begin{aligned}
&I_1 = I_1'\rho^1 h \\
&I_2 = I_2'\rho^1 h^2 \\
&I_3 = I_3'\rho^1 h^3 \\
&I_4 = I_4'\rho^1 h^4 \\
&I_5 = I_5'\rho^1 h^5 \\
&I_7 = I_7'\rho^1 h^7
\end{aligned}
\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (43)
$$

$\left(I_1', I_2', I_3', \ldots\ldots\ldots\right.$ are the non-dimensionalised quantities

Also

$$
\begin{aligned}
&\overline{I}_1 = I_1'\rho^1 h = \tilde{I}_1\rho^1 h \\
&\overline{I}_2 = \left[I_2' - \frac{4}{3}I_4'\right]\rho^1 h^2 = \tilde{I}_2\rho^1 h^2 \\
&\overline{I}_3 = \left[I_3' - \frac{8}{3}I_5' + \frac{16}{9}I_7'\right]\rho^1 h^3 = \tilde{I}_3\rho^1 h^3 \\
&\overline{I}_5 = \left[I_5' - \frac{4}{3}I_7'\right]\rho^1 h^5 = \tilde{I}_5\rho^1 h^5
\end{aligned}
\qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (44)
$$

After non –dimensionalising the terms, the equation [41] in matrix form can be written as given below

$$\left[\overline{\mathrm{H}}\right]\left\{\overline{\mathrm{X}}\right\}=\overline{\omega}^2\left\{\overline{\mathrm{X}}\right\} \tag{45}$$

Where,

$$\overline{\omega}^2=\frac{\rho b^2\omega^2}{Q_2}$$

$$\left[\overline{H}\right]=\left[\overline{M}\right]^{-1}\left[\overline{C}\right] \tag{46}$$

A non-trivial solution for the column matrix $\left\{\overline{X}\right\}$ will give the required eigenvalues, which are the values of the square of the frequency parameter $\overline{\omega}$ in the present case. The lowest value of $\overline{\omega}$ is of particular interest.

Chapter 4

Result and Discussion

CHAPTER- 4

NUMERICAL RESULTS AND DISCUSSIONS

4.1 INTRODUCTION

The frequency parameters are calculated by using a computer program for vibration of laminated composite spherical shells. The results obtained using the present theory are compared to earlier results and are tabulated. The numerical values of the lowest value of frequency parameter are presented for various shell parameters in this chapter to study the effect of number of layers, the orientation of layers and the ‘a/h’ ratio on frequency. The frequency envelopes are plotted as a function of ‘R/a’. For the all above studies, the results of the present theory are compared with those of the results obtained by a higher order shear deformation theory as proposed by J.N. Reddy.

4.2 SOLUTION OF EIGEN VALUE PROBLEM AND COMPUTERPROGRAM

The equation (41) represents a general Eigen value problem, where ω^2 is the Eigen value and {X} is the eigenvector. For convenience the equation (41) is non-dimensionalised. Then if the equation is pre multiplied by $\left[\overline{M}\right]^{-1}$, (where the bar indicates the non- dimensionalised form), one obtains the following standard eigenvalue problem,

$$\left[\overline{H}\right]\left\{\overline{X}\right\} = \overline{\omega}^2\left\{\overline{X}\right\}$$

Where,

$$\bar{\omega}^2 = \frac{\rho b^2 \omega^2}{Q_2}$$

$$\left[\bar{H}\right] = \left[\bar{M}\right]^{-1}\left[\bar{C}\right]$$

A non trivial solution for the column matrix $\{\bar{X}\}$ will give the required eigenvalues, which are the values of the square of the frequency parameter $\bar{\omega}$ in the present case. The lowest values of $\bar{\omega}$ is of particular interest (for a set of fixed shell parameters, many values of $\bar{\omega}^2$ can be obtained).

A standard subroutine in the computer program to find the eigenvalue of matrices has been used, which consists of root power method of iteration with Wielandt's deflection technique. The program will be called RTPM, which is capable of finding the required number of roots in descending order. The change of the sign of the determinant value is checked for values of one percent on either side of the root to verify the convergence. The RTPM program gives the highest value of eigenvalue first, so if the $\left[\bar{M}\right]$matrix is taken as $\left[\bar{C}\right]^{-1}\left[\bar{H}\right]$, then the highest value of $\left(\frac{1}{\bar{\omega}^2}\right)$is obtained, that is the lowest value of $\bar{\omega}^2$ and $\bar{\omega}$ which is of particular interest. These programs are written in FORTRAN language.

4.3 NUMERICAL RESULTS AND DISCUSSION

4.3.1 The Validation of the Formulation and Numerical Results

Using the formulation developed in the previous sections, numerical studies are carried out. The lowest value of the frequencies has been calculated at first for two layers , three layers and four layers laminated composite spherical shells for various values of R/a and a/h by first and

Higher order shear deformation theories. These results are compared with earlier available results in tables. This also serves as to check on the validity of the present theory, as the results are mostly agreeable.

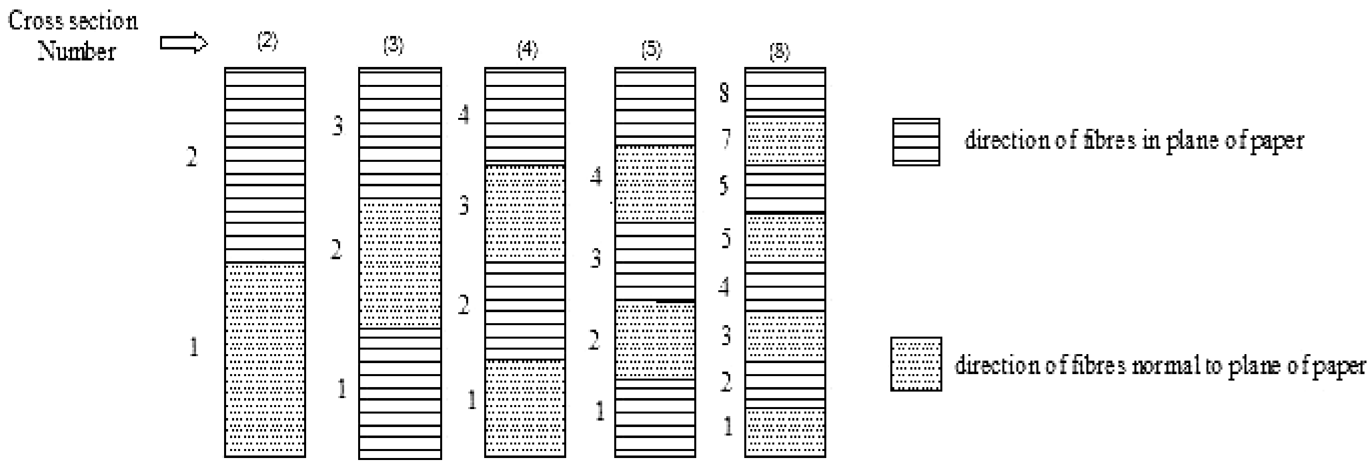

Figure 4. 1 : Direction and cross section of panels

Table 4.1 shows the comparison of present results and those of J .N. Reddy [15] for the non dimensional frequency parameter $\bar{\omega}^2 = \dfrac{\omega^2 a^4 \rho}{E_2 h^2}$ of [0/90], [0/90/0] and [0/90/90/0] simply supported spherical shell. The geometrical and material properties used are

$$\bar{E}_{11} = 25\bar{E}_{22}, \quad \bar{G}_{12} = \bar{G}_{13} = 0.5\bar{E}_{22}, \quad \bar{G}_{23} = 0.2\bar{E}_{22}, \quad \bar{\nu}_{12} = 0.25, \quad \rho = 1$$

These tables (4.1-4.6) shows that the present theory by higher order gives slightly higher values of frequency parameter when compared with the first order theory[15] for all R/a values and a/h values. For all the panels, a/b=1 and $R_1 = R_2 = R$.

Table 4. 1: Comparison of lowest nondimensional Frequency Parameter $\bar{\omega}^2 = \frac{\omega^2 a^4 \rho}{E_2 h^2}$ for a [0°/90°] simply Supported Laminated composite spherical Shell (a/h=100)

R/a	J N Reddy	Present Theory			
		First order	Error in %	Higher order	Error in %
1	125.930	125.930	0	126.041	0.09
2	67.362	67.362	0	67.506	0.21
3	46.002	46.000	-0.004	46.189	0.41
4	35.228	35.228	0	35.464	0.67
5	28.825	28.825	0	29.108	0.98
10	16.706	16.706	0	17.175	2.81
10^30	9.687	9.687	0	10.469	8.07

Error in Percentage = 100 * (Present Theory – J N Reddy)/ J N Reddy

Table 4. 2 : Comparison of lowest Nondimensional Frequency Parameter $\bar{\omega}^2 = \frac{\omega^2 a^4 \rho}{E_2 h^2}$ for a [0°/90°/0°] simply Supported Laminated composite spherical Shell (a/h=100)

R/a	J N Reddy	Present Theory			
		First order	Error in %	Higher order	Error in %
1	125.99	125.99	0	126.138	0.12
2	68.075	68.074	-0.001	68.161	0.13
3	47.265	47.216	-0.104	47.325	0.13
4	36.971	36.967	-0.011	37.026	0.15
5	30.993	30.988	-0.016	31.042	0.16
10	20.347	20.337	-0.049	20.389	0.21
10^30	15.183	15.169	-0.09	15.225	0.28

Table 4. 3: Comparison of lowest Nondimensional Frequency Parameter $\bar{\omega}^2 = \frac{\omega^2 a^4 \rho}{E_2 h^2}$ for a [0°/90°/90°/0°] simply Supported Laminated composite spherical Shell (a/h=100)

