# STUDY OF HYPERBOLIC SYSTEMS

**A THESIS**
**Submitted in partial fulfillment of the requirements for the award of the degree of**

**MASTER OF SCIENCE**
**IN**
**MATHEMATICS**

**By**

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# NATIONAL INSTITUTE OF TECHNOLOGY

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# DECLARATION

I here certify that the work which is being presented in the thesis entitled “STUDY OF HYPERBOLIC SYSTEMS” in partial fulfillment of the requirement for the award of the degree of master of Science, submitted in the Department of Mathematics, National Institute of Technology, Rourkela is a authentic record of my own work carried out under the supervision of Dr. R.S Tungala.

The matter embodied in this has not been submitted by me for the award of any other degree.

Date: (LAXMI BEHERA)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Finally all credits goes to my parents and my friends for their continued support.

And to all mighty, who made all things possible.

Date:

(LAXMI BEHERA)

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# CONTENTS

# ABSTRACT

In this report, we define the conservation form of PDF with initial data .We noticed that even though the initial data is smooth after finite time the solution will be collapsed, means we will not get smooth solution. We defined the weak solution for Conservation laws and then derived Rankine Hugoniot jump conditions. To obtain unique solution of Conservation law, we defined two types of entropy conditions. Finally, we discussed Riemann problem for Scalar Conservation law as well as systems and then discussed the solution for them.

# INTRODUCTION

Many practical problems in science and engineering involve conservative quantities and lead to partial differential equations of this class. In the first chapter we have discussed conservation laws and its solutions. There exists non-smooth solution when two characteristics drawn from two different intersect at a point. Then it is clear that if flux function is non-linear, discontinuities may develop. This concept introduces the weak solutions. We have given the Rankine-Hugonoit jump condition which describes the weak solution and speed of the discontinuity. Then we have given the Riemann problem and its solution.

In the second chapter we have consider the case of a linear hyperbolic system with constant coefficients for which the Riemann problem is easily solved. Next in the nonlinear case we introduce the notions of a rarefaction waves, shock waves, which play an important role in the explicit construction of the solution of the Riemann Problem.

# Chapter 1

# SCALAR CONSERVATION LAW

## 1.1 Conservation laws

Scalar conservation law in 1-D is of the form

$$u_t + f(u)_x = 0, \quad (x,t) \in R \times (0,\infty) \tag{1.1}$$

where $x, t$ are independent variables and $u$ is dependent variable.

$f: R \to R$ is a smooth function.

$u: R \times R^{+} \to R$

If $f'(u)$ is a function of $u$ then (1.1) is non linear partial differential equation.

If $f'(u)$ is constant then (1.1) is linear partial differential equation.

Initial condition of (1.1) $u(x,0) = u_0(x)$ , $x \in R$ (1.2)

If $u$ be a smooth solution of (1.1) then non-conservative form of (1.1) is

$$u_t + f'(u)\, u_x = 0 \tag{1.3}$$

The characteristic curve associated with (1.3) are the solution of the differential equation

$$\begin{cases} \dfrac{dx(t)}{dt} = f'(u(x(t),t)) \\ x(0) = x_0 \end{cases} \tag{1.4}$$

Solution for (1.4) exists at least on a small interval $[0, t_0)$. Along such a curve. $u$ is constant since

$$\frac{d}{dt} u(x(t),t) = \frac{\partial u}{\partial t}(x(t),t) + \frac{\partial u}{\partial x}(x(t),t)\,\frac{dx(t)}{dt}$$
$$\left(\frac{\partial u}{\partial \mathrm{t}} + f'(u)\,\frac{\partial u}{\partial \mathrm{x}}\right)(x(t),t) = 0$$

Solving (1.4 ) $x(t) = x_0 + t f'(u(x_0))$ (1.5)

Characteristic curves are straight lines where slopes are constants depending on initial data.

Solution is

$$u(x(t),t) = u_0(x_0) \tag{1.6}$$

If $u$ is smooth then $u$ can be found by the method of characteristics.

**EXAMPLE:**

$u_t + au_x = 0$ a is a real number

Characteristic equation

$$\frac{\mathrm{dx}}{\mathrm{dt}} = a \qquad \text{(A)}$$
$$x(\mathrm{o}) = x_0$$

solving (A) $x(t) = at + c$ ,at $t = 0$ , $c = x(0) = x_0$

$$x(t) = at + x_0$$
$$u(x(t), t) = u(x(0), 0) = u_0(x_0)$$
$$= u_0(x - at)$$

## 1.2 Existence of a non-smooth solution

Suppose $f$ is non linear i.e. $f'' > 0$

Assume that there are two points $x_1, x_2 \in R$ , $x_1 < x_2$ and $u_0$ is decreasing function

$u_0(x_2) < u_0(x_1)$

$\Rightarrow f'(u_0(x_2)) < f'(u_0(x_1))$

$$m_1 = \frac{1}{f'(u_0(x_1))} < m_2 = \frac{1}{f'(u_0(x_2))}$$

Along $\mathrm{x}_1$ speed is $f'(u_0(x_1))$

Along $\mathrm{x}_2$ speed is $f'(u_0(x_2))$ .Characteristic curve passing through $x_1$ is greater than the characteristic curve passing through $x_2$. So somewhere they will intersect . At intersection point we will get $u_0(x_1) = u_0(x_2)$

So $u$ is not continuous

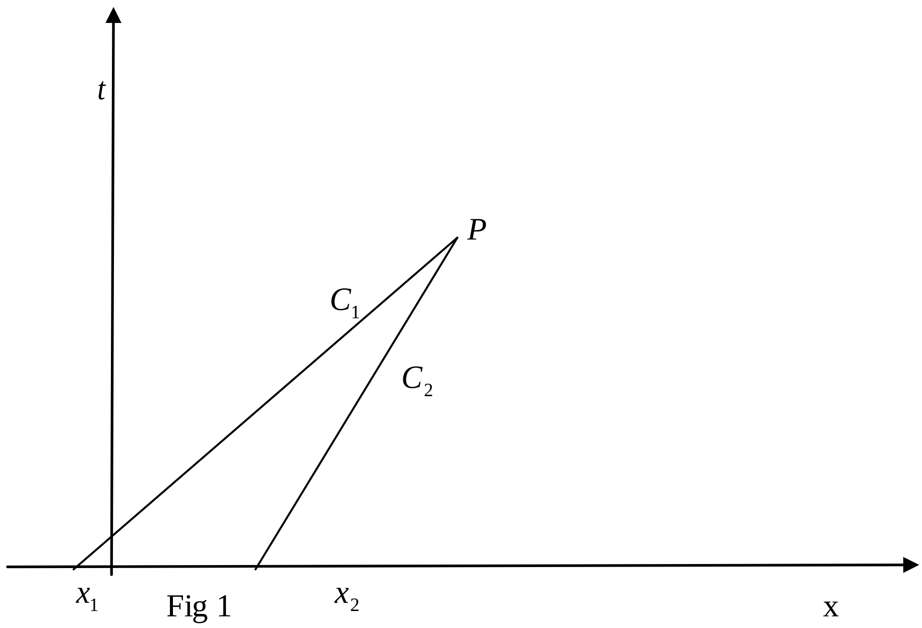



Fig 1

**Examples**

Consider the Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \text{ with } u(x,0) = u_o(x) = \begin{cases} 1 & \text{if } x < 0 \\ 1 - x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Solution-Here $\quad f(u) = \frac{u^2}{2} \quad f'(u) = u$

