Compactness and Convergence in the Space Of Analytic Functions

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By

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DECLARATION

*This is to certify that the thesis entitled "**Compactness and Convergence in the Space Of Analytic Functions**" submitted by Basundhara Goswami as one year project work for the requirement for the award of Master of Science in Mathematics, in the Department of Mathematics, National Institute of Technology, Rourkela is a record of authentic work carried out by her under my supervision and guidance.*

To the best of my knowledge, the matter embodied in this project work has not been submitted to any where for the award of any degree.

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Abstract:

Here, our main aim is to study compactness and convergence in the space of continuous functions defined on a fixed region G subset of complex plane. First, we will define an appropriate metric in which we will study compactness and convergence. For defining compactness we will introduce the concept of normal set and then we will prove that normal closure is compact. Subsequently, we will prove a variant of a famous theorem i.e. Arzela-Ascoli theorem. Then we will divert our attention in studying compactness and convergence in the space of analytic functions defined on a fixed region G. The analytic functions are having an exceptional importance as this class is sufficiently large. It includes the majority of functions which are encountered in the principal problems of mathematics and applications to science and technology. Here, in our discussion we visualize these analytic functions as points in a metric space. Also, here we have proved Hurwitz and Montel theorem. In the last section of this dissertation we have studied the space of meromorphic functions defined in a fixed region G.

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Chapter-1: Space of continuous functions

A mapping $f : X \to Y$ is said to be continuous at a point if G is an open subset of Y, then $f^{-1}(G)$ should be open in X .In terms of (ε, δ)-criterion,let $X = (X, d)$ and $Y = (Y, \tilde{d})$ be two metric spaces. Then a mapping $f : X \to Y$ is said to be continuous at a point $x_0 \in X$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\tilde{d}(f(x), f(x_0)) < \varepsilon \quad \forall x \in X$ satisfying $d(x, x_0) < \delta$.

Suppose, G be an open set in C and (Ω, d) is a complete metric space. Then C (G, Ω) is the set all continuous functions from G to Ω which will be never be empty as it contains the constant functions. If Ω is C, then apart from constant functions it also contain all analytic function. And if it is $C \bigcup \{\infty\}$, then it contains meromorphic functions which will be our concern of latter discussion.

<u>Proposition</u>: *(see [2]) If G is open in C, there is a sequence $\{K_n\}$ of compact subsets of G such that*

$$G = \bigcup_{n=1}^{\infty} K_n .$$

Moreover, the sets K_n can be chosen to satisfy the following conditions:

(a) $K_n \subset \operatorname{int} K_{n+1}$

(b) $K \subset G$ *and* K *compact implies* $K \subset K_n$ *for some n;*

(c) Every component of $C_\infty - K_n$ *contains a component of* $C_\infty - G$

Proof:

Consider

$$K_n = \{z : |z| \le n\} \cap \{z : d(z, C - G) \ge \frac{1}{n}\} \text{ for each positive } n .$$

Now $[\{z:|z|\le n\}\cap\{z:d(z,C-G)\ge\frac{1}{n}\}]\subset\{z:|z|\le n\}$ *.So ,* K_n *is bounded. Also, it is the intersection of two closed sets and we know that every closed and bounded subsets of* R^2 *is compact. So, it's compact. We have drawn the following figure to explain this theorem properly.*

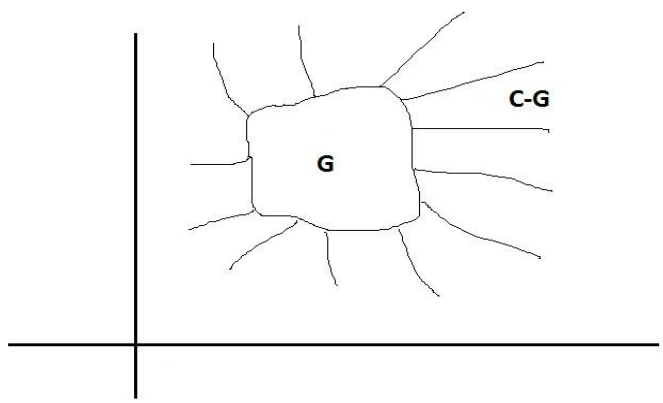


Fig - 1

G *is open in* C *,so* $C-G$ *is closed and so* $C-G=\overline{C-G}$ *. We know* $x\in\overline{A}\Leftrightarrow d(\{x\},A)=0$ **.** *So, we can draw the following figure as follows.*

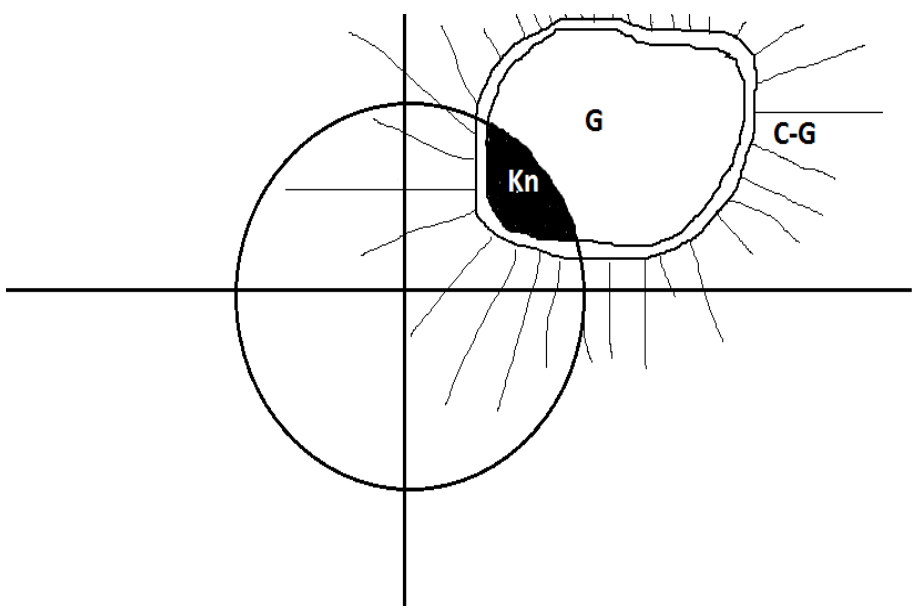


Fig-2

From the above figure, we can say that, $G = \bigcup_{n=1}^{\infty} K_n$. *Now we are required to prove the above mentioned (a), (b), (c).*

