PETURBATION TECHNIQUES

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CERTIFICATE

This is to certify that the dissertation entitled "**PERTURBATION TECHNIQUES**" submitted by Mamata Sahu of the Department of Mathematics, National Institute of Technology, Rourkela for the degree of Master of Science in Mathematics is based on the work in the bonafide project work carried out by her under my guidance and supervision.

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DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "PERTURBATION TECHNIQUE" in partial fulfillment of the requirement for the award of the degree of Master of Science, submitted in the Department of Mathematics, National Institute of Technology, Rourkela is an authentic record of my own work carried out under the supervision of Dr. B.K. OHJA

The matter embodied in this thesis has not been submitted by me for the award of any other degree.

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And to all mighty, who made all things.

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ABSTRACT

When we model any physical problems to mathematical form, the governing equations may be linear, nonlinear and nonlinear boundary conditions in complex for known and unknown boundaries. The exact solution of those equations may or may not be obtain. Those equations are not solving for an exact solution then their solutions are approximated by using some techniques namely numerical techniques, analytic techniques, and combinations of both. Foremost among the analytic techniques are the systematic methods of perturbations in terms of a small or a large parameter or co-ordinate. These are called perturbation technique.

CHAPTER 1

1.1 INTRODUCTION

In boundary value problems there are some equations and/or the boundary conditions involve with a parameter ϵ[1]. To find their solutions one particular perturbation method is used. In this method in general studied to be the problem for determining the solution of the equation for $\epsilon = 0$. In general if the order of the equations and the boundary condition remain fixed in the procedure than it is called to be an ordinary or regular perturbation. However if the order of the equation is lowered when $\epsilon = 0$ then the perturbation is called to be singular.

In this technique we expand the function 'f' for which solution is needed in increasing powers of the parameter of the perturbation parameter ϵ. If the equation is non linear then the main aim is to see that the perturbed equations are rendered linear and become solvable exactly. Since ϵ is a small positive real number the higher powers of ϵ are neglected and the perturbed equations after solution yield an approximate estimate for the function f. In many non Newtonian fluid flow problems it is seen that the number of boundary conditions is less than the order and in such cases the highest order differential coefficient is associated with the non-Newtonian parameter R_c. Here generally the expansion of the function whose solution is needed is done in ascending powers of R_c. By this order of the zeroth order perturbed equation yielding the solution for the viscous fluid ($R_c = 0$) is lowered. This enables one to solve this equation. The first order perturbed equation is then solved utilizing the solution for the zeroth order. Since generally the algebra involved becomes complicated one is forced to stop with the solution of the first order perturbed equation.

1.2 LITERATURE REVIEW

Ali Nayfeh, published “Introduction to perturbation Techniques, and Perturbation Methods” in 1981 [2]. J.Kevorkian and J.D.cole, studied Perturbation Methods in Applied Mathematics, in 1981[3]. Donald Smith studied “Singular Perturbation Theory in 1985[4]. Carl M. Bender and Steven A.Orszag, investigate about “Advanced Mathematical Methods for Scientists and Engineers, in 1978[5]. Erich Zauderer studied Partial Differential Equations of Applied Mathematics, in1983 [6]. S Padhi also studied perturbation methods of his PhD, Thesis[1]. M.Van Dyke, studied Perturbation Methods in Fluid Mechanics in 1975.J.Kevorkian, investigate “Perturbation Techniques for Oscilatory systems with slowly varing coefficients, in, 1987[7].

A. Erdelyi studied Asymptotic Expansions, in 1956[8]. M.C. Eckstein, Y.Y. Shi, and J. Kevorkian, investigate about” Satellite motion for all inclinations around a planet,” Procededing of symposium No.25, International Astronomical Union, in 1966[9]. J.Cochran, studied about A new approach to

Singular Perturbation Problem, in 1962[10]. Ali Hasan Nayfeh studied "Introduction to Perturbation Techniques" in 1981[11]. E. J. Hinch studied "Perturbation Methods" in 1991[12]. Mark.H. Holmes in 1995 studied "Introduction to Perturbation methods" [13].

Akulenko, L.D. and Shamayev, A.S in 1986 studied"Aproximated solution of some perturbed boundary value problem [14]. "Kath, W. L, Knessl, C. and Murkowski, B.J.in 1987 studied"A Variation approach to nonlinear singularity Perturbed boundary-value Problem, "studies in Appl. Math[15]. Kato, T. in 1980 investigates" Perturbation Theory for Linear Operators [16]. Keller, H. B. in 1992 studied Numerical Methods for two-point Boundary value problems [17]. Keller, J.B and Kogelman, S. in 1970 studied. "Asymptotic solutions of initial value problems for nonlinear Partial Differential Equation [18].

