Geometrically Nonlinear Vibration of Laminated Composite Plates Fitted With Piezoelectric Actuators and Subjected to Thermal Environments

A THESIS SUBMITTED IN PARTIAL FULFILLMENT FOR THE DEGREE OF

BACHELOR OF TECHNOLOGY

IN

CIVIL ENGINEERING

By

ABHISHEK GUPTA

Roll No. – 10601027

Department of Civil Engineering

National Institute of Technology, Rourkela

MAY, 2010

Geometrically Nonlinear Vibration of Laminated Composite Plates Fitted With Piezoelectric Actuators and Subjected to Thermal Environments

A THESIS SUBMITTED IN PARTIAL FULFILLMENT FOR THE DEGREE OF

BACHELOR OF TECHNOLOGY

IN

CIVIL ENGINEERING

By

ABHISHEK GUPTA

Roll No. – 10601027

Department of Civil Engineering

National Institute of Technology, Rourkela

MAY, 2010

National Institute of Technology Rourkela

CERTIFICATE

This is to certify that this report entitled, "**Geometrically Nonlinear Vibration of Laminated Composite Plates Fitted With Piezoelectric Actuators and Subjected to Thermal Environments**" submitted by Abhishek Gupta in partial fulfillments for the requirements for the award of Bachelor of Technology Degree in Civil Engineering at **National Institute of Technology**, **Rourkela** (Deemed University) is an authentic work carried out by him under my supervision and guidance.

To the best of my knowledge, the matter embodied in this report has not been submitted to any other University / Institute for the award of any Degree or Diploma

Date**:** **(Prof. S.K. Sahu)**

Dept. of Civil Engineering,

National Institute of Technology

Rourkela - 769008, Orissa

ACKNOWLEDGEMENT

I deem it a privilege to have been a student of Civil Engineering stream in National Institute of Technology, Rourkela

I express my deep sense of gratitude to my project guide Prof. S.K. Sahu for his invaluable guidance and inputs. I thank him for allowing me to work on this project and for his constant help and support throughout the making of this project. I consider myself fortunate to have worked under his supervision.

I would also like to thank Prof. S. Pradyumna, for being such an immense source of knowledge and encouragement.

Abhishek Gupta

Roll No- 10601027

Department of Civil Engineering

NIT Rourkela

CONFERENCES (Based on the present research work)

A.Gupta, D.Samal and S.K.Sahu, "Transient response of laminated composite plates in thermal environments", presented at the **International Conference on Challenges and Applications of Mathematics in Science and Technology (CAMIST-2010),** held at NIT Rourkela.

ABSTRACT

Laminated composite plates have found widespread applications in the construction of engineering structures due to the several attributes of the composites such as light weight, high specific strength, high specific stiffness as well as excellent fatigue and corrosion resistance properties. Plates are used in structural applications either as main structures or as structural components. All structures are exposed to varying thermal conditions during their service life. The degraded material properties as well as the residual stresses generated due to the elevated temperature conditions influence the vibration behavior considerably. As a result the effect of thermal environment on the static and dynamic behavior of laminated composite plates needs to be studied thoroughly.

When the transverse vibrations of a uniformly heated structure is studied using the linear theories, the thermal effect is converted into a body force and the results show negligibly small variation as compared to those of a structure without any temperature change. Therefore, nonlinear analysis methods need to be applied to analyze the behavior.

This project reports the nonlinear free vibration characteristics of laminated composite plates which are bonded with piezoelectric actuator layers. The plates are subjected to thermal environment in addition to the electric load. The finite element method (FEM) is employed for the analysis. An eight-noded isoparametric C^o continuity element with five degrees of freedom per node is used taking into consideration von Karman large deflection assumptions. The governing differential equations are obtained using the modified first-order shear deformation plate theory (MFSDT). The formulation includes the effects of transverse shear, in-plane and rotary inertia. The nonlinear matrix amplitude equation obtained by employing the Galerkin's weighted residual method is solved using the direct iteration technique. Detailed parametric studies are carried out to investigate the effect of different parameters on the free vibration characteristics of laminated plates.

The finite element codes are accordingly developed in MATLAB. The validity of the present finite element code is demonstrated by comparing the present results with the solutions available in the literature. Then the study is further extended to investigate the effect of different parameters such as temperature rise, control voltage, boundary conditions, fibre orientation and

stacking sequence on the free vibration behavior of laminated composite plates fitted with piezoelectric actuators and subjected to thermal environments.

Keywords: Laminated composites plates, Nonlinear free vibration, Finite element method, Piezoelectric actuators, Galerkin's weighted residual method

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 General introduction

Composites are engineered materials made from two or more constituent materials with significantly varied physical or chemical properties which remain distinctive on a macroscopic level within the finished structure. Laminated composite plates and shells consisting of several layers of different fibre reinforced laminates are bonded together to obtain desired structural properties. Varying the plate geometry, material properties, and stacking sequence the requirements of the particular structure to which they are to be applied can be fulfilled. Due to their high strength to weight ratio composite materials are ideally suited for use in weight sensitive applications of aerospace and ship building industries.

These composite materials are susceptible to temperature and moisture when operating in harsh environmental conditions. Elevated thermal exposure will result in large thermal deformations which may adversely affect the behavior of the composite materials.

Smart structures have been extensively studied in recent years. These consist of piezoelectric materials embedded and/or surface bonded to the laminated structures. By taking advantage of the direct and/or converse piezoelectric effect they behave as sensors and/or actuators. Composite structures can therefore combine the traditional performance advantages of composite laminates along with the inherent capability of piezoelectric materials to adapt to their current environments.

1.2 Thermal effects

The varying environmental conditions have an adverse effect on the stiffness and strength of the structural composites. Stiffness and strength are reduced with increase in temperature. Residual stresses might also be introduced due to dissimilar lamina expansion. It is therefore highly essential to study the effect of temperature on the static and dynamic response of composite structures.

1.3 Piezoelectric materials

Whenever a force is applied to a crystal, then the atoms are displaced slightly from their position in the lattice. In a piezoelectric crystal, this deformation of the crystal lattice results in the crystal acquiring a net charge. Thus the piezoelectric crystal gives a direct electrical output to an applied force. This property is known as the direct piezoelectric effect and can be utilized in piezoelectric sensing equipments.

The direct piezoelectric effect is reversible. Whenever a voltage is applied to the crystal it causes a mechanical displacement within the crystal. This converse piezoelectric effect is used in deflection control systems. Piezoelectric actuating systems employ this property of the material.

A major drawback of piezoelectric materials is that they show dielectric ageing and hence lack reproducibility of strains, i.e, a drift from zero state of strain is observed under cyclic electric field applications.

1.4 Importance of the present study

The present study reports the non linear free vibration analysis of composite plates and shells fitted with piezoelectric actuators. Composite laminates are subjected to a variety of forces. If the circular frequency of the forcing function is very near the natural circular frequency of the system, it results in very large amplitude of motion. This undesirable condition is known as resonance, and for this reason, the natural circular frequency of the system is often called as resonant frequency. Hence it becomes extremely essential to study the free vibration characteristics of all structural elements.

1.5 Outline of the present study

The present thesis comprises of eight chapters. In the first chapter a brief introduction of laminated composites and piezoelectric materials is presented. The importance of the present study is also discussed.

Chapter 2 contains a detailed review of the published important literature related to the present area of study.

Chapter 3 and Chapter 4 contain all the details of the theoretical and finite element formulations that have been used in the present study.

In Chapter 5 the present formulation and MATLAB code is validated by comparing the present results with those available in the literature.

The results obtained by conducting several parametric studies are tabulated in Chapter 6. The results are also analyzed in detail.

The major conclusions drawn from the present study are enumerated in Chapter 7.

Chapter 8 contains a brief idea of all the related areas to which the present investigation can be extended.

CHAPTER 2

LITERATURE REVIEW

Use of composite materials in the structural components of aerospace, automobile and other high performance applications resulted in reducing the weight to increase the performance. These light weighed and thin walled structural components may be subjected to varying thermal environments during service life. Temperature variations have a significant effect on the free vibration of fibre reinforced laminated plates. The study of the free vibration of composite laminates using linear theories has been considered earlier by several researchers. Some have also investigated the effects of additional thermal loading. Whitney and Ashton (1971) used the Ritz method to analyze the symmetric laminates and equilibrium equations of motion in the case of anti-symmetric angle-ply laminates, based upon the classical laminated plate theory. A few results were presented for only symmetric angle-ply laminates. Reddy (1979) presented the results for the free vibration of antisymmetric, angle-ply laminated plates including transverse shear deformation by the finite element method. Dhanaraj and Palaninathan (1989) used the semi-loof shell element to study the free vibration characteristics of composite laminates under initial stress, which may also arise due to temperature. Sai Ram and Sinha (1992) presented the effects of moisture and temperature on the linear free vibration of laminated composite plates. Maiti and Sinha (1995) studied the bending, free vibration and impact response of thick laminated composite plates. Lee and Lee (1994) conducted the finite element analysis of free vibration of delaminated composite plates. Parhi and Sinha (2001) investigated the hygrothermal effects on the dynamic response of multiple delaminated composite plates and shells using the first order shear deformation theory. Matsunga (2007) presented a two dimensional global higher order theory for the free vibration and stability problems of angle ply laminated composite and sandwich plates subjected to thermal loading. Alnefaie (2009) developed a three dimensional finite element model of delaminated fibre reinforced composite plates and analyzed their dynamics.