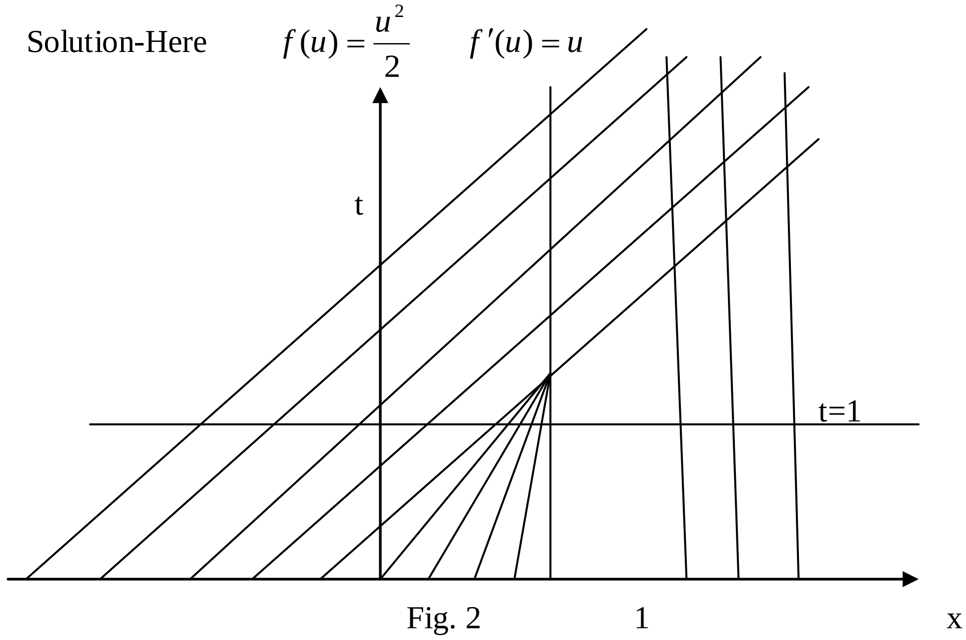



Fig. 2

$$\frac{dx}{dt} = u(x(t),t)$$

$\Rightarrow x = tu(x(t),t) + c$ at $t = 0, c = x_0$

$\Rightarrow x = tu_0(x_0) + x_0$

$$x(t)=\begin{cases} x_0+t & \text{if } x_0\le 0 \\ x_0+(1-x_0)t & \text{if } 0\le x_0\le 1 \\ x_0 & \text{if } x_0\ge 1 \end{cases}$$

$$x_0=\begin{cases} x-t & \text{if } x\le t \\ \dfrac{x-t}{1-t} & \text{if } t\le x\le 1 \\ x & \text{if } x\ge 1 \end{cases}$$

$$u(x(t),t)=u_0(x_0)$$

$$=\begin{cases} 1 & \text{if } x\le t\le 1 \\ 1-\left(\dfrac{x-t}{1-t}\right) & \text{if } t\le x\le 1 \\ 0 & \text{if } x\ge 1, t<1 \end{cases}$$

At t=1 , the characteristics intersects, So we can say at t=1,

$$u(x,1)=\begin{cases} 1 \ \text{ if } x<1 \\ 0 \text{ if } x>1 \end{cases}$$

## 1.3 WEAK SOLUTIONS: RANKINE -HUGONIOT JUMP CONDITION

Solution of conservation form $u$ will not be smooth if $f''>0$ and $u_0(x)$ is decreasing function. In this case $u$ have weak solution.

Let $\varphi: R\times[0,\infty)\to R$ is a smooth function with compact support

$$u_t+f(u)_x=0 \qquad u(x,0)=u_0(x)$$

Multiplying $\phi$ in above equation and then integrating

$$0=\int_0^\infty\int_{-\infty}^\infty (u_t+f(u)_x)\phi\, dxdt$$

$$=-\int_0^\infty\int_{-\infty}^\infty \phi_t u dxdt-\int_0^\infty\int_{-\infty}^\infty f(u)\,\phi_x dxdt-\int_{-\infty}^\infty (u\phi)_{t=0}\,dx$$

$$\Rightarrow \int_0^\infty\int_{-\infty}^\infty (\phi_t u+f(u)\,\phi_x)dxdt+\int_{-\infty}^\infty (u\phi)_{t=0}\,dx=0$$

**Definition:**

We say that $u \in L^{\infty}\left(R\times\left[0,\infty\right)\right)$ is a weak solution of (1.1) and (1.2)

provided $\int_0^{\infty}\int_{-\infty}^{\infty}(\phi_t u + f(u)\phi_x)dxdt + \int_{-\infty}^{\infty}(u\phi)_{t=0}dx = 0$ (1.7)

for all smooth function $\phi$ with compact support.

Solutions of (1.8) although not continuous has a simple structure.

Let $x(t)$ be a curve in the (x ,t) plane across which $u(x,t)$ fails to be continuous .

But $u$ has a limit as points (x, t) approaches the curve from either side.

Suppose in some open set $V \subset R\times(0,\infty)$, $u$ is smooth on either side of the smooth curve.

Let $V_l$ be the part of V on the left of the curve and $V_r$ that on the right.

Assume $u$ and its derivatives are uniformly continuous on $V_l$ and $V_r$.

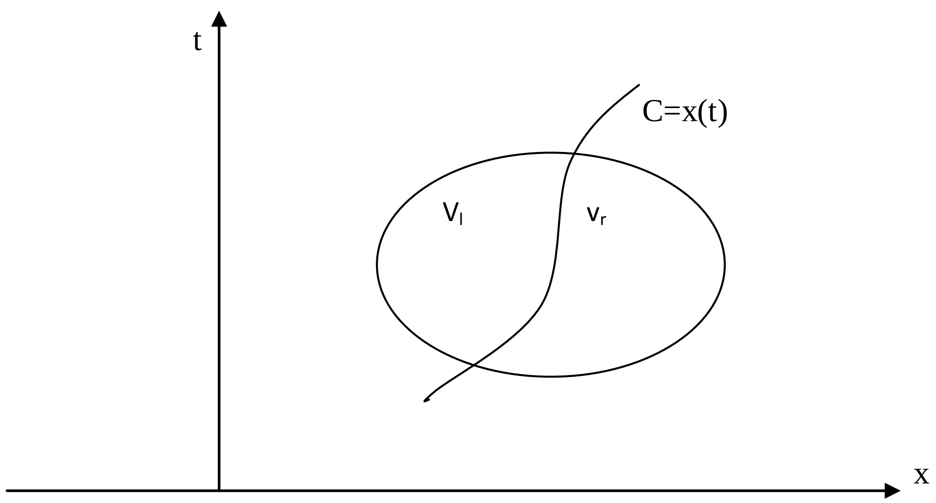



Fig. 3

First choose a test function $\phi$ with compact supportin $V_1$.Then (1.7) become

$$0 = \int_0^{\infty}\int_{-\infty}^{\infty}(u_t + f(u)_x)\phi\, dxdt$$

$$= \int_0^{\infty}\int_{-\infty}^{\infty}(\phi_t u + f(u)\phi_x)dxdt + \int_{-\infty}^{\infty}(u\phi)_{t=0}dx = 0$$

$$= -\int_0^\infty \int_{-\infty}^\infty \phi_t u dt dx - \int_0^\infty \int_{-\infty}^\infty f(u)\phi_x dt dx$$
$$= -\left[ \int_0^\infty \int_{-\infty}^\infty u\phi_t dx dt + \int_0^\infty \int_{-\infty}^\infty f(u)\phi_x dx dt \right]$$
$$= -\left[ \int_0^\infty \int_{-\infty}^\infty (u\phi_t + f(u)\phi_x) dx dt \right]$$
$$= 0$$

$$\Rightarrow \int_0^\infty \int_{-\infty}^\infty \left(u\phi_t + f(u)\phi_x\right) dx dt = -\int_0^\infty \int_{-\infty}^\infty \left(u_t + f(u)_x\right) \phi \, dx dt = 0 \tag{1.8}$$

