SPATIOTEMPORAL CHAOS IN COUPLED MAP LATTICE

By

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CERTIFICATE

This is to certify that the project thesis entitled " Spatiotemporal chaos in Coupled Map Lattice " being submitted by Itishree Priyadarshini in partial fulfilment to the requirement of the one year project course (PH 592) of MSc Degree in physics of National Institute of Technology, Rourkela has been carried out under my supervision. The result incorporated in the thesis has been produced by developing her own computer codes.

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ABSTRACT

The *sensitive dependence on initial condition*, which is the essential feature of chaos is demonstrated through simple Lorenz model. Period doubling route to chaos is shown by analysis of Logistic map and other different route to chaos is discussed. Coupled map lattices are investigated as a model for spatio-temporal chaos. Diffusively coupled logistic lattice is studied which shows different pattern in accordance with the coupling constant and the non-linear parameter i.e. frozen random pattern, pattern selection with suppression of chaos , Brownian motion of the space defect, intermittent collapse, soliton turbulence and travelling waves.

Contents

1 Introduction

The study of physics has changed in character, mainly due to the passage from the analyses of linear systems to the analyses of nonlinear systems. The importance of these systems is due to the fact that the major part of physical reality is nonlinear. Linearity appears as a result of the simplification of real systems, and often, is hardly achievable during the experimental studies.

The qualitative change took place and boldly evolved after the understanding of the nature of chaos in nonlinear systems.the solution to a deterministic equation can behave in an irregular or chaotic fashion. Small differences can give high consequences. The study of chaos is common in applied sciences like astronomy,meteorology, population biology, and economics.

My study of chaos begun with simple low dimensional system, Lorenz system and Logistic map. The study of these low dimensional dynamical system has been successful in explaining the basic properties of chaos and chaotic behaviour of the system. But there are serious theoretical and practical problem in extending those studies towards understanding dynamical systems that are noisy, non-stationary, inhomogeneous and spatio-temporal. So we go for complex model. In my project, I studied the dynamics of complex systems using coupled map lattice model.

2 Chaos

It is not hard to imagine that if a system is complicated (with many springs and wheels and so forth) and hence governed by complicated mathematical equations, then its behaviour might be complicated and unpredictable. What has come as a surprise to most scientists is that even very simple systems, described by simple equations, can have chaotic solutions. Chaotic processes are not random; they follow rules, but even simple rules can produce extreme complexity. The solutions are, in fact, uniquely determined by the initial conditions, but the effects of small changes in the initial conditions are so amplified by the equations of motion that any finite-precision information about the initial conditions provides no finite precision information about the state of the system at much later times.

An important element in the explanation of the chaotic behaviour of solutions of deterministic equations of motion is the sensitive dependence of solutions on initial condition.

People always misinterpret about deterministic and predictability. they are not the same. An example is the weather. The weather is governed by the atmosphere, and the atmosphere obeys deterministic physical laws. However, long-term weather predictions are not possible. It has been shown that the predictability horizon in weather forecasting cannot be more than two or three weeks. The reason for this unpredictability is that the weather exhibits extreme sensitivity to initial conditions. A tiny change in today's weather (the initial conditions) causes a larger change in tomorrow's weather and an even larger change in the next day's weather. This sensitivity to initial condition is popularly called as the "*Butterfly effect*". This effect is cited as "The flapping of a butterfly wings in Brazil can set off tornadoes in Texas".

A great deal of confusion has arisen from the failure to distinguish between uncertainty in the current state, uncertainty in the value of a parameter, and uncertainty regarding the model structure itself. Technically, chaos is a property of a dynamical system with fixed equations (structure) and specified parameter values. So the uncertainty that chaos acts on is only the uncertainty in the initial state.

The working definition of chaos can be as follows.
"Chaos is *aperiodic long term behaviour* in a *deterministic system* that exhibits *sensitive dependence on initial conditions.*"

"*Aperiodic long term behaviour*" means that the trajectories do not settle down to fixed points, periodic orbits or quasi periodic orbits as $t \to \infty$. "*Deterministic system*" means that The system has no random or noisy inputs or parameters the irregular behaviour arises from system's nonlinearity. "*Sensitive dependence on initial conditions*" means that the nearby trajectories separate exponentially fast, i.e. the system has a positive lyapunov exponent.