The large deflections caused due to the thermal loads make it essential to apply nonlinear theories for a more accurate study of the behavior of laminated composites. Bhimaraddi and

Chandrashekhara (1993) used the parabolic shear deformation theory to analyze the large amplitude vibrations, buckling and post buckling of heated angle-ply laminated plates. Chandrashekhara and Tenneti (1993) studied the nonlinear static and dynamic responses of laminated plates subjected to thermal or thermo-mechanical loads. Naidu and Sinha (2007) dealt with the nonlinear free vibration of laminated composite shells subjected to hygrothermal environments using the finite element method. The Green-Lagrange type nonlinear strains were incorporated into the first order shear deformation theory. Nanda and Bandyopadhyay (2007) analyzed the non linear free vibration of laminated composite shells with cut outs. Chia and Chia (1992) studied nonlinear vibration of moderately thick antisymmetrically laminated angle ply shallow spherical shell with rectangular plan form employing Galerkin's procedure. Sathyamoorthy (1995) investigated the effects of large amplitude on the free flexural vibrations of moderately thick orthotropic spherical shells. Ganapathi and Varadan (1995) carried out investigations to study the nonlinear free vibration of laminated anisotropic circular cylindrical shells using finite element method. Shin (1997) investigated large amplitude vibration behavior of symmetrically laminated moderately thick doubly curved shallow open shells with simply supported sides by applying Galerkin's approximation. Singha and Daripa (2007) presented the results for the large amplitude free flexural vibration behavior of symmetrically laminated composite skew plates using the finite element method. The nonlinear matrix amplitude equations obtained by the Galerkin's weighted residual method were solved using the direct iteration technique. Lee and Reddy (2005) used the third order shear deformation theory to study the nonlinear response of laminated compoite plates under thermomechanical loading.

In recent years, piezoelectric materials are extensively used as sensors and actuators to control vibration of elastic structures. The top piezoelectric layer in a composite plate/shell acts as an actuator and the bottom layer does as either a sensor or an actuator. The applications of piezoelectric materials include shape control, active dampening and vibration suppression. The piezoelectric actuators induce in-plane stresses that may significantly affect the dynamic behavior of the composite plates/shells. Hui-Shen Shen (2004) conducted an analytical study of the non-linear bending of unsymmetric cross-ply laminated plates with piezoelectric actuators in thermal environments. Huang and Shen (2005) did a similar study to obtain the results for non-linear free and forced vibrations of simply supported shear deformable laminated plates with piezoelectric actuators. Heidary and Eslami (2006) dealt with the piezo-control of forced vibrations of laminated thermoelastic plates. Dash and Singh (2009) addressed the nonlinear free vibration characteristics of laminated composite plates with embedded and/or surface bonded piezoelectric layers. The nonlinear governing equations were derived in the Green-Lagrange sense in the framework of a higher order shear deformation theory. To the best of the authors' knowledge the nonlinear free vibration analysis of laminated composite plates with piezoelectric actuators and subjected to thermal environments has not received much attention of the researchers.

In the present report the, the nonlinear free vibration behavior of laminated composite shells fitted with piezoelectric actuator and subjected to thermal environments is studied by employing the finite element method. A modified first order shear deformation theory (MFSDT), earlier developed by Tanov and Tabei (2000) is used to carry out the analysis. This MFSDT formulation considers parabolic distribution of both transverse shear strains and stresses and therefore the use of shear correction factor is eliminated. The nonlinear matrix amplitude equations obtained by employing the Galerkin's method are solved by direct iteration to obtain the free vibration results.

CHAPTER 3

THEORETICAL FORMULATION

This chapter presents the theoretical formulation for the free vibration problem of laminated composite plates. The strain-displacement as well as the stress-strain relations are furnished in this section.

3.1 Strain-Displacement Relations

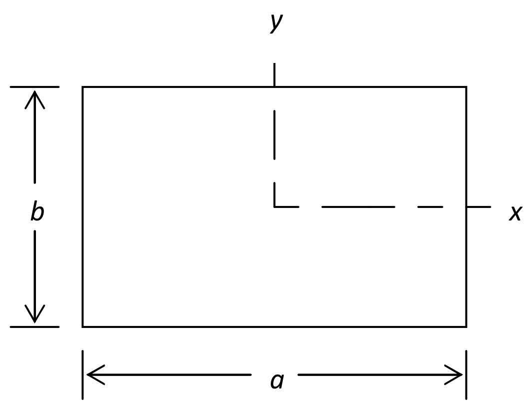

Figure 3.1 Plate geometry

The modified first order shear deformation theory, as earlier developed by Tanov and Tabei (2000) for plates is detailed here. Let us consider a laminated composite plate with the co-ordinate system (*x, y, z*) shown in Fig. (3.1), and chosen such that the plane *x-y* at $z = 0$ coincides with the mid-plane. In order to approximate the three-dimensional elasticity problem to a two-dimensional one, the displacement components $u(x,y,z)$, $v(x,y,z)$ and $w(x,y,z)$ at any point in the plate space are expanded in Taylor's series in terms of the thickness co-ordinates. The elasticity solution indicates that the transverse shear stresses vary parabolically through the thickness. This requires the use of a displacement field in which the in-plane displacements are expanded as cubic functions of the thickness co-ordinate. The displacement fields, which satisfy the above criteria, are assumed in the form as,

$$
\begin{aligned}
u &= u_0 + z\theta_y + z^2\varphi_y + z^3\psi_y \\
v &= v_0 - z\theta_x - z^2\varphi_x - z^3\psi_x \qquad (3.1) \\
w &= w_0
\end{aligned}
$$

where *u, v* and *w* are the displacements of a general point (*x, y, z*) in an element of the laminate along *x, y* and *z* directions, respectively. The parameters u_0, v_0, w_0, θ_x and θ_y are the displacements and rotations of the middle plane, while φ_x, φ_y, ψ_x and ψ_y are the higher-order displacement parameters defined at the mid-plane.

Using the strain-displacement relations

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3.2}$$

the components of the strain vector corresponding to the displacement field (3.1) are

$$\varepsilon_x = \frac{\partial u_0}{\partial x} + \frac{\partial \theta_y}{\partial x} z + \frac{\partial \varphi_y}{\partial x} z^2 + \frac{\partial \psi_y}{\partial x} z^3$$

$$\varepsilon_y = \frac{\partial v_0}{\partial y} - \frac{\partial \theta_x}{\partial y} z - \frac{\partial \varphi_x}{\partial y} z^2 - \frac{\partial \psi_x}{\partial y} z^3$$

$$2\varepsilon_{xy} = \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + \left(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x}\right) z + \left(\frac{\partial \varphi_y}{\partial y} - \frac{\partial \varphi_x}{\partial x}\right) z^2 + \left(\frac{\partial \psi_y}{\partial y} - \frac{\partial \psi_x}{\partial x}\right) z^3 \tag{3.3}$$

$$2\varepsilon_{yz} = \left(\frac{\partial w_0}{\partial y} - \theta_x\right) - 2\varphi_x z - 3\psi_x z^2$$

$$2\varepsilon_{xz} = \left(\frac{\partial w_0}{\partial x} + \theta_y\right) + 2\varphi_y z + 3\psi_y z^2$$

Vanishing of the transverse shear stresses at the top and bottom plate surfaces, $\sigma_{yz}(\pm h/2) = \sigma_{xz}(\pm h/2) = 0$, makes the corresponding strains zero, which yields,

$$\varphi_x = \varphi_y = 0,\ \psi_x = \frac{4}{3h^2}\left(\frac{\partial w_0}{\partial x} - \theta_x\right) \text{ and } \psi_y = \frac{4}{3h^2}\left(\frac{\partial w_0}{\partial y} + \theta_y\right)$$

in which *h* is the plate thickness. Using these expressions, Eqs. (3.1) and (3.3) are simplified as

$$u = u_0 + z\theta_y + z^3\psi_y$$
$$v = v_0 - z\theta_x - z^3\psi_x \qquad (3.4)$$
$$w = w_0$$

and

$$\varepsilon_x = \frac{\partial u_0}{\partial x} + \frac{\partial \theta_y}{\partial x} z + \frac{\partial \psi_y}{\partial x} z^3$$

$$\varepsilon_y = \frac{\partial v_0}{\partial y} - \frac{\partial \theta_x}{\partial y} z - \frac{\partial \psi_x}{\partial y} z^3$$

$$2\varepsilon_{xy} = \gamma_{xy} = \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + \left(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x}\right) z + \left(\frac{\partial \psi_y}{\partial y} - \frac{\partial \psi_x}{\partial x}\right) z^3 \qquad (3.5)$$

$$2\varepsilon_{yz} = \gamma_{yz} = \left(\frac{\partial w_0}{\partial y} - \theta_x\right)\left(1 - \frac{4}{h^2} z^2\right)$$

$$2\varepsilon_{xz} = \gamma_{xz} = \left(\frac{\partial w_0}{\partial x} + \theta_y\right)\left(1 - \frac{4}{h^2} z^2\right)$$

By assuming that the first two terms in the ε_x, ε_y and ε_{xy} expressions in Eq. (3.5) represent the distribution of the in-plane strains through the thickness with enough accuracy. This means that we can neglect the contribution of the derivatives of ψ_x and ψ_y with respect to x and y and simplify the strain expressions as,

$$\varepsilon_x = \frac{\partial u_0}{\partial x} + \frac{\partial \theta_y}{\partial x} z$$

$$\varepsilon_y = \frac{\partial v_0}{\partial y} - \frac{\partial \theta_x}{\partial y} z$$

$$\gamma_{xy} = \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + \left(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x}\right) z \qquad (3.6)$$

$$\gamma_{yz} = \left(\frac{\partial w_0}{\partial y} - \theta_x\right)\left(1 - \frac{4}{h^2} z^2\right)$$

$$\gamma_{xz} = \left(\frac{\partial w_0}{\partial x} + \theta_y\right)\left(1 - \frac{4}{h^2} z^2\right)$$

These expressions are identical to the strain expressions from the first-order shear deformation displacement field except for the transverse shear strain expressions.

