"AN APPROACH TO DYNKIN DIAGRAMS ASSOCIATED WITH KAC-MOODY SUPERALGEBRA"

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DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**An Approach to Dynkin Diagram associated with Kac-Moody Superalgebra**" in partial fulfilment of the requirement for the award of the degree of Master of Science, submitted in the Department of Mathematics, National Institute of Technology, Rourkela is an authentic record of my own work carried out under the supervision of Prof. K.C. Pati. The matter embodied in this thesis has not been submitted by me for the award of any other degree.

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Words at my command are inadequate to convey the profound to my parents whose love, affection and blessings has inspired me the most.

Karan Kumar Pradhan

ABSTRACT

In the present project report, a sincere report has been made to construct and study the basic portions related to Simple Lie Algebras, Lie superalgebras and Kac-Moody (super-) algebras and their corresponding Dynkin Diagrams.

In chapter-1, I have given the precise definitions of Lie Algebra and some of the terms related to Lie algebra, i.e. subalgebras, ideals, commutativity, solvability, nilpotency etc. Also, I have done the classifications of Classical Lie algebras.

In chapter-2, I made a review over the basics of Representation Theory, i.e. structure constants, modules, reflections in a Euclidean space, root systems (simple roots) and their corresponding root diagrams. Then I have discussed the formation of Dynkin Diagrams associated with the roots of the simple lie algebras.

After that, in chapter-3, the introductory parts of Lie superalgebras, i.e. Z_2 graded algebra, definition of lie superalgebra, modules, the killing form, root systems and simple root systems (distinguished root systems) are addressed. In this section, also the classifications of simple lie superalgebras are plotted and the Dynkin Diagrams related to the Basic Lie superalgebras are presented.

In chapter-4, I gave the necessary theory based on Kac-Moody Lie superalgebras and their classifications. Then the definition of the extended Dynkin diagrams for Affinization of Kac-Moody superalgebras and the Dynkin Diagrams associated with the affine Kac-Moody superalgebras are provided.

Contents

1

Preliminaries

Basic Definition

Before going to my concerned topic **KAC-MOODY SUPER ALGEBRA**, we need to get a precise definition of **LIE ALGEBRA**, i.e. A vector Space **L** over a field **F** with an operation **L** × **L** → **L** denoted by $(x , y) \longmapsto [x , y]$ and called the ***bracket*** or ***commutator*** of x and y is called a **Lie Algebra** if the following axioms are satisfied :

1. The bracket operation is ***bilinear***, i.e.

   $[\alpha x + \beta y, z] = \alpha\ [x, z] + \beta\ [y, z]$ for all scalars α, β in **F** and all elements x, y, z in **L**.

2. The bracket operation is ***skew-symmetric***, i.e.
   $[x, x] = \mathbf{0}$ for all x in **L**.

3. $[x, [\, y, z]] + [y, [z, x]] + [z, [x, y]] = \mathbf{0}$ for all x, y, z in **L**.

The axiom is called ***Jacobi's Identity***.[1]

The followings are some of the examples related to Lie algebra.

Example 1: Let A be an algebra over F (a vector space with an associative multiplication x · y). A is a Lie algebra A_L (also called A as Lie algebra*)* by defining [x , y] = x · y − y · x.

Example 2: Take for A the algebra of all operators (endomorphisms) of a vector space V ; the corresponding A_L is called the general Lie algebra of V , $\mathfrak{gl}$(V). Concretely, taking number space R_n as V, this is the general linear Lie algebra $\mathfrak{gl}$(n, R) of all n × n real matrices, with [x , y] = x . y – y . x. Similarly $\mathfrak{gl}$(n, C).
Example 3: The special linear Lie algebra $\mathfrak{sl}$(n,R) consists of all n × n real matrices with trace 0 (and has the same linear and bracket operations as $\mathfrak{gl}$(n,R)—it is a "sub Lie algebra"); similarly for C. For any vector space V we have $\mathfrak{sl}$(V), the special linear Lie algebra of V, consisting of the operators on V of trace 0.[8]

Subalgebras, Ideals

- A subset **K** of a Lie Algebra **L** is called a **subalgebra** of **L** if for all x, y in **K** and all α, β in **F**, one has $\alpha x + \beta y$ in **K** , $[x, y]$ in **K**.

- An **ideal I** of a Lie Algebra **L** is a sub-algebra of **L** with the property [**I**,**L**] is a subset of **I**, i.e. for all x in **I** and y in **L** one has $[x, y]$ in **I**. Every (non-zero) Lie Algebra has at least two Ideals, namely the Lie Algebra **L** itself and the sub-algebra **0** consisting of the **zero element** only. Both these ideals are called **Trivial**. All non-trivial ideals are called **Proper ideals**.[1]

Abelian, Solvability & Nilpotency

- The Lie Algebra **L** is called **abelian** or **commutative** if $[x, y]$ = **0** for all x, y in **L**.

- The solvability of a Lie Algebra **L** can be checked if in the sequence of ideals of **L**(the derived series)
  $\mathbf{L}^{(0)} = \mathbf{L}, \mathbf{L}^{(1)} = [\mathbf{L},\mathbf{L}], \mathbf{L}^{(2)}=[\mathbf{L}^{(1)},\mathbf{L}^{(1)}], \mathbf{L}^{(3)}=[\mathbf{L}^{(2)},\mathbf{L}^{(2)}], \ldots, \mathbf{L}^{(i)} = [\mathbf{L}^{(i-1)},\mathbf{L}^{(i-1)}]$
  $\mathbf{L}^{(n)} = 0$ for some n.
  e.g. Abelian implies solvable whereas simple algebras are definitely non-solvable.
  A lie algebra $t_{+}(n)$ of the upper triangular matrices is a prototype of solvable algebras.