3 Lorenz System

I begin my study of chaos by solving a very simple system known as Lorenz system. It originates from a model that aims to describe the stability and the onset of convective or turbulent motion in a fluid heated from the bottom and cooled from the top. The Lorenz model is based on three differential equations, three constants, and three initial conditions. The system represents the behaviour of gas at any given time, and its condition at any given time depends upon its condition at a previous time. The simple Lorenz equations is given by,

$$\dot{x} = \sigma(y - z) \tag{1}$$

$$\dot{y} = rx - y - z \tag{2}$$

$$\dot{x} = xy - bz \tag{3}$$

This simple looking deterministic system could have extremely erratic dynamics. Over a wide range of parameters, the solutions oscillate irregularly, never repeating but always remaining in a bounded region of phase space. When trajectories are plotted in three dimensions [Figure 1], it is seen that they settle down to a complicated set, called as *strange attractor*. The strange attractor is not a point or a curve or even a surface. it's a fractal with a fractional dimension between 2 and 3. It is a set of points with zero volume but infinite surface area.

For the parameter value $\sigma = 10$, b=8/3 , r=28 the solutions are plotted.

After an initial transient, the solution settles into an irregular oscillation that persists as $t \to \infty$, but never repeats exactly. The motion is aperiodic [Figure 2]. When x(t) is plotted against z(t), a butterfly pattern appears [Figure 3]

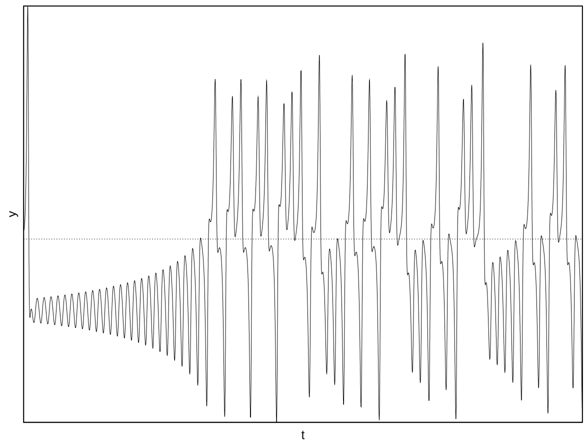

Figure 1: lorenz plot for $t \sim y$

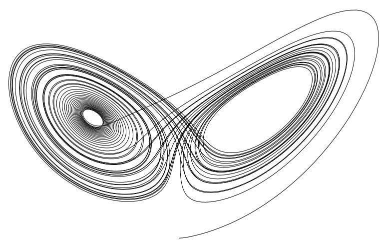

Figure 2: Lorenz plot in three dimension

Lorenz equation can exhibit transient chaos when r=21($\sigma = 10$, b=8/3). At first the trajectory seems to be tracing out a strange attractor, but eventually it stays

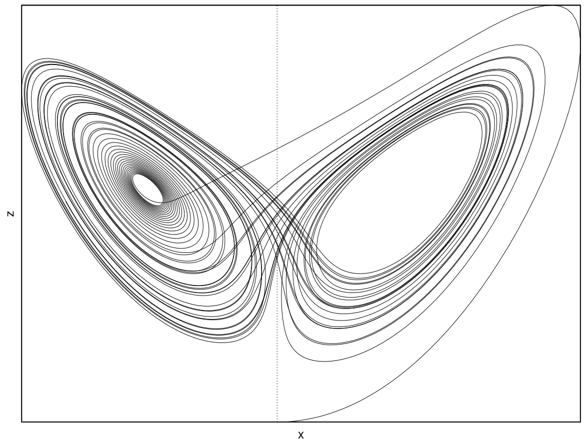

Figure 3: Butterfly pattern

on the right and spirals down towards the stable fixed point C^+. The time series of y vs. t shows an initially erratic solution ultimately damps down to equilibrium. Transient chaos shows that a deterministic system can be unpredictable, even if its final states are very simple.

4 Route to Chaos

Another important issue concerns the specific ways in which chaos sets in the evolution of nonlinear systems. We want to explore How does a (nonlinear) system change its behaviour from regular (either stationary or periodic) to chaotic (or vice versa) as the control parameters of the system are (slowly) changed.