Eq. (3.6) represents the linear components of the strains. The nonlinear components of the in-plane strains are

$$\{\varepsilon_{NL}\} = \begin{Bmatrix} \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) \end{Bmatrix} \tag{3.7}$$

This corresponds to the well known von Karman relationships for large displacements.

3.2 Stress-Strain Relations

The stress-strain relationship of any k^{th} orthotropic layer/lamina with reference to the fiber-matrix co-ordinate axis 1-2 (Figs. 3.2 and 3.3) may be expressed as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix}^k = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & Q_{55} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{Bmatrix}^k \tag{3.8}$$

Where, $Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}$, $Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}$, $Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}$, $Q_{66} = G_{12}$, $Q_{44} = G_{13}$, $Q_{55} = G_{23}$ and

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}$$

Appropriate transformation is required in order to obtain the elastic constant matrix corresponding to any arbitrary principal axis (*x*-*y*) with which the material principal axis makes an angle θ. Thus, the off-axis elastic constant matrix is obtained from the on-axis elastic constant matrix as,

$$\left[\overline{Q}_{ij}\right]=[T]^{T}\left[Q_{ij}\right][T] \tag{3.9}$$

Where [*T*] is the transformation matrix given by

$$[T]=\begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin\theta\cos\theta & 0 & 0 \\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta & 0 & 0 \\ -2\sin\theta\cos\theta & 2\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta & 0 & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta \\ 0 & 0 & 0 & -\cos\theta & \sin\theta \end{bmatrix} \tag{3.10}$$

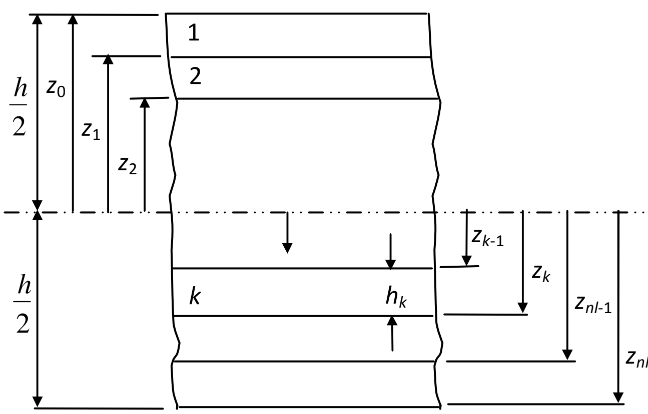

Figure 3.2 Details of lamination

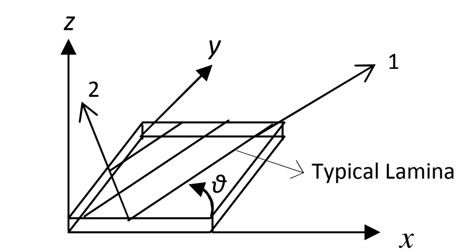

Figure 3.3 Lamina reference axis and fiber orientation

After substituting Eq. (3.10) into Eq. (3.9), constitutive relation of the lamina with reference to any arbitrary axis is,

$$\left[\overline{Q}_{ij}\right]=\begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} & 0 & 0 \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} & 0 & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} \\ 0 & 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix} \tag{3.11}$$

In which

$$\overline{Q}_{11} = Q_{11}l^4 + 2(Q_{12} + 2Q_{66})l^2m^2 + Q_{22}m^4$$

$$\overline{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})l^2m^2 + Q_{12}(l^2 + m^2)$$

$$\overline{Q}_{22} = Q_{11}m^4 + 2(Q_{12} + 2Q_{66})l^2m^2 + Q_{22}l^4$$

$$\overline{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})lm^3 + (Q_{12} - Q_{22} + 2Q_{66})l^3m \quad (3.12)$$

$$\overline{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})l^3m + (Q_{12} - Q_{22} + 2Q_{66})lm^3$$

$$\overline{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})l^2m^2 + Q_{66}(l^2 + m^2)$$

$$\overline{Q}_{44} = G_{13}l^2 + G_{23}m^2$$

$$\overline{Q}_{45} = (G_{13} - G_{23})lm$$

$$\overline{Q}_{55} = G_{13}m^2 + G_{23}l^2$$

Where $l = \cos\theta$ and $m = \sin\theta$. The stress-strain relationship of any k^{th} lamina with reference to the global axis (*x*-*y*) is given as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^k = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} & 0 & 0 \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} & 0 & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} \\ 0 & 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^k \quad (3.13)$$

Or, $\{\sigma\}_k = \left[\overline{Q}\right]_k \{\varepsilon\}_k$ (3.14)

In which,

$\{\sigma\} = [\sigma_x \quad \sigma_y \quad \tau_{xy} \quad \tau_{xz} \quad \tau_{yz}]^T$ and $\{\varepsilon\} = [\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy} \quad \gamma_{xz} \quad \gamma_{yz}]^T$ are the stress and linear strain vectors with respect to the laminate axis, respectively.

Integrating Eq. (3.14) through the laminate thickness, the stress-strain relation can be written as

$$\int \sigma \, dz = \int \overline{Q} \varepsilon \, dz \tag{3.15}$$

Or $\bar{\boldsymbol{\sigma}} = \mathbf{C} \bar{\boldsymbol{\varepsilon}}$ (3.16)

Where $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\varepsilon}}$ are the stress resultant vector and middle surface strain vector, respectively and **C** is the rigidity matrix consisting membrane (C_p), bending (C_b), coupling (C_c) and shear (C_s) rigidity matrices.

3.3 Constitutive Relations

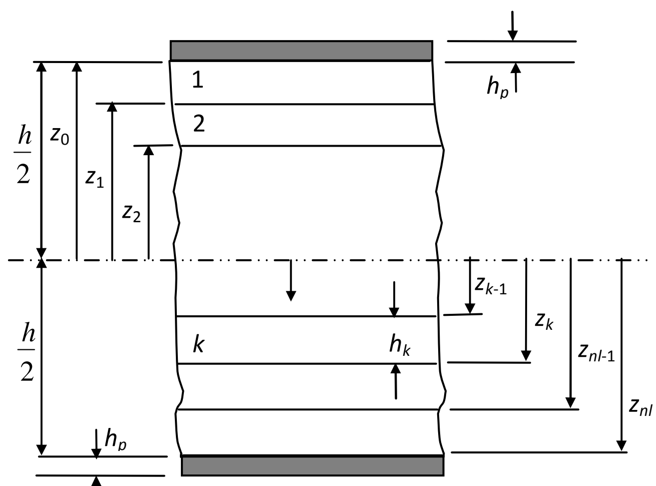

Figure 3.4 Details of lamination of a plate bonded with piezoelectric actuators on the top and bottom surface

The transformed stress-strain relations of an orthotropic lamina in a plane state of stress are

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^k = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^k + \begin{bmatrix} 0 & 0 & \bar{d}_{31} \\ 0 & 0 & \bar{d}_{32} \\ 0 & 0 & \bar{d}_{36} \end{bmatrix}^k \begin{Bmatrix} 0 \\ 0 \\ -\phi,_z \end{Bmatrix} \tag{3.17a}$$

$$\begin{Bmatrix} \tau_{yz} \\ \tau_{yz} \end{Bmatrix}^k = \begin{bmatrix} \overline{Q}_{44} & \overline{Q}_{45} \\ \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix}^k \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^k + \begin{bmatrix} \bar{d}_{14} & \bar{d}_{24} & 0 \\ \bar{d}_{15} & \bar{d}_{25} & 0 \end{bmatrix}^k \begin{Bmatrix} 0 \\ 0 \\ -\phi,_z \end{Bmatrix} \tag{3.17b}$$

Where $\bar{d}_{ij}$ are the transformed piezoelectric moduli.

For the plate type piezoelectric material, only thickness direction electric field $E_Z = -\phi,_z$ is dominant, where ϕ is the potential field. If the voltage applied to the actuator in the thickness only, then $E_Z = V_k / h_p$, where V_k is the applied voltage across the *k*th ply and h_p is the thickness of the ply (Fig. 3.4).

CHAPTER 4
FINITE ELEMENT FORMULATION

The element used in this work is an eight noded isoparametric element.

4.1 Shape function for second order rectangular element

The variation of displacement u can be expressed by the following polynomial in natural co-ordinates.