Since $u$ is smooth on $V_1$ so

$$u_t + f(u)_x = 0 \text{ in } V_l \tag{1.9}$$

Similarly $V_r$ $\quad u_t + f(u)_x = 0$ (1.10)

Now choosing a test function $\phi$ with compact support in V which does not vanish along the curve $x(t)$ . Again using (1.8) we have

$$0 = \int_0^\infty \int_{-\infty}^\infty \left(u\phi_t + f(u)\phi_x\right) dx dt$$
$$= \iint_{V_l} \left(u\phi_t + f(u)\phi_x\right) dx dt + \iint_{V_r} \left(u\phi_t + f(u)\phi_x\right) dx dt$$

Now we will calculate one by one

$$\iint_{V_l} \left(u\phi_t + f(u)\phi_x\right) dx dt =$$
$$\iint_{V_L} (\phi_x, \phi_t) \cdot (f(u), u) dx dt$$

Using Green's Identity

$$= -\iint_{V_l} (\phi, \phi) \cdot (f(u)_x, u_t dx dt) + \int_C (f(u)_l \phi_l, u_l \phi_l) \cdot \left( \frac{1}{\sqrt{1 + \left(\frac{dx}{dt}\right)^2}}, \frac{-\frac{dx}{dt}}{\sqrt{1 + \left(\frac{dx}{dt}\right)^2}} \right) dt$$

Since $\phi$ is test function so it will be same in in $v_l$ and $v_r$

$$= -\iint_{V_l} (\phi,\phi)\cdot(f(u)_x, u_t)dxdt + \int_C (f(u)_l\phi, \phi u_l)\cdot\left(\frac{1}{\sqrt{1+\left(\frac{dx}{dt}\right)^2}}, \frac{-\frac{dx}{dt}}{\sqrt{1+\left(\frac{dx}{dt}\right)^2}}\right)dt$$

$$= -\iint_{V_l} \phi(f(u)_x + u_t)dxdt + \int_{x(t)} \frac{\phi(x(t),t)(f(u)_l - u_l\frac{dx}{dt})}{\sqrt{1+\left(\frac{dx}{dt}\right)^2}}dt \tag{1.11}$$

Since $f(u)_x + u_t = 0$ in $V_1$ and let $\frac{dx}{dt} = s =$ speed of curve.

$$\iint_{V_l} (u\phi_t + f(u)\phi_x)dxdt = \int_{x(t)} \frac{\phi(x(t),t)(f(u)_l - u_l s)}{\sqrt{1+s^2}}dt$$

Similarly on $V_r$

$$\iint_{V_r} (u\phi_t + f(u)\phi_x)dxdt = \int_{x(t)} \frac{\phi(x(t),t)(f(u)_r - u_r s)}{\sqrt{1+s^2}}dt \tag{1.12}$$

So

$$\int_0^\infty\int_{-\infty}^\infty (u_t + f(u)_x)\phi dxdt = \int_{x(t)} \frac{\phi(x(t),t)(f(u)_l - u_l s)}{\sqrt{1+s^2}}dt + \int_{x(t)} \frac{\phi(x(t),t)(f(u)_r - u_r s)}{\sqrt{1+s^2}}dt = 0$$

$$\Rightarrow \int_{x(t)} \phi\,[f(u_l) - u_l s + f(u_r) - u_r s]dt = 0$$

$$f(u_l) - u_l s + f(u_r) - u_r s = 0$$
$$f(u_l) - f(u_r) - (u_l - u_r)s = 0$$
$$f(u_l) - f(u_r) = (u_l - u_r)\,s \tag{1.13}$$

Let $[u] = u_l - u_r =$ jump in u across the curve C

$[f(u)] = f(u_l) - f(u_r) =$ Jump in $f(u)$ across the curve C

$s = \dot{x}(t) =$ speed of discontinuity

$$\Rightarrow [\,f(u)\,] = [\,u\,]\,s \tag{1.14}$$

along discontinuity curve

This relation is called Rankine-Hugoniot jump (R-H) conditions.

**EXAMPLE**-Consider the Burgers Equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

$$u(x,0) = u_0(x) = \begin{cases} 1 & \text{if } x < 0 \\ 1 - x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Now we should define $u$ for $t > 1$ so that it satisfies (1.14). Solution

$$u(x,t) = \begin{cases} 1 & \text{if } x < \dfrac{1+t}{2} \\ 0 & \text{if } x > \dfrac{1+t}{2} \end{cases}$$

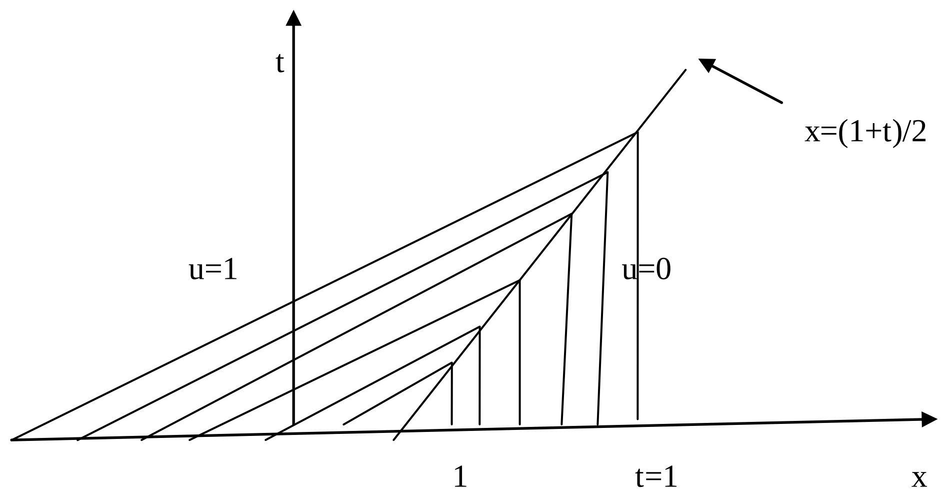
$$\Rightarrow u(x,1) = \begin{cases} 1 & \text{if } x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