(a) $K_{n+1} = \{z : |z| \le n+1\} \cap \{z : d(z, C-G) \ge \frac{1}{n+1}\}$;

$\text{int } K_{n+1} = \{z : |z| < n+1\} \cap \{z : d\,(z, C-G) > \frac{1}{n+1}\}$. *This implies* $K_n \subset \text{int } K_{n+1}$.

(b) *Given,* $K \subset G$ *is compact where* $G = \bigcup_{n=1}^{\infty} K_n$.*We can also get* $G = \bigcup_{n=1}^{\infty} \text{int } K_n$. *Hence,* $K \subset \bigcup_{n=1}^{\infty} \text{int } K_n$.*Since* K *is compact,so we can say* $\{\text{int } K_n\}$ *forms an open cover of* K *which has also a finite subcover.*

Here $K_n \subset \text{int } \mathrm{K}_{\mathrm{n+1}} \subset \text{int } \mathrm{K}_{\mathrm{N}} \subset K_N$. *So,* $K \subset K_N$ for some N.

(c) $\mathrm{C}_\infty - G$ is a subset of $C_\infty - K_n$.*The unbounded component of* $C_\infty - K_n$ *will contain* ∞ *and hence* $C_\infty - G$ *will contain the same. The unbounded component of* $C_\infty - K_n$ *also contains* $\{z : |z| > n\}$.*Assume, D be a bounded component of* $C_\infty - K_n$ *where* z *be a point* D *such that* $d(z, C-G) < \frac{1}{n}$; *So, there exist a point* w in $C - G$ with $|w - z| < \frac{1}{n}$. *This implies* $z \in B(w; \frac{1}{n}) \subset C_\infty - K_n$.

We have got $z \in D \subset C_\infty - K_n$ *and* $B(w; \frac{1}{n}) \subset C_\infty - K_n$.*So,* $B(w; \frac{1}{n}) \subset D$.*If* D_1 *is the component of* $C_\infty - G$ *that contains* w. *It follows that* $D_1 \subset D$.

Let $G = \bigcup_{n=1}^{\infty} K_n$, *where each* K_n *is compact and* $K_n \subset \text{int } K_{n+1}$, *then define*

$$\varsigma_n(f, g) = \sup\{d(f(z), g(z) : z \in K_n)\}$$

for all functions f *and* g *in* $C(G,\Omega)$. *Also define*

$$\varsigma\ (f,g)=\sum_{n=1}^{\infty}\left(\frac{1}{2}\ \right)^{n}\frac{\varsigma_n(f,g)}{1+\varsigma_n(f,g)}.$$

Since, we know $\frac{\varsigma_n(f,g)}{1+\varsigma_n(f,g)}\leq 1$, for $\mathrm{n}=1,\ldots,\infty$. *So,*

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\ \right)^{n}\frac{\varsigma_n(f,g)}{1+\varsigma_n(f,g)}\leq\sum_{\mathrm{n}=1}^{\infty}\left(\frac{1}{2}\right)^{n}.$$

But $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n}$ *is a geometric series, so it will converge. So, by comparison test*

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n}\frac{\varsigma_n(f,g)}{1+\varsigma_n(f,g)},$$

it will also converge. Here, $\varsigma(f,g)$ *is a metric.*

***Proposition*:** *(see [2])* $(C(G,\Omega),\varsigma)$ *is a metric space.*

Proof:

(a) $\varsigma\ (f,g)=\sum_{n=1}^{\infty}\left(\frac{1}{2}\ \right)^{n}\frac{\varsigma_n(f,g)}{1+\varsigma_n(f,g)}\geq 0.$

(b) $\varsigma(f,g)=0$ *.This implies*

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\ \right)^{n}\frac{\varsigma_n(f,g)}{1+\varsigma_n(f,g)}=0$$

$$i.e.\quad\left(\frac{1}{2}\right)^{n}\frac{\varsigma_n(f,g)}{1+\varsigma_n(f,g)}\ =0\ \textit{for each}\ n.$$

It implies $\sup\{d(f(z),g(z)):z\in K_n\}=0$.*So* $d(f(z),g(z))=0$ *and hence f=g for all* $z\in K_n$.

(c) $\varsigma(f,g)=\varsigma(g,f)$

(d) Here, we need to prove the triangle inequality. Let $0 < \alpha < \beta$.

This implies $\alpha + \alpha\beta < \beta + \alpha\beta$ *and so* $\dfrac{\alpha}{1+\alpha} < \dfrac{\beta}{1+\beta}$.