Transitions to Chaos

I Via Local Bifurcations

- A period-doubling
- B Quasi-periodicity
- C intermittency
  1. Type I (tangent bifurcation intermittency)
  2. Type II (Hopf bifurcation intermittency)
  3. Type III (period-doubling intermittency)
  4. Type IV (On-off intermittency

II Via Global Bifurcations

- A Chaotic transients
- B Crises

In the first category of transitions (via **local bifurcations**), a limit cycle occurs for a range of parameter values. As some control parameter of the system is changed, the limit cycle behaviour "disappears" and chaotic behaviour appears.

In the second category (via **global bifurcations**), the long-term behaviour of the system is influenced by unstable fixed points or cycles as well as by an attractor (or several attractors). As a parameter is changed, the transient trajectories, which would eventually end up approaching the fixed point (or cycle) become more and more complicated, producing what we call chaotic transients. These chaotic transients eventually last forever and the long-term behaviour of the system is chaotic.

4.1 Period Doubling

The period doubling route begins with limit cycle behaviour of the system. As some control parameter changes, this limit cycle becomes unstable. As the control parameter is changed further, this period-two limit cycle may become unstable and give birth to a period-four cycle with four Poincare intersection points. The period-doubling process may continue until the period becomes infinite; that is, the trajectory never repeats itself. The trajectory is then chaotic.

4.2 Quasi Periodicity

Quasi-periodic route begins with a limit cycle trajectory. As a control parameter is changed, a second periodicity appears in the behaviour of the system. This bifurcation event is a generalization of the Hopf bifurcation. So, it is also called a Hopf bifurcation. If the control parameter is changed further,the motion becomes chaotic.

4.3 Intermittency

The intermittency route to chaos is characterized by dynamics with irregularly occurring bursts of chaotic behaviour interspersed with intervals of apparently periodic behaviour. As some control parameter of the system is changed, the chaotic bursts become longer and occur more frequently until, eventually, the entire time record is chaotic.

4.4 Crisis

A crisis is a bifurcation event in which a chaotic attractor and its basin of attraction suddenly disappear or suddenly change in size as some control parameter is adjusted. Alternatively, if the parameter is changed in the opposite direction, the chaotic attractor can suddenly appear "out of the blue" or the size of the attractor can suddenly be reduced. A crisis event involves the interaction between a chaotic attractor and an unstable fixed point or an unstable limit cycle.

4.5 Chaotic Transients

In the global bifurcation category, this transition to chaos not (usually) marked by any change in the fixed points of the system or the fixed points of a Poincare section. The transition is due to the interaction of trajectories with various unstable fixed points and cycles in the state space. The common features are homoclinic orbits and heteroclinic orbits. These special orbits may suddenly appear as a control parameter is changed. More importantly, these orbits strongly influence the nature of other trajectories passing near them.

5 Logistic Map

one dimensional map are the discrete time dynamical system. These systems are known as difference equations, recursion relations, iterated maps or simply maps. one such map is "Logistic map" which is the discrete time analogue of population growth model. It is defined by a simple equation as follows,

$$x_{n+1} = ax_n(1 - x_n) \tag{4}$$

The successive points can be found out through iteration. These simple system has been seen to display many of the essential features of deterministic chaos.
for $0 \leq a \leq 4$, the map is a parabola with maximum value of $a/4$ at $x = 1/2$.

5.1 Period doubling

For $1 < a < 3, xn$ grows as n increases,reaching a non-zero steady state.
For larger $a(e.g.a = 3.3)x_n$ repeats every two iterations, called a **period-2 cycle**.
For still larger $a(e.g.a = 3.5)x_n$ repeats every four cycles. The previous cycle has doubled its period to **period 4**.
Further period doublings to cycles of period 8,16,32,... occurs as a increases. Computer experiment shows that

$a_1 =$	3	(period 2 is born)
$a_2 =$	3.44	4
$a_3 =$	3.544	8
$a_4 =$	3.5644	16
.	.	
.	.	
.	.	
$a_\infty =$	3.5699	∞

5.2 Orbit diagram/Bifurcation diagram

Orbit diagram shows the long term behaviour for all values of a at once. At $a = 3.4$, the attractor is a period 2 cycleas indicated by the two branches. As a increases, both branches split simultaneously yielding a period-4 cycle. A cascade of further period doublings occur as a increases, yielding period-8,period-16 and so on. At $a = a_\infty$ the map becomes chaotic and attractor changes from finite points to infinite set of points. For $a > a_\infty$ there is a mixture of order and chaos with *periodic windows* interpersed between chaotic cloud of points. A stable period-3 cycle is observed near $a = 3.83$.