1.3 AIM OF THE PROJECT

Some equations in mathematics do not possess the exact solution. Such type of equations gives the approximated solution by different methods available in mathematics. Perturbation method is one of them. The aim of this project is to investigate a few problems to find solution approximated by perturbation method. They are

- Incompressible fluid flow on a flat plate.
- The algebraic equation.
- The ordinary differential equation.
- The boundary value problem.

CHAPTER 2

2.1 DIMENSIONAL ANALYSIS

Because of nonlinearities, in homogeneities, and general boundary conditions, exact solutions are rare in many branches of fluid mechanics, solid mechanics, motion, and physics. Hence, engineers, physicists and mathematicians are facing many problems to determine the approximate solutions. These approximation solutions are purely numerical, purely analytical, or a combination of both.

This process involves keeping certain elements, neglecting some, and approximating yet others. To analyze this important step, one needs to decide the order of magnitude of the different elements of the system by comparing them with each other as well as with the basic elements of the system. This process is called non dimensionalization or making the variables dimensionless. Consiquently one should always introduce dimensionless variable before attempting to make any approximate solutions.

For example, if an element has a length of one centimeter, would this element be large or small? We cannot answer this question without knowing the problem being considered. If the problem involves the motion of a satellite in an orbit around the earth, then one centimeter is very small.

On the other hand, if the problem involves intermolecular distances then one centimeter is very large [2]. As a second example, is one gram small or large? Again one gram is very very small compared with the mass of a satellite but it is very large compared with the mass of an electron.

Therefore, expressing the equations in dimensionless form bring out the important dimensionless parameters that govern the behavior of the system. We give an example illustrating the process of non dimensionalization.

Example1:- Two dimensional steady incompressible fluid flow past on a flat plate. The equation of continuity and motion are is given by

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0 \qquad 2.1$$

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \qquad 2.2$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \qquad 2.3$$

And the boundary conditions are

$$u=v=0 \quad \text{at} \quad y=0$$

$$u\to U_\infty, v\to 0 \quad \text{as} \quad x\to-\infty \qquad 2.4$$

Where u and v are the velocity components in the x and y directions, respectively is the pressure, ρ is the density, and μ is the coefficient of viscosity. In this case u, v and p are the dependent variables and x and y is independent variables.

We cannot solve this equation directly. To make the equation from dimensional form to non dimensional form, we use L as a characteristic length, and use U_∞ as a characteristic velocity. We take ρU_∞^2 as a characteristic pressure. Thus we define dimensionless quantities according to

$$u^* = \frac{u}{U_\infty} \quad v^* = \frac{v}{U_\infty} \quad p^* = \frac{p}{\rho U_\infty^2} \quad x^* = \frac{x}{L} \quad y^* = \frac{y}{L} \qquad 2.5$$

With the help of (2.5), the equations (2.1) to (2.4) reduce to

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \qquad 2.6$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{R}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right) \qquad 2.7$$

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{R}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right) \qquad 2.8$$

$$u^* = v^* = 0 \quad \text{at} \quad y^* = 0 \qquad 2.9$$

$$u^* \to 1 \quad v^* \to 0 \quad \text{as} \quad x^* \to -\infty \qquad 2.10$$

Where $R = \frac{\rho U_\infty L}{\mu}$ is called the Reynolds number. Where, R is the dimensionless parameter. For the case of small viscosity, namely μ small compared with $\rho U_\infty L$, R is large and its inverse can be used as a perturbation parameter to determine an approximate solution of the present problem. When the flow is slow, namely $\rho U_{\infty} L$ is small compared with μ, R is small and it can be used as a perturbation parameter to construct an approximate solution of this problem.

CHAPTER 3

ALGEBRAIC EQUATIONS [2]

In this chapter we discuss the approximate solutions of algebraic equations that depend on a small parameter. The solution is represented as an asymptotic expansion in terms of the small parameter. Such expansions are called parameter perturbations. We can solve quadratic, cubic, etc by perturbation parameter methods. To describe the methods we begin by applying it to quadratic equations because their exact solutions are easily obtained for comparison.

3.1 **Quadratic Equations**:-

We begin with quadratic equations because their exact solutions available for comparison. Let us consider an example

Example:-

We have to determine the roots of $x^2 - (3 + 2\in)x + 2 + \in = 0$ 3.1

Where $\in$ is very small. When $\in = 0$ this 3.1 reduce to

$$x^2 - 3x + 2 = (x - 2)(x - 1) = 0 \qquad 3.2$$

Whose roots are $x = 1 \text{ and } 2$ .Equation (3.1) is called the perturbed equation, whereas (3.2) is called unperturbed or reduced equation. When $\in$ is small but finite, we expect the roots to deviate slightly from 1 and 2.

