EFFECT OF VIBRATION ON DELAMINATED BEAMS

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

BACHELOR OF TECHNOLOGY
IN
MECHANICAL ENGINEERING

BY

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DEPARTMENT OF MECHANICAL ENGINEERING
NATIONAL INSTITUTE OF TECHNOLOGY
ROURKELA - 769008
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National Institute of Technology
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CERTIFICATE

This is to certify that the thesis entitled **"EFFECT OF VIBRATION ON DELAMINATED BEAMS"** is submitted by **Sri AMIT KUMAR SINGH** in partial fulfillment of the requirements for the award of Bachelor of Technology degree in Mechanical Engineering at National Institute of Technology, Rourkela is an authentic work carried out by him under my supervision and guidance.

To the best of my knowledge, the matter embodied in the thesis has not been submitted to any other University/Institute for the award of any Degree or Diploma.

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ACKNOWLEDGEMENT

I wish to express my profound gratitude and indebtedness to **Prof. R. K. Behera**, Department of Mechanical Engineering, NIT, Rourkela for introducing the present topic and for their inspiring guidance, constructive criticism and valuable suggestion throughout the project work.

I am also thankful to Metallurgy Deptt., NIT Rourkela for the assistance and guidance which helped me in completing my project.

Last but not least, my sincere thanks to all our friends who have patiently extended all sorts of help for accomplishing this undertaking.

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ABSTRACT

The phenomenon of delamination is common in composite beams as the composite beams are having laminate structures. Delamination leads to development of cracks and reduces the strength of the material.

This paper deals to analyze the effect of vibration on the delaminated beam. A FEM model is constructed using Euler Bournouli beam theory. Basically the model is consisting of a single symmetric delamination at the centre of the beam. The beam is considered as group of beam and each beam satisfies the boundary and continuity conditions at the points of joining and at ends. This model gives the idea about the natural frequency of vibration of beam without subjecting the beam to actual conditions. Furthermore, the model can be modified according to the length and the position of delaminated surface.

In this project the modeling is done by finite element method using matlab as mathematical software tool.

CHAPTER 1

INTRODUCTION

Introduction:

1.1 Composite Materials:

Composite materials are composed of at least two elements working together to produce material properties that are different to the properties of those elements on their own.

There are of constituent materials: matrix and reinforcement. At least one portion (fraction) of each type is required. The matrix material surrounds and supports the reinforcement materials by maintaining their relative positions. The reinforcements impart special physical (mechanical and electrical) properties to enhance the matrix properties.

Composite materials are widely used because of their less weight o strength ratio. The most primitive composite materials comprised straw and mud in the form of bricks for building construction. Now composites are also used in aerospace industry, many jets and airplanes are made of composite materials that are stronger and lighter than the materials they were made from.

1.2 Delamination in Composite Beams:

Shocks, impact, loadings or repeated cyclic stresses can cause the laminate to separate at the interface between two layers, forming a mica-like structure of separate layers, with significant loss of mechanical toughness. This condition is known as delamination. Delamination is very common in composite because of they are made in the form of laminate. Basically, it occurs because of partial bonding, non bonding or debonding between the different layers of composite. Partial bonding and non bonding are manufacturing defect whereas debonding occurs because of sudden impact or repeated cyclic stress.

Delamination failure may be detected in the material by its sound; solid composite has bright sound, while delaminated part sounds dull. Other nondestructive testing methods are used, including embedding optical fibers coupled with optical time domain reflectometer testing of their state, testing with ultrasound, radiographic imagining, and infrared imaging.

1.3 Objective and Scope of Work:

1.3.1 Vibration and Fatigue Failure Problem in Planes:

The vibration behavior of a structure which is subjected to dynamic forces can be predicted using finite element analysis (FEA) modeling. The manner in which the structure behaves is determined by its stiffness, mass, damping, and constraints and it is important to determine this response since internal, oscillating stresses may exceed the strength or fatigue life of the material. Modes analysis is performed to determine the eigen values (natural frequencies) and mode shapes (eigenvectors) of the structure. Now, the model can be subjected to transient dynamic loads and or displacements to determine the time histories of nodal displacements, velocities, accelerations, stresses and reaction forces.

Random vibration is of prime concern because all vehicles (cars, trucks, trains) traveling over a relative rough surface are subjected to this source of vibration and stress. Aircraft, missiles, and rockets are subjected to random vibration excitation. This is due to the extreme turbulence of jet exhaust downstream of the jet and rocket engines, and aerodynamic buffeting.

The output from the Random vibration analysis can be interfaced with the fatigue/fracture analysis to estimate the durability of a structure.

Knowing the load history, the fatigue analysis; the crack initiation and crack propagation time is predicted. The load history is the number of cycles and the dynamic responses of the structure over its intended service life.

CHAPTER 2

LITERATURE REVIEW

2. LITRETURE REVIEW:

2.1 General Background:

Laminate composite beams have extensive use in aircraft, spacecraft and space structures because of less strength to weight ratio and stiffness to weight ratio. As the applications are very huge and expensive, they require non destructive testing or predictive testing. In effort to this Rajkumar et al. first proposed the vibration analysis model of composite beams. The model was based on four Timoshenko beams.

Wang et al. [2] have examined the free vibrations of an isotropic beam with a through-width delamination by using four Euler–Bernoulli beams connected at the delamination boundaries. In their formulation he considered the coupling effect of longitudinal and flexural motions in delaminated layers. He assumed that the delaminated layers deformed ‘freely’ without touching each other (‘free mode’), which was shown to be physically inadmissible.

Mujumdar and Suryanarayan [3] then proposed a ‘constrained mode’ where the delaminated layers are assumed to be in touch along their whole length all the time, but are allowed to slide over each other. This model was physically admissible and the results was also in the vicinity of the experimental results.

Tracy and Pardoen [4] presented similar constrained model based on a simply supported composite beam.

Hu and Hwu [5] presented constrained model based on a sandwich beam

Shu and Fan [6] presented constrained model based on a bimaterial beam.

Shen and Grady [7] in their experiments found the opening of mode shapes. However constrained model failed to explain this.