R/a	J N Reddy	Present Theory			
		First order	Error in %	Higher order	Error in %
1	126.330	126.323	-0.006	126.47	0.11
2	68.294	68.293	-0.001	68.385	0.13
3	47.415	47.411	-0.008	47.485	0.15
4	37.082	37.082	0	37.149	0.18
5	31.079	31.079	0	31.145	0.21
10	20.380	20.380	0	20.449	0.34
10^30	15.184	15.184	0	15.262	0.51

Table 4. 4: Comparison of lowest Nondimensional Frequency Parameter $\bar{\omega}^2 = \frac{\omega^2 a^4 \rho}{E_2 h^2}$ for a [0°/90°] simply Supported Laminated composite spherical Shell (a/h=10)

R/a	J N Reddy	Present Theory			
		First order	Error in %	Higher order	Error in %
1	14.481	14.452	-0.2	15.100	4.27
2	10.749	10.743	-0.06	11.288	5.01
3	9.961	9.715	-2.47	10.296	3.36
4	9.410	9.409	-0.01	9.926	5.48
5	9.231	9.230	-0.01	9.745	5.57
10	8.984	8.984	0	9.500	5.74
10^30	8.900	8.900	0	9.423	5.88

Table 4. 5: Comparison of lowest Nondimensional Frequency Parameter $\bar{\omega}^2 = \frac{\omega^2 a^4 \rho}{E_2 h^2}$ for a [0°/90°/0°] simply Supported Laminated composite spherical Shell (a/h=10)

R/a	J N Reddy	Present Theory			
		First order	Error in %	Higher order	Error in %
1	16.115	15.782	-2.07	16.586	2.92
2	13.382	12.889	-3.68	13.352	-0.22
3	12.731	12.182	-4.31	12.546	-1.45
4	12.487	11.925	-4.50	12.251	-1.89
5	12.372	11.800	-4.62	12.107	-2.14
10	12.215	11.629	-4.79	11.911	-2.49
10^30	12.162	11.572	-4.85	11.845	-2.61

Table 4. 6: Comparison of lowest Nondimensional Frequency Parameter $\bar{\omega}^2 = \frac{\omega^2 a^4 \rho}{E_2 h^2}$ for a [0°/90°/90°/0°] simply Supported Laminated composite spherical Shell (a/h=10)

R/a	J N Reddy	Present Theory			
		First order	Error in %	Higher order	Error in %
1	16.172	16.146	-0.16	16.957	4.85
2	13.447	13.440	-0.05	13.849	2.99
3	12.795	12.718	-0.60	13.081	2.24
4	12.552	12.551	-0.01	12.802	1.99
5	12.437	12.436	-0.01	12.666	1.84
10	12.280	12.280	0	12.479	1.62
10^30	12.226	12.227	0.01	12.416	1.55

From the results presented in this table, it is clear that the present FSDT results are in excellent agreement with those obtained by Reddy [25]. But for a/h=10 and [0°/90°/0°] layer composite shell, the values of frequency parameters by present formulation are slightly lower than values of frequency parameter as obtained by Reddy [15].

4.3.2 NUMERICAL RESULTS

4.3.2.1: Influence of layer configuration of composites

When the spherical shell is made of composite material, the influence of the layer configuration should be considered since it is one of the most important characteristics of a composite material. Usually layers are made of different isotropic materials, and their principal directions may also be oriented differently. For laminated composites, the fiber directions determine layer orientation. In this section, however, the discussion is made for simplified case where the spherical shells are composed of two ,three and four layer cross-ply laminated composite panels.

Table 4.7: Nondimensional frequency parameter $\bar{\omega}^2 = \frac{\omega^2 \rho^1 b^2}{Q_{22}}$ for laminated composite symmetric spherical panel for different layers.

R/a	a/h	2-layer		3-layer		4-layer	
		First	Higher	First	Higher	First	Higher
1	100	1.258	1.259	1.258	1.259	1.262	1.263
	10	1.443	1.508	1.576	1.657	1.613	1.694
2	100	0.673	0.675	0.679	0.68	0.682	0.683
	10	1.073	1.127	1.287	1.333	1.342	1.383
3	100	0.459	0.462	0.472	0.472	0.474	0.474

	10	0.97	1.029	1.217	1.253	1.270	1.307
4	100	0.352	0.354	0.369	0.37	0.37	0.372
	10	0.939	0.991	1.191	1.224	1.254	1.279
5	100	0.288	0.292	0.309	0.310	0.310	0.311
	10	0.922	0.973	1.179	1.209	1.242	1.265
10	100	0.167	0.170	0.203	0.204	0.204	0.204
	10	0.897	0.949	1.162	1.189	1.226	1.246
10^30	100	0.097	0.105	0.152	0.152	0.152	0.152
	10	0.889	0.941	1.156	1.183	1.221	1.240

In Table 4.7 two ,three and four layered anti-symmetric laminated composite spherical shell has been analysed. The nondimensionalised frequency parameter is $\bar{\omega}^2 = \frac{\omega^2 \rho^1 b^2}{Q_{22}}$ and 'a/h' value is 100 and 10. The result obtained by the present theory by first order shear deformation theory is slightly lower than that obtained by higher order shear deformation theory. From the table 4.7,the nondimensionalised frequency is increasing with increase in number of layers for each R/a and a/h values.

4.3.2.2 Influence of physical parameters

There are generally many physical and geometrical parameters which influence the frequency characteristics of spherical panels. Physical parameters include the material properties and boundary conditions. The major geometrical parameters include the length (a), radius (R), and thickness (h). In this section, discussions are made on the influence of the geometrical length ratio R/a and thickness ratio a/h on the frequency characteristics of the laminated composite spherical shells.

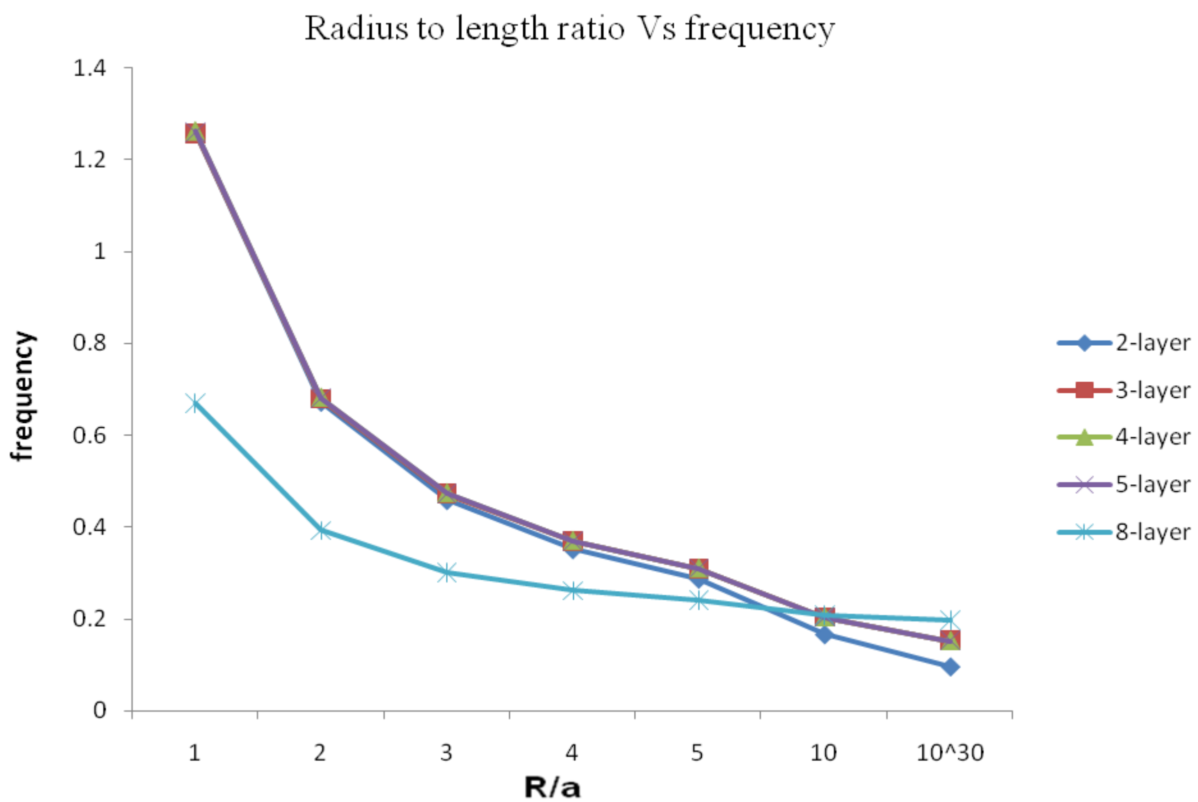

Figure 4. 2 : Variation of the natural frequencies ϖ with radius to length ratios (R/a) for cross ply laminated shell with a/b=1 , $R_1 = R_2 = R$ and a/h = 100

Figure 4.2 shows the variation of the natural frequency ‘ ϖ ’ with the radius-length ratios (R/a) and constant length-to-thickness (i.e. a/h =100) for the laminated composite spherical shell with the simply supported boundary condition at the edges. It is observed that for spherical shell the natural frequency parameter rapidly increases as the radius-length ratios of the shell decreases. Figure 4.3 shows the variation of the natural frequency ‘ ϖ ’ with various radius-length ratios (R/a) and constant length-to-thickness (i.e. a/h =10). It is observed that for spherical shell if the thickness of the shell is increased the natural frequency parameter slightly increases for the different radius -length ratios.