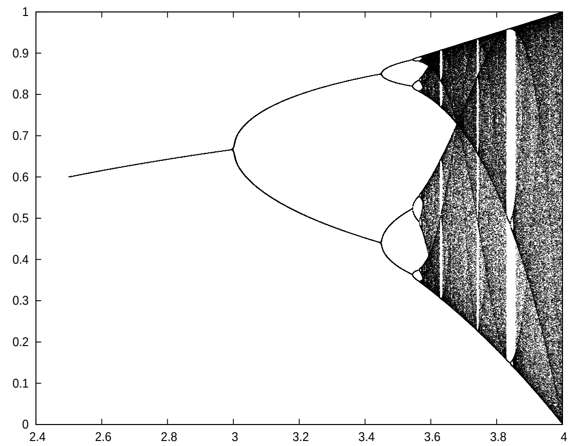

Figure 4: Orbit diagram

The lyapunov exponent λ of logistic map is computed. A graph results as a function of control parameter a , for $3 \leq a \leq 4$.

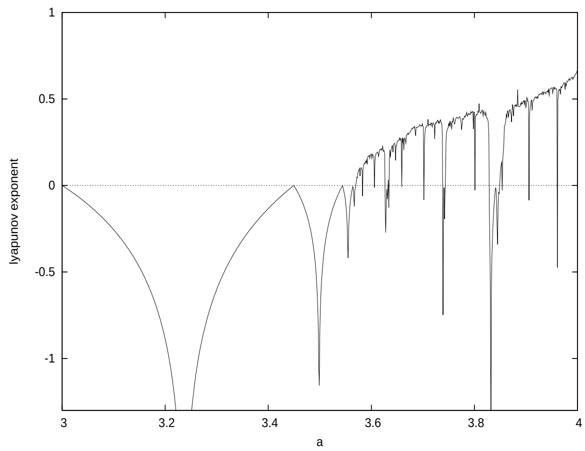

Figure 5: Lyapunov exponent of Logistic map

The interesting feature about orbit diagram is that it repeats itself over and over even at a smaller scale. This behaviour is found to be universal in many dynamical system. So studying the dynamics of logistic system can provide a better insight into a large class of complicated system.

6 Coupled Map Lattice (CML)

The nonlinear dynamics and the chaos theory was successful in understanding low dimensional complex dynamics (e.g. Lorenz system and Logistic map). For high dimensional complex system, a new model would be required. In recent years, study of higher dimensional spatially extended system has become an interesting and challenging subject. This is because, these systems can show some interesting behaviours such as spatiotemporal chaos, patterns formation, travelling waves, spiral waves, turbulence and so on. the control and synchronism of such behaviours have extensive and great potential of interdisciplinary applications, such as security communication, laser, many fluid dynamics, biological systems, crystal growth, information processing, chemical reactor, biochemistry, medicine and engineering.

Spatially extended systems are often modeled by partial differential equations (PDE), ordinary differential equations (ODE), cellular automata (e.g. cellular neural networks (CNNs)) and coupled map lattices (CML). Such systems are proven to be useful in studying qualitative properties of spatially extended dynamical systems. They can easily be simulated on a computer, and able to capture some essential qualitative feature of physical systems. Many remarkable results about coupled map lattices were obtained by researchers working in different areas of physics, biology, mathematics, and engineering.

Model	**Space**	**Time**	**State**
PDE	Continuous	Continuous	Continuous
Coupled ODE	Discrete	Continuous	Continuous
CML	Discrete	Discrete	Continuous
Cellular Automata	Discrete	Discrete	Discrete

CML is a dynamical system with discrete time (map), discrete space (lattice)and continuous state. It consists of dynamical elements on a lattice, which interact (coupled) with suitably chosen sets of other elements. They were introduced by K. Kaneko in 1983 as simple models with essential features of spatio-temporal chaos.