$$s = \frac{1}{2} \quad ,f(u_l)\text{-}f(u_r) = \frac{1}{2} - 0 = \frac{1}{2}$$

$$s(u_l\text{-}u_r) = \frac{1}{2}$$

Since it satisfies R-H condition along the curve $C = x(t)$

Hence this solution is a weak solution for $t \ge 1$



## 1.4 NON- UNIQUENESS OF WEAK SOLUTIONS

### EXAMPLE

Now consider the Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

$$u(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

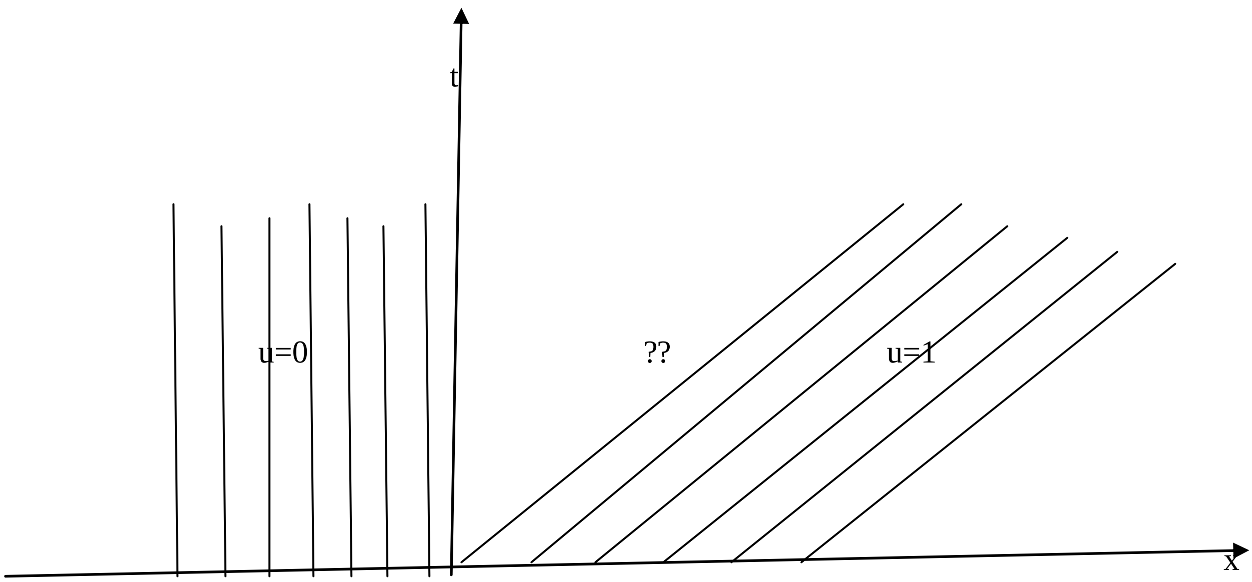



Fig. 4

We will fill up this gap by following equation (see Fig.5)

$$u_1(x,t) = \begin{cases} 0 & \text{if } \; x < \dfrac{t}{2} \\ 1 & \text{if } \; \dfrac{t}{2} < x \end{cases}$$

Along the curve C= $x(t) = \dfrac{t}{2}$

$$f(u_l)\text{-}f(u_r) = 0\text{-}\frac{1}{2}, \; s = \frac{1}{2}$$

$$u_l\text{-}u_r = 0 - 1 = -1$$

Therefore $u_l$ satisfies the R-H condition along C

On the other hand we can fill this gap $\{0 < x < t\}$ by (see Fig.6)

$$u_2(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ \dfrac{x}{t} & \text{if } 0 < x \le t \\ 1 & \text{if } x > t \end{cases}$$

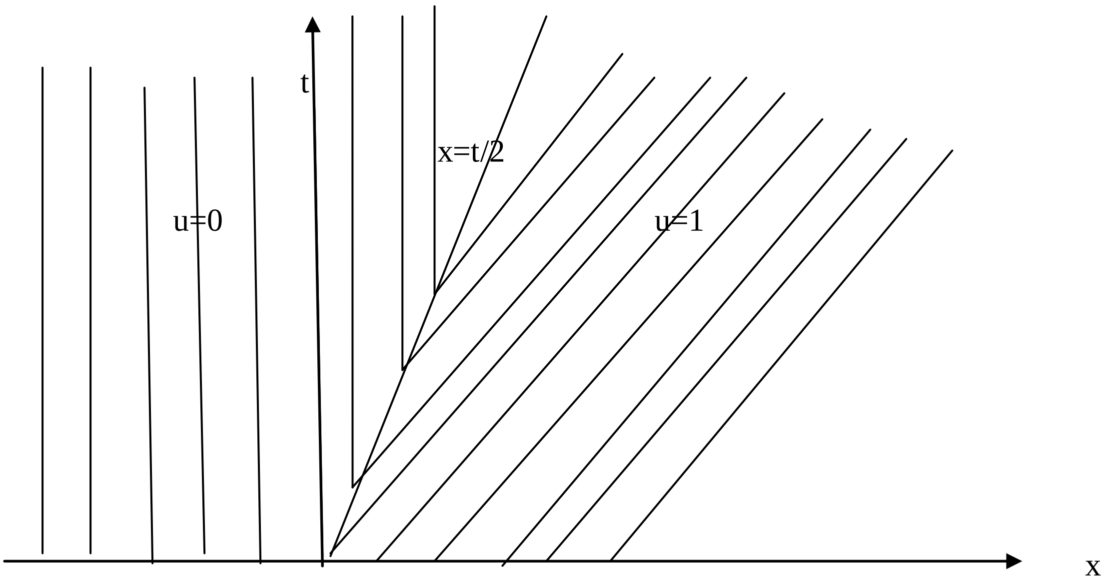



Fig. 5

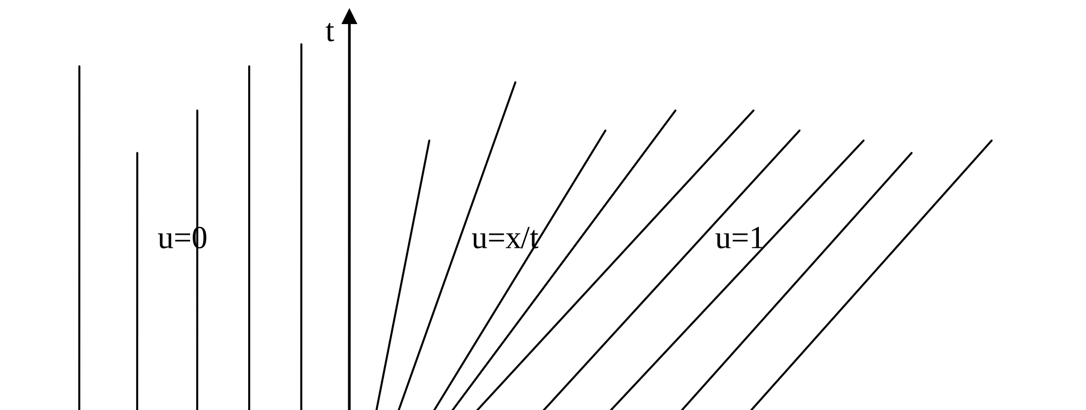



Fig. 6

solution $u_1$ is called a shock.

**Shock**: Shock is a weak solution of (1.1) satisfying R-H condition and Lax condition.

solution $u_2$ is called a Rarefaction wave solution .

**Rarefaction wave solution**: It is a continuous solution of (1.1) and the characteristic fields increases from left to right. i.e. $f'(u_l) < f'(u_r)$.