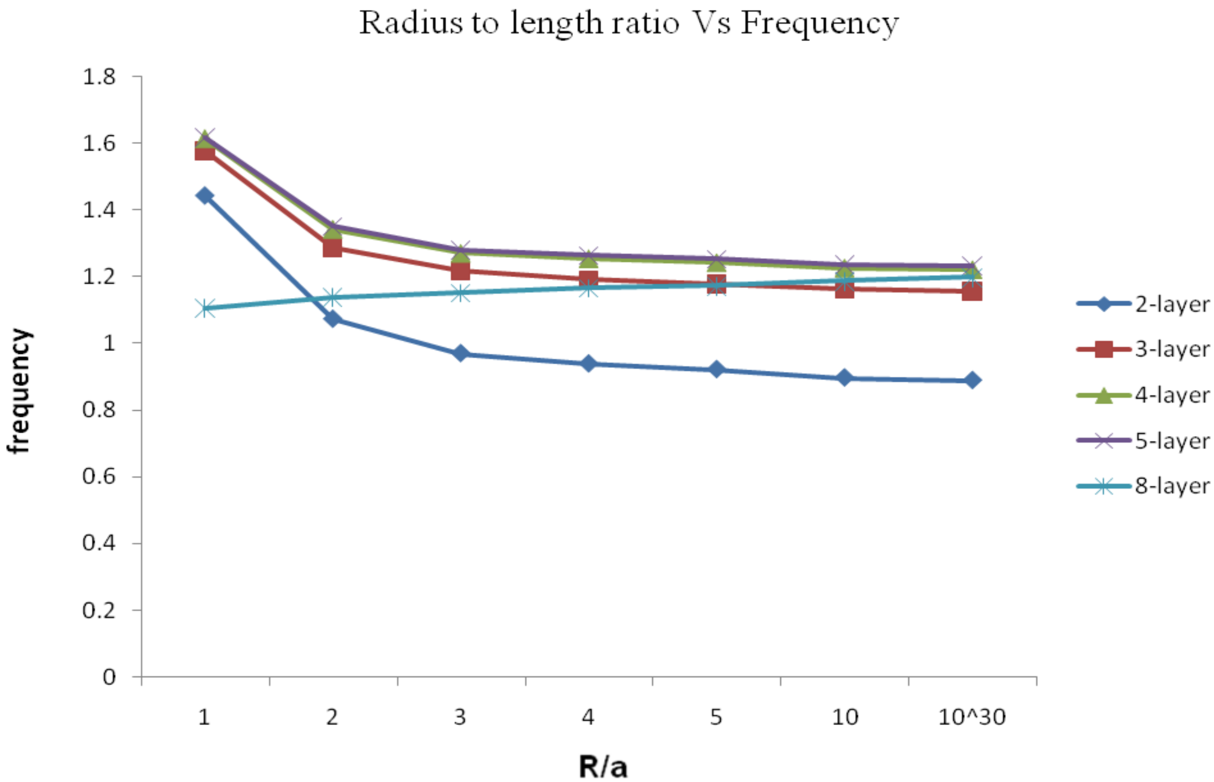

Figure 4. 3 : Variation of the natural frequencies $\overline{\omega}$ with radius to length ratios (R/a) for cross ply laminated shell with a/b=1 , $R_1 = R_2 = R$ and a/h = 10

4.3.3.3 First and Higher order shear deformation theory

The lowest value of the frequencies has been calculated at first for two layers , three layers and four layers laminated composite spherical shells for various values of 'R/a' and 'a/h' by first and Higher order theories. Figure 4.4 and figure 4.5 shows the variation of the natural frequencies $\overline{\omega}$ with the radius to length ratios (R/a) for different number of laminates and for constant a/h=100 for the spherical panel with the simply supported boundary condition at both edges. The overlapping of the graphs illustrate that the there is not much difference in frequency by the first and higher order shear deformation theory for a/h=100. There is a slight difference for a/h=10, with the higher order theory showing higher values. It is further observed that the influence of the length ratio R/a on the natural frequency of the spherical shell is larger than that of the thickness ratio a/h.

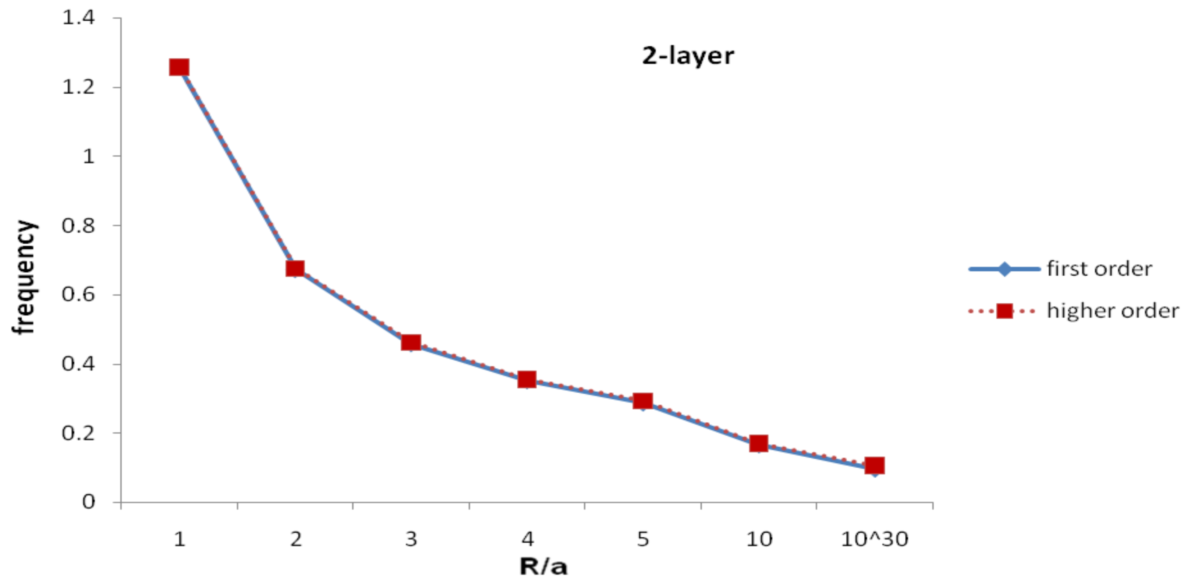

Figure 4. 4 : Variation of the natural frequencies $\overline{\varpi}$ with radius to length ratios (R/a) for cross ply laminated shell with a/b=1 , $R_1 = R_2 = R$ and a/h = 100

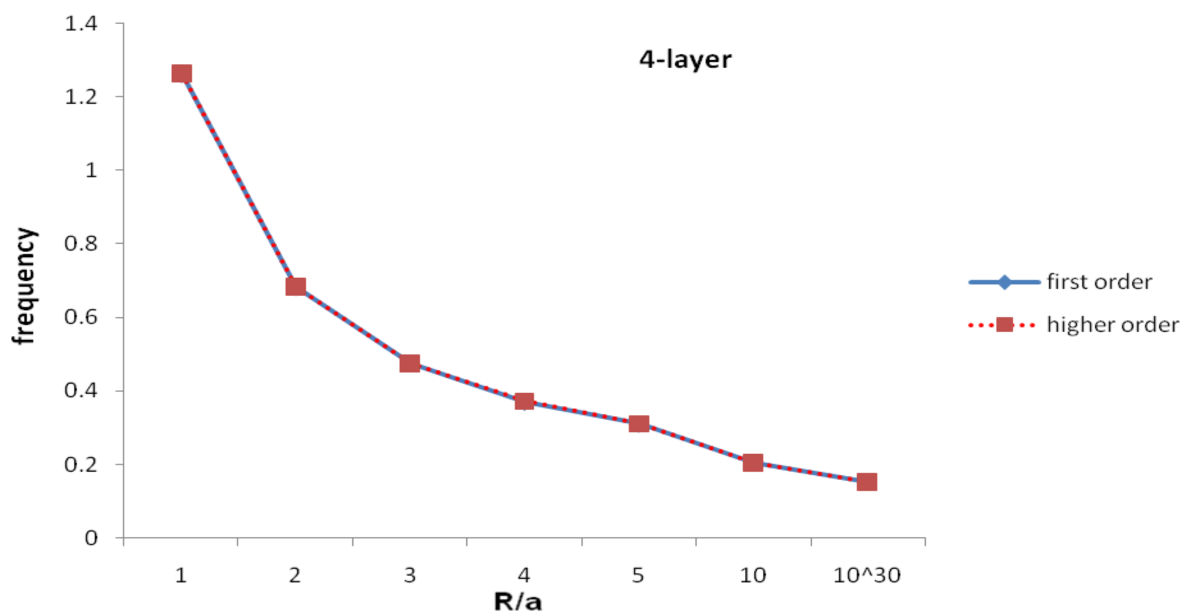

Figure 4. 5 : Variation of the natural frequencies $\overline{\varpi}$ with radius to length ratios (R/a) for cross ply laminated shell with a/b=1 , $R_1 = R_2 = R$ and a/h = 100