Luo and Hanagud [8] proposed an analytical model based on the Timoshenko beam theory, which uses piecewise-linear springs to simulate the ‘open’ and ‘closed’ behavior between the delaminated surfaces to capture the opening in the mode shapes found in the experiments. His model was combination of both free and constrained model. He considered that for free model the spring stiffness would be zero and infinity for constrained model.

Saravanos and Hopkins [9] developed an analytical solution for predicting natural frequencies, mode shapes and modal damping of a delaminated composite beam based on a general laminate theory which involves kinematic assumptions representing the discontinuities in the in-plane and through-the-thickness displacements across each delamination crack.

Chakraborty et al. [10] presented a finite element method to study the free vibration of delaminated asymmetric composite beams using refined locking free first-order shear deformable elements.

Zak et al. [11 and 12] presented model on Finite element methods based on the first-order shear deformation theory.

Chattopadhyay et al. [13], Radu and Chattopadhyay [14] and Hu et al. [15] presented the model on finite element methods based on the higher-order shear deformation theories.

Shu [16] presented an analytical solution to study a sandwich beam with double delaminations. His study emphasized on the influence of the contact mode, ‘free’ and ‘constrained’, between the delaminated layers and the local deformation near the two fronts of the delamination.

Shu and Mai [17] investigated the local deformation near the delamination fronts of delamination and identified the ‘rigid connector’ and the ‘soft connector’ conditions. A ‘rigid connector’ considers the differential stretching between the delaminated layers, while a ‘soft connector’ neglects the differential stretching.

Lestari and Hanagud [18] studied a composite beam with multiple delaminations by using the Euler–Bernoulli beam theory with piecewise-linear springs to simulate the ‘open’ and ‘closed’ behavior between the delaminated surfaces. Lee et al. [19] studied a composite beam with arbitrary lateral and longitudinal multiple delaminations by using the ‘free model’ and by assuming a constant curvature at the multiple-delamination tip.

Shu and Della [20] studied the composite beams with multiple delaminations by using the ‘free’ and ‘constrained’ models. Their study emphasized on the influence of the multiple delaminations on the first and second frequencies and the corresponding mode shapes of the beams.

Ju et al. [21] presented the finite element analysis using the Timoshenko beam theory and Lee [22] using the layer wise theory.

Finite element methods have been developed by Ju et al. [23] using the Mindlin plate theory, Cho and Kim [24] using the higher-order zigzag theory and Kim et al. [25 and 26] using the layer wise theory.

Dongwei Shu and Christian N. Della et al.[27] presented model based on finite element method with two non overlapping delamination. The objective of their research was to present an analytical solution based on the Euler–Bernoulli beam theory to study the free vibration of composite beams with two non-overlapping delaminations.

2.2 Objective of Present Work:

The objective of this project is to develop an analytical model with the help of finite element method for the vibration analysis of delaminated composite beams.

My work is comprised of:

1. Formulation of analytical model with single delamination using finite element method.

   The model is based on the assumptions:

   i. *Free model:* The delaminated layers can deform 'freely' without touching each other.

   ii. *Rigid connectors:* The cross-sections near the delamination fronts remain perpendicular to the deformed midplane of the beam and thus take full account of the differential stretching between the two delaminated layers of the beam.

2. Calculation of natural frequency for the model.

3. Analysis of result.

4. Discussion and conclusion on the result.

CHAPTER 3

THEORY

3. THEORY:

3.1. Formulation:

Euler–Bernoulli beam theory is a simplification of the linear theory of elasticity which provides a means of calculating the load-carrying and deflection characteristics of beams. This is also known as engineer's beam theory, classical beam theory or just beam theory.

The Euler-Bernoulli equation describes the relationship between the beam's deflection and the applied load:

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = q \tag{1}$$

Where, q is a force per unit length.

E is the elastic modulus.

I is the second moment of area.

The beam equation for the time-dependant loading is

$$m\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = 0$$

(2)

Where, m is the mass per unit area.

In the model, we are considering delaminated beam as combination of five different beams. Therefore the equation for whole delaminated beam will be

$$m_i \frac{\partial^2 w_i}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(E\mathrm{I}_i \frac{\partial^2 w_i}{\partial x^2}\right) = 0$$

(3)

Where, $E\mathrm{I}_i$ is the reduced bending stiffness of the *i*th beam.

m_i is the mass per unit length.

The value of EI_i can be calculated for each beam using classical laminate theory, which comes as

$$EI_i = D_{11}^{(i)} - \frac{\left(B_{11}^{(i)}\right)^2}{A_{11}^{(i)}}$$

(4)

Where,

$$A_{11}^{(i)} = b \sum_{k=1}^{n} \left(Q_{11}^{-k}\right)_k (z_k - z_{k-1})$$

(5)

$$B_{11}^{(i)} = \frac{b}{2}\sum_{k=1}^{n}\left(Q_{11}^{-k}\right)_k (z_k^2 - z_{k-1}^2)$$

(6)

$$D_{11}^{(i)} = \frac{b}{3}\sum_{k=1}^{n}\left(Q_{11}^{-k}\right)_k (z_k^3 - z_{k-1}^3)$$

(7)

$$Q_{11}^{-k} = Q_{11}^{k} cos^4\emptyset + Q_{22}^{k} sin^4\emptyset + 2\left(Q_{11}^{k} + 2Q_{66}^{k}\right)cos^2\emptyset sin^2\emptyset$$

(8)

$$Q_{11} = \frac{E_{11}}{1 - \upsilon_{12}\upsilon_{21}}$$

(9)

$$Q_{22} = \frac{E_{22}}{1 - \upsilon_{12}\upsilon_{21}}$$

(10)

$$Q_{66} = G_{12},$$

(11)

Where, $A_{11}^{(i)}$ is the extensional stiffness,

$B_{11}^{(i)}$ is the coupling stiffness,

$D_{11}^{(i)}$ is the bending stiffness

b is the width,

E_{11} and E_{22} are the longitudinal and transverse Young's moduli, respectively,

G_{12} is the in-plane shear modulus,

υ_{12} and υ_{21} are the longitudinal and transverse Poisson's ratio, respectively,

Fig 1: Showing lamination orientation and location with respect to midplane.