## 1.5 Entropy Condition (for convex flux function):

A discontinuity propagating with a speed s defined by (1.14) satisfies the entropy condition if

$$f'(u_l) > s > f'(u_r)$$

For a convex flux $f$ above condition is equivalent to $u_l > u_r$

In example (3) $u_1(x,t)$ solution does not satisfy entropy condition

So we reject this solution and keep the continuous solution $u_2$

## 1.6 Entropy condition (Oleinik) for all flux functions:

A discontinuity propagating with a speed s defined by (1.14) satisfies the entropy condition if

$$\frac{f(v) - f(u_l)}{v - u_l} \geq s \geq \frac{f(v) - f(u_r)}{v - u_r} \quad \text{for all } v \text{ between } u_l \text{ and } u_r$$

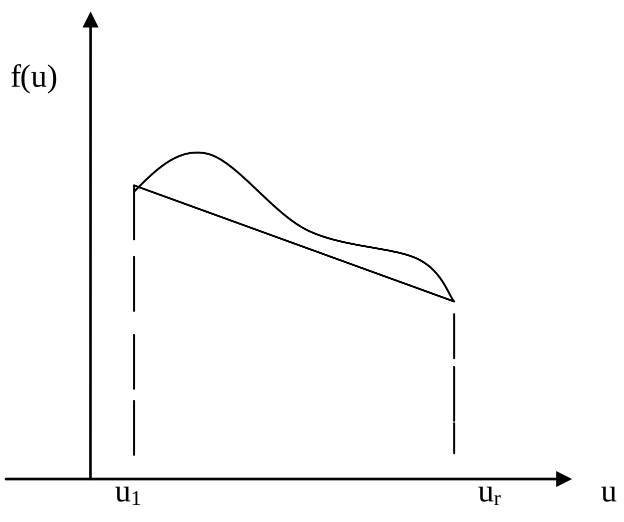



Fig 1.6a

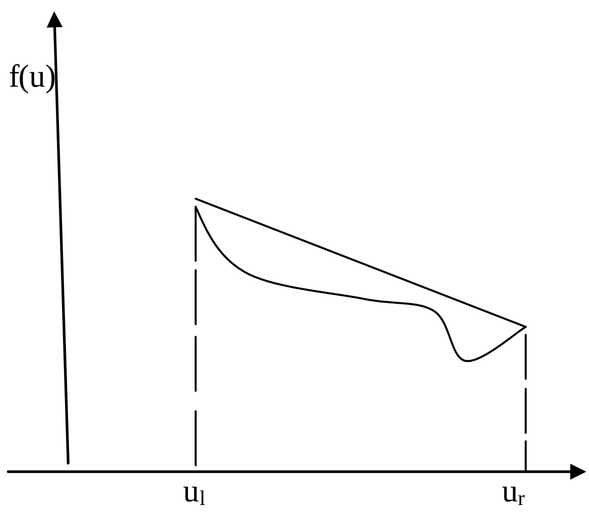



Fig. 1.6b

## 1.7 RIEMANN PROBLEM

A conservation law together with piecewise constant data having a single discontinuity is known as the **Riemann problem** i.e

$$\cdot \begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases} \end{cases}$$

Here $u_l, u_r \in R$ are the left and right initial states

## 1.8 RIEMANN PROBLEM FOR CONVEX FLUX

Theorem- Let $f$ be strictly convex and $G = (f')^{-1}$

(1) If $u_l > u_r$, then the weak solution satisfying entropy condition is given by

$$u(x,t) = \begin{cases} u_l & \text{if } x < st \\ u_r & \text{if } x > st \end{cases} \tag{1.15}$$

Where $s = \dfrac{f(u_l) - f(u_r)}{u_l - u_r}$

(2) If $u_l < u_r$, then weak solution satisfying entropy condition is given by

$$u(x,t) = \begin{cases} u_l & \text{if } x < tf'(u_l) \\ G(\frac{x}{t}) & \text{if } tf'(u_l) < x < tf'(u_r) \\ u_r & \text{if } x > f'(u_r) \end{cases} \tag{1.16}$$

Remark-In the first case the states $u_l$ and $u_r$ are separated by a shock wave with with constant speeds. In the second case the states $u_l$ and $u_r$ are separated by a rarefaction wave

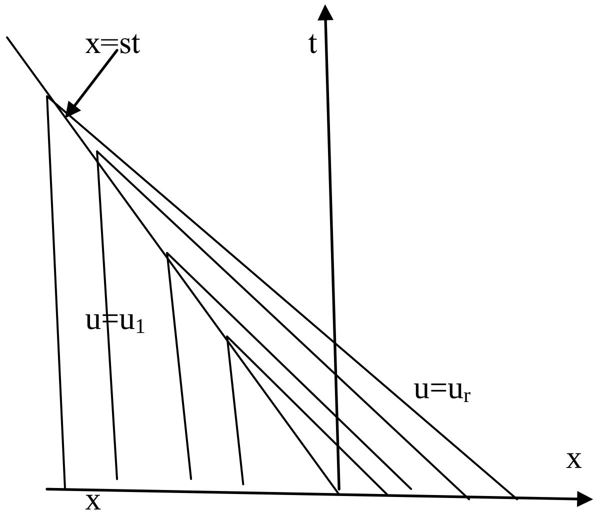



Fig 1.7(a)

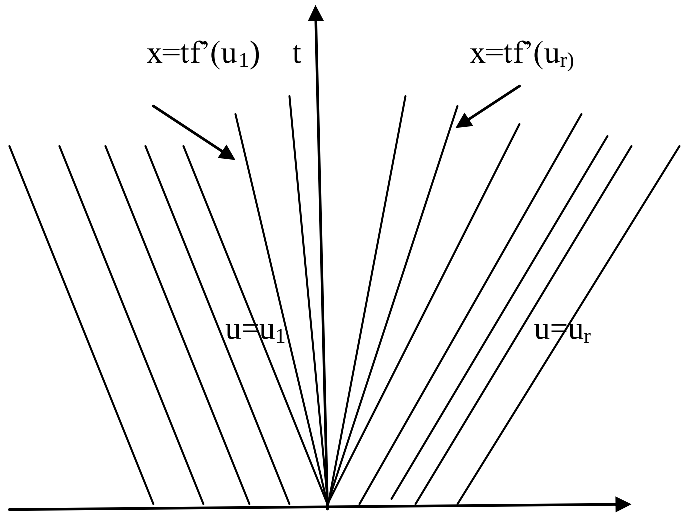



Fig 1.7(b)

(1) Across the shock $u(x,t)$ satisfies the R-H condition. If $x \neq st$ the constant states

$u_l$ and $u_r$ are the solution of left and right part of the shock .Since $u_l > u_r$, the entropy condition holds as well.