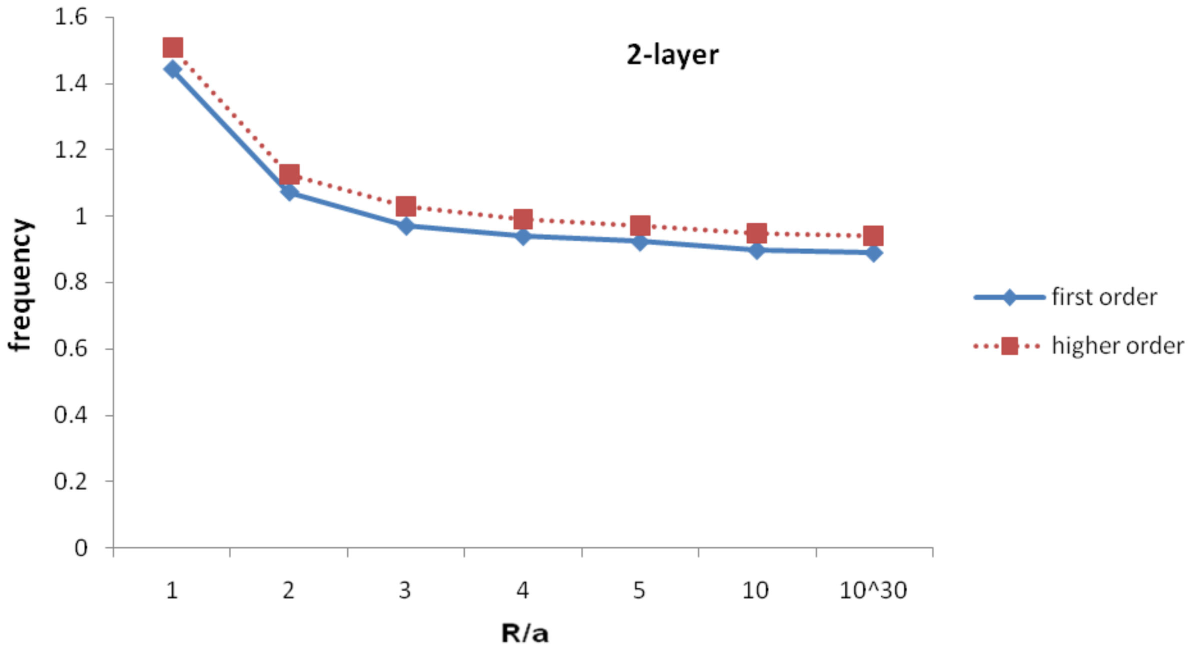

Figure 4. 6 : Variation of the natural frequencies $\overline{\omega}$ with radius to length ratios (R/a) for cross ply laminated shell with a/b=1 , $R_1 = R_2 = R$ and a/h = 10

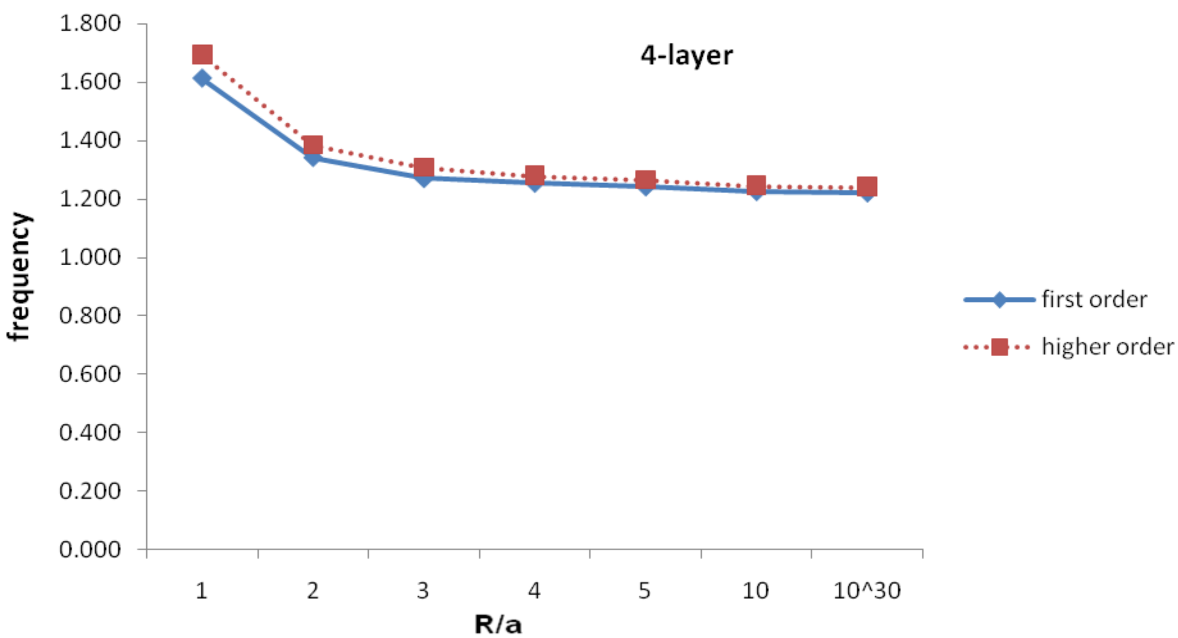

Figure 4. 7 : Variation of the natural frequencies $\overline{\omega}$ with radius to length ratios (R/a) for cross ply laminated shell with a/b=1 , $R_1 = R_2 = R$ and a/h = 10

Figure 4.6 and figure 4.7 shows the variation of the natural frequencies $\overline{\omega}$ with the radius to length ratios (R/a) for different number of laminates and for constant a/h=10. It is observed that the nondimensionalised frequency obtained by using higher order shear deformation theory is slightly higher than that obtained by first order shear deformation theory.

4.4 FREQUENCY ENVELOPES FOR CROSS –PLY SPHERICAL PANELS

The frequency envelopes for m=1 and for two-layer ,three-layer, four-layer, five-layer and eight-layer cross ply spherical panels for a length to thickness ratio (a/h) of 100, as the function of radius to length ratios (R/a) are plotted in figure 4.8. The first layer properties are given in table 4.8.

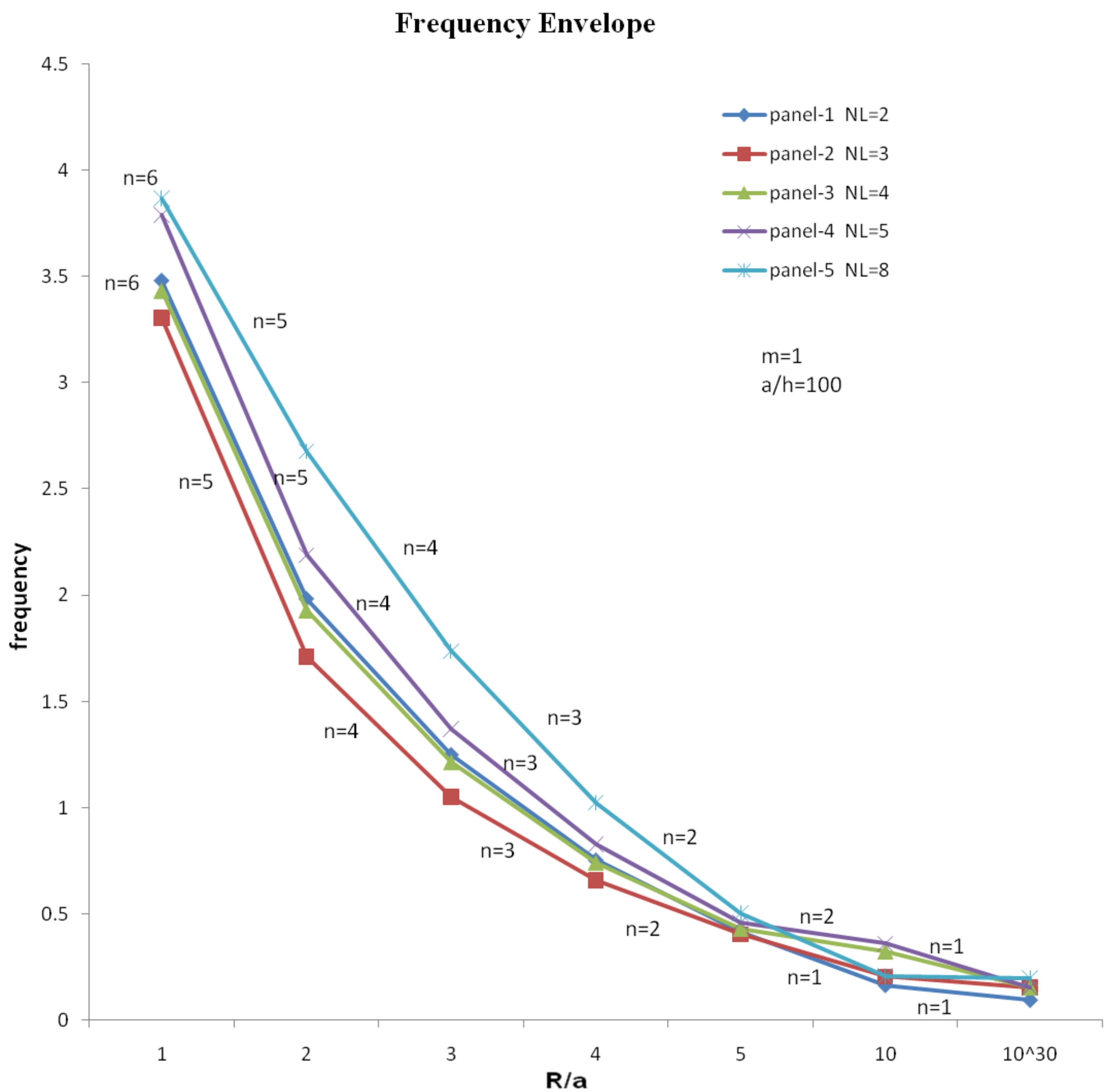

Figure 4. 8 : Frequency envelopes of cross-ply spherical panels a/h=100

.