$u = \alpha_1 + \alpha_2 r + \alpha_3 s + \alpha_4 r^2 + \alpha_5 rs + \alpha_6 s^2 + \alpha_7 r^2 s + \alpha_7 r s^2$ (4.1)

In the above expressions the cubic terms r^3 and s^3 are omitted and 'geometric invariance' is maintained.

$\{\phi\}^T = [1 \quad r \quad s \quad r^2 \quad rs \quad s^2 \quad r^2 s \quad rs^2]$ (4.2)

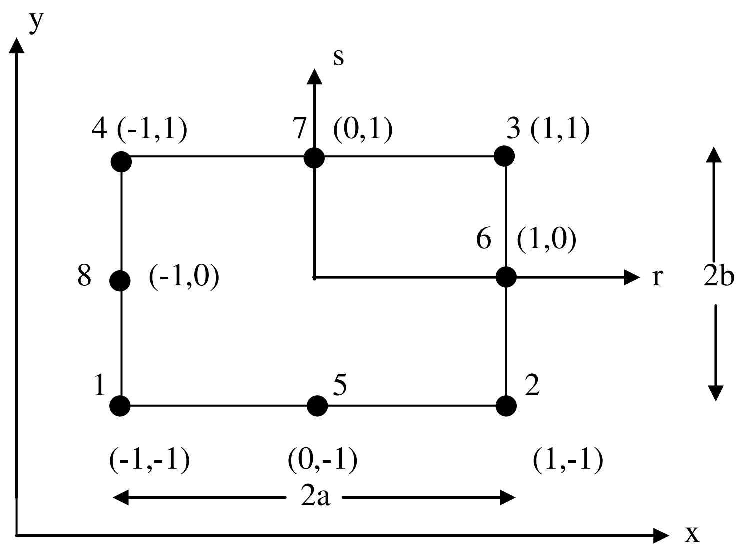

Figure 4.1 Eight noded rectangular element

$[N]^T = [¼ (1- r) (1- s)(-r - s - 1), ¼ (1+ r) (1- s)(r - s - 1), ¼ (1+ r) (1+ s)(r + s - 1),$
$¼ (1+ r) (1+ s)(r + s - 1), ¼ (1- r) (1+ s)(-r + s - 1), ½ (1+ r) (1- r)(1- s),$
$½ (1+ r) (1+ s)(1 - s), ½ (1+ r) (1- s)(1 + s), ½ (1- r) (1+ s)(1 - s)]$ (4.3)

or can be expressed in concise form as

$\{N\}^T = [N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5 \quad N_6 \quad N_7 \quad N_8]$ (4.4)

The displace ment components are given by

$$u = \sum_{i=1}^{8} N_i u_i$$
$$v = \sum_{i=1}^{8} N_i v_i$$
$$w = \sum_{i=1}^{8} N_i w_i \qquad (4.5)$$
$$\theta_x = \sum_{i=1}^{8} N_i \theta_{xi},$$
$$\theta_y = \sum_{i=1}^{8} N_i \theta_{yi},$$

Where u_i and v_i are displacements of the nodes 1, 2, 3,…..8.

The linear strain vector $\{\varepsilon\}_L$ can be evaluated as

$$\{\varepsilon\}_L = [B]_p \{d\} \qquad (4.6)$$

where $[B]_p$ is the linear strain displacement relation given by

$$[B]_p = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix} \qquad (4.7)$$

i =1,2…8. N_1 , N_2 , N_3 …. N_8 are given by Eq. (4.3)

The nonlinear strain vector $\{\varepsilon\}_{NL}$ can be written as

$$\{\varepsilon_{NL}\} = \frac{1}{2}[B_{NL}]\{d\} \qquad (4.8)$$

$[B_{NL}]$ is the nonlinear strain displacement matrix.

$[B_{NL}] = [A][Q]$ (4.9)

Matrices $[A]$ and [Q] are

$$[A]=\begin{bmatrix}\frac{\partial w}{\partial x} & 0\\ 0 & \frac{\partial w}{\partial y}\\ \frac{\partial w}{\partial y} & \frac{\partial w}{\partial x}\end{bmatrix} \;;\; [Q]=\begin{bmatrix}0 & 0 & \frac{\partial}{\partial x} & 0 & 0\\ 0 & 0 & \frac{\partial}{\partial y} & 0 & 0\end{bmatrix} \quad (4.10)$$

Total strain is written in the form of linear and nonlinear parts as

$$\{\varepsilon\}=\{\varepsilon_L\}+\{\varepsilon_{NL}\} \quad (4.11)$$

4.2 Energy Expressions

The strain energy over the entire plate volume due to initial stresses and strains can be written as

$$U_0=\frac{1}{2}\int\int\int_V\{\varepsilon\}^T\{\sigma\}dv=\frac{1}{2}\int\int\int_V\{\varepsilon\}^T[C]\{\varepsilon\}dv \quad (4.12)$$

$$U_0=\frac{1}{2}\int_{A_e}\{d\}^T[B]^T[C][B]\{d\}dA$$

$$U_0=\frac{1}{2}\{d\}^T(\int_{A_e}[B]^T[C][B]dA)\{d\}$$

$$or \quad (4.13)$$

$$U_0=\frac{1}{2}\{d\}^T(\int_{-1}^{1}\int_{-1}^{1}[B]^T[C][B]|J|.dr.ds)\{d\}=\frac{1}{2}\{d\}^T[k]\{d\}$$

Assuming constant thickness h, the element plane stiffness matrix is given by

$$[k]_p=abh\int_{-1}^{1}\int_{-1}^{1}[B]^T[C][B]\,\mathrm{dr\;ds} \quad (4.14)$$

Bending Strain Energy

The expression for bending strain energy for the plate is given by the relation:

$$U_1 = \int_A \{ \frac{D}{2}\left[\left(\frac{\partial \theta_y}{\partial x} \right)^2 + \left(\frac{\partial \theta_x}{\partial y} \right)^2 - 2\nu \left(\frac{\partial \theta_y}{\partial x} \right)\left(\frac{\partial \theta_x}{\partial y} \right) + (1-\nu)\left(\frac{\partial \theta_x}{\partial x} - \frac{\partial \theta_y}{\partial y} \right)^2 \right]$$
$$+ \frac{\kappa E h}{2(1+\nu)}\left[\left(\frac{\partial w}{\partial x} - \theta_y \right)^2 + \left(\frac{\partial w}{\partial y} - \theta_x \right)^2 \right] \} dy.dx \qquad (4.15)$$

Kinetic Energy

When the plate undergoes a time dependent motion, it possesses some kinetic energy. The displacement, w, θ_x and θ_y in this case are the functions of time. Considering the effects of rotary inertia and neglecting in-plane inertia, the kinetic energy of the plate is given by

$$T = \int_A \left[\frac{h}{2}\left(\frac{\partial w}{\partial t} \right)^2 + \frac{h^3}{24}\left\{ \left(\frac{\partial \theta_x}{\partial t} \right)^2 + \left(\frac{\partial \theta_y}{\partial t} \right)^2 \right\} \right] .dy.dx \qquad (4.16)$$

4.3 Derivation of Element Matrices

Element matrices are derived as:

Element plane elastic stiffness matrix

$$[k]_p = \int\int [B]_p^T [C]_p [B].dA \qquad (4.17)$$

Where $[B]_p$ is given by Eq. (4.7).

Element linear elastic stiffness matrix

$$[k]_B = \int [B]_B^T [C]_B [B]_B .dx.dy \qquad (4.18)$$

Where linear strain displacement matrix $[B]_B$ can be written as

$$[B]_B = \begin{bmatrix} \frac{-\partial N_i}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 & 0 & 0 \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial N_i}{\partial x} \\ 0 & 0 & 0 & \frac{-\partial N_i}{\partial y} & 0 \\ 0 & 0 & 0 & \frac{-\partial N_i}{\partial x} & \frac{-\partial N_i}{\partial y} \\ 0 & 0 & \frac{\partial N_i}{\partial x} & 0 & N_i \\ 0 & 0 & \frac{\partial N_i}{\partial y} & -N_i & 0 \end{bmatrix} i = 1, 2..8$$

Generalized element mass matrix or consistent matrix

$$[m]_e = \int\int [N]^T [P][N] dx.dy \qquad (4.19)$$

Where the shape function matrix

$$[N] = \begin{bmatrix} N_i & 0 & 0 & 0 & 0 \\ 0 & N_i & 0 & 0 & 0 \\ 0 & 0 & N_i & 0 & 0 \\ 0 & 0 & 0 & N_i & 0 \\ 0 & 0 & 0 & 0 & N_i \end{bmatrix} i = 1, 2..8 \qquad (4.20)$$

$$[P]=\begin{bmatrix} P_1 & 0 & 0 & P_2 & 0 \\ 0 & P_1 & 0 & 0 & P_2 \\ 0 & 0 & P_1 & 0 & 0 \\ P_2 & 0 & 0 & P_3 & 0 \\ 0 & P_2 & 0 & 0 & P_3 \end{bmatrix} \tag{4.21}$$

and

$$(P_1, P_2, P_3)=\sum_{k=1}^{n}\int_{z_k-1}^{z_k}(\rho)_k(1, z, z^2).dz \tag{4.22}$$