(2) It is enough to check that , solution defined in the regions $\left\{ f'(u_l) < \frac{x}{t} < f'(u_r) \right\}$

solves the equation of Riemann problem. Let $u(x,t) = v\left(\frac{x}{t}\right)$ then

$$0 = u_t + f(u)_x = u_t + f'(u)u_x$$
$$= -v'\left(\frac{x}{t}\right)\frac{x}{t^2} + f'(v)\, v'\left(\frac{x}{t}\right).\frac{1}{t}$$
$$= v'\left(\frac{x}{t}\right)\frac{1}{t}\left[ f'(v) - \frac{x}{t} \right]$$
$$f'\left(v\left(\frac{x}{t}\right)\right) = \frac{x}{t}$$
$$\Rightarrow u(x,t) = v\left(\frac{x}{t}\right)$$
$$= (f')^{-1}\left(\frac{x}{t}\right) = G\left(\frac{x}{t}\right)$$

Now $v\left(\frac{x}{t}\right) = u_1$ provided $\frac{x}{t} = f'(u_l)$ and similarly $v\left(\frac{x}{t}\right) = u_r$ if $\frac{x}{t} = f'(u_r)$

$\Rightarrow$ Solutions defined in (1.16) is a continuous solution

## 1.9 RIEMANN PROBLEM FOR GENERAL NON CONVEX FLUX

Let $f$ be a non convex function.

Case1 (See Fig.1.7) $.u_l > u_r$

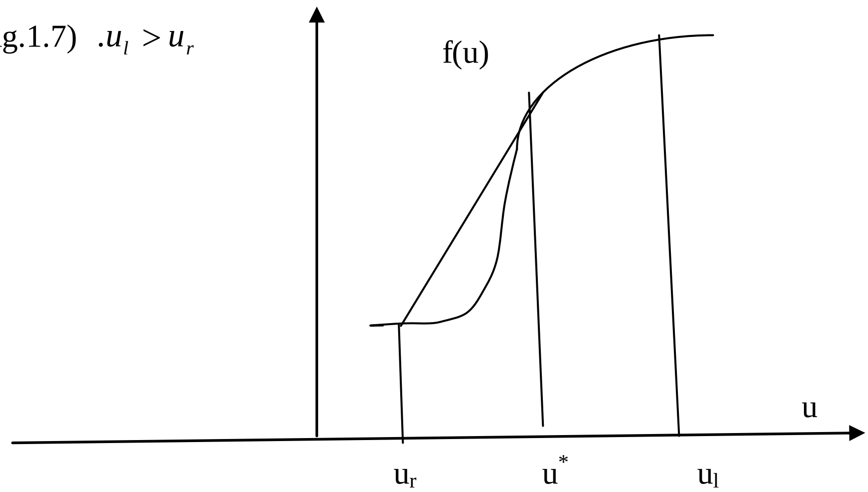



Fig. 1.7

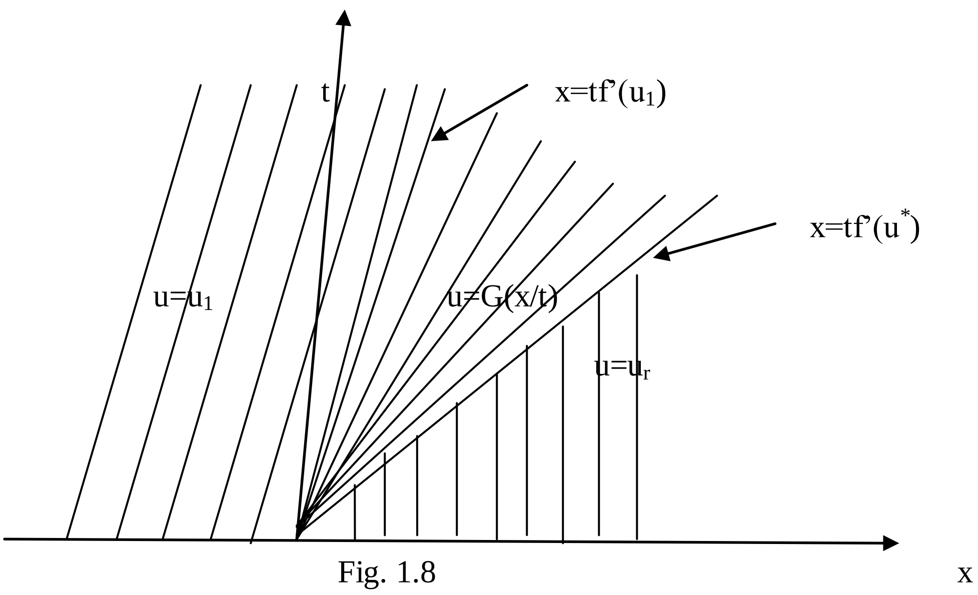



Fig. 1.8

Consider the convex hull of the set $A=\{(x,y):u_r \le x \le u_l, y \le f(x)\}$

At the upper boundary of the set, it is composed of straight line segment

$(u_r,\ f(u_r))$ to $(u^*,\ f(u^*))$ and then $y=f(x)$ upto $(u_l,\ f(u_l))$

Connect $u_r$ to $u^*$ by a shock , $u^*$ can be calculated by $f'(u^*)=\dfrac{f(u^*)\text{-}f(u_r)}{u^*-u_r}$ and then connect $u^*$ and $u_l$ by a rarefaction wav(see Fig.1.8)

Across the line $x=tf'(u_l)$ solution is continuous and across the line $x=tf'(u^*)$ solution is discontinuous with the left state $u^*$ and the right state $u_r$.

$$u(x,t)=\begin{cases} u_l & \text{if } x \le t\ f'(u_l) \\ (f')^{-1}\left(\dfrac{x}{t}\right) & \text{if } t\ f'(u_l) \le x \le t\ f'(u^*) \\ u_r & \text{if } x > t\ f'(u^*) \end{cases}$$

Case2(see Fig. 1.9): $u_l < u_r$

In this case consider the convex hull of the set $A=\{(x,y):u_l \le x \le u_r, y \ge f(x)\}$

Connect $u_l$ and $u^*$ by a rarefaction wave and then connect $u^*$ and $u_r$ by a shock where $u^*$ is given by $f'(u_*)=\dfrac{f(u_*)\text{-}f(u_r)}{u_*-u_r}$

$$u(x,t)=\begin{cases} u_l & \text{if } x \le t\ f'(u_l) \\ (f')^{-1}\left(\dfrac{x}{t}\right) & \text{if } t\ f'(u_l) \le x \le t\ f'(u_*) \\ u_r & \text{if } x > t\ f'(u_r) \end{cases}$$

(See Fig.1.10)