- The Nilpotency of a Lie Algebra **L** can be verified if in the lower central series of **L**
  $\mathbf{L}^0 = \mathbf{L}, \mathbf{L}^1 = [\mathbf{L},\mathbf{L}], \mathbf{L}^2 = [\mathbf{L},\mathbf{L}^1], \ldots, \mathbf{L}^i = [\mathbf{L},\mathbf{L}^{i-1}]$
  $\mathbf{L}^n = 0$ for some n.
  e.g. any abelian algebra is nilpotent.
  A lie algebra $t_{++}(n)$ of strictly upper triangular matrices is the prototype of nilpotent algebras.

Clearly, $L^{(i)} \subset L^i$ for all I, so nilpotent algebras are solvable. But the converse is false.[1]

Simple & Semisimple Lie Algebra

- A Lie algebra **L** is **simple** if it has no proper ideals and is not abelian.
- A Lie Algbera L is said to be **semisimple** if its radical is zero. Equivalently, **L** is semisimple if it does not contain any non-zero abelian ideals. In particular, a simple Lie algebra is semisimple.[1]

Classical Lie Algebras

The Classical Lie algebras are broadly divided into four families, i.e $\boldsymbol{A_l}$, $\boldsymbol{B_l}$, $\boldsymbol{C_l}$ and $\boldsymbol{D_l}$. Let V be a finite-dimensional vector space over **F** and denotes End V, the set of linear transformations $V \to V$ (**endomorphisms of** V).

- $\boldsymbol{A_l}$: **Special linear lie algebra**, denoted by $\mathfrak{sl}(V)$ or $\mathfrak{sl}(l+1, \mathbf{F})$. It is the End V having trace 0.
  The dimension of $\boldsymbol{A_l}$ is at most $(l+1)^2 - 1$.

- $\boldsymbol{B_l}$: **Orthogonal lie algebra**, denoted by $\mathfrak{o}(V)$ or $\mathfrak{o}(2l+1, \mathbf{F})$. It consists of End V satisfying the following property :
  $$f(x(v), w) = -f(v, x(w))$$
  or, $$sx = -x^t s$$
  where $s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix}$ and $x = \begin{pmatrix} a & b_1 & b_2 \\ c_1 & m & n \\ c_2 & p & q \end{pmatrix}$

  The dimension of $\boldsymbol{B_l}$ is $2l^2 + l$.

- $\boldsymbol{C_l}$: **Sympletic Lie algebra**, denoted by $\mathfrak{sp}(V)$ or $\mathfrak{sp}(2l, \mathbf{F})$. The End V satisfies the following property :

  $$f\ (x(v), w) = -\ f\ (v,, x(w))$$

  or, $sx = -\ x^{t}s$

  where $s = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$ and $x = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$

  The dimension of $\boldsymbol{C_l}$ is $2l^2 + l$.

- $\boldsymbol{D_l}(\boldsymbol{l \geq 2})$: **Orthogonal lie algebra**, denoted by $\mathfrak{o}(V)$ or $\mathfrak{o}(2l, \mathbf{F})$. The construction of $\boldsymbol{D_l}$ is identical that for $\boldsymbol{B_l}$, except that dim $V = 2l$ is even and s has the simpler form $\begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}$.[1]

2

Representations

In representation theory, a **Lie algebra representation** or **representation of a Lie algebra** is a way of writing a Lie algebra as a set of matrices (or endomorphisms of a vector space) in such a way that the Lie bracket is given by the commutator.

Structure Constants:

Let $\mathfrak{g}$ be a lie algebra and take a basis $\{X_1, X_2, \ldots, X_n\}$ for (the vector space) $\mathfrak{g}$. By bilinearity [,] – operation in $\mathfrak{g}$ is completely determined once the values $[X_i, X_j]$ are known. We know them by writing them as linear combinations of X_i. The coefficients c_{ij}^k in the relations $[X_i, X_j] = c_{ij}^k X_k$ are called the **structure constants** of $\mathfrak{g}$ (with respect to the given basis).[1]

Modules:

Let $\mathfrak{g}$ be a Lie algebra. It is often convenient to use the language of modules along with the language of representations.

A vector space V, endowed with an operation $\mathfrak{g} \times V \longrightarrow V$ (denoted by $(x.v) \longmapsto x.v$) is called an $\mathfrak{g}$-module if the following conditions are satisfied:

- $(ax + by).v = a(x.v) + b(y.v)$
- $x.(av + bw) = a(x.v) + b(x.w)$
- $[x, y].v = x.y.v - .x.v$, where $x, y \in \mathfrak{g}$; $v, w \in V$; $a, b \in \mathbf{F}$.[1]

Representations:

A representation of a lie algebra $\mathfrak{g}$ on a vector space V is a homomorphism (say φ) of $\mathfrak{g}$ into the general linear lie algebra $\mathfrak{gl}(V)$ of V. φ assigns to each X in $\mathfrak{g}$ an operator $\varphi(X): V \to V$ depending linearly on X (thus, $\varphi(aX + bY) = a\varphi(X) + b\varphi(Y)$) and satisfying $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$. A vector space V, together with the representation φ, is called an $\mathfrak{g}$***-space*** or $\mathfrak{g}$***-module***.

For example, the adjoint representation ad : $\mathfrak{g} \to \mathfrak{gl}(V)$, where ad x $(y) = [x, y]$. It preserves the bracket as follows:

$$
\begin{aligned}
[\text{ad}\ , \text{ad}\ y](z) &= \text{ad}\ x\ \text{ad}\ y\ (z) - \text{ad}\ y\ \text{ad}\ x\ (z) \\
&= \text{ad}\ x\ ([y, z]) - \text{ad}\ y\ ([x, z]) \\
&= [x, [y, z]] - [y, [x, z]] \\
&= [x, [y, z]] + [[x, z], y] \\
&= [[x, y], z] \\
&= \text{ad}\ [x, y]\ (z).
\end{aligned}
$$

[1]

Reflections in a Euclidean Space:

Let us consider a fixed euclidean space **E**, i.e. a finite dimensional vector space over **R** equipped with a positive definite symmetric bilinear form (α , β). Geometrically, a reflection in **E** is an invertible linear transformation leaving pointwise fixed some hyperplane (subspace of codimension one) and sending any vector orthogonal to that hyperplane into its negative.