For the isotropic element mass matrix

$$(P_1, P_2, P_3)=(\rho h, 0, \frac{\rho h^3}{12}) \tag{4.23}$$

The element mass matrix can be expressed in local natural coordinates of the element as

$$[m]_e=\int_{-1}^{1}\int_{-1}^{1}[N]^T[P][N]\left|J\right|.dr.ds \tag{4.24}$$

Geometric stiffness matrix

The Strain energy due to initial stresses is

$$U_2=\int_v\left[\sigma\ \right]^T\{\varepsilon_{nl}\}dv \tag{4.25}$$

Using the non-linear strains, the strain energy can be written as

$$U_2 = \int_A \frac{h}{2}[\sigma_x \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 \right\}$$

$$+\sigma_y \left\{ \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right\}$$

$$+2\tau_{xy} \left\{ \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \right\}]dx.dy \tag{4.26}$$

$$+\int_A \frac{h^3}{24}\left[\sigma_x \left\{ \left(\frac{\partial \theta_x}{\partial x}\right)^2 + \left(\frac{\partial \theta_y}{\partial x}\right)^2 \right\} + \sigma_y \left\{ \left(\frac{\partial \theta_y}{\partial y}\right)^2 + \left(\frac{\partial \theta_y}{\partial y}\right)^2 \right\}\right.$$

$$\left. +2\tau_{xy} \left\{ \left(\frac{\partial \theta_y}{\partial x}\frac{\partial \theta_y}{\partial y}\right) + \left(\frac{\partial \theta_x}{\partial x}\frac{\partial \theta_x}{\partial y}\right) \right\}\right] dx.dy$$

This can also be expressed as

$$U_2 = \frac{1}{2}\int_v \{\delta_e\}[G]^T[S][G]\{\delta_e\}dv = \frac{1}{2}\{\delta_e\}^T[K]_G[G]\{\delta_e\} \tag{4.27}$$

Where element geometric stiffness matrix

$$[k]_G = \int_{-1}^{1}\int_{-1}^{1}[G]^T[S][G]\left|J\right|dr.ds \tag{4.28}$$

Where $\{\delta_e\}$=[u v w θ_x θ_y]^T (4.29)

$$[S] = \begin{bmatrix} s & 0 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 0 \\ 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 0 & s \end{bmatrix}$$

$$[s] = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \tag{4.30}$$

and

$$[G] = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 & 0 & 0 \\ \frac{\partial N_i}{\partial y} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial x} & 0 & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_i}{\partial x} & 0 \\ 0 & 0 & 0 & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial N_i}{\partial x} \\ 0 & 0 & 0 & 0 & \frac{\partial N_i}{\partial y} \end{bmatrix} i = 1, 2, 3..8 \tag{4.31}$$

4.4 Derivation of governing equations

For nonlinear free vibration problem governing equilibrium is

$$\mathbf{M}\ddot{\delta} + \left[\mathbf{K} + \mathbf{K}_{\mathbf{ET}} + \frac{1}{2}\mathbf{N}_1 + \frac{1}{3}\mathbf{N}_2 \right] \delta = 0 \tag{4.32}$$

Where $\mathbf{K}$ and $\mathbf{M}$ are elastic stiffness and mass matrices, respectively. $\mathbf{K}_{\mathbf{ET}}$ is the initial stress stiffness matrix due to electro-thermal loads. $\mathbf{N}_1$ and $\mathbf{N}_2$ are nonlinear stiffness matrices. δ and $\ddot{\delta}$ are the displacement and acceleration vectors.

The solution of Eq. (4.32) is assumed to be $\delta = \delta_{\max} \sin \omega t$

Therefore, Eq. (4.32) reduces to

$$-\omega^2 \mathbf{M}\delta_{max} \sin \omega t + \left[\mathbf{K} + \mathbf{K_{ET}} + \frac{1}{2}\mathbf{N_1}(\delta) + \frac{1}{3}\mathbf{N_2}(\delta,\delta) \right] \delta_{max} \sin \omega t = 0 \tag{4.33}$$

$$-\omega^2 \mathbf{M}\delta + \left[\mathbf{K} + \mathbf{K_{ET}} + \frac{1}{2}\mathbf{N_1}(\delta) + \frac{1}{3}\mathbf{N_2}(\delta,\delta) \right] \delta = 0 \tag{4.34}$$

Since Eq. (4.34) does not satisfy the governing equation at all the points, rewriting Eq. (4.34) as,

$$-\omega^2 \mathbf{M}\delta + \left[\mathbf{K} + \mathbf{K_{ET}} + \frac{1}{2}\mathbf{N_1}(\delta) + \frac{1}{3}\mathbf{N_2}(\delta,\delta) \right] \delta = \{R\} \tag{4.35}$$

Taking the weighted residual method along the path

$$\int_0^{T/4} R \sin \omega t \, \mathrm{d}t = 0 \tag{4.36}$$

Since $\int_0^{T/4} \sin^2 \omega t \, \mathrm{d}t = T/8$; $\int_0^{T/4} \sin^3 \omega t \, \mathrm{d}t = T/3\pi$ and $\int_0^{T/4} \sin^3 \omega t \, \mathrm{d}t = 3T/32$, we have,

$$-\omega^2 \mathbf{M}\delta_{max} + \left[\mathbf{K} + \mathbf{K_{ET}} + \frac{4}{3\pi}\mathbf{N_1}(\delta_{max}) + \frac{1}{4}\mathbf{N_2}(\delta_{max},\delta_{max}) \right] \delta_{max} = 0 \tag{4.37}$$

4.5 Solution Procedure

Equation (4.37) is a standard eigenvalue problem and can be solved to obtain eigenvalues and eigenvectors. The solution of Eq. (4.37) is obtained using the direct iteration method. The steps involved are:

Step 1: The fundamental linear frequency and corresponding linear mode shape is calculated by solving Eq. (4.37) with all the nonlinear terms being set to zero.

Step 2: The mode shape is normalized by appropriately scaling the eigenvector ensuring that the maximum displacement is equal to the desired amplitude, $c = w_{max}/h$.

Step 3: The nonlinear terms in the stiffness matrix is computed using the normalized mode shape.

Step 4: The equations are then solved to obtain new eigenvalues and corresponding eigenvectors.

Step 5: Steps 2 to 4 are repeated for a few cycles till the value of the frequency converges up to the desired accuracy (say 10^{-4}).

CHAPTER 5
COMPARISON STUDIES

The present formulation and MATLAB code is first validated for the linear analysis of static and dynamic response of laminated composite plates subjected to either a thermal load or an electrical load or a combination of both. The validation is done by comparing the present results with those available in the literature. It is found that the present results are in very good agreement with those obtained by researchers previously.

5.1 Comparison of Static Response

First, linear static analysis of a [0/90] cross-ply simply supported square plate subjected to a linearly varying temperature field, earlier solved by Chandrashekhara and Tenneti, is taken up. The temperature distribution through the thickness is considered to be $T=T_2$ z/h. The a/h ratio is taken as 50. The thermoelestic properties of the unidirectional ply are the same as those assumed by Chandrashekhara and Tenneti (1993). Fig 5.1 shows the central deflection as obtained by the author along with those of Chandrashekhara and Tenneti.

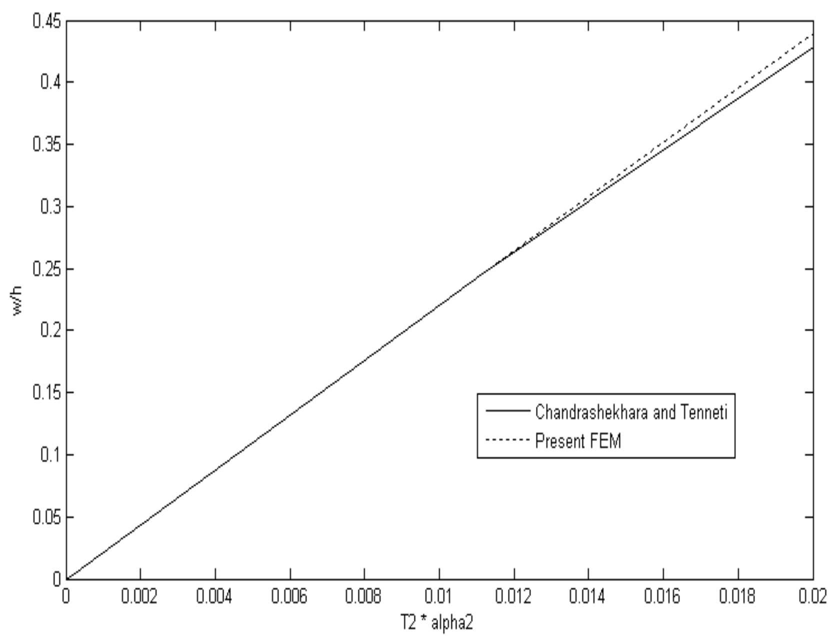

Figure 5.1 Deflection of simply supported laminated composite plate.

5.2 Comparison of linear free vibration and transient responses

Table 5.1 shows the comparison of the results obtained by the present formulation with those obtained by the Ritz Method and those obtained by Sai Ram and Sinha. The material properties used are the same as those taken by Sai Ram and Sinha (1992).