Step1-To determine an approximate solution let us assume the form of the expansion. We assume that the roots have expansions in the form

$$x = x_0 + \in x_1 + \in^2 x_2 + \cdots \qquad 3.3$$

Where the ellipses stand for all terms with powers of $\in^n$ for which $n \geq 3$.We should note that in many physical problems, especially nonlinear problems, determination of higher- order terms is not straightforward . We concerned only with the first few terms in this expansion. Usually, one refers to the first term x_0 as the zeroth-order term, the second term $\in x_1$ as the first-order term, and the third term $\in^2 x_2$ as the second order term.

Step2-

Substituting the expansion (2.3) into the governing equation (2.1).It becomes

$$(x_0 + \in x_1 + \epsilon^2 x_2 + \cdots)^2 - (3 + 2\epsilon)(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) + 2 + \epsilon = 0 \qquad 3.4$$

Step3-

The third step involves carrying out elementary operations such as addition, subtraction, multiplication, exponential and so on and then, collecting coefficients of like powers of ε. Here only terms up to $O(\epsilon^2)$ have been retained, consistent with the assumed expansion. We get

$$({x_0}^2 - 3x_0 + 2) + \epsilon(2x_0x_1 - 3x_1 - 2x_0 + 1) + \epsilon^2(2x_0x_2 + {x_1}^2 - 3x_2 - 2x_1) + \ldots = 0 \quad 3.5$$

Step4-

Equating the coefficient of each power of ϵ to zero. We let $\epsilon \to 0$ in (3.5). The result is

$$x_0{}^2 - 3x_0 + 2 = 0 \qquad 3.6$$

And (3.5) becomes

$$\epsilon(2x_0x_1 - 3x_1 - 2x_0 + 1) + \epsilon^2(2x_0x_2 + x_1{}^2 - 3x_2 - 2x_1) + \cdots = 0$$

Dividing by ϵ gives

$$(2x_0x_1 - 3x_1 - 2x_0 + 1) + \epsilon(2x_0x_2 + x_1{}^2 - 3x_2 - 2x_1) + \cdots = 0 \qquad 3.7$$

Which, upon letting $\epsilon \to 0$, yields $2x_0x_1 - 3x_1 - 2x_0 + 1 = 0$ 3.8

Then, (3.7) becomes

$\epsilon(2x_0x_2 + x_1{}^2 - 3x_2 - 2x_1) + \cdots = 0$ Which when divided by ϵ , yields

$$2x_0x_2 + x_1{}^2 - 3x_2 - 2x_1 + O(\epsilon) = 0 \qquad 3.9$$

Letting $\epsilon \to 0$ in (5.9), we have $2x_0x_2 + x_1{}^2 - 3x_2 - 2x_1 = 0$ 3.10

Step5-

This step involves solving the simplified equations (3.6), (3.8), and (3.10) can be obtained directly from (3.5) by equating the coefficient of each power of ϵ to zero. Equation (3.6) is same as the reduce equation (3.2) and hence are, its solutions are $x_0 = 1,2$.

With x_0 known, (3.8) can be solved for x_1. We know that (5.8) is linear in x_1. In many problems all the perturbation equations are linear, except the first one. When $x_0 = 1$, (3.8) becomes

$x_1 + 1 = 0$ or $x_1 = -1$

By using x_0 and x_1 we can solve the equation (3.10) for x_2. When $x_0 = 1, x_1 = -1$ and (3.10) becomes

$x_2 - 3 = 0$ or $x_2 = 3$

When $x_0 = 2$, (3.8) becomes

$x_1 - 3 = 0$ or $x_1 = 3$

Then, (3.10) becomes $x_2 + 3 = 0$ or $x_2 = -3$

Step6:-

This step involves substituting the values for x_0, x_1, x_2 into the expansion (3.3). When $x_0 = 1, x_1 = -1$ and $x_2 = 3$; therefore (3.3) becomes

$$x = 1 - \epsilon + 3\epsilon^2 + \cdots \qquad 3.11$$

When $x_0 = 2, x_1 = 3$ and $x_2 = -3$ then (3.3) becomes

$$x = 2 + 3\epsilon - 3\epsilon^2 + \cdots \qquad 3.12$$

3.11 and 3.12 are the two approximate solutions of 3.1. We can compare them with the exact solution

$$x = \frac{1}{2}[3 + 2\epsilon \mp \sqrt{\{(3 + 2\epsilon)^2 - 4(2 + \epsilon)\}}]$$

Or $x = \frac{1}{2}\left[3 + 2\epsilon \mp \sqrt{(1 + 8\epsilon + 4\epsilon^2)}\right]$ 3.13