A reflection is orthogonal, i.e. preserves the inner product on **E**. Any non-zero vector $\boldsymbol{\alpha}$ determines a reflection σ_α, with **reflecting hyperplane** $P_\alpha = \{\beta \in E : (\beta , \alpha) = 0\}$. An explicit formula for $\sigma_\alpha(\beta) = \beta - 2\frac{(\beta ,\boldsymbol{\alpha})}{(\boldsymbol{\alpha} ,\boldsymbol{\alpha})}\alpha$. (This is because it sends α to - α and fixes all points in P_α).

We replace $2\frac{(\beta ,\boldsymbol{\alpha})}{(\boldsymbol{\alpha} ,\boldsymbol{\alpha})}$ by $< \beta, \alpha >$ and is linear only in the first variable β.[1]

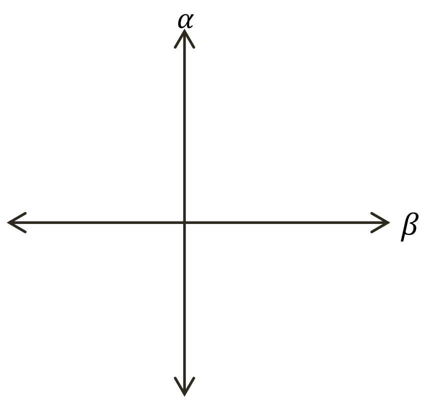
Root Systems:

A subset Φ of the Euclidean space E is called a root system in E if the following axioms are satisfied:

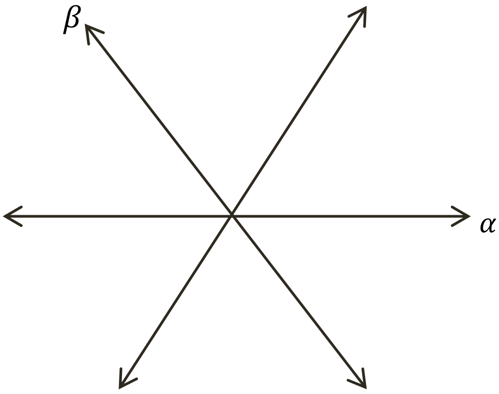
- Φ is finite, spans E, and doesn't contain 0.
- If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.
- If $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant .
- If $\alpha, \beta \in \Phi$, then $< \beta, \alpha > \in \mathbf{Z}$.

The ***simple roots*** can be defined as are the positive roots that cannot be written as the linear combination of other positive roots. If there are $\boldsymbol{r}$ simple roots for an algebra Lof rank $\boldsymbol{r}$ and they form a basis of the root system.[1]

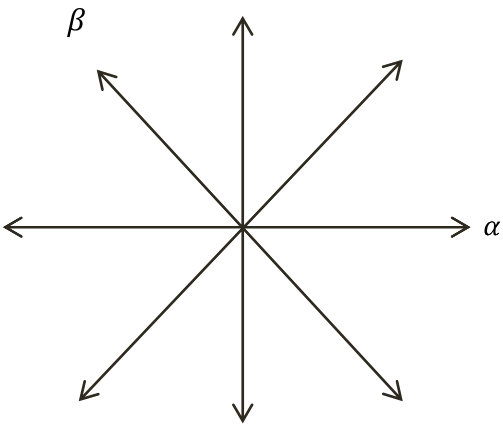
Root Diagrams of Lie Algebras[1].



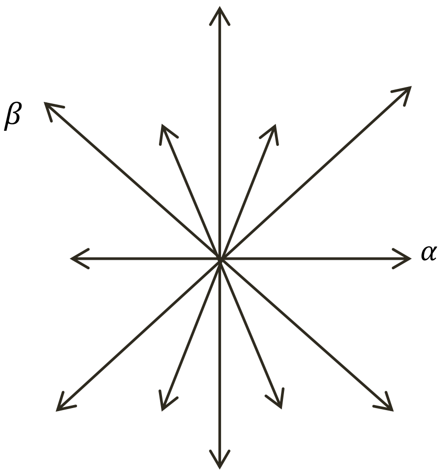
Root Diagram of $A_1 \times A_1$



Root Diagram of A_2



Root Diagram of B_2



Root Diagram of G_2

Cartan Matrix:

A **generalised Cartan matrix (**GCM**)** is a square matrix A = (a_{ij}) with integer entries such that

- For diagonal entries, a_{ii} = 2.
- For non-diagonal entries, $a_{ij} \leq 0$.
- a_{ij} = 0 if and only if a_{ji} = 0.
- A can be written as DS, where D is a diagonal matrix, and S is a symmetric matrix.

We can always choose D with positive diagonal entries and if S is positive definite, then A is said to be a **Cartan Matrix**.

The Cartan Matrix of a simple lie algebra is the matrix whose elements are the scalar products

$$a_{ij} = 2\frac{(\beta\, ,\boldsymbol{\alpha}\,)}{(\boldsymbol{\alpha}\, ,\boldsymbol{\alpha}\,)}\,.$$ [5]

Killing Form:

Let g be any lie algebra. If x , y ∈ g, define $\kappa(x\, , y) = Tr(\text{ad}\ x\, , ad\ y)$. Then κ is a symmetric bilinear form on g, called the **Killing form**.

κ is also **associative**, i.e. $\kappa([x\, , y], z) = \kappa(x\, , [y\, , z])$.