Table 5.2 compares the authors' results of the linear non-dimensional frequency of laminated plates with piezoelectric actuators with those obtained by Huang and Shen. The material properties, plate dimensions, and non dimensioning parameter used are the same as in Huang and Shen (2005). Laminated plates bonded with piezoelectric layers on the top and bottom surfaces have been considered.

Fig 5.2 shows the transient response of (P/0/90/90/0/P) laminated composite plate bonded with piezoelectric actuators and subjected to a sudden uniform load of 0.5 MPa. The results are compared with those of Huang and Shen. The material properties and plate dimensions are the same as in Huang and Shen (2005). A voltage of -50 volts is applied to both the piezoelectric layers. A temperature rise of 100 K is considered.

Having validated the formulation studying the linear static and dynamic response of composite plates, the present work is extended to validate the free vibration results when geometric nonlinearity is taken into consideration.

Table 5.1 Verification of non-dimensional frequency by comparison with Ritz method results and results obtained by Sai Ram and Sinha; a/b = 1, a/h = 100, (0/90/90/0), T=325 K.

Mode number (m)	Present FEM	Sai Ram and Sinha	Ritz method
1	8.5824	8.088	8.068
2	19.5209	19.196	18.378
3	39.7136	39.324	38.778
4	45.8484	45.431	44.778

Table 5.2 Comparison of non-dimensional frequency of laminated plates with piezoelectric actuators with the results obtained by Huang and Shen

Stacking Sequence	Control Voltage (Volts) $V_{lower}=V_{upper}$	Temperature rise=0 K		Temperature rise=100 K		Temperature rise=300 K	
		Present FEM	Huang and Shen	Present FEM	Huang and Shen	Present FEM	Huang and Shen
(P/0/90/0/90/P)	**-50 V**	10.8569	10.664	10.3327	10.133	9.1949	8.975
	0 V	10.8098	10.617	10.2832	10.082	9.1393	8.918
	50 V	10.7625	10.568	10.2334	10.032	9.0833	8.862
(P/0/90/90/0/P)	**-50 V**	10.8696	10.861	10.3461	10.343	9.2103	9.217
	0 V	10.8226	10.841	10.2967	10.293	9.1548	9.163
	50 V	10.7753	10.768	10.2470	10.244	9.0989	9.106

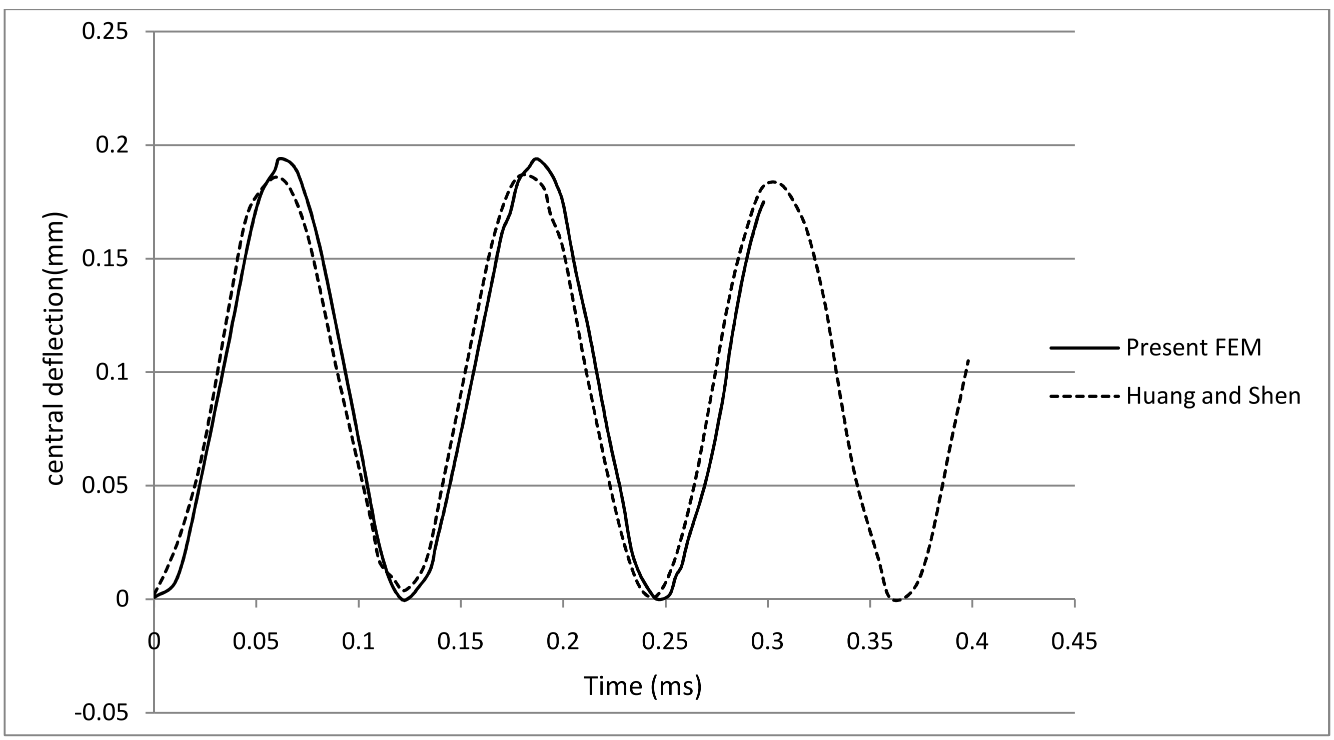

Figure 5.2 Dynamic response of (P/0/90/90/0/P) laminated composite plates

5.3 Comparison of nonlinear free vibration results

Table 5.3 compares the authors' results studying the effects of temperature rise on the nonlinear to linear frequency ratio for shear deformable laminated plates with piezoelectric actuators with those obtained by Huang and Shen. The material properties, plate dimensions, and non dimensioning parameter used are the same as in Huang and Shen (2005).

Table 5.4 compares the authors' results studying the effects of control voltage on the nonlinear to linear frequency ratio for shear deformable laminated composite plates fitted with piezoelectric actuators with those obtained by Huang and Shen. The material properties, plate dimensions, and non dimensioning parameter used are the same as in Huang and Shen (2005).

The results of Table 5.3 and Table 5.4 are obtained by iteratively solving Eq. (4.32). The changes introduced by applying the Galerkin's weighted residual method have not been considered in this case.

Table 5.5 gives the comparison of nonlinear frequency ratio by iteratively solving Eq. (4.37) which employs the Galerkin's weighted residual method. It is seen that by applying Galerkin's method the obtained results compare very well with the analytical solutions of Huang and Shen (2005). The use of this formulation throughout the present study is therefore justified.

Table 5.3 Effects of temperature rise on the nonlinear to linear frequency ratio for shear deformable laminated plates with piezoelectric actuators.

Stacking sequence	**Temperature rise**	**W_{max}/h**				
		0.2	**0.4**	**0.6**	**0.8**	**1.0**
(P/0/90/0/90/P)	**0 K**	1.0167 (1.019)	1.0655 (1.073)	1.1427 (1.157)	1.2436 (1.266)	1.3636 (1.393)
	100 K	1.0185 (1.021)	1.0721 (1.080)	1.1566 (1.173)	1.2664 (1.292)	1.3963 (1.430)
	300 K	1.0233 (1.026)	1.0905 (1.102)	1.1948 (1.277)	1.3283 (1.363)	1.4840 (1.529)
(P/0/90/90/0/P)	**0 K**	1.017 (1.018)	1.067 (1.070)	1.1457 (1.151)	1.2483 (1.256)	1.3700 (1.379)
	100 K	1.0189 (1.020)	1.0737 (1.077)	1.1599 (1.166)	1.2715 (1.280)	1.4032 (1.413)
	300 K	1.0238 (1.025)	1.0924 (1.096)	1.1987 (1.206)	1.3343 (1.344)	1.4921 (1.504)

Bracketed values are the results obtained by Huang and Shen (2005)

Table 5.4 Effects of voltage on the nonlinear to linear frequency ratio for shear deformable laminated plates with piezoelectric actuators.

Stacking sequence	Voltage	W_{max}/h				
		0.2	**0.4**	**0.6**	**0.8**	**1.0**
(P/0/90/0/90/P)	**-50 V**	1.0166 (1.02)	1.0649 (1.079)	1.1415 (1.170)	1.2417 (1.286)	1.3608 (1.423)
	0 V	1.0167 (1.02)	1.0655 (1.08)	1.1427 (1.171)	1.2436 (1.289)	1.3636 (1.426)
	50 V	1.0169 (1.03)	1.0661 (1.088)	1.1438 (1.173)	1.2455 (1.292)	1.3663 (1.430)
(P/0/90/90/0/P)	**-50 V**	1.017 (1.019)	1.0664 (1.075)	1.1445 (1.163)	1.2464 (1.275)	1.3672 (1.406)
	0 V	1.017 (1.019)	1.067 (1.076)	1.1457 (1.164)	1.2483 (1.277)	1.3700 (1.410)
	50 V	1.0173 (1.02)	1.0675 (1.077)	1.1469 (1.166)	1.2503 (1.280)	1.3728 (1.413)

Bracketed values are the results obtained by Huang and Shen (2005)

Table 5.5 Comparison of nonlinear frequency ratio (ω_{NL} / ω_L) of [P/0^0/90^0/90^0/0^0/P] laminated plate with piezoelectric actuators

Applied Voltage	W_{max}/h	0.2	0.4	0.6	0.8	1
-50 V	Present	1.019	1.075	1.161	1.272	1.402
	Huang and Shen (2005)	1.019	1.075	1.163	1.275	1.406
0 V	Present	1.019	1.075	1.163	1.274	1.405
	Huang and Shen (2005)	1.019	1.076	1.164	1.277	1.410
50 V	Present	1.020	1.076	1.164	1.276	1.408
	Huang and Shen (2005)	1.020	1.077	1.166	1.280	1.413

Having validated all the present codes the study is extended to investigate the effect of different parameters such as temperature rise, control voltage, boundary conditions, fibre orientation and stacking sequence on the free vibration behavior of laminated composite plates fitted with piezoelectric actuators and subjected to thermal environments.