Now, $\varsigma_n(f,g) = \sup\{d(f(z),g(z)) : z \in K_n\}$ *which will be less than equal to* $\sup\{d(f(z),p(z))\} + \sup\{d(p(z),g(z))\}$. *So, we can write*

$$\frac{\sup\{d\ (f(z),g(z))\}}{1+\sup\{d\ (f(z),g(z))\}} \le \frac{\sup\{d(f(z),p(z))\} + \sup\{d(p(z),g(z))\}}{1+\sup\{d(f(z),p(z))\} + \sup\{d(p(z),g(z))\}}$$

$$\le \frac{\sup\{d(f(z),p(z))\}}{1+\sup\{d(f(z),p(z))\}} + \frac{\sup\{d(p(z),g(z))\}}{1+\sup\{d(p(z),g(z))\}}$$

and hence finally we get

$$\varsigma(f,g) \le \varsigma(f,p) + \varsigma(p,z).$$

Lemma: *(see [2])*

Let the metric ς *be defined as before. If* $\varepsilon > 0$ *is given then there is a* $\delta > 0$ *and a compact set* $K \subset G$ *such that for* f and g in $C(G,\Omega)$, $\sup\{d(f(z),g(z)) : z \in K\} < \delta$ *will imply* $\rho(f,g) < \varepsilon$.

Conversely, if $\delta > 0$ *and a compact set* K *are given, there is an* $\varepsilon > 0$ *such that for* f and g in $C(G,\Omega)$, $\rho(f,g) < \varepsilon$, *implies* $\sup\{d(f(z),g(z)) : z \in K\} < \delta$

Proof:

Choose $\varepsilon > 0$ *be fixed and coonsider* p *be a positive integer such that* $\sum_{n=p+1}^{\infty}\left(\frac{1}{2}\right)^n < \frac{\varepsilon}{2}$. *Assume* $K = K_p$.

Take $\delta > 0$ *such that* $0 \le t < \delta$ *gives* $\dfrac{t}{1+t} < \dfrac{\varepsilon}{2}$. *Let* f and g *be functions in* $C(G,\Omega)$ *which satisfy* $\sup\{d(f(z),g(z)) : z \in K\} < \delta$. *For* $1 \le n \le p$, *since* $K_n \subset K_p = K$,

so $\sup\{d(f(z),g(z)) : z \in K_n\} < \delta$. *That means,* $\varsigma_n(f,g) < \delta$, *for* $1 \le n \le p$. *So,* $\dfrac{\varsigma_n(f,g)}{1+\varsigma_n(f,g)} < \dfrac{\varepsilon}{2}$ for $1 \le n \le p$. *Therefore*

$$\varsigma\ (f,g)=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n}\frac{\varsigma_n(f,g)}{1+\varsigma_n(f,g)}.$$

This expression can also be written as

$$\varsigma(f,g)=\sum_{n=1}^{p}\left(\frac{1}{2}\right)^{n}\frac{\varsigma_n(f,g)}{1+\varsigma_n(f,g)}+\sum_{n=p+1}^{\infty}\left(\frac{1}{2}\right)^{n}\frac{\varsigma_n(f,g)}{1+\varsigma_n(f,g)}$$

and it is less than $\frac{\varepsilon}{2}\sum_{n=1}^{p}\left(\frac{1}{2}\right)^{n}+\sum_{n=p+1}^{\infty}\left(\frac{1}{2}\right)^{n}$ *and it is less than* ε *.Thus one part of the lemma is proved.*

<u>*Converse part:*</u> *Let* K *and* δ *are given. Since* $G=\bigcup_{n=1}^{\infty}K_n=\bigcup_{n=1}^{\infty}\operatorname{int}K_n$ *and* K *is compact, so, there exists an integer* $r\geq 1$ *such that* $K\subset K_r$ *. So,*

$$\sup\{d(f(z),g(z)):z\in K_r\}\geq\sup\{d(f(z),g(z)):z\in K\}.$$

That shows $\varsigma_r(f,g)$ *is greater than equal to* $\sup\{d(f(z),g(z)):z\in K\}$ *. Choose* $\varepsilon>0$ *appropriately so that* $0\leq p\leq 2^r\varepsilon$ *implies* $\frac{p}{1-p}<\delta$ *. For any number* $\frac{t}{1+t}<2^r\varepsilon$ *implies* $\frac{t/1+t}{1-t/1+t}<\delta$ *i.e.* $t<\delta$ *. Let* $\varsigma(f,g)<\varepsilon$ *,then it will automatically imply* $\sup\{d(f(z),g(z)):z\in K\}<\delta$ *.*

Proposition: *(see [2])*

(a) A set $O\subset(C(G,\Omega),\varsigma)$ *is open iff for each* f in O *there is a compact set* K and $\delta>0$ *such that* $O\supset\{g:d(f(z),g(z))<\delta,z\in K\}$.

(b) A sequence $\{f_n\}$ *in* $(C(G,\Omega),\varsigma)$ *converges* f *iff* $\{f_n\}$ *converges to* f *uniformly on all compact subsets of* G *.*

Proof*: Let* O *is open and let* $f\in O$ *.Then for some* $\varepsilon>0$ *,* $\{g:\rho(f,g)<\varepsilon\}\subset O$ *.So, by using the preceding lemma we can write* $[\sup\{d(f(z),g(z)):z\in K\}<\delta]\subset O$*. Hence the result follows. Conversely let for each* f *in* O *there is a compact set* K *and a* $\delta>0$ *such that*

$$O \supset \{g : d(f(z), g(z)) < \delta, z \in K\}.$$

Since for all z *it happens so* $\sup\{g : d(f(z), g(z)) < \delta, z \in K\} \subset O$.