Table 4. 8: **Material properties (first layer) of laminated spherical panels.**

$E_1 \times 10^6\ psi$	$E_2 \times 10^6\ psi$	υ_{12}	$G_{12} \times 10^6\ psi$	$G_{23} \times 10^6\ psi$	$G_{13} \times 10^6\ psi$	ρ
25.00	1.0	0.25	0.5	0.2	0.5	1.0

Table 4. 9: **Comparison of dimensionless frequency parameters of spherical panels**

R/a	Values of dimensionless frequency Parameters $\bar{\omega}$	Shell cross-section	% increase in frequency
1	Maximum = 3.869 Minimum = 3.307	Panel-5 Panel-3	17
2	Maximum = 2.677 Minimum = 1.711	Panel-5 Panel-3	56.46
3	Maximum = 1.738 Minimum = 1.051	Panel-5 Panel-3	65.37
4	Maximum = 1.024 Minimum = 0.657	Panel-5 Panel-3	55.86
5	Maximum = 0.503 Minimum = 0.402	Panel-5 Panel-3	25.12
10	Maximum = 0.359 Minimum = 0.167	Panel-4 Panel-1	115
10^30	Maximum = 0.197 Minimum = 0.097	Panel-5 Panel-1	103

It is seen from figure 4.8 that the eight layered panel gives the highest values of frequency parameter when compared with all other panels except for a value of R/a=10, where three layer gives the maximum value. The three layered panel (panel-2) gives the minimum frequency parameter of all panels for R/a<5. For values of R/a>8, the two layered panel (panel-1) gives the minimum value of frequency parameter for all panels. In fact, the frequency parameters of panel-2, panel-3 and panel-4 are equal for values of R/a=10^30. Panel 1 and 3 shows close frequency envelopes for R/a<5.

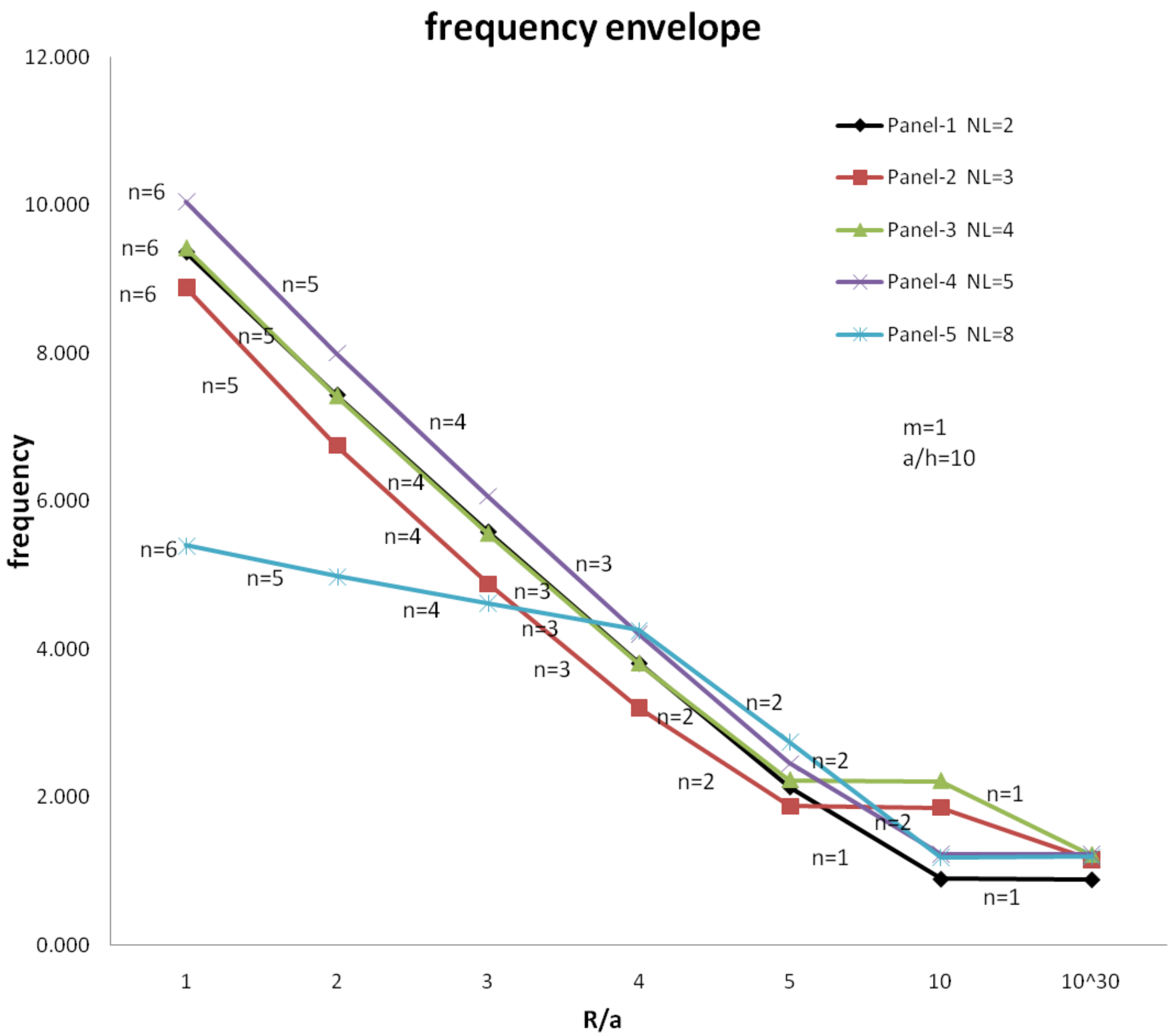

Figure 4. 9 : Frequency envelopes of cross-ply spherical panels a/h=10

Table 4. 10: **Comparison of dimensionless frequency parameters of spherical panels**

R/a	Values of dimensionless frequency Parameters $\bar{\omega}$	Shell cross-section	% increase in frequency
1	Maximum = 10.048 Minimum = 5.400	Panel-4 Panel-5	86.10
2	Maximum = 8.002 Minimum = 4.978	Panel-4 Panel-5	60.75
3	Maximum = 6.069 Minimum = 4.621	Panel-4 Panel-5	31.34
4	Maximum = 4.252 Minimum = 3.210	Panel-5 Panel-2	32.46
5	Maximum = 2.751 Minimum = 1.878	Panel-5 Panel-2	46.50
10	Maximum = 2.222 Minimum = 0.897	Panel-3 Panel-1	147.7
10^30	Maximum = 1.232 Minimum = 0.889	Panel-4 Panel-1	38.58

From figure 4.9, the five layered panel gives the highest values of frequency parameter when compared with all other panels except for a value of R/a=4, 5, 10, where eight and four layer gives the maximum value. The eight layered panel (panel-5) gives the minimum frequency parameter of all panels for R/a<3.5. For values of R/a between 3.5 and 5.5, the three layered panel (panel-2) gives the minimum value of frequency parameter for all panels. The two layered panel (panel-1) gives the minimum frequency parameter of all panels for R/a>5.

The panel giving maximum and minimum frequency parameters at selected R/a values with the percentage increase in frequencies are tabulated in table 4.9 and 4.10 for a/h=100,10. By simply changing the orientation and number of layers, the values of frequency parameter may be varied. In table 4.9, at R/a =10, panel-4 has higher value of frequency parameter compared to panel-1.

4.4.1 Comparison of results by First and Higher order shear deformation theory

The dimensionless frequencies have been calculated for different laminated composite spherical shells for various values of 'R/a' and 'a/h' by the first and higher order theories. Figure 4.10 and figure 4.11 shows the variation of the dimensionless natural frequencies $\overline{\omega}$ with the radius to length ratios (R/a) by both theories. In figure 4.10, the overlapping of the graphs illustrate that there is not much difference in frequency by the first and higher order shear deformation theory for a/h=100 (4-layer).In figure 4.10, the frequency envelope shows the values of dimensionless frequency obtained by first order shear deformation theory is slightly higher than higher order deformation theory for 4-layer laminated composite spherical panel ,this is because of the influence of a/h ratio.