φ is the angle of the *k*th lamina orientation and z_k and z_{k-1} are the locations of the *k*th lamina with respect to the midplane of the *i*th beam.

For vibration,

$$w_i(x_i,t)=W_i(x_i)\sin(\omega t), \tag{12}$$

Where, ω is the natural frequency.

W_i is the mode shape.

Substituting ω in eqn we get the equation of mode shape

$$W_i(x)=C_i \cos(\lambda_i x)+S_i \sin(\lambda_i x)+CH_i \cosh(\lambda_i x)+SH_i \sinh(\lambda_i x), \tag{13}$$

where

$$\lambda_i^4 = \frac{\omega^2 m_i}{EI_i} \tag{14}$$

where λ_i is the non-dimensional frequency.

For the support at x = x_1 boundary conditions applied is (simply supported)

$$W_i=0 \text{ and } W_i''=0$$

The continuity conditions for deflection, slope and shear at $x=x_2$ are

$$W_1=W_2, \tag{15}$$

$$W_1=W_3, \tag{16}$$

$$W_1'=W_2', \tag{17}$$

$$W_1'=W_3'. \tag{18}$$

From the continuity for shear and bending moments at $x=x_2$ are

$$EI_1W_1'''=EI_2W_2'''+EI_3W_3''', \tag{19}$$

$$EI_4 \times W_4'' + \frac{H_1^2}{4a_1}\left(\frac{A_{11}^{(2)}A_{11}^{(3)}}{A_{11}^{(2)}+A_{11}^{(3)}}\right)\left(W_4'(x_3)-W_1'(x_2)\right) = EI_2W_2''+EI_3W_3'' \tag{20}$$

The continuity conditions for deflection, slope and shear at $x=x_3$ are

$$W_4=W_2, \tag{21}$$

$$W_4 = W_3, \tag{22}$$

$$W_4' = W_2', \tag{23}$$

$$W_4' = W_3'. \tag{24}$$

And continuity condition for shear and bending moments at $x = x_3$ are

$$EI_4 W_4''' = EI_2 W_2''' + EI_3 W_3''', \tag{25}$$

$$EI_4 \times W_4'' + \frac{H_1^2}{4a_1}\left(\frac{A_{11}^{(2)} A_{11}^{(3)}}{A_{11}^{(2)} + A_{11}^{(3)}}\right)\left(W_1'(x_2) - W_4'(x_3)\right) = EI_2 W_2'' + EI_3 W_3'' \tag{26}$$

And for the support at x = x_4 boundary conditions applied is (simply supported)

$$W_i = 0 \text{ and } W_i'' = 0 \tag{27}$$

A non-trivial solution for the coefficients is obtained by equating the determinant of the coefficient matrix with 0. The frequencies and mode shapes can be obtained as eigenvalues and eigenvectors, respectively.

CHAPTER 4

RESULT AND DISCUSSION

RESULT AND DISCUSSION:

Considering a composite beam of carbon fiber reinforced plastic of dimension 150 × 12.6 × 1 mm^3. The following assumptions and conditions are considered:

i. Free model :
ii. Rigid supports
iii. Single delamination
iv. Symmetric mid plane delamination
v. Bending extension coupling is considered

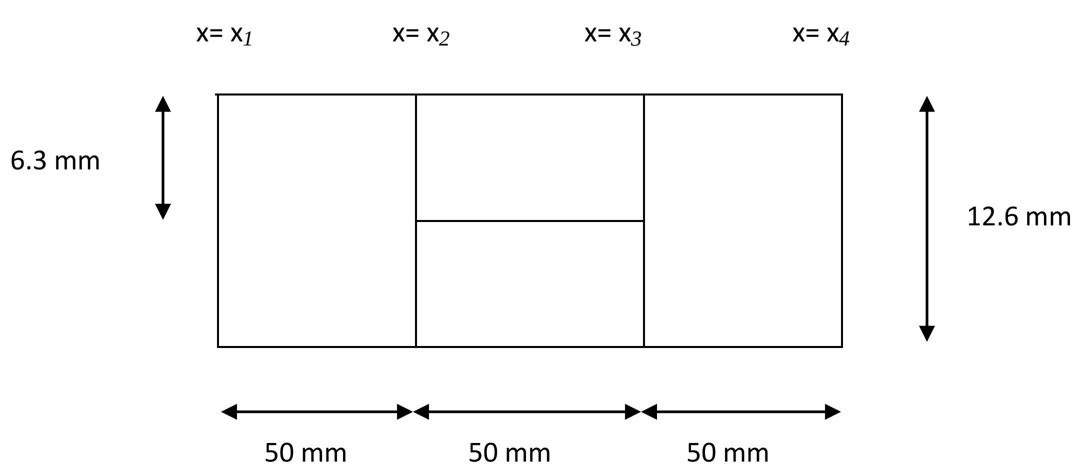

Fig 2: Beam dimension.

Properties of carbon fiber reinforced plastic:

Longitudinal modulus, E_{11} = 140 Gpa

Transverse modulus, E_{12} = 92 Gpa

Fiber volume fraction, γ_f = 0.58

Laminate density, ρ = 1.57 g/cm^3

Poisson's ratio, υ = 0.28

Mass per unit area, m = 0.019 kg/m^3

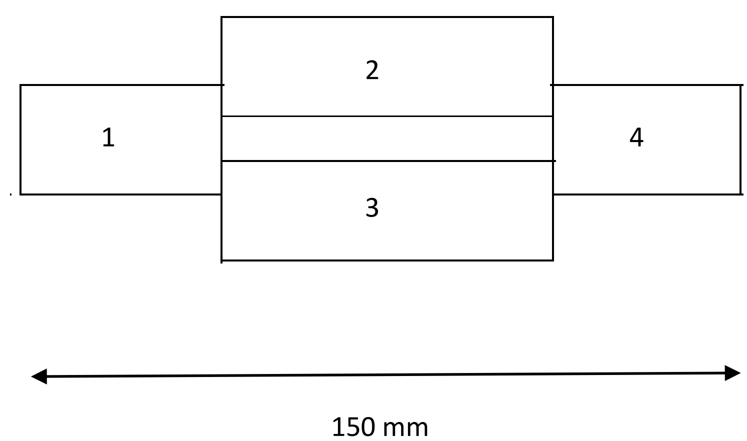

Fig 3: Equivalent interconnected beam diagram.