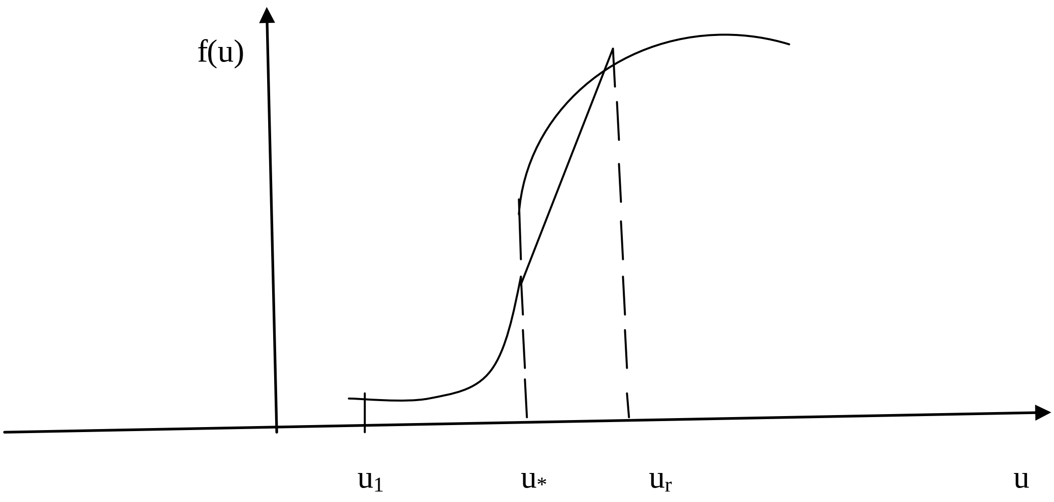



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Fig.1.9

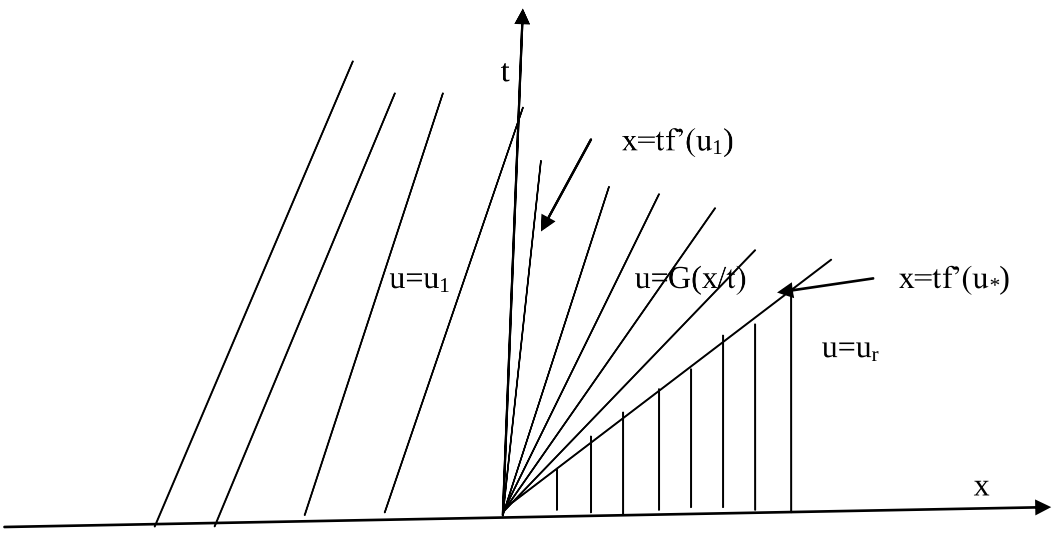


Fig.1.10

# CHAPTER-2

# SYSTEM OF CONSERVATION LAWS

## 2.1 Linear Hyperbolic systems with constant coefficients

Consider the first order linear system

$$u_t + A u_x = 0 \tag{2.1}$$

where $u = (u_1, u_2, ..., u_p)^T$

A is a $p \times p$ matrix

We assume that the system is strictly hyperbolic i.e. the matrix has $p$ distinct real eigenvalues arranged in increasing order $\lambda_1 < \lambda_2 < ... < \lambda_p$ with each eigenvalue $\lambda_k$ we associate a right eigen vector $r_k \in R^p$

$$Ar_k = \lambda_k r_k$$

and a left eigenvector $l_k^T A = \lambda_k \, l_k^T$

i.e. $l_k$ is an eigenvector of $A^T$

Since the eigenvalues are distinct , the eigenvectors $r_k, 1 \le k \le p$, form a basis of $R^p$ and we have

$$l_j^T r_k = l_j . r_k = 0, \;\; j \neq k$$

We can normalize the vectors $l_k^T$ in such a way that $l_k^T r_k = 1$

Hence using the Kronecker delta symbol, we obtain

$$l_j^T \; r_k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

Now setting $u = \sum_{k=1}^{p} \alpha_k r_k, \;\; \alpha_k = l_k^T u$

We have

$$\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = \sum_{k=1}^{p} \left( \frac{\partial \alpha_k}{\partial t} + \lambda_k \frac{\partial \alpha_k}{\partial x} \right) r_k$$

So that the first order system (2.1) is equivalent to $p$ independent scalar first order equations

$$\frac{\partial \alpha_k}{\partial t} + \lambda_k \frac{\partial \alpha_k}{\partial x} = 0 \ ,1 \le k \le p$$

Thus we can derive an explicit expression for the solution u of the Cauchy problem.

$$\begin{cases} \dfrac{\partial \mathrm{u}}{\partial t} + \mathrm{A}\dfrac{\partial \mathrm{u}}{\partial \mathrm{x}} = 0 \\ \mathrm{u}(x,0) = \mathrm{u}_0(x) \end{cases}$$

setting

$$\alpha_{k0}(x) = l_k^T \mathrm{u}_0(x)$$

We obtain

$$\begin{aligned} \alpha_k(x,t) &= \alpha_{k0}(x - \lambda_k t) \\ &= \mathrm{l}_k^T \mathrm{u}_0(x - \lambda_k t) \quad 1 \le k \le p \end{aligned}$$

and therefore

$$u(x,t) = \sum_{\mathrm{k}=1}^{\mathrm{p}} I_k^T \mathrm{u}_0 \ (x - \lambda_k t) r_k$$

**Example1:**

In the Riemann problem for the system (2.1) with initial condition

$$\mathrm{u}_0(x) = \begin{cases} \mathrm{u}_l & \mathrm{x} < 0 \\ \mathrm{u}_r & \mathrm{x} > 0 \end{cases}$$

If we define $\alpha_{\mathrm{kl}}$ and $\alpha_{kr}$ , $1 \le k \le p$

$$\mathrm{u}_\mathrm{l} = \sum_{k=1}^{p} \alpha_{kl} r_k$$

$$\mathrm{u}_\mathrm{r} = \sum_{k=1}^{p} \alpha_{kr} r_k$$

we obtain $\alpha_\mathrm{k}(x,t) = \begin{cases} \alpha_{kl} & \mathrm{x} < \lambda_\mathrm{k} t \\ \alpha_{kr} & \mathrm{x} > \lambda_\mathrm{k} t \end{cases}$

So that the solution u of the Cauchy problem is of the form

$$\mathrm{u}(x,t) = w_R\left(\frac{x}{t};\mathrm{u}_1,\mathrm{u}_r\right)$$

for $\lambda_m < \frac{x}{t} < \lambda_{m+1}$

u takes the constant value.

$$w_m = \sum_{k=1}^{m} \alpha_{kr} r_k + \sum_{k=m+1}^{p} \alpha_{kl} r_k$$

with the convection $\lambda_0 = -\infty \quad \lambda_{\mathrm{p+1}} = +\infty$

Hence we have

$$\mathrm{w}_{\mathrm{R}}\left(\frac{x}{t};u_L,u_R\right) = \begin{cases} w_0 = \mathrm{u}_L & \frac{x}{t} < \lambda_1 \\ w_1 & \lambda_1 < \frac{x}{t} < \lambda_2 \\ \vdots & \\ w_p = \mathrm{u}_R & \frac{x}{t} > \lambda_p \end{cases}$$

Which shows the initial discontinuity breaks up into $p$ discontinuity waves , which propagate with characteristic speeds $\lambda_{\mathrm{K}}$

The intermediate states $\mathrm{w}_{\mathrm{m}}$ satisfy

$$w_m - w_{m-1} = (\alpha_{mr} - \alpha_{ml})r_m$$
$$A(w_m - w_{m-1}) = \lambda_m (w_m - w_{m-1})$$