By using binomial theorem, we have

$$(1 + 8\epsilon + 4\epsilon^2)^{\frac{1}{2}} = 1 + \frac{1}{2}(8\epsilon + 4\epsilon^2) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(8\epsilon + 4\epsilon^2)^2 + \cdots$$

$$= 1 + 4\epsilon + 2\epsilon^2 - \frac{1}{8}(64\epsilon^2 + \cdots) = 1 + 4\epsilon - 6\epsilon^2 + \cdots$$

Then 3.13 gives

$$x = \frac{1}{2}(3 + 2\epsilon + 1 + 4\epsilon - 6\epsilon^2 + \cdots)$$

$$= \frac{1}{2}(3 + 2\epsilon - 1 - 4\epsilon + 6\epsilon^2 + \cdots)$$

Or $x = 2 + 3\epsilon - 3\epsilon^2 + \cdots$

$$= 1 - \epsilon + 3\epsilon^2 + \cdots$$

This solution agreement with (3.11) and (3.12).when $\varepsilon = 0$ it is closed with exact solution.

CHAPTER 4

BOUNDARY LAYER PROBLEM [2]

There are number of methods available for treating boundary layer problems including the methods of matched asymptotic expansions, the method of composite expansions, the methods of multiple scales and the Langer transformation. In this chapter we introduce the method of multiple scales only to one example.

We consider the simple boundary value

$$\epsilon y'' + (1+\epsilon^2)y' + (1-\epsilon^2)y = 0 \qquad 4.1$$

$$y(0) = \alpha\,, y(1) = \beta \qquad 4.2$$

Where ϵ a small dimensionless positive number. It is assumed that the equation and boundary conditions have been made dimensionless. We assumed a straight forward expansion in the form $y(x;\epsilon) = y_0(x) + \epsilon y_1(x) + \cdots$ 4.3

Substituting (4.3) into (4.1) and (4.2) it becomes

$$\epsilon(y_0'' + \epsilon y_1'' + \cdots) + (1+\epsilon^2)(y_0' + \epsilon y_1' + \cdots) + (1-\epsilon^2)(y_0 + \epsilon y_1 + \cdots) = 0$$

$$y_0(0) + \epsilon y_1(0) + \cdots = \alpha$$

$$y_0(1) + \epsilon y_1(1) + \cdots = \beta$$

Equating coefficients of like power of t we have

Order of $\epsilon^0 y_0' + y_0 = 0$ 4.4

$$y_0(0) = \alpha, y_0(1) = \beta \qquad 4.5$$

Order of $\epsilon y_1' + y_1 = -y_0''$ 4.6

$y_1(0) = 0, y_1(1) = 0$ 4.7

The general solution of equation (4.4) is

$y_0 = c_0 e^{-x}$ 4.8

Where $c_0$is an arbitrary constant. Here in equation (4.5), there are two boundary conditions on y_0, whereas the general solution of y_0 contains only one arbitrary constant. Thus y_0 cannot satisfy both boundary conditions.

If we impose the condition $y_0(0) = \alpha$ we obtain from equation (8), that

$\alpha = c_0$ so that $y_0 = \alpha e^{-x}$ 4.9

Then imposing the boundary condition $y_0(1) = \beta$, we obtain from (4.8) that

$c_0 = \beta e$ 4.10

Putting (6.10) in (6.8) so that $y_0 = \beta e e^{-x}$

$\Rightarrow y_0 = \beta e^{1-x}$ 4.11

Comparing (6.9) and (6.10), we find that the boundary conditions depends two different value forc_0, i, e $c_0 = \alpha$ and $c_0 = \beta e$, which are inconsistent unless if happens by coincidence that $\alpha = \beta e$.

Comparing equation (4.1) and (4.4), note that the order of the differential equation is reduce from 2nd order, which can cope with two boundary conditions, to 1st order, which can cope with only one boundary condition.

Hence one of the boundary condition cannot be satisfied and must be dropped. Consequently the resulting expansion is not expected to be valid at or near the end point, where the boundary conditions must be dropped. The question arises as to which of the boundary condition must be dropped. This question can be answered using either physical or mathematical arguments as discuss below. When the coefficient ofy' in equation (4.1) is positive, the boundary condition at the left end must be dropped.