It follows from $Tr([x\, , y], z) = Tr(x\, , [y\, , z])$ for endomorphisms $x\, , y\, , z$ of a finite dimensional vector space.[1]

Coxeter Graphs and Dynkin Diagrams:

If $\alpha\, , \beta$ are distinct positive roots, then we know that $< \alpha, \beta > \; < \beta, \alpha >$ = 0, 1, 2 or 3. Define the **Coxeter graph** of ℵ to be a graph having l vertices, the i^{th} joined to the $j^{th}(i \neq j)$ by $< \alpha_i\, , \alpha_j >< \alpha_j, \alpha_i >$ edges.

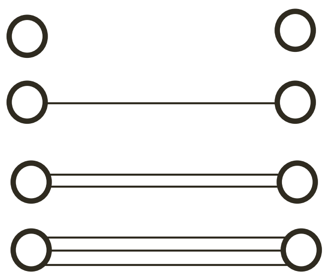
Examples[1]:

$A_1 \times A_1$

A_2

B_2

G_2



The coxeter graph determines the numbers $< \alpha_i\, , \alpha_j >$ in case all roots have equal lengths, since then $< \alpha_i\, , \alpha_j > \; = \; < \alpha_j\, , \alpha_i >$. In case more than one root length occurs(Ex. B_2 or G_2), the graph fails to tell us which of a pair of vertices should correspond to a short simple root, which to alone.

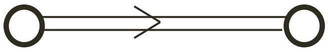
When a double or triple edge occurs in the coxeter graph, we can add an arrow pointing to the shorter of the two roots. This additional information gives us the cartan integers, and resulting figure **Dynkin Diagram**.[1]

To a cartan matrix is associated a Dynkin Diagram, consisting of vertices representing the simple roots and (oriented) lines connecting them. The Dynkin Diagram an algebra L of rank $\boldsymbol{r}$ is constructed using the following rules:

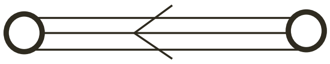
1. Draw $\boldsymbol{r}$ vertices, one for each simple root α_i.
2. Connect the vertices i and j with number of lines equal to max$\{|A_{ij}|, |A_{ji}|\}$, or equivalently to the product $A_{ij}A_{ji}$.
3. If $|A_{ij}| > |A_{ji}|$, then draw an arrow pointing towards j from i, i.e. from the biggest to the smallest root.[9]

Examples[1]:

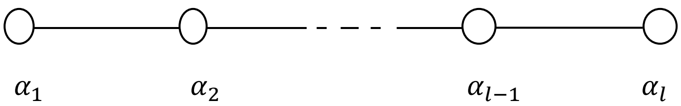
B_2



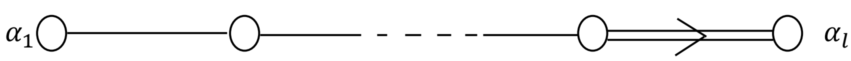
G_2



Dynkin Diagrams of Simple Lie Algebra[1]

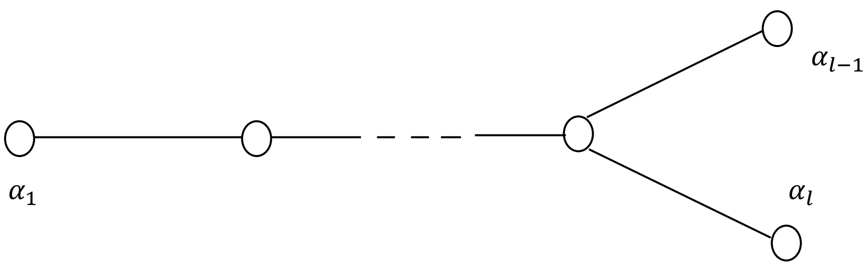


$\boldsymbol{A_l(l \leq 1)}$

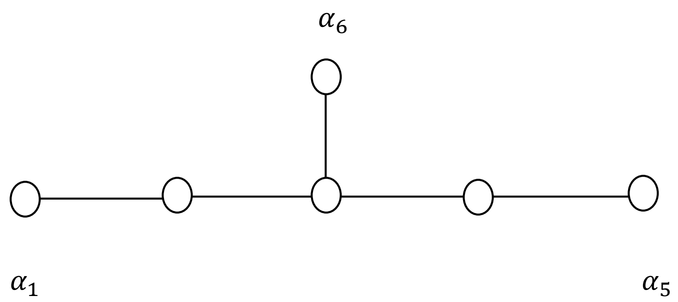


$\boldsymbol{B_l(l \geq 2)}$

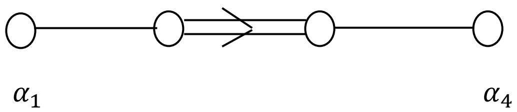
$\boldsymbol{C_l(l \geq 3)}$



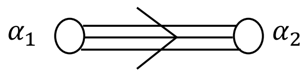
$\boldsymbol{D_l(l \geq 4)}$



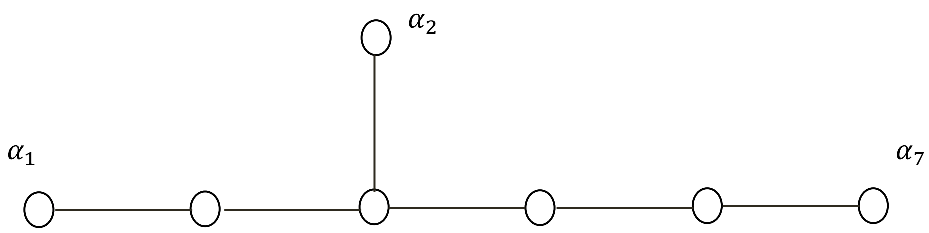
$\boldsymbol{E_6}$



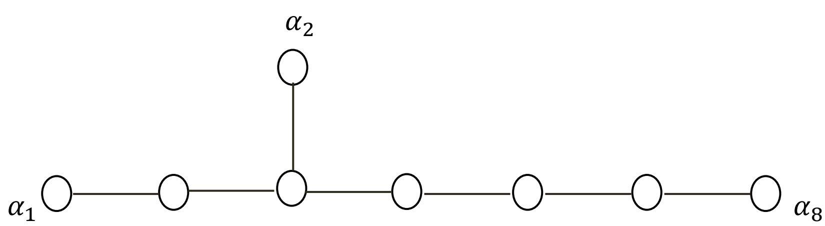
$\boldsymbol{F_4}$



$\boldsymbol{G_2}$



$\boldsymbol{E_7}$



$\boldsymbol{E_8}$

3

Lie Superalgebra

In this section we attempt to discuss the definition and classification of Kac-Moody superalgebras, their root systems and the Dynkin Diagrams associated with the algebras. But before defining Kac-Moody superalgebra, we need to define what Z_2 graded algebra and Lie superalgebra are.