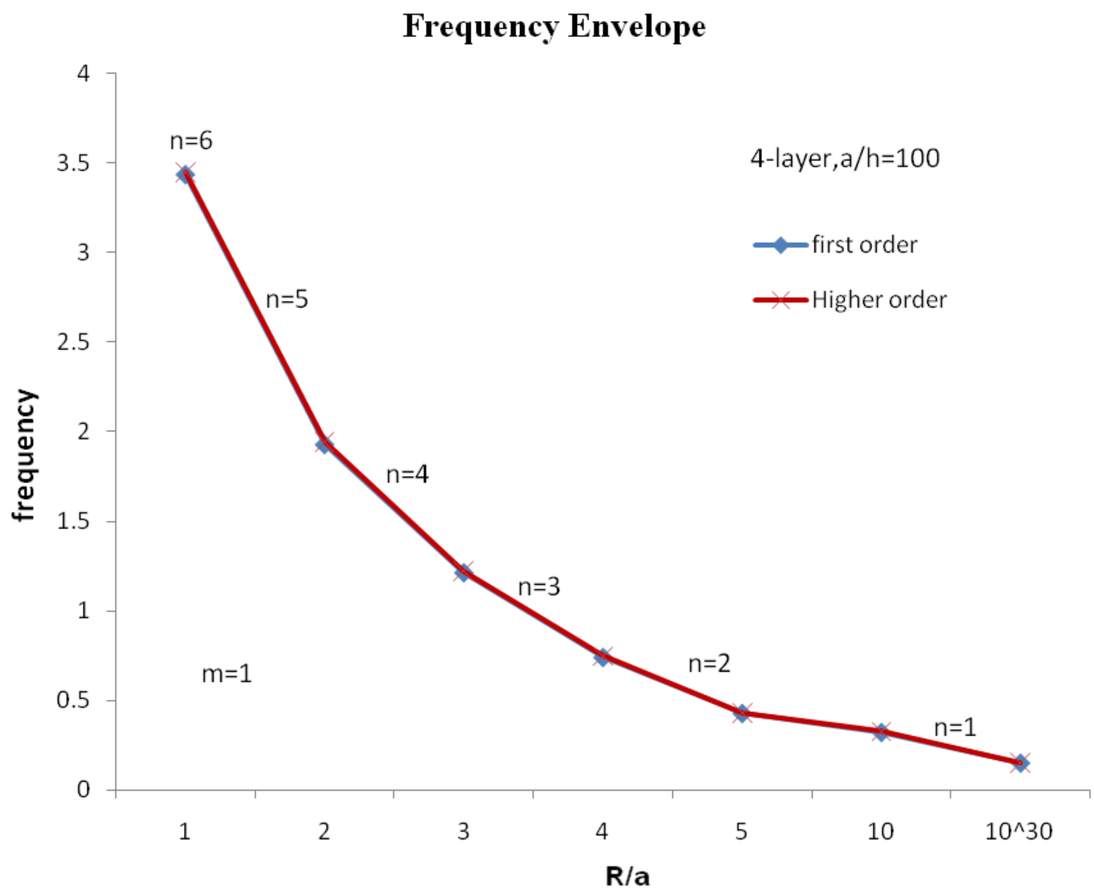

Figure 4. 10 : Frequency envelopes of cross-ply (4-layer) spherical panels a/h=100

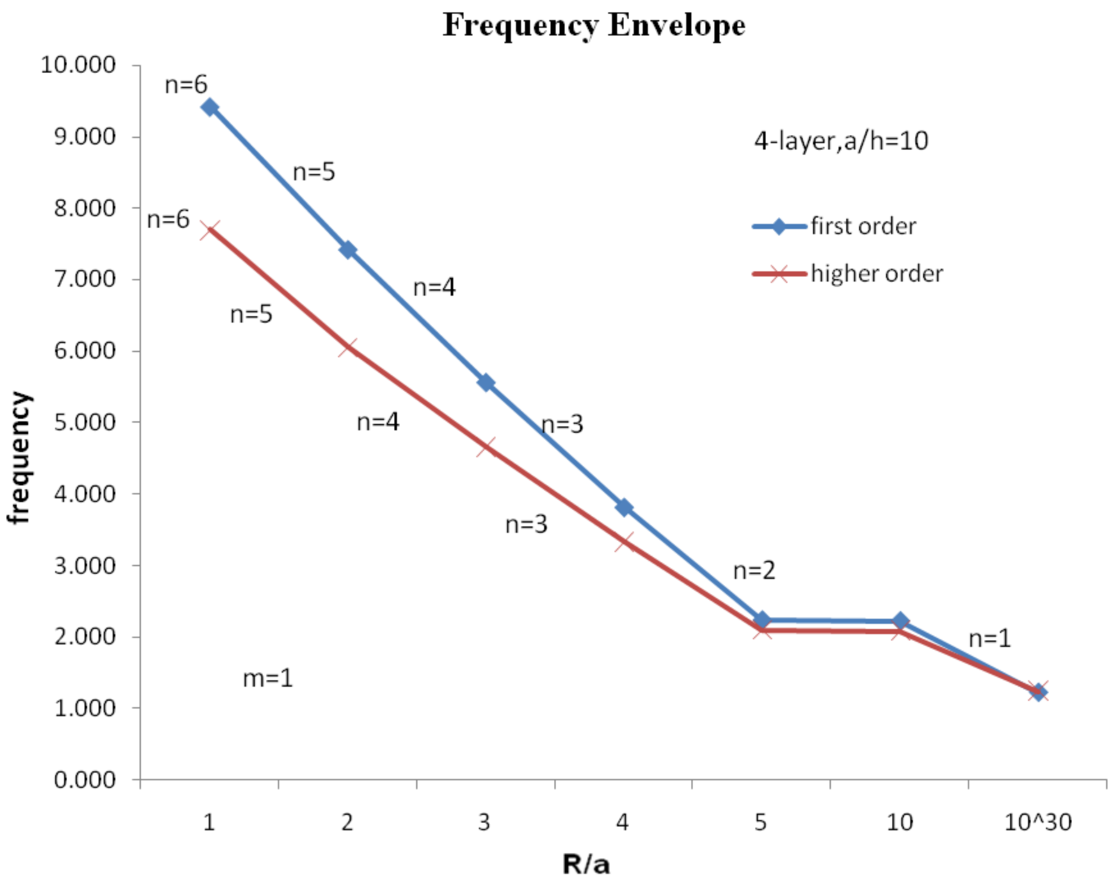

Figure 4. 11 : Frequency envelopes of cross-ply (4-layer) spherical panels a/h=10

Chapter 5

Conclusion

CHAPTER-5

CONCLUSION

The free vibration problem of laminated thin spherical panels is analyzed in this study by using both the first order shear deformation theory and a higher order shear deformation theory as proposed by Reddy and Liu. The higher order shear deformation theory is based on a displacement field in which the displacements of the middle surface are expanded as cubic functions of the thickness coordinate and the transverse displacement is assumed to be constant through the thickness. This displacement field leads to parabolic distribution of the transverse shear stresses and zero transverse normal strain and hence no shear correction factors are used.

The thickness coordinate multiplied by the curvature is assumed to be small in comparison to unity and hence negligible. The governing equations have been developed. These equations are then reduced to the equations of motion for spherical panel and the Navier solution has been obtained for cross-ply laminated composite spherical panels. The resulting equations are suitably nondimensionalised. The eigen value problem is then solved to obtain the free vibration frequencies. The lowest frequency is considered in all cases.

The illustrative examples of spherical shells with simply supported boundary conditions for convenience are considered. Versatility and validity of first order shear deformation and higher order shear deformation theory are also aptly illustrated by comparing the results from the present work with the corresponding results in previous studies using quite different alternative approaches. It is observed that there is a very good agreement between various sets of results.

The effects of the radius to length ratio, circumferential wave number, length to thickness ratio, composite lamination, and geometrical properties were investigated on the vibration characteristics of thin spherical panels. From the analysis of the result presented in previous section the following concluding remarks can be drawn:

1. For the simply supported laminate the effect of E_{11} is most dominant on dispersion in the natural frequencies and effect of υ_{12} is least dominant. Out of all the natural frequencies, the fundamental frequency is most sensitive to changes in E_{11}.
2. The natural frequencies rapidly increases with circumferential mode number 'n',obtained from frequency envelope curve.
3. The influence of radius to length ratio on natural frequency of the spherical shell is larger than that of length to thickness ratio.
4. The natural frequencies obtained by first order shear deformation theory is close to the values that obtained by higher order shear deformation theory for a/h=100. But in the case of a/h=10, the value obtained by first order shear deformation theory is slightly lower than higher order shear deformation theory.
5. For a/h =100, the eight layered panel gives the highest values of frequency parameter when compared with all other panels except for a value of R/a =10, where three layer gives the maximum value. The three layered panel (panel-2) gives the minimum frequency parameter of all panels for R/a<5. For values of R/a>8, the two layered panel (panel-1) gives the minimum value of frequency parameter for all panels. In fact, the frequency parameters of panel-2, panel-3 and panel-4 are equal for values of R/a=10^30. Panel 1 and 3 shows close frequency envelopes for R/a<5.

6. For a/h =10, the five layered panel gives the highest values of frequency parameter when compared with all other panels except for a value of R/a=4, 5, 10, where

eight and four layer gives the maximum value. The eight layered panel (panel-5) gives the minimum frequency parameter of all panels for R/a<3.5. For values of R/a between 3.5 and 5.5, the three layered panel (panel-2) gives the minimum value of frequency parameter for all panels. The two layered panel (panel-1) gives the minimum frequency parameter of all panels for R/a>5.

FUTURE WORK

The formulation in the present study can be used to evaluate the vibration characteristics for forced vibration of spherical shells with different boundary conditions. The present study is discussed about the open type spherical shells/panels, so it can be extended for closed type cross ply laminated spherical shell. It can also be used to analyse the buckling load of the simply supported spherical panels.

References

REFERENCES

1) A. V. Singh," Free vibration analysis of deep doubly curved sandwich panels" Computers & Structures Volume 73, Issues 1-5, 12 October 1999, Pages 385-394.

2) Ahmet Sinan Oktem, Reaz A. Chaudhuri (2009): "Higher-order theory based boundary-discontinuous Fourier analysis of simply supported thick cross-ply doubly curved panels," Composite Structures, 89, pp.448–458.

3) Collins, J. D., and Thomson, W. T. (1969): "The eigen value problem for structural system with statistical properties," American Institute of Aeronautics and Astronautics Journal 7, pp.170-173.

4) Dasgupta, A., and Huang, K.H ., (1997): "**A** layer-wise analysis for free vibration of thick composite spherical panels", Journal of Composite materials, 31, pp.658.

5) De Souza, V. C. M. and Croll, C. G. A. (1981): "Free Vibrations of Orthotropic Spherical Shells Engineering Structures," Vol. 3, No. 2, pp. 71-84.

6) Dong, S. B., (1968):"Free Vibration of laminated Orthotropic Cylindrical Shells", Journal of the Acoustical Society of America, Vol.44, No.6, pp. 1628-1635.

7) Engin, A. E. and Liu, Y. K. (1970):"Axisymmetric Response of A Fluid-Filled Spherical Shell in Free Vibrations," Journal of Biomechanics, Vol. 3, No. 1, pp. 11-16.