For each beam apply Euler Bournoulli beam theory and find the mode shape equation in terms of ω and x.

For beam 1:

$$Q_{66} = G_{12}$$

Q_{66} = 5 Gpa

$$Q_{11} = \frac{E_{11}}{1 - \upsilon_{12}\upsilon_{21}}$$

Q_{11} = 142.14 Gpa

$$Q_{22} = \frac{E_{22}}{1 - \upsilon_{12}\upsilon_{21}}$$

Q_{22} = 9.44 Gpa

$$Q_{11}^{-k} = Q_{11}^{k} cos^4\emptyset + Q_{22}^{k} sin^4\emptyset + 2\left(Q_{11}^{k} + 2Q_{66}^{k}\right)cos^2\emptyset sin^2\emptyset$$

Q_{11}^{-k} = 142.14 Gpa

$$D_{11}^{(i)} = \frac{b}{3} \sum_{k=1}^{n} \left(Q_{11}^{-k}\right)_k (z_k^3 - z_{k-1}^3)$$

$D_{11}^{(1)}$ = 12.325 Pa

$$B_{11}^{(i)} = \frac{b}{2} \sum_{k=1}^{n} \left(Q_{11}^{-k}\right)_k (z_k^2 - z_{k-1}^2)$$

$B_{11}^{(1)}$ = 2911.6 Pa

$$A_{11}^{(i)} = b \sum_{k=1}^{n} \left(Q_{11}^{-k}\right)_k (z_k - z_{k-1})$$

$A_{11}^{(1)}$ = 9.17 $\times 10^5$ Gpa

$$EI_i = D_{11}^{(i)} - \frac{\left(B_{11}^{(i)}\right)^2}{A_{11}^{(i)}}$$

EI_1 = 3.08 Pa

$$W_i(x) = C_i \cos(\lambda_i x) + S_i \sin(\lambda_i x) + CH_i \cosh(\lambda_i x) + SH_i \sinh(\lambda_i x),$$

$$W_1(x) = C_1 \cos(0.28 \times \sqrt{\omega} \times x) + S_1 \sin(0.28 \times \sqrt{\omega} \times x) + CH_1 \cosh(0.28 \times \sqrt{\omega} \times x) + SH_1 \sinh(0.28 \times \sqrt{\omega} \times x),$$

$$W'_1(x) = 0.28 \times \sqrt{\omega}\{-C_1 \sin(0.28 \times \sqrt{\omega} \times x) + S_1 \cos(0.28 \times \sqrt{\omega} \times x) + CH_1 \sinh(0.28 \times \sqrt{\omega} \times x) + SH_1 \cosh(0.28 \times \sqrt{\omega} \times x)\},$$

$$W''_1(x) = 0.08 \times \omega\{-C_1 \cos(0.28 \times \sqrt{\omega} \times x) - S_1 \sin(0.28 \times \sqrt{\omega} \times x) + CH_1 \cosh(0.28 \times \sqrt{\omega} \times x) + SH_1 \sinh(0.28 \times \sqrt{\omega} \times x)\},$$

$$W'''_1(x) = 0.02 \times \sqrt{\omega^3}\{-C_1 \sin(0.28 \times \sqrt{\omega} \times x) + S_1 \cos(0.28 \times \sqrt{\omega} \times x) + CH_1 \sinh(0.28 \times \sqrt{\omega} \times x) + SH_1 \cosh(0.28 \times \sqrt{\omega} \times x)\},$$

For beam 2:

$$Q_{66} = G_{12}$$

Q_{66} = 5 Gpa

$$Q_{11} = \frac{E_{11}}{1 - v_{12}v_{21}}$$

Q_{11} = 142.14 Gpa

$$Q_{22} = \frac{E_{22}}{1 - v_{12}v_{21}}$$

Q_{22} = 9.44 Gpa

$$Q_{11}^{-k} = Q_{11}^{k}cos^4\emptyset + Q_{22}^{k}sin^4\emptyset + 2\left(Q_{11}^{k} + 2Q_{66}^{k}\right)cos^2\emptyset sin^2\emptyset$$

Q_{11}^{-k} = 142.14 Gpa

$$D_{11}^{(i)} = \frac{b}{3} \sum_{k=1}^{n} \left(Q_{11}^{-k}\right)_k (z_k^3 - z_{k-1}^3)$$

$D_{11}^{(2)}$ = 6.16 Pa

$$B_{11}^{(i)} = \frac{b}{2} \sum_{k=1}^{n} \left(Q_{11}^{-k}\right)_k (z_k^2 - z_{k-1}^2)$$

$B_{11}^{(2)}$ = 1455.8 Pa

$$A_{11}^{(i)} = b \sum_{k=1}^{n} \left(Q_{11}^{-k}\right)_k (z_k - z_{k-1})$$

$A_{11}^{(2)}$ = 4.58 $\times 10^5$ Pa

$$EI_i = D_{11}^{(i)} - \frac{\left(B_{11}^{(i)}\right)^2}{A_{11}^{(i)}}$$

EI_2 = 1.54 Pa

$$W_i(x) = C_i \cos(\lambda_i x) + S_i \sin(\lambda_i x) + CH_i \cosh(\lambda_i x) + SH_i \sinh(\lambda_i x),$$

$$W_2(x) = C_2 \cos(0.28 \times \sqrt{\omega} \times x) + S_2 \sin(0.28 \times \sqrt{\omega} \times x) + CH_2 \cosh(0.28 \times \sqrt{\omega} \times x) + SH_2 \sinh(0.28 \times \sqrt{\omega} \times x),$$