Z_2 graded algebra

A Z_2 graded algebra is an algebra defined on a super vector space (or Z_2 graded vector space), i.e. a vector space

$$V = V_0 \oplus V_1$$

such that $V_i V_j \subseteq V_{i+j(mod\ 2)}$. The vectors in V_0 are called even and the vectors in V_1 are odd. Now we can define a function on the set of homogenous vectors of V,

$$V_0 \cup V_1 \rightarrow Z_2$$

$$a \longmapsto \overline{a}$$

where $\overline{a} = 0$ if a is even and $\overline{a} = 1$ if a is odd.[3]

Lie Superalgebra

Lie superalgebra is a (non-associative) Z_2 graded algebra, or superalgebra ($\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$) with a linear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, called the lie super-bracket or supercommutator, satisfies the two conditions (analogous of the lie algebra axioms, with grading):

- Super skew-symmetry:
  $[x , y] = - (-1)^{|x|.|y|}[y , x]$

- Super Jacobi Identity:
  $(-1)^{|z|.|x|}[x , [y , z]] + (-1)^{|x|.|y|}[y , [z , x]] + (-1)^{|y|.|z|}[z , [x , y]] = 0$

where x, y, z ∈ $\mathfrak{g}$. Here, |x| denotes the degree of x (either 0 or 1) and the degree of [x , y] is the sum of the degrees of x and y modulo 2.[2]

Modules for Lie Superalgebras

Let $\mathfrak{g}$ be a commutative superalgebra.

- A left $\mathfrak{g}$-module is a super vector space $\mathfrak{h}$ with a morphism $\mathfrak{g} \otimes \mathfrak{h} \longrightarrow \mathfrak{h}$, $a \otimes m \longrightarrow am$ of super vector spaces obeying the following rules:
  i. $a(x + y) = ax + ay$
  ii. $(a + b)x = ax + bx$
  iii. $(ab)x = a(bx)$

iv. $1x = x$

For all $a, b \in \mathfrak{g}$, and $x, y \in \mathfrak{h}$.

- A right $\mathfrak{g}$-module is defined in the same manner as that of left $\mathfrak{g}$-module.

NOTE: Since $\mathfrak{g}$ is commutative, a left $\mathfrak{g}$-module is also a right $\mathfrak{g}$-module if we define

$$m.a = (-1)^{|m||a|}a.m$$

For all $a \in \mathfrak{g}$, and $m \in \mathfrak{h}$.[5]

Representation

A representation of a Lie superalgebra $\mathfrak{g}$ is a super vector space V with a lie algebra homomorphism, i.e. a linear map with respect to the super-bracket $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$.

The adjoint representation of Lie superalgebra $\mathfrak{g}$ on its own vector space is defined by

$$\text{ad } x(y) = [x , y]$$

for all $x , y \in \mathfrak{g}$.[5]

The Killing Form

The **Killing form** of a Lie superalgebra $\mathfrak{g}$ can be defined as a bilinear form κ associated with a representation of $\mathfrak{g}$ as $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ such that

$$\kappa(x , y) = \text{str (ad , ad } y)$$

for all $x , y \in \mathfrak{g}$.

The bilinear form κ on $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ satisfies the following properties:

- κ is **consistent** :
  $\kappa(x , y) = 0$ for all $x \in \mathfrak{g}_{\bar{0}}$ and $y \in \mathfrak{g}_{\bar{1}}$.
- κ is **super-symmetric** :
  $\kappa(x , y) = (-1)^{|x|.|y|} \kappa(y , x)$
- κ is **invariant** :
  $\kappa([x , y] , z) = \kappa(x , [y , z])$ for all $x , y , z \in \mathfrak{g}$
- κ is **non-degenerate** :
  $\kappa(x , y) = 0$ for some $x \in \mathfrak{g}$ and for all $y \in \mathfrak{g}$, then $x = 0$.[3]

Root Systems

The roots of Lie superalgebra don't satisfy many of the properties related to the roots of the Simple Lie algebra. The roots of the Lie superalgebra can be categorised into three important classes given as follows:

- Roots α such that $(\alpha , \alpha) \neq 0$ and 2α is not a root. Such roots are said to be even roots.
- Roots α such that $(\alpha , \alpha) \neq 0$ and 2α is still a root. Such roots are called odd roots of non-zero length.

- Roots α such that $(\alpha, \alpha) = 0$. Such roots are said to be odd roots of zero length.