Dropping the boundary condition $y(0) = \alpha$

We conclude that $c_0 = \beta e$ and that y_0 is by (4.11). Then equation (4.6) becomes,

$y_1{}' + y_1 = -\beta e^{1-x}$ 4.12

The general solution is $y_1 = c_1 e^{-x} - \beta x e^{1-x}$ 4.13

Again (4.12) is of 1st order. Hence y_1 contains only one arbitrary constant and cope with the two boundary conditions. Now we drop the boundary condition at $x = 0$, use the boundary condition $y_1(1) = 0$, from equation (4.13) we find that $c_1 = \beta e$

Hence $y_1 = \beta(1 - x)e^{1-x}$ 4.14

Substituting the equation (4.11) and (4.14) into (4.13),we obtain

$y = \beta e^{1-x} + \epsilon\beta(1 - x)e^{1-x} + \cdots$ 4.15

At the origin $y = \beta e(1 + \epsilon)$ which is in general different from the α in equation (6.2). Next we determine the exact solution

Exact solution:-

Since equation (4.1) is linear, having constant coefficients its solution can be found by putting $y = e^{sx}$ and we get $\epsilon s^2 + (1+\epsilon^2)s + 1 - \epsilon^2 = 0$

$(\epsilon s + 1 - \epsilon)(s + 1 + \epsilon) = 0$

Hence $s = -(1+\epsilon)$, or $s = -{}^{1}/_{\epsilon} + 1$

and the general solution of equation (4.1) is

$$y = a_1 e^{-(1+\epsilon)x} + a_2 e^{-\left[\left(\frac{1}{\epsilon}\right)-1\right]x} \quad 4.16$$

Where a_1 and a_2 are two arbitrary constant. Using the boundary condition (4.2) in equation (6.16), we have when $x = 0, y = \alpha \therefore \alpha = a_1 + a_2$

$x = 1, \beta = a_1 e^{-(1+\epsilon)} + a_2{}^{-\left(\frac{1}{\epsilon}-1\right)}$

Whose solution is

$$a_1 = \frac{\beta - \alpha e e^{-\left[\frac{1}{\epsilon}-1\right]}}{e^{-(1+\epsilon)-e^{-\left[\frac{1}{\epsilon}-1\right]}}}$$

$$a_2 = \frac{\alpha e^{-(1+\epsilon)-\beta}}{e^{-(1+\epsilon)} - e^{-\left(\frac{1}{\epsilon}-1\right)}}$$

Therefore the exact solution of equation (4.1) is

$$y = \frac{\left[\beta - \alpha e^{-\left[\left(\frac{1}{\epsilon}\right)-1\right]}\right] e^{-(1+\epsilon)x} + \left[\alpha e^{-(1+\epsilon)} - \beta\right] e^{-\left[\frac{1}{\epsilon}-1\right]x}}{e^{-(1+\epsilon)} - e^{-\left[\frac{1}{\epsilon}-1\right]}} \quad 4.17$$

To understand the nature of the no uniformity at the origin in the straight forward expansion, we expand the exact solution (4.17) for smallϵ. We note that $\exp\left(-\frac{1}{\epsilon}\right)$ is smaller than any power of ϵ as $\epsilon \to 0$.Hence equation (4.17) can be written as

$$y = \beta e^{(1+\epsilon)(1-x)} + [\alpha - \beta e^{1+\epsilon}]e^{-\frac{x}{\epsilon}+x} + EST \quad 4.18$$

Where EST stands for exponentially small terms. In deriving the straight forward expansion (4.15) we assumed that x is fixed at a value different from zero and then expanded y for small ϵ.If we keep x is fixed and positive, then $\exp\left(-\frac{x}{\epsilon}\right)$ is exponentially small and (4.18) can be rewritten as

$$y = \beta e^{(1+\epsilon)(1+x)} + EST \quad 4.19$$

Expanding (4.19) for small ϵ, we have

$$y = \beta e^{1-x} e^{\epsilon(1-x)} + EST$$

$$= \beta e^{1-x}\left[1 + \epsilon(1-x) + \frac{1}{2!}\epsilon^2(1-x)^2 + \cdots\right]$$

Hence,

$$y = \beta e^{1-x} + \epsilon\beta(1-x)e^{1-x} + \cdots \quad 4.20$$

As in the case of the straight forward expansion,(4.20) is not valid at or near the origin because $y(0) = \beta e(1+\epsilon)$, which is in general different from the α in the boundary condition in (4.2). Therefore, in the process of expanding (4.17) for small ϵ, we must have performed one or more operations that caused the no uniformity. 1st we assumed that $exp\left(-\frac{1}{\epsilon}\right)$ is exponentially small and arrived at(4.18), which is uniform because at $x = 0$, it gives $y(0) = \alpha$, the imposed boundary condition in equation (4.2). Next we fixed at a positive value that is different from zero, concluded that $\exp\left(-\frac{x}{\epsilon}\right)$ is exponentially small, and obtained (4.19) putting $x = 0$ in equation (6.19), we find that $y(0) = \beta \exp(1+\epsilon)$, which is different from α in (4.2). Hence this step is one responsible for the no uniformity.