So, by preceding lemma there is an $\varepsilon > 0$ *such that for* f *and* g *in* $C(G,\Omega)$, $\rho(f, g) < \varepsilon$. *So,* O *is open.*

(b) Let a sequence $\{f_n\}$ *in* $(C(G,\Omega), \rho)$ *converges to* f. *Then for* $\varepsilon > 0$ *there exist* $N > 0$ *such that* $\rho(f_n, f) < \varepsilon$ *for all* $n \geq N$. *Let* K *be a compact subset of* G. *It will imply that there is a positive integer* n *such that* $K \subset K_n$. *Now from* $\rho(f_n, f) < \varepsilon$, we can get

$$\frac{\rho_n(f_n, f)}{1 + \rho_n(f_n, f)} < 2^n \varepsilon \quad (= \eta \text{ say}).$$

Then $\rho_n(f_n, f) < \dfrac{\eta}{1-\eta}$. Hence, $\sup\{d(f_n(z), f(z)) : z \in K\} < \dfrac{\eta}{1-\eta}$. *So,* $\{f_n\}$ *converges to* f *uniformly on all compact subsets of* G.

Conversely, let K *be a arbitrary compact set. Let* $\{f_n\}$ *converges to* f *uniformly on* K. *Then* $\rho(f_n(z), f(z)) \to 0$ as n *tend to infinity and for all* z *in* K. *It will imply* $\rho_n(f_n(z), f(z)) \to 0$ as n *tend to infinity* .So, we can write $\sup\{d(f_n(z), f(z)), z \in K\}$ *tend to zero* as n *tend to infinity*. *Also we can say, for given* $\delta > 0$, $\sup\{d(f_n(z), f(z)), z \in K\} < \delta$ *for all* $n \geq N$. *Hence,* there is as an ε such that $\rho(f_n, f) < \varepsilon$.

Proposition: *(see [2])* $C(G,\Omega)$ *is a complete metric space.*

Proof*: Let* $\{f_m\}$ *be a Cauchy sequence in* $C(G,\Omega)$. *So,* $\rho(f_m, f_p)$ *tend to zero as* m, p *tend to infinity. Then* $\sup\{d(f_m(z), f_p(z)) : z \in K_n\}$ *tend to zero. Let* $K \subset G$ be compact .*Then* $K \subset K_n$ *for* some n and so $\sup\{d(f_m(z), f_p(z)) : z \in K\}$ *also tend to zero as* m, p *tend to infinity..Hence* $\{f_m\}$ *is a Cauchy sequence in* K .*For every* $\delta > 0$, *there is an integer* n *such that* $\sup\{d(f_m(z), f_p(z)) : z \in K\} < \delta$ *for* $m, p \geq n$. *But* $\{f_m(z)\}$ *is a sequence in* Ω *which is complete. So,* $f_m(z)$ *converges to* $f(z)$ *and* $f(z)$ *also belongs to* Ω. *Hence,* $\rho(f_m, f)$ *tend to zero as* $m \to \infty$. *So for an* $\eta > 0$ *there exist* $N > 0$ *such that* $\sup\{d(f_m(z), f(z)) : z \in K\} < \eta$ *for all* $m \geq N$. *So,* f_m *converges to* f *uniformly on all compact set.Hence* f *is continuous and so* f *belongs to* $C(G,\Omega)$ **.**

Proposition: *(see [2])* *A set $\Im \subset C(G,\Omega)$ is normal iff its closure is compact.*

Proof: *First let $\Im \subset C(G,\Omega)$ is normal .Then by definition of normality , each sequence in $\Im$ has a subsequence which converges to a function f in $C(G,\Omega)$.Our claim is that $\overline{\Im}$ is compact. Since $\overline{\Im}$ is the set containing the set $\Im$ as well as all the limit points of $\Im$ .So, each converging function f of the subsequence of each sequence in $\Im$ is in $\overline{\Im}$ and hence the result follows.*

Now, let us assume $\overline{\Im}$ is compact. Let $\{f_n\}$ is sequence in $\overline{\Im}$ .So, it has a convergent subsequence $\{f_{n_k}\}$ and suppose $f_{n_k} \to f$, $f \in \overline{\Im}$.The set $\Im$ is a subset of $C(G,\Omega)$ and since $C(G,\Omega)$ is complete, so $\overline{\Im}$ is also contained in $C(G,\Omega)$.Hence $f_{n_k} \to f$, $f \in \overline{\Im} \subset C(G,\Omega)$,i.e. f is continuous and so it is normal.

Proposition*: (see [2])* *A set $\Im \subset C(G,\Omega)$ is normal iff for every compact set $K \subset G$ and $\delta > 0$ there are functions $f_1, f_2,..., f_n$ in $\Im$ such that for f in $\Im$ there is at least one $k, 1 \le k \le n$, with*

$$\sup\{d(f(z), f_k(z)) : z \in K\} < \delta .$$

Proof: *Suppose $\Im \subset C(G,\Omega)$ is normal.Let $K \subset G$ be compact and $\delta > 0$ be given. $\Im$ is normal so, $\overline{\Im}$ is compact and so it is totally bounded. $\Im \subset \overline{\Im} \subset \bigcup_{k=1}^{n}\{f : \varsigma(f, f_k) < \varepsilon\}$. Now $\varsigma(f, f_k) < \varepsilon$ it will imply $\dfrac{\varsigma_n(f, f_k)}{1+\varsigma_n(f, f_k)} < 2^n\varepsilon = \delta(say)$.So,we can say $\sup\{d(f(z), f_k(z)) : z \in K\} < \delta$.*

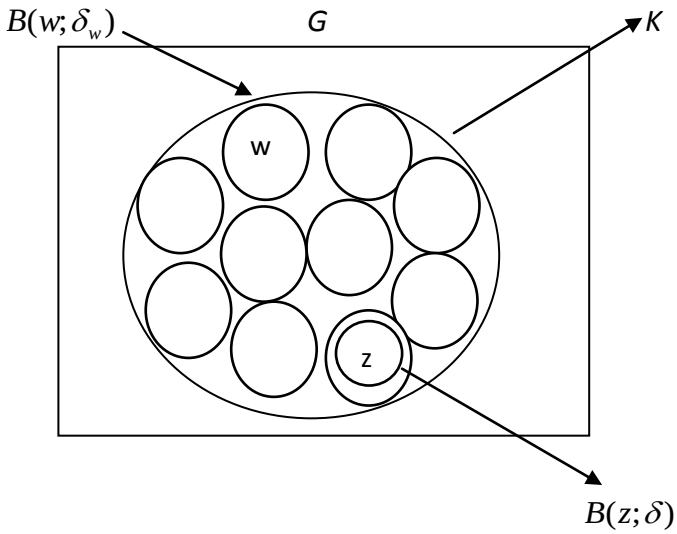
For the converse, suppose $\Im$ has the above stated property.