8) Ganapathi, M. (1995): "Large- Amplitude Free Flexural Vibrations of Laminated Composite Curved Panels Using Shear-Flexible Shell Element," Defence Science Journal, Vol. 45, l, pp 55-60.

9) Girish, J., and Ramachandra, L. S. (2007): "Nonlinear Static Response and Free Vibration Analysis of Doubly Curved Cross-Ply Panels," Journal of Aerospace Engineering, Vol. 20, No. 1

10) H. Nguyen-Van, N. Mai-Duy, T. Tran-Cong." Free vibration analysis of laminated plate/shell structures based on FSDT with a stabilized nodal-integrated quadrilateral element", Journal of Sound and Vibration 313 (2008) 205–223.

11) Hu, X. X., Tsuiji T., (1999): "Free Vibration Analysis of Curved and Twisted Cylindrical Thin Panels", Journal of Sound and Vibration, 219, pp. 52-77.

12) Jose Simoes Moita, Cristovao M. Mota Soares and Carlos A. Mota Soares**,"** Buckling and dynamic behaviour of laminated composite structures using a discrete higher-order displacement model" Computers & Structures Volume 73, Issues 1-5, 12 October 1999, Pages 407-423.

13) Kabir, H. R. H., (1996): "On boundary value problems of moderately thick shallow cylindrical panels with arbitrary laminations," Composite Structures, 34, pp.169- 184.

14) Kraus.H."Thin elastic shells",wiley, New York,1967.

15) Lam, K. Y. and Loy, C. T., (1995): "Influence of boundary conditions and fibre orientation on the natural frequencies of thin orthotropic laminated cylindrical shells", Journal Composite Structures, 31, pp.21-30.

16) Latifa, S .K., and Sinha, P. K., (2005): "Improved Finite Element Analysis of Multilayered, Doubly Curved Composite Shells," Journal of Reinforced Plastics and Composites; 24; pp.385.

17) Lee, J. J., Oh, I. K., Lee, I. and Rhiou J. J.(2003): " Non-linear Static and Dynamic Instability of Complete Spherical Shells Using Mixed Finite Element Formulation," International Journal of Non-Linear Mechanics, Vol. 38, pp. 923 – 934.

18) Liew, K.M., Peng, L.X., Ng, T.Y., (2002): "Three-dimensional vibration analysis of spherical shell panels subjected to different boundary conditions," International Journal of Mechanical Sciences, 44, pp. 2103–2117.

19) Naghieh, M. and Hayek, S. J. (1971): "Vibration of Fiber-Reinforced Spherical Shells, Fibre Science and Technology," Vol. 4, No. 2, pp. 115-137.

20) Narasimhan, M. C. and Alwar, R. S. (1991): "Free Vibration Analysis of Laminated Orthotropic Spherical Shells", Journal of Sound and Vibration, Vol. 154, No. 3, pp. 515-529.

21) Niordson, F. I., (1984): "Free Vibrations of Thin Elastic Spherical Shells", International Journal of Solids and Structures, Vol. 20, No. 7, pp. 667-687.

22) Partha Bhattacharya, Hassan Suhail, Sinha, P. K., (2002): "Finite element analysis and distributed control of laminated composite shells using LQR/IMSC approach," Aerospace Science and Technology, 6, pp.273–281.

23) Ram, K. S. S. and Babu, T. S. (2002): "Free Vibration of Composite Spherical Shell Cap with and without A Cutout", Computers and Structures, Vol. 80, pp. 1749-1756.

24) Reddy, J. N. and Liu, C. F. (1985):"A higher-order shear deformation theory of laminated elastic shells," International journal of engineering science, 23, pp.319-330.

25) Reddy, J.N., (1984): "Laminated shells", Journal of engineering mechanics, vol.110, No.5.

26) S.K Panda and B.N Singh . "Nonlinear free vibration of spherical shell panel using higher order shear deformation theory – A finite element approach," International Journal of Pressure Vessels and Piping 86 (2009) 373–383

27) Sathyamoorthy, M. (1995): "Nonlinear vibrations of moderately thick orthotropic shallow spherical shells," Computer and Structures, Vol. 57, No. I, pp. 59-65.

28) Soldatos, K. P. (1984): “A comparison of some shell theories used for the dynamic analysis of cross-ply laminated circular cylindrical panels”, Journal of Sound and Vibration, 97, pp. 305–319.

29) To, C.W.S., Liu, M.L., (2001): “Geometrically nonlinear analysis of layerwise anisotropic shell structures by hybrid strain based lower order elements,” Finite Elements in Analysis and Design, 37, pp. 1-34.

30) Umut Topal,(2006):” Mode-Frequency Analysis of Laminated Spherical Shell” Department of Civil Engineering Karadeniz Technical University.

31) Yu, S. D., Cleghorn, W. L., and Fenton, R. G. (1995): “On the Accurate Analysis of Free Vibration of Open Circular Cylindrical Shells”, Journal of Sound and Vibration, Vol.188, no.3, pp. 315-336.

Appendix

APPENDIX

The non-dimensionalised coefficients of the [C_{ij}] and [M_{ij}] matrices of First order shear deformation theory are given below

$$C(1,1) = -\left(\bar{A}_{11}\bar{\lambda}_m{}^2 S^2 + \bar{A}_{66}\bar{\lambda}_n{}^2 + K_{22}^2\bar{A}_{55}J^2\right)$$

$$C(1,2) = -\bar{\lambda}_m\overline{\lambda_n}S(\bar{A}_{12} + \bar{A}_{66})$$

$$C(1,3) = J\,S\,\bar{\lambda}_m(\bar{A}_{11} + \bar{A}_{12} + K_{22}^2\bar{A}_{55})$$

$$C(1,4) = -\left(\bar{B}_{11}\bar{\lambda}_m{}^2 S^2 + \bar{B}_{66}\bar{\lambda}_n{}^2 - K_{22}^2\bar{A}_{55}\frac{J^2}{H}\right)$$

$$C(1,5) = -\bar{\lambda}_m\bar{\lambda}_n S(\bar{B}_{12} + \bar{B}_{66})$$

$$C(2,1) = C(1,2)$$

$$C(2,2) = -\left(\bar{A}_{66}\bar{\lambda}_m^2 S^2 + \bar{A}_{22}\bar{\lambda}_n^2 + K_{11}^2\bar{A}_{44}J^2\right)$$

$$C(2,3) = (\bar{A}_{21} + \bar{A}_{22} + K_{11}^2\bar{A}_{44})\bar{\lambda}_n J$$

$$C(2,4) = -\bar{\lambda}_m\bar{\lambda}_n S(\bar{B}_{66} + \bar{B}_{21})$$

$$C(2,5) = -\left(\bar{B}_{66}\bar{\lambda}_m^2 S^2 + \bar{B}_{22}\bar{\lambda}_n^2 - K_{11}^2\bar{A}_{44}\frac{J^2}{H}\right)$$

$$C(3,1) = C(1,3)$$

$$C(3,2) = C(2,3)$$

$$C(3,3) = -\left(K_{22}^2\bar{A}_{55}\bar{\lambda}_m^2 S^2 + K_{11}^2\bar{A}_{44}\bar{\lambda}_n^2 + (\bar{A}_{11} + \bar{A}_{12} + \bar{A}_{21} + \bar{A}_{22})J^2\right)$$

$$C(3,4) = -\left(K_{22}^2\bar{A}_{55}S\frac{J}{H} - J\,S(B_{11} + B_{21})\right)\bar{\lambda}_m$$

$$C(3,5) = -\left(K_{11}^2\bar{A}_{44}\frac{J}{H} - J(B_{12} + B_{22})\right)\bar{\lambda}_n$$

$$C(4,1) = C(1,4)$$

$$C(4,2) = C(2,4)$$

$$C(4,3) = C(3,4)$$

$$C(4,4) = \left((\bar{B}_{11} + \bar{B}_{12}) J\, S - \mathrm{K}_{22}^2 \bar{A}_{55} S \frac{J}{H} \right) \bar{\lambda}_m$$

$$C(4,5) = -(\bar{D}_{12} + \bar{D}_{66}) \bar{\lambda}_m \bar{\lambda}_n S$$

$$C(5,1) = C(1,5)$$

$$C(5,2) = C(2,5)$$

$$C(5,3) = C(3,5)$$

$$C(5,4) = C(4,5)$$

$$C(5,5) = -\left(\mathrm{K}_{11}^2 \bar{A}_{44} \frac{J^2}{H^2} - \bar{D}_{66} S^2 \bar{\lambda}_m^2 - \bar{D}_{22} \bar{\lambda}_n^2 \right)$$

$M(1,1) = I_1'$ $\quad M(2,2) = I_1'$ $\quad M(3,3) = I_1'$ $\quad M(4,4) = I_3'$ $\quad M(5,5) = I_3'$

$S = \frac{b}{a}$ $\quad J = \frac{b}{R}$ $\quad H = \frac{h}{R}$

The elements of the M matrices, which are not given above, are zero. The value of shear correction factor K_{11}^2, K_{22}^2 is 5/6