The methods of multiple scales:-

In this section we have discussed $y(x,\epsilon)$ depends x and ϵ in the combinations $\frac{x}{\epsilon}$ and ϵx in addition with x and ϵalone. To make the problem ideal for the application of the method of multiple scale and no uniformities will not arise from the presence of the scalar terms $\epsilon x, \epsilon^2 x^2$, ...with the case of infinite domains. Thus it is sufficient to introduce the scale $\varphi = \frac{x}{\epsilon}$, which is this case is same as the inner variable, and $x_0 = x$ which in this case is the outer variable.

In terms of this scales

$$\frac{d}{dx} = \frac{1}{\epsilon}\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial x_0} \qquad 4.21a$$

$$\frac{d^2}{dx^2} = \frac{1}{\epsilon^2}\frac{\partial^2}{\partial\varphi^2} + \frac{2}{\epsilon}\frac{\partial^2}{\partial\varphi\partial x_0} + \frac{\partial^2}{\partial {x_0}^2} \qquad 4.21b$$

Then equation (4.1) becomes

$$\frac{1}{\epsilon}\frac{\partial^2 y}{\partial\varphi^2} + 2\frac{\partial^2 y}{\partial\varphi\partial x_0} + \epsilon\frac{\partial^2 y}{\partial {x_0}^2} + (1+\epsilon^2)\left(\frac{1}{\epsilon}\frac{\partial y}{\partial\varphi} + \frac{\partial y}{\partial x_0}\right) + (1-\epsilon^2)y = 0 \qquad 4.22$$

Here we take first order uniform expansion for y in the form

$$y = y_0(\varphi, x_0) + \epsilon y_1(\varphi, x_0) + \cdots \qquad 4.23$$

As in the nonlinear problem we need to find the term $O(\epsilon)$ to determine the arbitrary functions that appear in y_0. substituting the equation (4.23) into (4.22) and equating coefficients of like powers of ϵ we have

$$\frac{\partial^2 y_0}{\partial \varphi^2} + \frac{\partial y_0}{\partial \varphi} = 0 \quad 4.24$$

$$\frac{\partial^2 y_1}{\partial \varphi^2} + \frac{\partial y_1}{\partial \varphi} = -2\frac{\partial^2 y_0}{\partial \varphi \partial x_0} - \frac{\partial y_0}{\partial x_0} - y_0 \quad 4.25$$

The general solution of the equation (4.24) is given by

$$y_0 = A(x_0) + B(x_0)e^{-\varphi} \quad 4.26$$

Where A and B are undetermined at the level of approximate. putting the value of y_o in (4.25)

It gives

$$\frac{\partial^2 y_1}{\partial \varphi^2} + \frac{\partial y_1}{\partial \varphi} = -(A' + A) + (B' - B)e^{-\varphi} \quad 4.27$$

The particular solution of (4.27) is given by

$$y_{1p} = -(A' + A)\varphi - (B' - B)\varphi e^{-\varphi} \quad 4.28$$

Which makes ϵy_1 much bigger than y_0 as $\varphi \to \infty$. Hence for a uniform expansion, the coefficients of φ and $\varphi e^{-\varphi}$ in (4.28) must be vanish independently. so

$$A' + A = 0 \quad 4.29a$$

$$B' - B = 0 \quad 4.29b$$

The solutions of (4.29) are

$$A = ae^{-x_0}, \;\; B = be^{-x_0} \quad 4.30$$

Where a and b are arbitrary constants. Then (4.26) becomes

$$y_0 = ae^{-x_0} + be^{-\varphi} + x_0$$

In terms of the original variable,

$$y_0 = ae^{-x} + be^{-\left(\frac{x}{\epsilon}\right)} + x \qquad 4.31$$

And imposing the boundary conditions (4.2) becomes

$$\alpha = a + b, \qquad \beta = ae^{-1} + be^{-\left(\frac{1}{\epsilon}\right)+1}$$

Neglecting the small term $e^{\left(-\frac{1}{\epsilon}\right)}$and solving for $a = \beta e, \quad b = \alpha - \beta e$ .Hence from equation (4.31) it follows that, the first approximation,

$$y = \beta e^{1-x} + (\alpha - \beta e)e^{-(x/\epsilon)+x} + \cdots$$

is close agreement with the exact solution.

CHAPTER 5

INTEGRALS

There are many differential and difference equations whose solutions can not be expressed in terms of elementary functions but can be expressed in the form of integrals. There are many methods that can be used to represent the solutions of differential equations as integrals. Before discuss methods of determining approximations of integrals, we show how to represent the solution of a simple differential equation as an integrals.

Let us consider the general solution of the following first order linear ordinary differential equation:

$$y^1 + y = \frac{1}{x} \qquad 5.1$$

The general solution of equation (5.1) is

$$y = ae^{1-x} + e^{-x}\int_1^x \frac{e^t}{t}\,dt \qquad 5.2$$

In this chapter we discuss number of methods for determining approximations to integrals such as ‘’expansions of integrands’ ‘integration by parts, Laplace’s method, the method of stationary phase and the method of steepest descent.