$$W'_2(x) = 0.28 \times \sqrt{\omega}\{-C_2 \sin(0.28 \times \sqrt{\omega} \times x) + S_2 \cos(0.28 \times \sqrt{\omega} \times x) + CH_2 \sinh(0.28 \times \sqrt{\omega} \times x) + SH_2 \cosh(0.28 \times \sqrt{\omega} \times x)\},$$

$$W''_2(x) = 0.08 \times \omega\{-C_2 \cos(0.28 \times \sqrt{\omega} \times x) - S2 \sin(0.28 \times \sqrt{\omega} \times x) + CH_2 \cosh(0.28 \times \sqrt{\omega} \times x) + SH_2 \sinh(0.28 \times \sqrt{\omega} \times x)\},$$

$$W'''_2(x) = 0.02 \times \sqrt{\omega^3}\{-C_2 \sin(0.28 \times \sqrt{\omega} \times x) + S_2 \cos(1.07 \times \sqrt{\omega} \times x) + CH_2 \sinh(0.28 \times \sqrt{\omega} \times x) + SH_2 \cosh(0.28 \times \sqrt{\omega} \times x)\},$$

For beam 3:

$$Q_{66} = G_{12}$$

Q_{66} = 5 Gpa

$$Q_{11} = \frac{E_{11}}{1 - \upsilon_{12}\upsilon_{21}}$$

Q_{11} = 142.14 Gpa

$$Q_{22} = \frac{E_{22}}{1 - \upsilon_{12}\upsilon_{21}}$$

Q_{22} = 9.44 Gpa

$$Q_{11}^{-k} = Q_{11}^{k} cos^4\emptyset + Q_{22}^{k} sin^4\emptyset + 2\left(Q_{11}^{k} + 2Q_{66}^{k}\right) cos^2\emptyset sin^2\emptyset$$

Q_{11}^{-k} = 142.14 Gpa

$$D_{11}^{(i)} = \frac{b}{3} \sum_{k=1}^{n} \left(Q_{11}^{-k}\right)_k (z_k^3 - z_{k-1}^3)$$

$D_{11}^{(3)}$ = 6.16 Pa

$$B_{11}^{(i)} = \frac{b}{2} \sum_{k=1}^{n} \left(Q_{11}^{-k}\right)_k (z_k^2 - z_{k-1}^2)$$

$B_{11}^{(3)}$ = 1455.8 Pa

$$A_{11}^{(i)} = b \sum_{k=1}^{n} \left(Q_{11}^{-k}\right)_k (z_k - z_{k-1})$$

$A_{11}^{(3)}$ = 4.58 $\times 10^5$ Pa

$$EI_i = D_{11}^{(i)} - \frac{\left(B_{11}^{(i)}\right)^2}{A_{11}^{(i)}}$$

EI_3 = 1.54 Pa

$$W_i(x) = C_i \cos(\lambda_i x) + S_i \sin(\lambda_i x) + CH_i \cosh(\lambda_i x) + SH_i \sinh(\lambda_i x),$$

$$W_3(x) = C_3 \cos(0.28 \times \sqrt{\omega} \times x) + S_3 \sin(0.28 \times \sqrt{\omega} \times x) + CH_3 \cosh(0.28 \times \sqrt{\omega} \times x) + SH_3 \sinh(0.28 \times \sqrt{\omega} \times x),$$

$$W'_3(x) = 0.28 \times \sqrt{\omega}\{-C_3 \sin(0.28 \times \sqrt{\omega} \times x) + S_3 \cos(0.28 \times \sqrt{\omega} \times x) + CH_3 \sinh(0.28 \times \sqrt{\omega} \times x) + SH_3 \cosh(0.28 \times \sqrt{\omega} \times x)\},$$

$$W''_3(x) = 0.08 \times \omega\{-C_3 \cos(0.28 \times \sqrt{\omega} \times x) - S_3 \sin(0.28 \times \sqrt{\omega} \times x) + CH_3 \cosh(0.28 \times \sqrt{\omega} \times x) + SH_3 \sinh(0.28 \times \sqrt{\omega} \times x)\},$$

$$W'''_3(x) = 0.02 \times \sqrt{\omega^3}\{-C_3 \sin(0.28 \times \sqrt{\omega} \times x) + S_3 \cos(0.28 \times \sqrt{\omega} \times x) + CH_3 \sinh(0.28 \times \sqrt{\omega} \times x) + SH_3 \cosh(0.28 \times \sqrt{\omega} \times x)\},$$

For beam 4:

$$Q_{66} = G_{12}$$

Q_{66} = 5 Gpa

$$Q_{11} = \frac{E_{11}}{1 - \upsilon_{12}\upsilon_{21}}$$

Q_{11} = 142.14 Gpa

$$Q_{22} = \frac{E_{22}}{1 - \upsilon_{12}\upsilon_{21}}$$

Q_{22} = 9.44 Gpa

$$Q_{11}^{-k} = Q_{11}^{k} cos^4\emptyset + Q_{22}^{k} sin^4\emptyset + 2\left(Q_{11}^{k} + 2Q_{66}^{k}\right) cos^2\emptyset sin^2\emptyset$$

Q_{11}^{-k} = 142.14Gpa

$$D_{11}^{(i)} = \frac{b}{3}\sum_{k=1}^{n}\left(Q_{11}^{-k}\right)_k (z_k^3 - z_{k-1}^3)$$

$D_{11}^{(4)}$ = 12.325 Pa

$$B_{11}^{(i)} = \frac{b}{2}\sum_{k=1}^{n}\left(Q_{11}^{-k}\right)_k (z_k^2 - z_{k-1}^2)$$