Simple Root System:

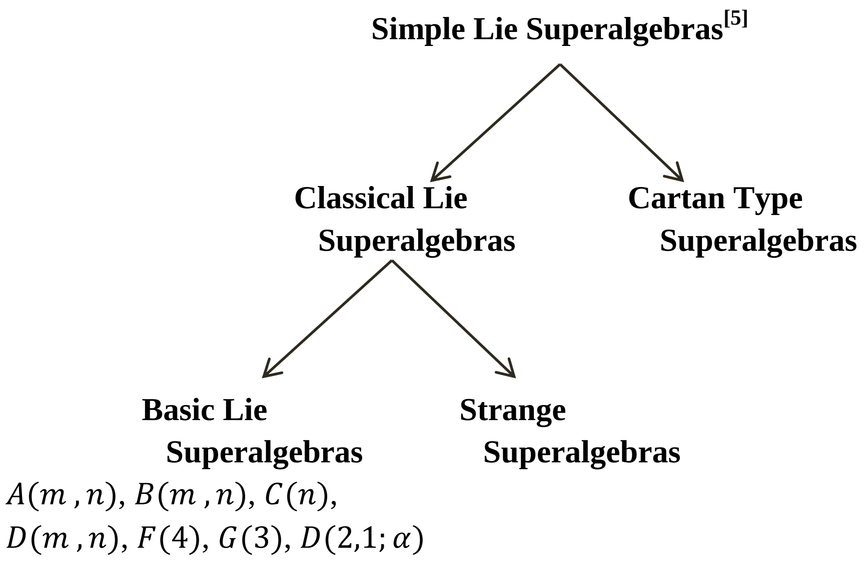
For each basic Lie superalgebra, there exists a simple root system for which the no. of odd roots is the smallest one. In this root system, the even simple roots are given by the even part of $\mathfrak{g}_{\bar{0}}$ and the odd simple root the lowest weight representation $\mathfrak{g}_{\bar{1}}$ of $\mathfrak{g}_{\bar{0}}$. Such simple root system is said to be *distinguished simple root system*, denoted by Δ_0.[5]

Simple Lie Superalgebra

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a non-abelian lie superalgebra. The lie superalgebra $\mathfrak{g}$ is called simple if it doesn't contain any of the nontrivial ideal. The Lie superalgebra $\mathfrak{g}$ is called semisimple if it doesn't contain any nontrivial solvable ideal.

A necessary condition for the lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ (with $\mathfrak{g}_{\bar{1}} \neq \emptyset$) to be simple is that the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is faithful and $\{\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}\} = \mathfrak{g}_{\bar{0}}$. If the representation of $\mathfrak{g}_{\bar{0}}$ is irreducible, then $\mathfrak{g}$ is simple.

1. **Classical Lie Superalgebra:** A simple lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is called classical if the representation of the even subalgebra $\mathfrak{g}_{\bar{0}}$ on the odd part $\mathfrak{g}_{\bar{1}}$ is completely reducible.
2. **Basic Lie superalgebra:** A classical lie superalgebra $\mathfrak{g}$ is called basic if there exists a non-degenerate invariant bilinear form on $\mathfrak{g}$. The classical lie superalgebra which aren't basic are called starnge.[5]



If X be any arbitrary matrix in Simple lie superalgebras that splits into homogeneous blocks according to the decomposition:

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A is $m \times m$, B is $m \times n$, C is $n \times m$, D is $n \times n$ matrices. We can define the supertrace of X to be:

$$\text{str}\,(X) = \text{tr}(A) - \text{tr}(D).$$ [4]

Dynkin Diagrams of Simple Lie Superalgebra

Let $\mathfrak{g}$ be a lie superalgebra of rank r and dimension n with Cartan subalgebra $\mathfrak{h}$. Let $\Delta_0 = (\alpha_1, \alpha_2, \dots, \alpha_r)$ be a simple root system of $\mathfrak{g}$ and $C^s = (c_{ij})$ be the corresponding symmetric cartan matrix, defined by $c_{ij} = (\alpha_i, \alpha_j)$. Now, we can associate to Δ_0 a Dynkin Diagram according to the following rules:

- We denote each simple even root as the white dot, each simple odd root of non-zero length ($c_{ii} \neq 0$) as black dot and each simple odd root of zero-length ($c_{ii} = 0$) as dot with cross mark.
- The ith and jth dots will be joined by n_{ij} lines where

$$n_{ij} = \frac{2|c_{ij}|}{\min(|c_{ii}|,|c_{ij}|)} \quad \text{if } c_{ii}.c_{ij} \neq 0$$

$$n_{ij} = \frac{2|c_{ij}|}{min_{c_{kk}\neq 0}|c_{kk}|} \quad \text{if } c_{ii} \neq 0 \text{ and } c_{jj} = 0$$

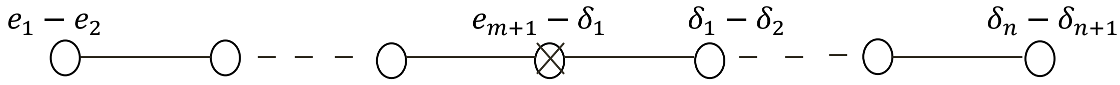
$$n_{ij} = |c_{ij}| \quad \text{if } c_{ii} = c_{ij} = 0$$

- Now add an arrow on the lines connecting the ith and jth dots when $n_{ij} > 1$, pointing from i to j if $c_{ii}.c_{jj} \neq 0$ and $|c_{ii}| \geq |c_{jj}|$ or if $c_{ii} = 0$, $c_{jj} \neq 0$, $|c_{jj}| < 2$, and pointing from j to i if $c_{ii} = 0$, $c_{jj} \neq 0$, $|c_{jj}| > 2$.[5]

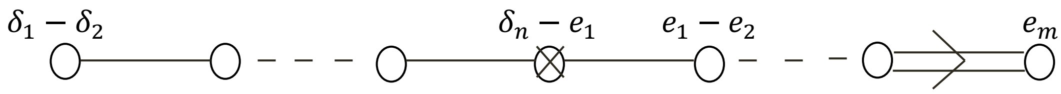
Distinguished Simple root systems of the **basic** Lie superalgebras[5]