Thus across the line of discontinuity $x = \lambda_m t$ R-H condition is satisfied

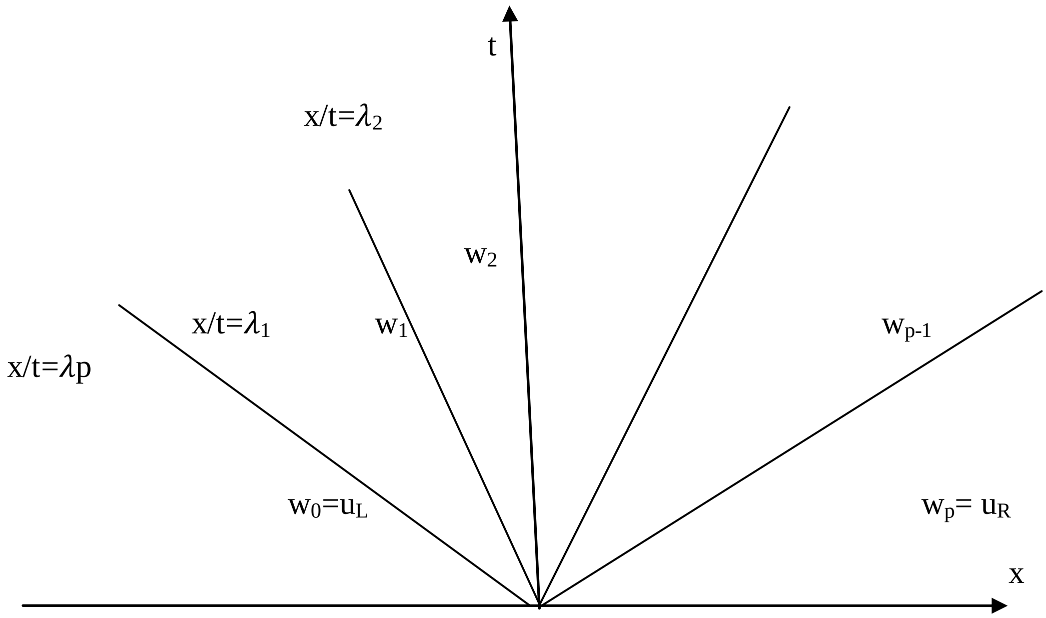


Fig.2.1

## 2.2 NON LINEAR CASE-

Let $\Omega$ be open subset of $R^p$.

$f:\Omega \to R^P$

We consider the nonlinear system of conservation laws

$$\frac{\partial \mathrm{u}}{\partial t}+\frac{\partial}{\partial x}\mathrm{f}(\mathrm{u})=0$$

where $\mathrm{u}=\left(u_1...,u_p\right)^T$

and $\mathrm{f(u)}=\left(f_1(\mathrm{u}),...f_p(\mathrm{u})\right)^T$

$A(\mathrm{u})=\left(\frac{\partial f_i}{\partial u_j}(\mathrm{u})\right)$ has $p$ distinct real eigen values

$\lambda_1(\mathrm{u})<\lambda_2(\mathrm{u})...<\lambda_{\mathrm{p}}(\mathrm{u})$

with each eigenvalue $\lambda_k(\mathrm{u})$ we associate a right vector $r_k(\mathrm{u})$

$A(\mathrm{u})\,r_k(\mathrm{u})=\lambda_k(\mathrm{u})\;r_k(\mathrm{u})$ and a left eigenvector $\mathrm{I_k}(\mathrm{u})$ such that

$\mathrm{I_k}(\mathrm{u})^T A(\mathrm{u})=\lambda_k(\mathrm{u})\mathrm{I_k}(\mathrm{u})^T$

Since the eigenvalues are distinct

$I_J(\mathrm{u})^T r_k(\mathrm{u}) = I_J(\mathrm{u}).r_k(\mathrm{u})$

**DEFINATION:**

**Genuinely Nonlinear**: The kth characteristic field is said to be genuinely nonlinear

If $D\lambda_k(\mathrm{u}).r_k(\mathrm{u}) \neq 0 \quad \forall u \in \Omega$

**Linearly degenerate**: The kth characteristic field is said to be linearly degenerate

If $D\lambda_k(\mathrm{u}).r_k(\mathrm{u}) = 0 \quad \forall \mathrm{u} \in \Omega$

**Example**

**1**: $u_t + f'(u)u_x = 0$

Comparing with $u_t + Au_x = 0$ we get $A = f'(u)$.

So eigen equation $A\mathrm{x} = \lambda\mathrm{x}$

So $\lambda = f'(\mathrm{u})$ and x=1

$$\nabla\lambda = \frac{\partial}{\partial u}\lambda = \frac{\partial f'(\mathrm{u})}{\partial u} = \mathrm{f}''(\mathrm{u})$$

if $\mathrm{f}''(\mathrm{u}) \neq 0 \rightarrow$ genuinely nonlinear

**2**: Consider the following system of equations

$$v_t - u_x = 0$$
$$u_t + p(v)_x = 0$$

**Solution-**

$$\mathrm{w}_t + f(\mathrm{w})_x = 0$$

$$\mathrm{w} = \begin{pmatrix} v \\ u \end{pmatrix}$$

$$\mathrm{f(w)} = \begin{pmatrix} -v \\ p(v) \end{pmatrix}$$

$$\mathrm{A(w)} = \mathrm{A}(v) = \begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix}$$

valid if $p'(v) < 0$

$$\lambda_1 = -\sqrt{-p'(v)}$$
$$\lambda_2 = \sqrt{-p'(v)}$$

Corresponding to eigenvalues

eigenvectors $r_1 = \left(1\ , \sqrt{-p'(v}\right)$ $\quad r_2 = \left(1, -\sqrt{-p'(v)}\right)$

$$\nabla\lambda_1 = \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)\ \left(-\sqrt{-p'(v)}\right)$$

$$= \left(0, \frac{\text{-1}\quad \text{-}p''(v)}{2\sqrt{-p'(v)}}\right)$$

$$\nabla\lambda_1.r_1 = \left(0, \frac{p''(v)}{2\sqrt{-p'(v)}}\right).\left(1, \sqrt{-p'(v)}\ \right) = \frac{p''(v)}{2\sqrt{-p'(v)}}$$

$$\nabla\lambda_2 = \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)\ \left(\sqrt{-p'(v)}\ \right)$$

$$= \left(0, \frac{(\text{-}p''(v))}{2\sqrt{-p'(v)}}\right)$$

$$\nabla\lambda_2.r_2 = \left(0, \frac{(\text{-}p''(v))}{2\sqrt{-p'(v)}}\ \right)\ .\left(1, \text{-}\sqrt{\text{-}\,p'(v)}\ \ \right) = \frac{p''(v)}{2\sqrt{-p'(v)}}$$

If $p''(v) \neq 0$ then genuinely linear.

# CONCLUSION

We have discussed about the conservation laws and its solutions .We have observed the formation of discontinuity when the flux function is nonlinear. We have derived the Rankine-Hugoniot jump conditions. We have shown the Entropy condition for both convex functions and general flux functions.

We have discussed the Riemann Problem and its solutions. We have discussed the Hyperbolic systems of Conservation Laws. We have shown both linear and nonlinear Hyperbolic Systems.

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