Now from $\sup\{d(f(z), f_k(z)) : z \in K\} < \delta$ we get $\rho(f, f_k) < \varepsilon$ .So, it is totally bounded. So, $\overline{\Im}$ *is compact. Hence $\Im$ is normal.*

Proposition*: (see [2])* *$(\prod_{n=1}^{\infty} X_n, d)$ where d is defined by $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \frac{d_n(x_n, y_n)}{1+d_n(x_n, y_n)}$ is a metric space.If $\xi^k = \{x_n^k\}_{n=1}^{\infty}$ is in X= $\prod_{n=1}^{\infty} X_n$ then $\xi^k \to \xi = \{x_n\}$ iff $x_n^k \to x_n$ for each n .Also,if each (X_n, d_n) is compact then X is compact.*

Proposition*: (see [2])* *Suppose $\Im \subset C(G,\Omega)$ is equicontinuous at each point of G ; then $\Im$ is equicontinuous over each compact subset of G .*

Proof: *Fix a* $\varepsilon > 0$ *and let* K *be a compact subset of* G *.Using the equicontinuity concept, for any* $w \in K$ *there exist* $\delta_w > 0$ *such that whenever* $|w - w'| < \delta_w$ *then* $|f(w) - f(w')| < \frac{\varepsilon}{2}$ *for all* f *in* $\Im$ *i.e* $w' \in B(w;\delta_w)$ *forming an open cover of* K *.*



We know if K *is compact and it has a cover then,it has Lebesgue no,that means for each* $z \in K$ *there exist* $\delta > 0$ *such that* $B(z;\delta) \subset B(w;\delta_w)$ *.If* z *and* z' *are in* K *with* $|z - z'| < \delta$ *,then there will be a* $w \in K$ *such that* $z' \in B(z;\delta) \subset B(w;\delta_w)$ *.So, we have got* $|f(z) - f(w)| < \frac{\varepsilon}{2}$ *and*

$|f(z') - f(w)| < \frac{\varepsilon}{2}$ *.Finally we got* $|f(z') - f(z)| < \varepsilon$ *and hence* $\Im$ *is equicontinuous over* K *.*

Arzela - Ascoli Theorem: *(see [2]) A set* $\Im \subset C(G,\Omega)$ *is normal iff the following two conditions are satisfied:*

(a) for each $z \text{ in } \mathrm{G}$ *,* $\{f(z) : f \in \Im\}$ *has compact closure in* Ω *;*

(b) $\Im$ *is equicontinuous at each point of* G *.*

Proof: *First suppose that* $\Im \subset C(G,\Omega)$ *is normal. Let* $A_z = \{f(z) : f \in \Im\} \subset \Omega$ *.We need to show that* $\overline{A}_z$ *is compact.Define a mapping* $T : C(G,\Omega) \to \Omega$ *by the rule* $T(f) = f(z)$ *i.e*

$T(\Im) = A_z$ *.Since* $\Im$ *is normal, so* $\overline{\Im}$ *is compact .Let {* f_n *} be a sequence in* $\overline{\Im}$ *.Let* $f_n \to f_0$ *i.e.*

$\varsigma(f_n, f_0) \to 0$ *as* $n \to \infty$ *.We are having*

$$\varsigma(f_n, f_0) = \sum \left(\frac{1}{2}\right)^n \frac{\varsigma_n(f_n, f_o)}{1+\varsigma_n(f_n, f_0)}$$

and so, $\varsigma_n(f_n, f_0) \to 0$ *as* $n \to \infty$*.*

It means $\sup\{d(f_n(z), f_0(z)) : z \in K\} \to 0$ *as* $n \to \infty$ *and hence*

$$\{d(f_n(z), f_0(z)) : z \in K\} \to 0 \text{ as } n \to \infty$$

So, T *is continuous .Hence* $\overline{A}_z$ *is compact.*

Secondly, we need to show $\Im$ *is equicontinuous at each point of* G *.Let* $z_0 \in G$ *.We are required to show that for every* $\varepsilon > 0$ *there exist* $\delta > 0$ *such that* $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$ *for all*

$f \in \Im$*. That means when* $z \in B(z_0;\delta)$*, then* $d(f(z), f(z_0)) < \varepsilon$ *.*

Let $K = \overline{B}(z_0; R)$ *where R>0. Then K is compact. Using one of the previous proposition for* $\varepsilon > 0$ *there are functions* $f_1, f_2, \ldots, f_n$ *in* $\Im$ *such that for* f *in* $\Im$ *there is at least one* $\mathrm{k}, 1 \le \mathrm{k} \le \mathrm{n}$, *with*

$$\sup\{d(f(z), f_k(z)) : z \in K\} < \frac{\varepsilon}{3}.$$

So we can write $d(f(z), f_k(z)) < \frac{\varepsilon}{3}$*.Also each* f_k *is continuous, so there is a* δ, $0 < \delta < \mathrm{R}$ *such that whenever* $|z - z_0| < \delta$, $d(f_k(z), f_k(z_0)) < \frac{\varepsilon}{3}$ *for* $1 \le k \le n$*.*