$B_{11}^{(4)}$ = 2911.6 Pa

$$A_{11}^{(i)} = b\sum_{k=1}^{n}\left(Q_{11}^{-k}\right)_k (z_k - z_{k-1})$$

$A_{11}^{(4)}$ = 9.17 $\times 10^5$ Pa

$$EI_i = D_{11}^{(i)} - \frac{\left(B_{11}^{(i)}\right)^2}{A_{11}^{(i)}}$$

EI_4 = 3.08 Pa

$$W_i(x) = C_i \cos(\lambda_i x)+S_i \sin(\lambda_i x)+CH_i \cosh(\lambda_i x)+SH_i \sinh(\lambda_i x),$$

$$W_4(x)=C_4 \cos(0.28 \times \sqrt{\omega} \times x)+S_4 \sin(0.28 \times \sqrt{\omega} \times x)+CH_4 \cosh(0.28 \times \sqrt{\omega} \times x)+SH_4 \sinh(0.28 \times \sqrt{\omega} \times x),$$

$$W'_4(x) = 0.28 \times \sqrt{\omega}\{-C_4 \sin(0.28 \times \sqrt{\omega} \times x)+S_4 \cos(0.28 \times \sqrt{\omega} \times x)+CH_4 \sinh(0.28 \times \sqrt{\omega} \times x)+SH_4 \cosh(0.28 \times \sqrt{\omega} \times x)\},$$

$$W''_4(x) = 0.08\times\omega\{-C_4 \cos(0.28 \times \sqrt{\omega} \times x)-S_4 \sin(0.28 \times \sqrt{\omega} \times x)+CH_4 \cosh(0.28 \times \sqrt{\omega} \times x)+SH_4 \sinh(0.28 \times \sqrt{\omega} \times x)\},$$

$$W'''_4(x) = 0.02 \times \sqrt{\omega^3}\{-C_4 \sin(0.28 \times \sqrt{\omega} \times x)+S_4 \cos(0.28 \times \sqrt{\omega} \times x)+CH_4 \sinh(0.28 \times \sqrt{\omega} \times x)+SH_4 \cosh(0.28 \times \sqrt{\omega} \times x)\},$$

Appling Boundary Conditions:

Since the beam is simply supported, thus

At x = x_1 = 0,

1. $W_1 = 0$

$$W_1(0) = C_1 + CH_1 = 0, \tag{16}$$

$$C_1 + CH_1 = 0 \tag{17}$$

2. $W''_1 = 0$

$$W''_1(0) = 1.14\times\omega(-C_1 + CH_1) = 0, \tag{18}$$

$$-C_1 + CH_1 = 0 \tag{19}$$

At x = x_4 = 0.15,

3. $W_4 = 0$

$$W_4(0.15)=C_4\cos(0.04\times\sqrt{\omega})+S_4\sin(0.04\times\sqrt{\omega})+CH_4\cosh(0.04\times\sqrt{\omega})+SH_4\sinh(0.04\times\sqrt{\omega}),$$

$$C_4\cos(0.04\times\sqrt{\omega})+S_4\sin(0.04\times\sqrt{\omega})+CH_4\cosh(0.04\times\sqrt{\omega})+SH_4\sinh(0.04\times\sqrt{\omega}) = 0 \tag{20}$$

4. $W''_4 = 0$

$$W''_4(0.15) = 0.08\times\omega\{-C_4\cos(0.04\times\sqrt{\omega})-$$
$$S_4\sin(0.04\times\sqrt{\omega})+CH_4\cosh(0.04\times\sqrt{\omega})+SH_4\sinh(0.04\times\sqrt{\omega})\},$$

$$-C_4\cos(0.04\times\sqrt{\omega})-S_4\sin(0.04\times\sqrt{\omega})+CH_4\cosh(0.04\times\sqrt{\omega})+SH_4\sinh(0.04\times\sqrt{\omega}) = 0 \tag{21}$$

Continuity Conditions:

At $x = x_2 = 0.05$,

5. $W_1 = W_2$

$$C_1\cos(0.01\times\sqrt{\omega})+S_1\sin(0.01\times\sqrt{\omega})+CH_1\cosh(0.01\times\sqrt{\omega})+SH_1\sinh(0.01\times\sqrt{\omega}) =$$
$$C_2\cos(0.01\times\sqrt{\omega})+S_2\sin(0.01\times\sqrt{\omega})+CH_2\cosh(0.01\times\sqrt{\omega})+SH_2\sinh(0.01\times\sqrt{\omega})$$

$$(C_1 - C_2)\cos(0.01\times\sqrt{\omega})+(S_1 - S_2)\sin(0.01\times\sqrt{\omega})+(CH_1 - CH_2)\cosh(0.01\times\sqrt{\omega})+(SH_1 - SH_2)$$
$$\sinh(0.01\times\sqrt{\omega}) = 0 \tag{22}$$

6. $W_1 = W_3$

$$C_1 \cos(0.01 \times \sqrt{\omega}) + S_1 \sin(0.01 \times \sqrt{\omega}) + CH_1 \cosh(0.01 \times \sqrt{\omega}) + SH_1 \sinh(0.01 \times \sqrt{\omega}) = C_3 \cos(0.01 \times \sqrt{\omega}) + S_3 \sin(0.01 \times \sqrt{\omega}) + CH_3 \cosh(0.01 \times \sqrt{\omega}) + SH_3 \sinh(0.01 \times \sqrt{\omega})$$

$$(C_1 - C_3) \cos(0.01 \times \sqrt{\omega}) + (S_1 - S_3) \sin(0.01 \times \sqrt{\omega}) + (CH_1 - CH_3) \cosh(0.01 \times \sqrt{\omega}) + (SH_1 - SH_3) \sinh(0.01 \times \sqrt{\omega}) = 0 \tag{23}$$

7. $W'_1 = W'_2$

$$-C_1 \sin(0.01 \times \sqrt{\omega}) + S_1 \cos(0.01 \times \sqrt{\omega}) + CH_1 \sinh(0.01 \times \sqrt{\omega}) + SH_1 \cosh(0.01 \times \sqrt{\omega}) = -C_2 \sin(0.01 \times \sqrt{\omega}) + S_2 \cos(0.01 \times \sqrt{\omega}) + CH_2 \sinh(0.01 \times \sqrt{\omega}) + SH_2 \cosh(0.01 \times \sqrt{\omega})$$

$$(C_2 - C_1) \sin(0.01 \times \sqrt{\omega}) + (S_1 - S_2) \cos(0.01 \times \sqrt{\omega}) + (CH_1 - CH_2) \sinh(0.01 \times \sqrt{\omega}) + (SH_1 - SH_2) \cosh(0.01 \times \sqrt{\omega}) = 0 \tag{24}$$