CHAPTER 6

NEW RESULTS AND DISCUSSIONS

The increasing use of laminated composites in engineering structures has resulted in the need for more information on their behavior. Elevated temperature conditions have an adverse effect on the performance of the composites. In this chapter, large amplitude free vibrations of laminated composite plates with bonded piezoelectric layers subjected to thermal environments will be studied using nonlinear finite element methods. The numerical results are presented for different temperature conditions, control voltages, boundary conditions, fibre orientations and stacking sequences.

6.1 Boundary conditions

In the present investigation two different simply supported boundary conditions are considered:

For both cases w= 0 at x= 0, a and y= 0, b.

In addition to the above constraints on transverse displacement, the following conditions are considered here:

Immovable (SS1) u_0= v_0= 0 at x= 0, a and y= 0, b.

Partially movable (SS2) v_0= 0 at x= 0, a; u_0= 0 at y= 0, b.

6.2 Material properties

Unless otherwise mentioned, graphite/epoxy composite material and PZT-5A were selected for the substrate and piezoelectric layers, respectively. The material properties for graphite-epoxy orthotropic layers of the substrate are $E_1 = 150$ GPa, $E_2 = 9$ GPa, $G_{12} = G_{13} = 7.1$ GPa, $G_{23} = 2.5$ GPa, $\nu_{12} = 0.3$, $\rho_0 = 1580$ kg/m^3 and for PZT-5A piezoelectric layers E = 63.0 GPa, G = 24.2 GPa, $\nu = 0.3$, $\rho = 1580$ kg/m^3 and $d_{31} = d_{32} = 2.54\times10^{-10}$ m/V .

6.3 Plate dimensions

Free vibration of simply supported cross-ply laminated plate having piezoelectric layers at its top and bottom is considered. The plate is having size of $a = b = 24$ mm and the total thickness of the plate is 1.2 mm. All orthotropic layers of the substrate are of equal thickness, whereas the thickness of piezoelectric layers is 0.1 mm.

The value of linear fundamental frequency is non-dimensionalised as $\lambda = \omega\left(a^2 / h\right)\sqrt{\rho_0 / E_1}$.

6.4 Cross-ply laminates

Table 6.1- 6.8 gives the nonlinear to linear frequency ratio for both symmetric and unsymmetric cross-ply laminates under uniform temperature rise and under different control voltages applied equally on both the upper and lower piezoelectric actuator layers. From the linear non-dimensional frequency values it is observed that increase in temperature causes the fundamental natural frequency to reduce. This can be explained by a reduction in stiffness of the composite plates with increasing temperature. On the other hand a negative voltage applied to the plate causes the fundamental natural frequency to increase. This may be explained by the pinching action of the piezoelectric actuator layer which causes the composite plate to be stretched, thereby increasing its stiffness.

It can also be seen that the temperature rise and positive control voltage increase the nonlinear to linear frequency ratios. This is in perfect accordance with the results obtained by Huang and Shen (2005).

Comparing the results for the two boundary conditions used, it is observed that the vibration amplitude` has a smaller effect on the nonlinear to linear frequency ratio of an SS2 plate as compared to an SS1 plate.

Table 6.1 Comparison of nonlinear frequency ratio (ω_{NL} / ω_L) of [P/0^0/90^0/90^0/0^0/P] laminated plate, SS1 boundary

W_{max}/h	Temperature Rise		
	0 K	100 K	300 K
0.2	1.0194	1.0214	1.027
0.4	1.0754	1.083	1.104
0.6	1.1627	1.1784	1.2212
0.8	1.2745	1.2999	1.3682
1.0	1.4049	1.4407	1.5362
1.2	1.549	1.5955	1.7188
λ	10.8318	10.306	9.1642

Table 6.2 Comparison of nonlinear frequency ratio (ω_{NL} / ω_L) of [P/0^0/90^0/90^0/0^0/P] laminated plate, SS1 boundary

W_{max}/h	Control Voltage		
	-100 V	0 V	100 V
0.2	1.0191	1.0194	1.0197
0.4	1.0742	1.0754	1.0767
0.6	1.1601	1.1627	1.1654
0.8	1.2703	1.2745	1.2788
1.0	1.399	1.4049	1.411
1.2	1.5412	1.549	1.5569
λ	10.9256	10.8318	10.7371

Table 6.3 Comparison of nonlinear frequency ratio (ω_{NL} / ω_L) of [P/0^0/90^0/90^0/0^0/P] laminated plate, SS2 boundary

W_{max}/h	Temperature Rise		
	0 K	100 K	300 K
0.2	1.0092	1.0111	1.0194
0.4	1.0361	1.0437	1.0754
0.6	1.0793	1.0956	1.1623
0.8	1.1367	1.164	1.2735
1.0	1.2062	1.2459	1.4027
1.2	1.2853	1.3385	1.5447
λ	10.8318	9.8266	7.4182

Table 6.4 Comparison of nonlinear frequency ratio (ω_{NL} / ω_L) of [P/0^0/90^0/90^0/0^0/P] laminated plate, SS2 boundary

W_{max}/h	Control Voltage		
	-100 V	0 V	100 V
0.2	1.0089	1.0092	1.0095
0.4	1.035	1.0361	1.0373
0.6	1.0769	1.0793	1.0819
0.8	1.1327	1.1367	1.141
1.0	1.2002	1.2062	1.2125
1.2	1.2774	1.2853	1.2938
λ	11.0065	10.8318	10.6542

Table 6.5 Comparison of nonlinear frequency ratio (ω_{NL} / ω_L) of [P/0^0/90^0/0^0/90^0/P] laminated plate, SS1 boundary

W_{max}/h	Temperature Rise		
	0 K	100 K	300 K
0.2	1.0191	1.021	1.0265
0.4	1.0742	1.0817	1.1023
0.6	1.1603	1.1758	1.2181
0.8	1.2709	1.296	1.3637
1.0	1.4003	1.4358	1.5306
1.2	1.5435	1.5897	1.7123
λ	10.8194	10.2928	9.1491

Table 6.6 Comparison of nonlinear frequency ratio (ω_{NL} / ω_L) of [P/0^0/90^0/0^0/90^0/P] laminated plate, SS1 boundary

W_{max}/h	Control Voltage		
	-100 V	0 V	100 V
0.2	1.0187	1.0191	1.0194
0.4	1.073	1.0742	1.0755
0.6	1.1577	1.1603	1.1629
0.8	1.2668	1.2709	1.2752
1.0	1.3944	1.4003	1.4063
1.2	1.5358	1.5435	1.5514
λ	10.9133	10.8194	10.7246

Table 6.7 Comparison of nonlinear frequency ratio (ω_{NL} / ω_L) of [P/0^0/90^0/0^0/90^0/P] laminated plate, SS2 boundary

W_{max}/h	Temperature Rise		
	0 K	100 K	300 K
0.2	1.0096	1.0117	1.0214
0.4	1.0378	1.0461	1.083
0.6	1.0829	1.1008	1.1781
0.8	1.1427	1.1727	1.299
1.0	1.2149	1.2585	1.4386
1.2	1.2971	1.3554	1.5913
λ	10.5854	9.5546	7.0547

Table 6.8 Comparison of nonlinear frequency ratio (ω_{NL} / ω_L) of [P/0^0/90^0/0^0/90^0/P] laminated plate, SS2 boundary

W_{max}/h	Control Voltage		
	-100 V	0 V	100 V
0.2	1.0093	1.0096	1.0099
0.4	1.0365	1.0378	1.0391
0.6	1.0803	1.0829	1.0857
0.8	1.1383	1.1427	1.1474
1.0	1.2085	1.2149	1.2218
1.2	1.2885	1.2971	1.3063
λ	10.764	10.5854	10.4037

6.5 Angle-ply laminates

Table 6.9-6.16 gives the nonlinear to linear frequency ratio for both symmetric and unsymmetric angle-ply laminates under uniform temperature rise and under different control voltages applied equally on both the upper and lower piezoelectric actuator layers. The trends observed for angle-ply laminates is similar to those observed for cross-ply laminates. However, for the material properties, plate geometry and boundary conditions employed in the present investigation, the angle-ply laminates have higher fundamental frequencies as compared to the cross ply laminates.