1st we discuss the integration by parts.

Let us consider the incomplete factorial function defined by

$$\mathrm{I}(x) = \int_x^\infty \frac{e^{-t}}{t^2}\,dt \qquad 5.3$$

The method of integration by parts on the identity

$$d(uv) = udv + vdu \qquad 5.4$$

$$\Rightarrow u\,dv = d(uv) - v\,du \qquad 5.5$$

If u and v are function of t, then integrating both sides (5.5) from $t = t_1$ to $t = t_2$

$$\int_{t_1}^{t_2} u\,dv = \int_{t_1}^{t_2} d(uv) - \int_{t_1}^{t_2} v\,du$$

$$\Rightarrow \int_{t_1}^{t_2} u\,dv = uv]_{t_1}^{t_2} - \int_{t_1}^{t_2} v\,du \qquad 5.6$$

To apply equation (5.6),we express the quantity under the integral sign in equation (5.3)as $u\,dv$

i.e $\frac{e^{-t}}{t^2}dt = u\,dv$ To illustrate these points we try two choices

First we let $u = e^{-t}, dv = \frac{dt}{t^2}$ 5.7

Hence $du = -e^{-t}dt, v = -\frac{1}{t}$ 5.8

Substituting (5.7) and (5.8) into (5.6) we get

$$\Rightarrow \int_x^{\infty} \frac{e^{-t}}{t^2}dt = \int_x^{\infty} u\,dv = -\frac{e^{-t}}{t}]_x^{\infty} - \int_x^{\infty} \frac{e^{-t}}{t}dt$$

$$\Rightarrow \int_x^{\infty} \frac{e^{-t}}{t^2}dt = \frac{e^{-x}}{x} - \int_x^{\infty} \frac{e^{-t}}{t}dt \qquad 5.9$$

To continue this process we put

$$u = e^{-t}, dv = \frac{dt}{t} \qquad 5.10$$

Then $du = -e^{-t}dt\ , v = lnt$ 5.11

Then equation (5.6) becomes

$$\int_x^{\infty} \frac{e^{-t}}{t} dt = e^{-t}lnt]_x^{\infty} + \int_x^{\infty} e^{-t}lnt\ dt \qquad 5.12$$

Since

$$\lim_{t\to\infty} e^{-t}lnt = \lim_{t\to\infty} \frac{lnt}{e^t}\left(\frac{0}{0}\right)$$

By L-hospital rule, $\lim_{t\to\infty} \frac{1}{te^t} = 0$

Equation (5.11) becomes

$$\Rightarrow \int_x^{\infty} \frac{e^t}{t} dt = -e^{-x}lnx + \int_x^{\infty} e^{-t}lnt\ dt \qquad 5.13$$

Now substituting the value of $\int_x^{\infty} \frac{e^{-t}}{t} dt$ in equation (5.9) it becomes

$$\int_x^{\infty} \frac{e^{-t}}{t^2} dt = \frac{e^{-x}}{x} + e^{-x}lnx - \int_x^{\infty} e^{-t}lnt\ dt \qquad 5.14$$

At $x \to \infty$ the second term of the right hand side of equation (5.12), is much bigger than the first term. Therefore the above choices (5.7) and (5.9) do not get an asymptotic expansion. Secondly we choose

Let $u = \frac{1}{t^2}\ ,\ dv = e^{-t}dt$ 5.15

$$du = \frac{-2}{t^3} dt\ , v = -e^{-t} \qquad 5.16$$

Now putting (5.15) and (5.16) into (5.6)

$$\int_x^{\infty} \frac{1}{t^2} e^{-t} = -\frac{1}{t^2} e^{-t}]_x^{\infty} - \int_x^{\infty} -e^{-t} \frac{-2}{t^3} dt$$

$$\Rightarrow \int_x^{\infty} \frac{e^{-t}}{t^2} dt = \frac{e^{-x}}{x^2} - 2\int_x^{\infty} \frac{e^{-t}}{t^3} dt \qquad 5.17$$

To continue this process, we let

$$u = \frac{1}{t^3} \ , \ \ dv = e^{-t} dt \qquad 5.18$$

$$du = \frac{-3}{t^4} dt \ , v = -e^{-t} \qquad 5.19$$

Hence equation (5.6) becomes

$$\int_x^{\infty} \frac{1}{t^3} e^{-t} dt = -\frac{e^{-t}}{t^3}]_x^{\infty} - \int_x^{\infty} -e^{-t} \frac{-3}{t^4} dt$$

$$\Rightarrow \int_x^{\infty} \frac{e^{-t}}{t^3} dt = \frac{e^{-x}}{x^3} - 3\int_x^{\infty} \frac{e^{t}}{t^4} dt \qquad 5.20$$