$d(f(z), f(z_0)) \le d(f(z), f_k(z)) + d(f_k(z), f_k(z_0)) + d(f_k(z_0), f(z_0))$ *which is less than*

$\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$ *i .e. less than* ε *.Hence proved. Note that for proof of converse part one may refer [1].*

Chapter-2: Spaces of analytic functions

A function which is differentiable at each point of a domain D is called analytic in that domain. Here, we are putting a metric on the set of all analytic functions on a fixed region and then our main aim is to discuss compactness and convergence with respect to this metric. Here, our discussion will be the analytic functions defined from G to Ω .Let it be denoted

by $H(G,\Omega)$. We are considering here $H(G)$ as a subset of $C(G,C)$ where $H(G)$ is the collection of analytic functions on G .

***Theorem**: (see [2]) If $\{f_n\}$ is a sequence in $H(G)$ and f belongs to $C(G,C)$ such that $f_n \to f$ then f*

is analytic and $f^k{}_n \to f^k$ for each integer $k \geq 1$.

Proof: We will use the application of Morera's Theorem for the first part of the proof the theorem. Let G be an open set in C and T is a tringle inside G . Since T is compact and $f_n \to f$. So, f_n converges to f uniformly on T . Hence $\int_T f = \lim \int_T f_n = 0$ So, f is analytic in G ,since each f_n is analytic.

To prove the second part, let $D = \overline{B}(a\,;r)$ be a closed ball contained in G . Then there exist a number $R > r$ such that $\overline{B}(a\,;R) \subset G$ .Consider a circle γ which is $|z-a| = R$.

Using Cauchy's Integral formula, we will get,

$$f_n{}^k(z) - f^k(z) = \frac{k!}{2\Pi i}\int_\gamma \frac{f_n(w) - f(w)}{(w-z)^{k+1}}dw \text{ for all } z \text{ in } D.$$

Then,

$$\left|f_n^k(z) - f^k(z)\right| \leq \frac{k!}{2\Pi}\int_\gamma \frac{\left|f_n(w) - f(w)\right|}{\left|(w-z)^{k+1}\right|}\left|dw\right|.$$

Take $M_n = \sup\{\left|f_n(w) - f(w)\right| : \left|w-a\right| = R\}$ and we can write

$$\left|(w-z)^{k+1}\right| = \left|\{(w-a)-(z-a)\}^{k+1}\right| = \left|(R-r)^{k+1}\right|.$$

So, $\left|f_n^k(z) - f^k(z)\right| \le \dfrac{k!M_nR}{(R-r)^{k+1}}$ *for* $|z-a| \le R$. *Since* $f_n \to f$. *So* $\lim M_n = 0$. *Hence,* $f^k{}_n \to f^k$ *uniformly on* $\overline{B}(a;R)$. *For an arbitrary compact subset* K *in* G *and for* $0 < r < d(k, \partial G)$ *we are having some* $a_1, a_2, \ldots, a_n$ *in* K *such that* $K \subset \bigcup_{j=1}^{n} B(a_j;r)$. *Since* $f^k{}_n \to f^k$ *uniformly on each* $B(a_j;r)$, *hence the convergence is uniform on* K.

Corollary: *(see [2])* $H(G)$ *is a complete metric space.*

Proof: *From the above theorem we got that* $H(G)$ *is closed and it is the subset of* $C(G,C)$ *which is complete. We know that closed subset of a complete set is complete. Hence,* $H(G)$ *is complete.*

Corollary: *(see [2])* *If* $f_n : G \to C$ *is analytic and* $\sum_{n=1}^{\infty} f_n(z)$ *converges uniformly on compact sets to* $f(z)$ *then* $f^k(z) = \sum_{n=1}^{\infty} f_n^k(z)$.

Hurwitz's Theorem: *(see [2])* *Let* G *be a region and suppose the sequence* $\{f_n\}$ *in* $H(G)$ *converges to* f. *If* $f \ne 0$, $\overline{B}(a;r) \subset G$ *and* $f(z) \ne 0$ *for* $|z-a| = R$ *then there is an integer* N *such that for* $n \ge N$, f *and* f_n *have the same number of zeros in* $B(a;r)$.

Proof:

Given, $f(z) \ne 0$ *for* $|z-a| = R$. *Take* $\eta = \inf\{|f(z) : |z-a| = R|\} > 0$. *If* $f_n \to f$ *in* $|z-a| = R$, *then this convergence will be uniform on* $|z-a| = R$. *So, there exist an integer* N *such that if* $n \ge N$ *and* $|z-a| = R$, *then* $|f(z) - f_n(z)| < \dfrac{\eta}{2}$. *Also* $|f(z)| \le |f(z)| + |f_n(z)|$ *and* $\dfrac{\eta}{2} < |f(z)|$. *So we can write*

$$|f(z) - f_n(z)| < \frac{\eta}{2} < |f(z)| \le |f(z)| + |f_n(z)|.$$

Hence using Rouche's theorem,we can say f *and* f_n *have same number of zeros in* $B(a;r)$.

Corollary: *(see [2])* *If* $\{f_n\} \subset H(G)$ *converges to* f *in* $H(G)$ *and each* f_n *never vanishes on* G *then either* $f = 0$ *or never vanishes.*

***Lemma**: (see [2]) A set $\Im$ in $H(G)$ is locally bounded iff for each compact set $K \subset G$ there is a constant M such that $|f(z)| \le M$ for all f in $\Im$ and z in K.*

Proof:

Let K be a compact subset of G and let there is a constant M such that $|f(z)| \le M$ for all f in $\Im$ and z in K. Since, K is compact, so it is totally bounded and complete. And we know the definition of locally bounded is that a set $\Im$ in $H(G)$ is locally bounded if for each point a in G there are constants M and $r > 0$ such that for all f in $\Im$, $|f(z)| \le M$, for $|z-a| < R$. Since for every compact set it happens, so the set $\Im$ in $H(G)$ is locally bounded.