8. $W'_1 = W'_3$

$$-C_1 \sin(0.01 \times \sqrt{\omega}) + S_1 \cos(0.01 \times \sqrt{\omega}) + CH_1 \sinh(0.01 \times \sqrt{\omega}) + SH_1 \cosh(0.01 \times \sqrt{\omega}) = -C_3 \sin(0.01 \times \sqrt{\omega}) + S_3 \cos(0.01 \times \sqrt{\omega}) + CH_3 \sinh(0.01 \times \sqrt{\omega}) + SH_3 \cosh(0.01 \times \sqrt{\omega})$$

$$(C_3 - C_1)\sin(0.01 \times \sqrt{\omega}) + (S_1 - S_3)\cos(0.01 \times \sqrt{\omega}) + (CH_1 - CH_3)\sinh(0.01 \times \sqrt{\omega}) + (SH_1 - SH_3)\cosh(0.01 \times \sqrt{\omega}) = 0 \qquad (25)$$

9. $EI_1 W_1''' = EI_2 W_2''' + EI_3 W_3'''$,

$$2 \times \{-C_1 \sin(0.01 \times \sqrt{\omega}) + S_1 \cos(0.01 \times \sqrt{\omega}) + CH_1 \sinh(0.01 \times \sqrt{\omega}) + SH_1 \cosh(0.01 \times \sqrt{\omega})\} = \{-C_2 \sin(0.01 \times \sqrt{\omega}) + S_2 \cos(0.01 \times \sqrt{\omega}) + CH_2 \sinh(0.01 \times \sqrt{\omega}) + SH_2 \cosh(0.01 \times \sqrt{\omega})\} + \{-C_3 \sin(0.01 \times \sqrt{\omega}) + S_3 \cos(0.01 \times \sqrt{\omega}) + CH_3 \sinh(0.01 \times \sqrt{\omega}) + SH_3 \cosh(0.01 \times \sqrt{\omega})$$

$$(2C_1 - C_2 - C_3)\sin(0.01 \times \sqrt{\omega}) + (S_2 + S_3 - 2S_1)\cos(0.01 \times \sqrt{\omega}) + (CH_2 + CH_3 - 2CH_1)\sinh(0.01 \times \sqrt{\omega}) + (SH_2 + SH_3 - 2SH_1)\cosh(0.01 \times \sqrt{\omega}) = 0 \qquad (26)$$

10. $EI_1 \times W_1'' + \frac{H_1^2}{4a_1}\left(\frac{A_{11}^{(2)}A_{11}^{(3)}}{A_{11}^{(2)}+A_{11}^{(3)}}\right)\left(W_1'(x_2) - W_4'(x_3)\right) = EI_2 W_2'' + EI_3 W_3''$

$$0.11\times \omega\{-C_1 \cos(0.01 \times \sqrt{\omega})-S_1 \sin(0.01 \times \sqrt{\omega})+CH_1 \cosh(0.01 \times \sqrt{\omega})+SH_1 \sinh(0.01 \times \sqrt{\omega})\}$$
$$+ 0.35\times \sqrt{\omega}\{-C_1 \sin(0.01 \times \sqrt{\omega})+S_1 \cos(0.01 \times \sqrt{\omega})+CH_1 \sinh(0.01 \times \sqrt{\omega})+SH_1 \cosh(0.01 \times$$
$$\sqrt{\omega}) +C_4 \sin(0.03 \times \sqrt{\omega})-S_4 \cos(0.03 \times \sqrt{\omega})+CH_4 \sinh(0.03 \times \sqrt{\omega})-SH_4 \cosh(0.03 \times \sqrt{\omega}) \} =$$
$$0.11\times \omega\{-C_2 \cos(0.01 \times \sqrt{\omega})- S_2 \sin(0.01 \times \sqrt{\omega})+CH_2 \cosh(0.01 \times \sqrt{\omega})+SH_2 \sinh(0.01 \times \sqrt{\omega})$$
$$- C_3 \cos(0.01 \times \sqrt{\omega})- S_3 \sin(0.01 \times \sqrt{\omega})+CH_3 \cosh(0.01 \times \sqrt{\omega})+SH_3 \sinh(0.01 \times \sqrt{\omega})\}$$

(27)

At x = x_3 = 0.1,

11. $W_4 = W_2$

$$C_4 \cos(0.03 \times \sqrt{\omega})+S_4 \sin(0.03 \times \sqrt{\omega})+CH_4 \cosh(0.03 \times \sqrt{\omega})+SH_4 \sinh(0.03 \times \sqrt{\omega}) =$$
$$C_2 \cos(0.03 \times \sqrt{\omega})+S_2 \sin(0.03 \times \sqrt{\omega})+CH_2 \cosh(0.03 \times \sqrt{\omega})+SH_2 \sinh(0.03 \times \sqrt{\omega})$$

$$(C_4 - C_2) \cos(0.03 \times \sqrt{\omega})+(S_4 - S_2)\sin(0.03 \times \sqrt{\omega})+(CH_4 - CH_2)\cosh(0.03 \times \sqrt{\omega})+(SH_4 - SH_2)$$
$$\sinh(0.03 \times \sqrt{\omega}) = 0$$

(28)

12. $W_4 = W_3$

$$C_4 \cos(0.03 \times \sqrt{\omega})+S_4 \sin(0.03 \times \sqrt{\omega})+CH_4 \cosh(0.03 \times \sqrt{\omega})+SH_4 \sinh(0.03 \times \sqrt{\omega}) = C_3 \cos(0.03 \times \sqrt{\omega})+S_3 \sin(0.03 \times \sqrt{\omega})+CH_3 \cosh(0.03 \times \sqrt{\omega})+SH_3 \sinh(0.03 \times \sqrt{\omega})$$

$$(C_4 - C_3) \cos(0.03 \times \sqrt{\omega})+(S_4 - S_3)\sin(0.03 \times \sqrt{\omega})+(CH_4 - CH_3)\cosh(0.03 \times \sqrt{\omega})+(SH_4 - SH_3) \sinh(0.03 \times \sqrt{\omega}) = 0 \tag{29}$$