Modelling of physical phenomena through CML is done by following steps.
1. Choose a (set of) field variable on a lattice.
2. Decompose the phenomenon of interest independent units (e.g. convection ,reaction, diffusion etc).
3. Replace each unit by simple parallel dynamics on a lattice, where the dynamics consists of a nonlinear transformation of the field variable at each lattice point.
4. Finally carry out each unit dynamics successively.

The CML equation is,

$$x_{n+1}(i) = (1-\epsilon)f(x_n(i)) + \epsilon/2[f(x_n(i+1)) + f(x_n(i-1))] \tag{5}$$

Here 'n' is discrete time step and 'i' is the discrete lattice points (i=1,2,3,...N), where 'N' is the system size.
f(x) is the local logistic map.

$$f(x) = 1 - ax^2 \tag{6}$$

As shown by the basic model, the dynamics of a CML is governed by two competing terms; an individual temporal nonlinear reaction represented by f and a spatial interaction (coupling) with variable intensity ϵ . The coupling factor ϵ tries to homogenize the system while the nonlinear parameter a tries to make the system chaotic. When these parameters are varied then different patterns are formed which are the universality classes in CML model.

6.1 Unique Universality classes in CML

The unique qualitative classes discovered in CML system are as follows.
1. Frozen random pattern
2. Pattern selection with suppression of chaos
3. Selection of zigzag pattern and chaotic diffusion of defects
4. Spatiotemporal intermittency
5. Soliton Turbulence
6. Travelling waves

6.1.1 Frozen random pattern

Here parts of lattice are chaotic and parts of lattice are periodic. As the nonlinear parameter of logistic map a is increased, the system exhibits period doubling. This leads to formation of domains of various sizes. Distribution of domain sizes depend upon the initial condition. In large domain motion is chaotic and in smaller domains period 8,4,2 are found. The orbits are frozen or fixed in space. [Figure 6]

6.1.2 Pattern selection with suppression of chaos

Here the nonlinearity parameter a is increased slightly. so that that the larger domain split up into smaller domains where chaos is suppressed. the domain sizes are almost uniform and the size depends on the parameter value. [Figure 7 & 8]

6.1.3 Selection of zigzag pattern and chaotic diffusion of defects

During the transient time regime, defects exist as a domain boundary between two zigzag patterns with different phases of oscillations. Defects pair annihilate and the domain size of a connected zigzag region increases with time. A defect is localized, but moves in space. Its motion is well described by Brownian motion. [Figure 9]

6.1.4 Spatiotemporal intermittency

Here the value of coupling term is reduced. Each site transits between coherent state and chaotic state intermittently. Intermittency denotes irregular alterations between temporally simple and irregular phases. space-time point can be classified as either laminar or turbulent. Laminar is a regular pattern whereas in Turbulent there is no regularity either in space or time. This intermittency is the route to Spatiotemporal chaos. [Figure 10]

6.1.5 Soliton turbulence

Here turbulent burst occurs. A fully developed spatiotemporal chaos is observed. Almost all the orbits or sites oscillate chaotically independent of one another. [Figure 11]

6.1.6 Travelling waves

When the coupling constant is large, the domain structures are no longer fixed in space, but move with some velocity. At nonlinearity parameter value corresponding to frozen random pattern (i.e. $a = 1.5$), the motion is irregular. In pattern selection regime (i.e. $a = 1.8$), the motion is regular. [Figure 12 & 13]

7 Conclusion

The attractor of the isolated one dimensional logistic map shows period doubling route to chaos with periodic windows interspersed which is a universal feature of hump like map. Diffusion tends to homogenize a system while chaotic system makes the systems in homogeneous. These two tendencies conflict with each other. Diffusively coupled logistic lattice shows different pattern in accordance with the coupling constant and the non-linear parameter i.e. frozen random pattern, pattern selection with suppression of chaos , Brownian motion of the space defect, intermittent collapse, soliton turbulence and travelling waves.