Conversely, let a set $\Im$ in $H(G)$ is locally bounded on a compact set K. Let $|f(z)| \le M_1$, for $|z-a_1| < R$; $|f(z)| \le M_2$, for $|z-a_2| < R$ and so on. Take supremum on all the upper bounds and let it be M. Then $|f(z)| \le M$. Proved.

***Montel's Theorem**: (see [2]) A family $\Im$ in $H(G)$ is normal iff $\Im$ is locally bounded.*

***Proof**: Suppose $\Im$ in $H(G)$ is normal and it is not locally bounded. Then for a compact set K in G we can have*

$\sup\{|f_n(z)| : z \in K,\ f_n \in \Im\} = \infty$. *We can also say* $\sup\{|f_n(z)| : z \in K,\ f_n \in \Im\} \ge n$. *Again since $\Im$ in $H(G)$ is normal, so $\{f_n\}$ has a converging subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$. So, we can also have the same expression for $\{f_{n_k}\}$, i. e.* $\sup\{|f_{n_k}(z)| : z \in K,\ f_{n_k} \in \Im\} \ge n_k$. *Now* n_k $\le \sup\{|f_{n_k}(z)| : z \in K\}$ *and it is less than equal to*

$$\sup\{|f_{n_k}(z) - f(z)| : z \in K\} + \sup\{f(z) : z \in K\}.$$

Let $\sup\{f(z) : z \in K\} \le \mathrm{M}$. *Then* $n_k \le \sup\{|f_{n_k}(z) - f(z)| : z \in K\} + M$. *When* $n_k \to \infty$, *then the expression will be $\infty \le M$ which is a contradiction. Hence it is locally bounded.*

Now let $\Im$ is locally bounded. We will use now the Ascoli- Arzela theorem to prove that it is normal. It easily satisfies the first condition of the theorem and so here we need to prove only the equicontinuous part. Let a be a fixed point in G and let $\overline{B}(a,r) \subset G$. Also let there is a $M > 0$ such that $|f(z)| \le M$ for all z in $\overline{B}(a,r)$ and for all f in $\Im$. Let $|z-a| < \frac{r}{2}$ and f is in $\Im$. Then using Cauchy's Formula with $\gamma(t) = a + re^{it}, 0 \le t \le 2\pi$,

$$
\begin{aligned}
|f(a)-f(z)| &= \frac{1}{2\pi}\left|[\int_\gamma \frac{f(w)}{w-a}\,dw - \int_\gamma \frac{f(w)}{w-z}\,dw]\right| \\
&= \frac{1}{2\pi}\left|\int_\gamma \frac{f(w)(a-z)}{(w-a)(w-z)}\,dw\right| \\
&= \frac{1}{2\pi}|a-z|\left|\int_\gamma \frac{f(w)}{(w-a)(w-z)}\,dw\right| \\
&\le \frac{4\mathrm{M}}{r}|a-z|
\end{aligned}
$$

By taking $\lambda < \min\{\frac{r}{2}, \frac{r}{4M}\varepsilon\}$ *it follows that* $|z-a| < \lambda$ *gives* $|f(z)-f(a)| < \varepsilon$ *for all* f *in* $\Im$.

***Corollary**: (see [2]) A set* $\Im$ *in* $H(G)$ *is compact iff it is closed and locally bounded.*

***Proof:** Suppose* $\Im$ *in* $H(G)$ *is compact. So it is closed and totally bounded. And we know if it is totally bounded then it will be locally bounded.*

<u>Converse part:</u> Let $\Im$ *in* $H(G)$ *is closed and locally bounded. We are required to prove its compactness, i.e. only boundedness of* $\Im$. *By using the previous lemma we are having if* $\Im$ *in* $H(G)$ *is locally bounded, then for each compact set* K *in* G, $|f(z)| \le M$ *for all* f *in* $\Im$ *and for all* z *in* K. *Since for each compact set in* G *it happens so we can conclude that* $\Im$ *is bounded on whole* G. *Hence proved.*

Chapter-3: Spaces of meromorphic functions

We know the definition of a meromorphic function as, if G is open and f is a function defined and analytic in G except for poles , then f is a meromorphic function on G . The set of all meromorphic functions is denoted by $M(G)$ and it is a subset of the space of continuous functions $C(G, C_\infty)$. Defining a metric on $M(G)$, we will discuss here this metric space. The metric defined on C_∞ is

$$d(z_1, z_2) = \frac{2|z_1 - z_2|}{[(1+|z_1|^2)(1+|z_2|^2)]^{\frac{1}{2}}},$$

where $z_1, z_2 \in C$ *and* $d(z,\infty) = \dfrac{2}{(1+|z|^2)^{\frac{1}{2}}}$. *We can check that* $d(z_1, z_2) = d\left(\dfrac{1}{z_1}, \dfrac{1}{z_2}\right)$ *and* $d(z,0) = d(\dfrac{1}{z}, \infty)$ *for* $z \neq 0$.

Proposition: *(see [2])*

(a) If a is in C and $r > 0$ then there is a number $\rho > 0$ such that $B_\infty(a;\rho) \subset B(a;r)$.

(b) If $\rho > 0$ is given and a is in C then there is a number $r > 0$ such that $B(a;r) \subset B_\infty(a;\rho)$.