The non-dimensionalised coefficients of the [C_{ij}] and [M_{ij}] matrices of Higher order shear deformation theory are given below,

$$C(1,1) = -\left(\bar{\lambda}_m^2 \bar{A}_{11} S^2 + \bar{A}_{66} \bar{\lambda}_n^2\right)$$

$$C(1,2) = -(\bar{A}_{12} + \bar{A}_{66}) S \bar{\lambda}_m \bar{\lambda}_n$$

$$C(1,3) = (\bar{A}_{11} + \bar{A}_{12}) S\, J \bar{\lambda}_m + \frac{4}{3}\frac{H}{J} S^3 \bar{\lambda}_m^3 \bar{E}_{11} + \frac{4}{3}(\bar{E}_{12} + 2\bar{E}_{66})\frac{H}{J} S \bar{\lambda}_n^2 \bar{\lambda}_m$$

$$C(1,4) = -\left(\bar{B}_{11} - \frac{4}{3}\bar{E}_{11}\right) S^2 \bar{\lambda}_m^2 - \left(\bar{B}_{66} - \frac{4}{3}\bar{E}_{66}\right) \bar{\lambda}_n^2$$

$$C(1,5) = -\left(\bar{B}_{12} - \frac{4}{3}\bar{E}_{12}\right) S \bar{\lambda}_m \bar{\lambda}_n - \left(\bar{B}_{66} - \frac{4}{3}\bar{E}_{66}\right) S \bar{\lambda}_m \bar{\lambda}_n$$

$$C(2,1) = C(1,2)$$

$$C(2,2) = -\left(\bar{A}_{66} \bar{\lambda}_m^2 S^2 + \bar{A}_{22} \lambda_n^2\right)$$

$$C(2,3) = (\bar{A}_{21} + \bar{A}_{22}) J\, \bar{\lambda}_n + \frac{4}{3}\bar{E}_{22}\frac{H}{J}\bar{\lambda}_n^3 + (2\bar{E}_{66} + \bar{E}_{21})\frac{4}{3} S^2 \frac{H}{J}\bar{\lambda}_m^2 \bar{\lambda}_n$$

$$C(2,4) = -\left(\bar{B}_{66} - \frac{4}{3}\bar{E}_{66} + \bar{B}_{21} - \frac{4}{3}\bar{E}_{21}\right) S\, \bar{\lambda}_m \bar{\lambda}_n$$

$$C(2,5) = -\left(\bar{B}_{66} - \frac{4}{3}\bar{E}_{66}\right) S^2 \bar{\lambda}_m^2 - \left(\bar{B}_{22} - \frac{4}{3}\bar{E}_{22}\right) \bar{\lambda}_n^2$$

$$C(3,1) = C(1,3)$$

$$C(3,2) = C(2,3)$$

$$\begin{aligned} C(3,3) = {} & -\bar{A}_{55} S^2 \bar{\lambda}_m^2 + 8\bar{D}_{55}\bar{\lambda}_m^2 - 16\bar{F}_{55} S^2 \bar{\lambda}_m^2 - \frac{4}{3}(2\bar{E}_{11} + \bar{E}_{12} + \bar{E}_{21}) S^2 H \bar{\lambda}_m^2 - \bar{A}_{44}\bar{\lambda}_n^2 \\ & + 8\bar{D}_{44}\bar{\lambda}_n^2 - 16\bar{F}_{44}\bar{\lambda}_n^2 - \frac{4}{3}(\bar{E}_{21} + 2\bar{E}_{22} + \bar{E}_{12}) H \bar{\lambda}_n^2 \\ & - \frac{16}{9}(\bar{H}_{12} + 4\bar{H}_{66} + \bar{H}_{21}) S^2 \frac{H^2}{J^2}\bar{\lambda}_m^2 \bar{\lambda}_n^2 - \frac{16}{9}\left(\bar{H}_{11} S^4 \frac{H^2}{J^2}\bar{\lambda}_m^4 + \bar{H}_{22}\frac{H^2}{J^2}\bar{\lambda}_n^4\right) \\ & - (\bar{A}_{11} + \bar{A}_{12} + \bar{A}_{21} + \bar{A}_{22}) J^2 \end{aligned}$$

$$C(3,4) = (-\bar{A}_{55} + 8\bar{D}_{55})S\frac{J}{H}\bar{\lambda}_m - 16\bar{F}_{55}S\frac{J}{H}\bar{\lambda}_m + (\bar{B}_{11} + \bar{B}_{21})S\,J\bar{\lambda}_m - \frac{4}{3}(\bar{E}_{11} + \bar{E}_{21})S\,J\bar{\lambda}_m - \frac{16}{9}\bar{H}_{11}S^3\frac{H}{J}\bar{\lambda}_m^3 + \frac{4}{3}(2\bar{F}_{66} + \bar{F}_{21})S\frac{H}{J}\bar{\lambda}_m\bar{\lambda}_n^2 - \frac{16}{9}(2\bar{H}_{66} + \bar{H}_{21})S\frac{H}{J}\bar{\lambda}_m\bar{\lambda}_n^2 + \frac{4}{3}\bar{F}_{11}S^3\frac{H}{J}\bar{\lambda}_m^3$$

$$C(3,5) = (-\bar{A}_{44} + 8\bar{D}_{44})\frac{J}{H}\bar{\lambda}_n + \frac{4}{3}(\bar{F}_{12} + 2\bar{F}_{66})\frac{H}{J}S^2\bar{\lambda}_m^2\bar{\lambda}_n - \frac{16}{9}(\bar{H}_{12} + 2\bar{H}_{66})\frac{H}{J}S^2\bar{\lambda}_m^2\bar{\lambda}_n + \frac{4}{3}\bar{F}_{22}\frac{H}{J}\bar{\lambda}_n^3 - \frac{16}{9}\bar{H}_{22}\frac{H}{J}\bar{\lambda}_n^3 + (\bar{B}_{12} + \bar{B}_{22})J\bar{\lambda}_n - \frac{4}{3}(\bar{E}_{12} + \bar{E}_{22})J\bar{\lambda}_n - 16\bar{F}_{44}\bar{\lambda}_n\frac{J}{H}$$

$$C(4,1) = C(1,4)$$

$$C(4,2) = C(2,4)$$

$$C(4,3) = C(3,4)$$

$$C(4,4) = -\bar{D}_{11}S^2\bar{\lambda}_m^2 + \frac{4}{3}\bar{F}_{11}S^2\bar{\lambda}_m^2 - \bar{D}_{66}\bar{\lambda}_n^2 + \frac{4}{3}\bar{F}_{66}\bar{\lambda}_n^2 - \bar{A}_{55}\frac{J^2}{H^2} + 4(2\bar{D}_{55} - 4\bar{F}_{55})\frac{J^2}{H^2} + \frac{4}{3}\bar{F}_{11}S^2\bar{\lambda}_m^2 - \frac{16}{9}\bar{H}_{11}S^2\bar{\lambda}_m^2 + \frac{4}{3}\bar{F}_{66}\bar{\lambda}_n^2 - \frac{16}{9}\bar{H}_{66}\bar{\lambda}_n^2$$

$$C(4,5) = -\bar{D}_{12}S\bar{\lambda}_m\bar{\lambda}_n + \frac{4}{3}\bar{F}_{12}S\bar{\lambda}_m\bar{\lambda}_n - \bar{D}_{66}S\bar{\lambda}_m\bar{\lambda}_n + \frac{4}{3}\bar{F}_{66}S\bar{\lambda}_m\bar{\lambda}_n + \frac{4}{3}(\bar{F}_{12} + \bar{F}_{66})S\bar{\lambda}_m\bar{\lambda}_n - \frac{16}{9}(\bar{H}_{12} + \bar{H}_{66})S\bar{\lambda}_m\bar{\lambda}_n$$

$$C(5,1) = C(1,5)$$

$$C(5,2) = C(2,5)$$

$$C(5,3) = C(3,5)$$

$$C(5,4) = C(4,5)$$

$$C(5,5) = -\bar{D}_{66}S^2\bar{\lambda}_m^2 + \frac{4}{3}\bar{F}_{66}S^2\bar{\lambda}_m - \bar{D}_{22}\bar{\lambda}_n^2 + \frac{4}{3}\bar{F}_{22}\bar{\lambda}_n^2 - \bar{A}_{44}\frac{J^2}{H^2} + 4(2\bar{D}_{44} - 4\bar{F}_{44})\frac{J^2}{H^2} + \frac{4}{3}\bar{F}_{66}S^2\bar{\lambda}_m^2 - \frac{16}{9}\left(\bar{H}_{66}S^2\bar{\lambda}_m^2 + \bar{H}_{22}\bar{\lambda}_n^2\right) + \frac{4}{3}\bar{F}_{22}\bar{\lambda}_n^2$$

$M(1,1) = -\tilde{I}_1$ $\quad$ $M(1,3) = \frac{4}{3}I_4'S\frac{H}{J}\bar{\lambda}_m$ $\quad$ $M(1,4) = -\tilde{I}_2$ $\quad$ $M(2,2) = -\tilde{I}_1$

$M(2,3) = \frac{4}{3} I_4' \frac{H}{J} \bar{\lambda}_n$ $\quad$ $M(2,5) = -\tilde{I}_2$ $\quad$ $M(3,1) = M(1,3)$ $\quad$ $M(3,2) = M(2,3)$

$M(3,3) = -\tilde{I}_1 - \frac{16}{9} I_7' \left(\bar{\lambda}_m^2 S^2 + \bar{\lambda}_n^2\right) \frac{H^2}{J^2}$ $\quad$ $M(3,4) = \frac{4}{3} \tilde{I}_5 S \frac{H}{J} \bar{\lambda}_m$ $\quad$ $M(3,5) = \frac{4}{3} \tilde{I}_5 \frac{H}{J} \bar{\lambda}_n$

$M(4,1) = M(1,4)$ $\quad$ $M(4,3) = M\ (3,4)$ $\quad$ $M(4,4) = -\tilde{I}_3$ $\quad$ $M(5,2) = M(2,5)$

$M(5,3) = M(3,5)$ $\quad$ $M(5,5) = -\tilde{I}_3$

The elements of the M matrices, which are not given above, are zero.