Table 6.9 Comparison of nonlinear frequency ratio (ω_{NL}/ω_L) of [P/45^0/-45^0/-45^0/45^0/P] laminated plate, SS1 boundary

W_{max}/h	Temperature Rise		
	0 K	100 K	300 K
0.2	1.0132	1.0143	1.0172
0.4	1.0517	1.056	1.067
0.6	1.1128	1.1218	1.1451
0.8	1.1928	1.2077	1.2459
1.0	1.2879	1.3094	1.3639
1.2	1.3949	1.4234	1.495
λ	11.9959	11.5194	10.5014

Table 6.10 Comparison of nonlinear frequency ratio (ω_{NL} / ω_L) of [P/45^0/-45^0/-45^0/45^0/P] laminated plate, SS1 boundary

W_{max}/h	Control Voltage		
	-100 V	0 V	100 V
0.2	1.013	1.0132	1.0134
0.4	1.051	1.0517	1.0524
0.6	1.1113	1.1128	1.1143
0.8	1.1903	1.1928	1.1953
1.0	1.2843	1.2879	1.2917
1.2	1.3901	1.3949	1.3999
λ	12.0813	11.9959	11.9098

Table 6.11 Comparison of nonlinear frequency ratio (ω_{NL} / ω_L) of [P/45^0/-45^0/-45^0/45^0/P] laminated plate, SS2 boundary

W_{max}/h	Temperature Rise		
	0 K	100 K	300 K
0.2	1.0086	1.0098	1.0139
0.4	1.0339	1.0388	1.0543
0.6	1.0747	1.0851	1.1182
0.8	1.1292	1.1467	1.2017
1.0	1.1953	1.221	1.3009
1.2	1.2704	1.305	1.4111
λ	11.9959	11.2128	9.4504

Table 6.12 Comparison of nonlinear frequency ratio (ω_{NL} / ω_L) of [P/45^0/-45^0/-45^0/45^0/P] laminated plate, SS2 boundary

W_{max}/h	Control Voltage		
	-100 V	**0 V**	**100 V**
0.2	1.0084	1.0086	1.0088
0.4	1.0332	1.0339	1.0347
0.6	1.0731	1.0747	1.0764
0.8	1.1264	1.1292	1.1321
1.0	1.1912	1.1953	1.1996
1.2	1.2649	1.2704	1.2762
λ	12.1342	11.9959	11.8559

Table 6.13 Comparison of nonlinear frequency ratio (ω_{NL} / ω_L) of [P/45^0/-45^0/45^0/-45^0/P] laminated plate, SS1 boundary

W_{max}/h	Temperature Rise		
	0 K	**100 K**	**300 K**
0.2	1.0126	1.0136	1.0162
0.4	1.0494	1.0533	1.0633
0.6	1.1079	1.1163	1.1375
0.8	1.185	1.1988	1.2337
1.0	1.2771	1.297	1.3472
1.2	1.381	1.4075	1.4737
λ	12.1423	11.6754	10.6807

Table 6.14 Comparison of nonlinear frequency ratio (ω_{NL} / ω_L) of [P/45^0/-45^0/45^0/-45^0/P] laminated plate, SS1 boundary

W_{max}/h	Control Voltage		
	-100 V	0 V	100 V
0.2	1.0124	1.0126	1.0127
0.4	1.0487	1.0494	1.05
0.6	1.1065	1.1079	1.1094
0.8	1.1826	1.185	1.1874
1.0	1.2737	1.2771	1.2805
1.2	1.3765	1.381	1.3856
λ	12.2261	12.1423	12.0578

Table 6.15 Comparison of nonlinear frequency ratio (ω_{NL} / ω_L) of [P/45^0/-45^0/45^0/-45^0/P] laminated plate, SS2 boundary

W_{max}/h	Temperature Rise		
	0 K	100 K	300 K
0.2	1.0082	1.0093	1.0129
0.4	1.0323	1.0367	1.0506
0.6	1.0712	1.0808	1.1106
0.8	1.1234	1.1395	1.1895
1.0	1.1866	1.2103	1.2829
1.2	1.2596	1.2917	1.3888
λ	12.1314	11.3608	9.6344

Table 6.16 Comparison of nonlinear frequency ratio (ω_{NL} / ω_L) of [P/45^0/-45^0/45^0/-45^0/P] laminated plate, SS2 boundary

W_{max}/h	Control Voltage		
	-100 V	0 V	100 V
0.2	1.008	1.0082	1.0084
0.4	1.0316	1.0323	1.033
0.6	1.0697	1.0712	1.0728
0.8	1.1208	1.1234	1.1261
1.0	1.1828	1.1866	1.1905
1.2	1.2544	1.2596	1.2649
λ	12.2677	12.1314	11.9936

CHAPTER 7

CONCLUSIONS

In the present investigation a finite element procedure is formulated for studying the nonlinear free vibration of laminated composite plates fitted with piezoelectric actuators and subjected to thermal environments. Modified first order shear deformation theory is employed using an 8 noded isoparametric element. The geometric nonlinearity is introduced through von Karman strains. The Galerkin's weighted residual method is used to obtain the final governing equation that is solved by direct iteration.

The important observations from the numerical results for the nonlinear free vibrations are presented in this section.

- An increase in temperature causes a reduction in the fundamental frequency due to reduction in stiffness of the composite plate.
- Increase in the minus voltage increases the fundamental frequency. This can be explained by the pinching action of the piezoelectric layer which causes stretching of the plate, thereby increasing its stiffness.
- For the material properties, plate geometry and boundary conditions used in the present, angle-ply laminates are found to have higher fundamental frequencies as compared to cross-ply laminates.
- Amplitude of vibration is found to have a greater impact on the nonlinear to linear frequency ratios of plates with SS1 boundary conditions as compared to those with SS2 boundary conditions.

CHAPTER 8

SCOPE FOR FUTURE RESEARCH

There are several related problems where either the present investigation can be extended. This section defines some such problems that can be taken up for future investigation.

- The current formulation can be extended to study the geometrically nonlinear free vibration of shells.
- Thermal post-buckling analysis can be carried out for composite plates and shells.
- The analysis of plates and shells bonded with piezoelectric sensors and actuators can be taken up.
- The present analysis can also be extended to material nonlinear problems.

REFERENCES

1. J.M Whitney and J.E.Ashton 1971 American Institute of Aeronautics of Astronautics Journal 9,1708-1713. Effect of environment on the elastic response of layered composite plates.
2. R.D.Dhanaraj and Palaninathan 1989 Proceedings of the Seminar on Science and Technology of Composites, Adhesives and Sealants, Bangalore, India, 245-251. Free vibration characteristics of composite laminates under initial stress.
3. K.S.Sai Ram and P.K.Sinha, Journal of Sound and Vibration(1992) 158(1),133-148. Hygrothermal effects on the free vibration of laminated composite plates.
4. P.K. Parhi, S.K. Bhattacharyya and P.K. Sinha, Hygrothermal effects on the dynamic behavior of multiple delaminated composite plates and shells, Journal of Sound and Vibration (2001) **248**(2), 195-214.
5. K.Chandrashekhara and R.Tenneti, Composites Science and Technology 51 (1994) 85-95. Non-linear static and dynamic analysis of heated laminated plates: A finite element approach.
6. Bathe, K.J. (1996). "Finite element procedures". Prentice Hall, New Jersey.
7. Srinath, L.S (2009). "Advanced Mechanics of Solids". Tata McGraw-Hill, New Delhi.
8. H.-S. Shen, Nonlinear bending analysis of unsymmetric cross-ply laminated plates with piezoelectric actuators in thermal environments, Composite Structures 63 (2004), pp. 167–177.
9. Xiao-Lin Huang and Hui-Shen Shen, Nonlinear free and forced vibration of simply supported shear deformable laminated plates with piezoelectric actuators, International Journal of Mechanical Sciences, Volume 47, Issue 2, February 2005, Pages 187-208.
10. N. Nanda and J. N. Bandyopadhyay, Nonlinear Transient Response of Laminated Composite Shells, Journal of Engineering Mechanics, ASCE, 2007.
11. N. Nanda and J. N. Bandyopadhyay, Nonlinear Free Vibration Analysis Of Laminated Composite Cylindrical Shells With Cutouts, Journal of Reinforced Plastics and Composites, 2007.

12. M.K.Singha and R. Daripa, Nonlinear vibration and dynamic stability analysis of composite plates, Journal of Sound and Vibration 328 (2009) 541-554.

13. M.K.Singha and R. Daripa, Nonlinear vibration of symmetrically laminated composite skew plates by finite element method, International Journalof Non-Linear Mechanics 42 (2007) 1144-1152.

14. P. Dash and B.N.Singh, Nonlinear free vibration of piezoelectric laminated composite plate, Finite Elements in Analysis and Design 45 (2009) 686-694.

15. N.V. Swamy Naidu and P.K. Sinha, Nonlinear finite element analysis of laminated composite shells in hygromthermal environments, Composite Structures 69 (2005) 387-395.

16. N.V. Swamy Naidu and P.K. Sinha, Nonlinear free vibration analysis of laminated composite shells in hygrothermal environments, Composite Structures 77 (2007) 475-483.

17. J.N. Reddy, Mechanics of laminated composite plates and shells: theory and analysis. Second Edition, CRC Press, Boca Raton, (2004).

18. R. Tanov and A. Tabiei, A simple correction to the first-order shear deformation shell finite element formulations, Finite Elements in Analysis and Design **35** (2000), 189-197.