Superalgebras $\mathfrak{g}$	Distinguished simple root systems Δ_0
$A(m,n)$	$\delta_1 - \delta_2, \dots, \delta_n - \delta_{n+1}, \delta_{n+1} - e_1, e_1 - e_2, \dots, e_m - e_{m+1}$
$B(m,n)$	$\delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, \delta_n - e_1, e_1 - e_2, \dots, e_{m-1} - e_m, e_m$
$B(0,n)$	$\delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, \delta_n$
$C(n)$	$e - \delta_1, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, 2\delta_m$
$D(m,n)$	$\delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, \delta_n - e_1, e_1 - e_2, \dots, e_{m-1} - e_m, e_{m-1} + e_m$
$F(4)$	$(e_1 + e_2 + e_3 + \delta)/_2, -e_1, e_1 - e_2, e_2 - e_3$
$G(3)$	$e_3 + \delta, e_1, e_2 - e_1$
$D(2,1;\alpha)$	$e_1 - e_2 - e_3, 2e_2, 2e_3$

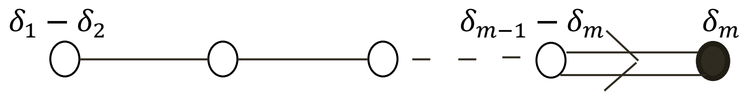
$A(m,n)$



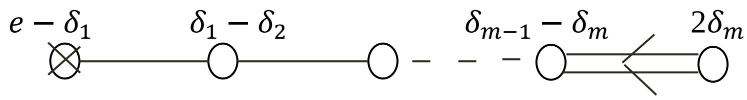
$B(m,n)$



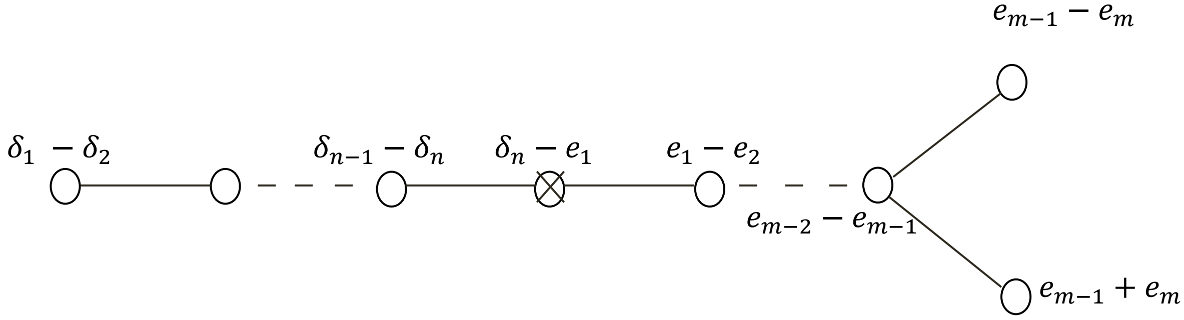
$B(0,n)$



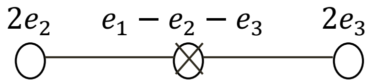
$C(n)$



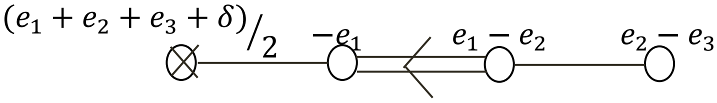
$D(m,n)$



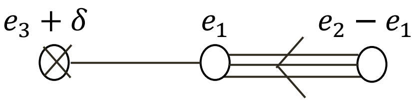
$\boldsymbol{D(2,1;\alpha)}$



$\boldsymbol{F(4)}$



$\boldsymbol{G(3)}$



4

Kac-Moody superalgebra

Basic Definition

Let $\mathfrak{g}$ be a Lie superalgebra with the respective even and odd parts $\mathfrak{g}_0$ and $\mathfrak{g}_1$, characterised by a **Cartan matrix** $C^S = (c_{ij})$. Let I be a finite subset $\{1, 2, 3, \ldots, r\}$ or a countable set in $\mathbb{Z}_+$ and S be a subset of I. The set S contains the elements of $\mathfrak{g}_1$, i.e. only the odd generators. The set S is supposed to consist of only one element in the *distinguished basis*.

Let $\mathfrak{h}$ be a vector space with a non-degenerate symmetric bilinear form (,) containing non-trivial vectors h_i indexed by the set I.

By setting $a_{ij} = (h_i, h_j)$ and assume that

- $a_{ij} = a_{ji}$
- $a_{ij} \leq 0$ if $i \neq j$
- $2\frac{a_{ij}}{a_{ii}} \in \mathbb{Z}$ if $a_{ii} \geq 0$
- $\frac{a_{ij}}{a_{ii}} \in \mathbb{Z}$ if $a_{ii} \geq 0$ and $i \in S$.

If [,] – operation stands for the graded product defined by $[x, y] = -(-1)^{|x|.|y|}[y, x]$ and

denote the adjoint operation by ad x as ad $x(y) = [x, y]$. The algebra $\mathfrak{g}$ can be constructed by the generators e_i, f_i, $i \in I$ satisfying the following axioms:

- $[e_i, e_j] = \delta_{ij} h_i$, $[h_i, h_j] = 0$
- $[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = - a_{ij} f_j$
- $|h_i| = 0$, if $|e_i| = |f_i| = 0$, $i \notin S$
- $|h_i| = 0$, if $|e_i| = |f_i| = 1$, $i \in S$

By the above axioms, $\mathfrak{g} = (C, S, \mathfrak{h})$ is said to be the **generalised Kac-Moody superalgebra**.[2]

Types of Kac-Moody superalgebra

Given a class of symmetrizable generalised cartan matrix (GCM) and their associated algebras, we can consider three types of superalgebras:

(i) **Simple Lie superalgebras:** These are the finite dimensional Kac-Moody superalgebras.
(ii) **Affine Kac-Moody superalgebras:** These are the sets of infinite dimensional Lie superalgebras and are of two types:
- Untwisted affine Kac-Moody superalgebras that corresponds to identity automorphisms of corresponding simple lie superalgebras,

- Twisted affine Kac-Moody superalgebras corresponding to outer automorphisms of order 2 or 4 of the corresponding simple lie superalgebras.