Substituting the value of $\int_x^{\infty} \frac{e^{-t}}{t^3} dt$ from equation (5.20) into (5.17), it becomes

$$\int_x^{\infty} \frac{e^{-t}}{t^2} dt = \frac{e^{-x}}{x^2} - 2\left[\frac{e^{-x}}{x^3} - 3\int_x^{\infty} \frac{e^{t}}{t^4} dt\right]$$

$$\Rightarrow \int_x^{\infty} \frac{e^{-t}}{t^2} dt = \frac{e^{-x}}{x^2} - 2\frac{e^{-x}}{x^3} + 3!\int_x^{\infty} \frac{e^{-t}}{t^4} dt \qquad 5.21$$

Continuing this process, we obtain

$$\int_x^{\infty} \frac{e^{-t}}{t^2} dt = \frac{e^{-x}}{x^2} - \frac{2!e^{-x}}{x^3} + \frac{3!e^{-x}}{x^4} - \frac{4!e^{-x}}{x^5} + \cdots + \frac{(-1)^{n-1} n! e^{-x}}{x^{n+1}} + (-1)^n (n+1)! \int_x^{\infty} \frac{e^{-t}}{t^{n+2}} dt \qquad 5.22$$

Since $t^{n+2} \geq x^{n+2}$ when $x \leq t < \infty$

$$\Rightarrow \frac{1}{t^{n+2}} \leq \frac{1}{x^{n+2}}$$

And $\int_x^\infty \frac{e^{-t}}{t^{n+2}} dt < \int_x^\infty \frac{e^{-t}}{x^{n+2}} dt$

$$\Rightarrow \int_x^\infty \frac{e^{-t}}{t^{n+2}} dt < \frac{1}{x^{n+2}} \int_x^\infty e^{-t} dt$$

$$= \frac{1}{x^{n+2}} [-e^{-t}]_x^\infty$$

$$= \frac{e^{-x}}{x^{n+2}}$$

Then the equation (5.22) can be written as

$$I(x) = e^{-x} \sum_{n=1}^{N} \frac{(-1)^{n-1} n!}{x^{n+1}} + e^{-x} O\left(\frac{1}{x^{N+2}}\right) \qquad 5.23$$

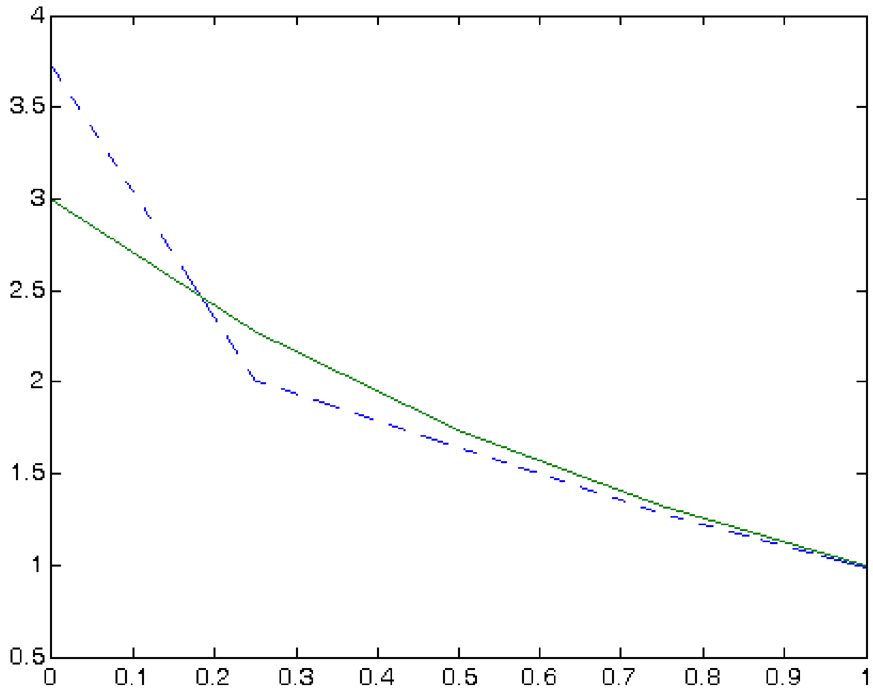
And hence it is an asymptotic expansion. Series (5.23) diverges because

$$\lim_{m\to\infty} \frac{mth\ tern}{(m-1)th\ term}$$

$$= \lim_{m\to\infty} \frac{(-1)^{m-1} m!\, x^m}{(-1)^{m-2} (m-1)!\, x^{m+1}}$$

$$= -\infty$$

RESULT AND DISCUSSION



This graph only for the boundary value problem. The line denotes for the exact solution and doted line denotes for the method of multiple scale.

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