13. $W'_4 = W'_2$

$$-C_4 \sin(0.03 \times \sqrt{\omega})+S_4 \cos(0.03 \times \sqrt{\omega})+CH_4 \sinh(0.03 \times \sqrt{\omega})+SH_4 \cosh(0.03 \times \sqrt{\omega}) = -C_2 \sin(0.03 \times \sqrt{\omega})+S_2 \cos(0.03 \times \sqrt{\omega})+CH_2 \sinh(0.03 \times \sqrt{\omega})+SH_2 \cosh(0.03 \times \sqrt{\omega})$$

$$(C_2 - C_4)\sin(0.03 \times \sqrt{\omega})+(S_4 - S_2)\cos(0.03 \times \sqrt{\omega})+(CH_4 - CH_2)\sinh(0.03 \times \sqrt{\omega})+(SH_4 - SH_2) \cosh(0.03 \times \sqrt{\omega}) = 0 \tag{30}$$

14. $W'_4 = W'_3$

$$-C_4 \sin(0.03 \times \sqrt{\omega})+S_4 \cos(0.03 \times \sqrt{\omega})+CH_4 \sinh(0.03 \times \sqrt{\omega})+SH_4 \cosh(0.03 \times \sqrt{\omega}) = -C_3 \sin(0.03 \times \sqrt{\omega})+S_3 \cos(0.03 \times \sqrt{\omega})+CH_3 \sinh(0.03 \times \sqrt{\omega})+SH_3 \cosh(0.03 \times \sqrt{\omega})$$

$$(C_3 - C_4)\sin(0.03\times\sqrt{\omega})+(S_4 - S_3)\cos(0.03\times\sqrt{\omega})+(CH_4 - CH_3)\sinh(0.03\times\sqrt{\omega})+(SH_4 - SH_3)\cosh(0.03\times\sqrt{\omega}) = 0 \tag{31}$$

15. $EI_4W_4''' = EI_2W_2''' + EI_3W_3'''$,

$$2\times\{-C_4\sin(0.03\times\sqrt{\omega})+S_4\cos(0.03\times\sqrt{\omega})+CH_4\sinh(0.03\times\sqrt{\omega})+SH_4\cosh(0.03\times\sqrt{\omega})\} =$$
$$\{-C_2\sin(0.03\times\sqrt{\omega})+S_2\cos(0.03\times\sqrt{\omega})+CH_2\sinh(0.03\times\sqrt{\omega})+SH_2\cosh(0.03\times\sqrt{\omega})\} +$$
$$\{-C_3\sin(0.03\times\sqrt{\omega})+S_3\cos(0.03\times\sqrt{\omega})+CH_3\sinh(0.03\times\sqrt{\omega})+SH_3\cosh(0.03\times\sqrt{\omega})$$

$$(2C_4 - C_2 - C_3)\sin(0.03\times\sqrt{\omega})+(S_2 + S_3 - 2S_4)\cos(0.03\times\sqrt{\omega})+(CH_2 + CH_3 - 2CH_4)\sinh(0.03\times\sqrt{\omega})+(SH_2 + SH_3 - 2SH_4)\cosh(0.03\times\sqrt{\omega}) = 0 \tag{32}$$

16. $EI_4 \times W_4'' + \frac{H_1^2}{4a_1}\left(\frac{A_{11}^{(2)}A_{11}^{(3)}}{A_{11}^{(2)}+A_{11}^{(3)}}\right)\left(W_4'(x_3) - W_1'(x_2)\right) = EI_2W_2'' + EI_3W_3''$

$$0.11\times\omega\{-C_4\cos(0.03\times\sqrt{\omega})-S_4\sin(0.03\times\sqrt{\omega})+CH_4\cosh(0.03\times\sqrt{\omega})+SH_4\sinh(0.03\times\sqrt{\omega})\}$$
$$+\,0.35\times\sqrt{\omega}\{-C_4\sin(0.03\times\sqrt{\omega})+S_4\cos(0.03\times\sqrt{\omega})+CH_4\sinh(0.03\times\sqrt{\omega})+SH_4\cosh(0.03\times\sqrt{\omega})+C_1\sin(0.01\times\sqrt{\omega})-S_1\cos(0.01\times\sqrt{\omega})+CH_1\sinh(0.01\times\sqrt{\omega})-SH_1\cosh(0.01\times\sqrt{\omega})\} =$$

$$0.6\times \omega\{-C_2 \cos(0.03 \times \sqrt{\omega})- S_2 \sin(0.03 \times \sqrt{\omega})+CH_2 \cosh(0.03 \times \sqrt{\omega})+SH_2 \sinh(0.03 \times \sqrt{\omega}) - C_3 \cos(0.03 \times \sqrt{\omega})- S_3 \sin(0.03 \times \sqrt{\omega})+CH_3 \cosh(0.03 \times \sqrt{\omega})+SH_3 \sinh(0.03 \times \sqrt{\omega})\} \tag{33}$$

For the non trivial solution, the determinant of coefficient of C_i, S_i, CH_i, *and* SH_i must be zero. On equating the determinant of matrix to 0 we get the value of ω.

Using the presented model, the value of primary frequency for the beam with the considered conditions and assumptions is found to be 12.89 Hz, 9534.66 Hz, 38840.37 Hz, 55933.2 Hz and 12.408.6 Hz.

This model can be used for finding the natural frequency of delaminated composite beams.

CHAPTER 5

CONCLUSION

CONCLUSION:

A model using FEM is presented having single symmetric delamination at the mid span of the beam is constructed. This shows that the FEM can be used for the determination of natural frequency of the delaminated beams. This method is less time consuming than the conventional method and also the results are in good agreement with experimental results in literature. With the little modification this model can be used for finding the natural frequency and primary frequency of any kind of delaminated beams. Using this method we can predict the life span of beams without actually doing any experiments on the beam. This is very useful where the material is not available at inspection for regular interval as in case of aeroplanes. Also the physical presence of material is not necessary.

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