(iii) **Hyperbolic Kac-Moody superalgebras:** This is a subclass of indefinite Kac-Moody superalgebra. Every leading principal submatrix of the generalised cartan matrix of the algebra decomposes into constituents of finite or affine type or equivalently, deletion of a vertex of the Dynkin Diagram of finite or affine type.[4]

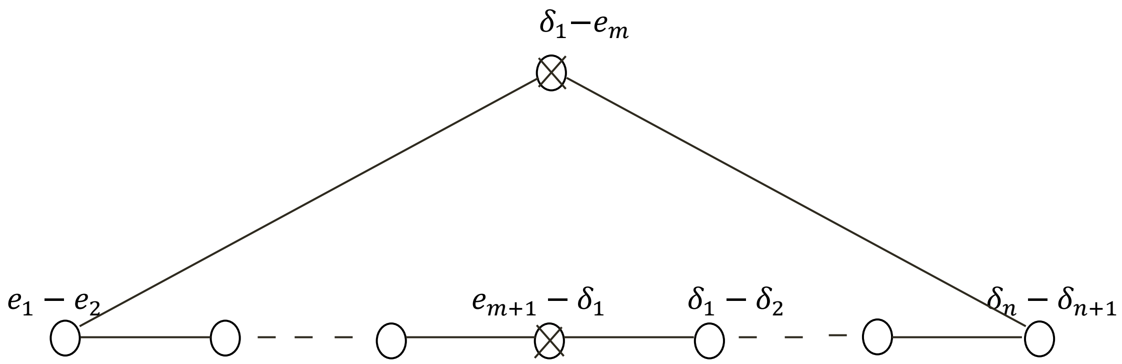
Extended Dynkin Diagram

Let $\Delta_0 = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ be the simple root system of a simple lie superalgebra $\mathfrak{g}$ of rank r. Let $-\alpha_0$ be the highest root with respect to Δ_0, i.e. the unique root of the maximal height. We can define the extended Dynkin diagram by simply adding an extra dot to the root $+\alpha_0$. This diagram is especially important in the sense that it allows us to determine all regular subalgebras of $\mathfrak{g}$.

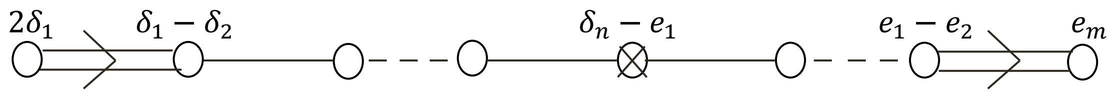
However, it is the Dynkin diagram of the so-called **affinization** $\tilde{\mathfrak{g}}$ of the lie superalgebra $\mathfrak{g}$.[5]

Dynkin Diagrams of Affine Kac-Moody Superalgebra[5]

$\boldsymbol{A^{(1)}(m, n)}$

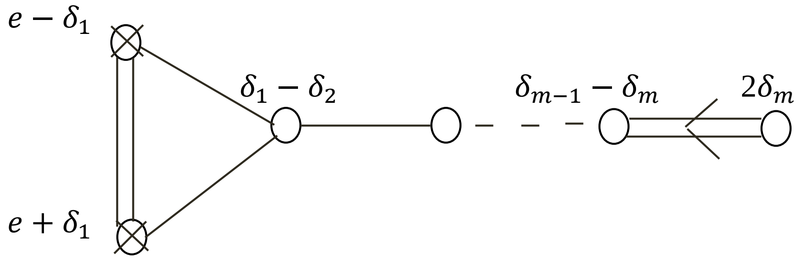


$\boldsymbol{B^{(1)}(m, n)}$

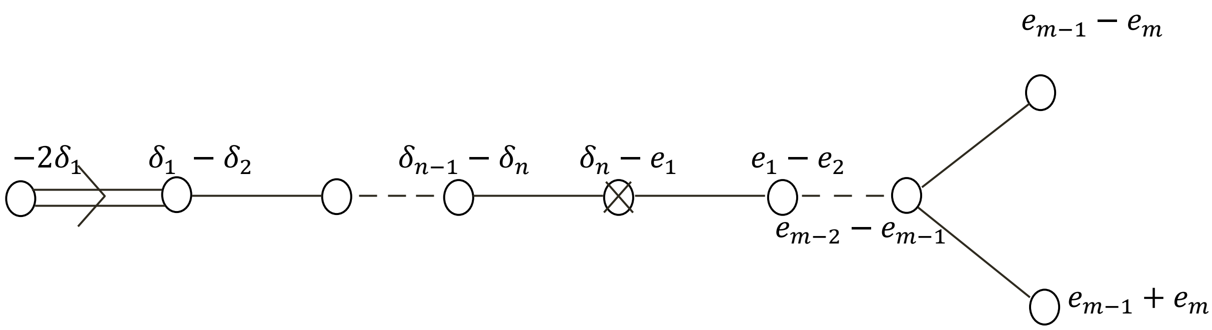


$\boldsymbol{B^{(1)}(0, n)}$

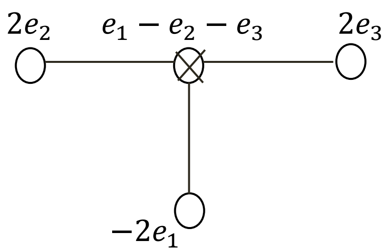
$\boldsymbol{C^{(1)}(n)}$



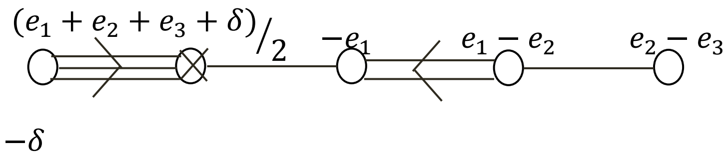
$\boldsymbol{D^{(1)}(m\,,n)}$



$\boldsymbol{D^{(1)}(2,1;\alpha)}$



$F^{(1)}(4)$



$G^{(1)}(3)$

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