(c) If $\rho > 0$ is given then there is a compact set $K \subset C$ such that $C_\infty - K \subset B_\infty(\infty;\rho)$.

(d) If a compact set $K \subset C$ is given, there is a number $\rho > 0$ such that $B_\infty(\infty;\rho) \subset C_\infty - K$.

Theorem: *(see [2]) Let $\{f_n\}$ be a sequence in $M(G)$ and suppose $f_n \to f$ in $C(G, C_\infty)$. Then either f is meromorphic or $f \equiv \infty$. If each f_n is analytic then either f is analytic or $f \equiv \infty$.*

Proof: *Let we are having a point $a \in G$ such that $f(a) \neq \infty$ and let $M = |f(a)|$. From the above proposition we can get a number $\rho > 0$ such that $B_\infty(f(a);\rho) \subset B(f(a);M)$. But as we know*

$f_n \to f$, *so for an integer* n_0, $d(f_n(a), f(a)) < \frac{\rho}{2}$ *for all* $n \geq n_0$. *We also have* $\{f, f_1, f_2, ...\}$ *is compact in* $C(G, C_\infty)$ *and so,it is equicontinuous. So, there is an* $r > 0$ *such that whenever* $|z-a| < r$ *then* $d(f_n(z), f(z)) < \frac{\rho}{2}$ *and hence* $d(f_n(z), f(a)) \leq \rho$ *for* $|z-a| \leq r$ *and for* $n \geq n_0$. *But choose* ρ *in such a way that,* $|f_n(z)| \leq |f_n(z) - f(a)| + |f(a)| \leq 2M$ *for all* z *in* $\overline{B}(a,r)$ *and* $n \geq n_0$. *Then ,*

$$\frac{2}{(1+4M^2)}|f_n(z) - f(z)| \leq d(f_n(z), f(z)) \text{ for } z \text{ in } \overline{B}(a,r) \text{ and } n \geq n_0.$$

As $d(f_n(z), f(z)) \to 0$ *uniformly for* z *in* $\overline{B}(a,r)$, *it implies that* $|f_n(z) - f(z)| \to 0$ *uniformly for* z *in* $\overline{B}(a,r)$. *The sequence* $\{f_n\}$ *is bounded on* $B(a,r)$, *so* f_n *has no poles and must be analytic near* $z = a$ *for* $n \geq n_0$. *Hence* f *is analytic in a disk about* a.

Again let there be a point a *in* G *with* $f(a) = \infty$. *Define* $\frac{1}{g}$ *by* $(\frac{1}{g})(z) = \frac{1}{g(z)}$ *if* $g(z) \neq 0$ *or* ∞; $(\frac{1}{g})(z) = 0$ *if* $g(z) = \infty$ *and* $(\frac{1}{g})(z) = \infty$ *if* $g(z) = 0$ *where* g *is a function in* $C(G, C_\infty)$. *Hence* $\frac{1}{g} \in C(G, C_\infty)$. *Also,* $f_n \to f$ *in* $C(G, C_\infty)$ *and so* $\frac{1}{f_n} \to \frac{1}{f}$ *in* $C(G, C_\infty)$. *Since each function* $\frac{1}{f_n}$ *is meromorphic on* G, *so we can have a number* $r > 0$ *and an integer* n_0 *such that* $\frac{1}{f}$ *and* $\frac{1}{f_n}$ *are analytic on* $B(a,r)$ *for* $n \geq n_0$ *and* $\frac{1}{f_n}$ *converges to* $\frac{1}{f}$ *uniformly on* $B(a,r)$. *Using Hurwitz's Theorem we are having either* $\frac{1}{f} \equiv 0$ *or* $\frac{1}{f}$ *has isolated zeros in* $B(a,r)$. *So if* $f \neq \infty$ *then* $\frac{1}{f} \neq 0$ *and* f *must be meromorphic in* $B(a,r)$. *Hence the result follows.*

Consider each f_n *is analytic then* $\frac{1}{f_n}$ *is having no zeros in* $B(a,r)$. *Then from the corollary of Hurwitz's theorem that either* $\frac{1}{f} \equiv 0$ *or* $\frac{1}{f}$ *never vanishes. But as* $f(a) = \infty$ *we have that* $\frac{1}{f}$ *has at least one zero; thus* $f \equiv \infty$ *in* $B(a,r)$. *Noting the first part of the proof we see that* $f \equiv \infty$ *or* f *is analytic.*

Corollary: *(see [2])* $M(G)$ *is not complete but* $M(G) \cup \{\infty\}$ *is a complete metric space.*

Proof: *Let* $\{f_n(z)\} = \{n\}$ *be a Cauchy sequence in* $M(G)$*. When* $n \to \infty$ *,it will converge to infinity in* $C(G, C_\infty)$*. So, it will not be meromorphic and hence* $M(G)$ *is not complete. But if we consider* $M(G) \cup \{\infty\}$ *then it will be a complete metric space*

Corollary: *(see [2])* $H(G) \cup \{\infty\}$ *is closed in* $C(G, C_\infty)$

Definition*: If* f *is a meromorphic function on the region* G *then define* $\mu(f) : G \to R$ *by*

$$\mu(f)(z) = \frac{2|f'(z)|}{1 + |f(z)|^2}$$

whenever z *is not a pole of* f *, and*

$$\mu(f)(a) = \lim_{z \to a} \frac{2|f'(z)|}{1 + |f(z)|^2}$$

if a *is a pole of* f *.*

Theorem: *(see [2])* *A family* $\Im \subset M(G)$ *is normal in* $C(G, C_\infty)$ *if and only if* $\mu(\Im) \equiv \{\mu(f) : f \in \Im\}$ *is locally bounded.*

Proof-: *One may refer [2].*

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