8 References

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3. J.R. Chazottes, B. Fernandez, "Dynamics of Coupled Map Lattices and of Related Spatially Extended Systems", Springer, 2005

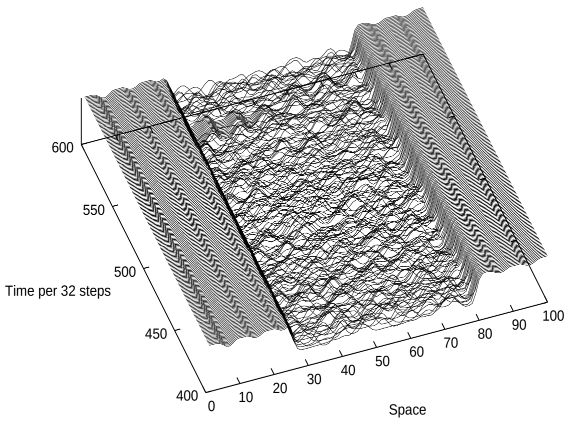

Figure 6: Frozen random pattern

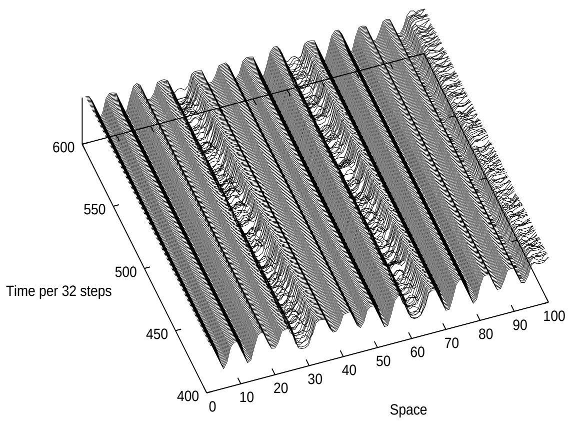

Figure 7: Pattern selection(1)

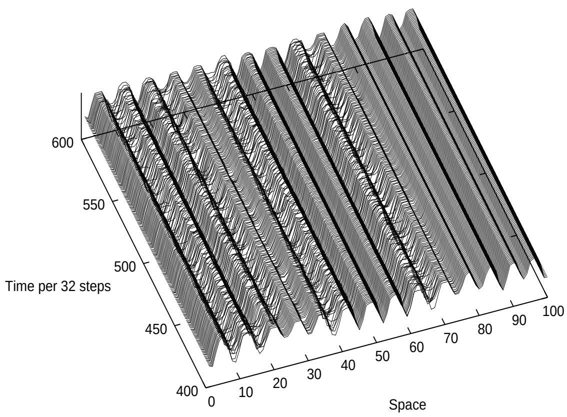

Figure 8: Pattern selection(2)

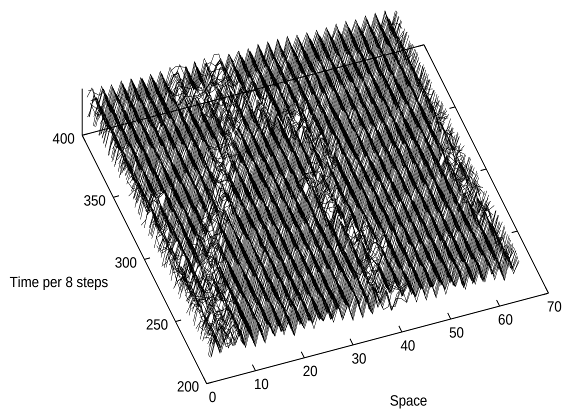

Figure 9: zigzag pattern

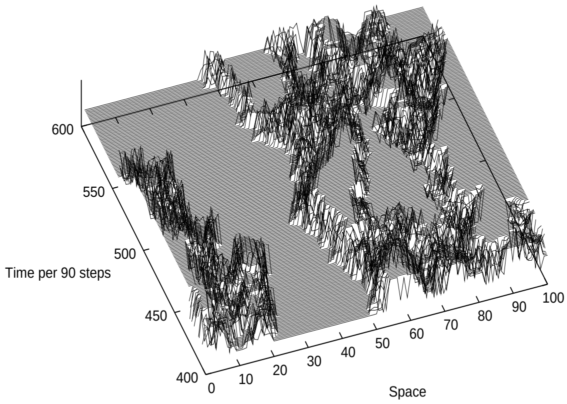

Figure 10: Spatiotemporal intermittency

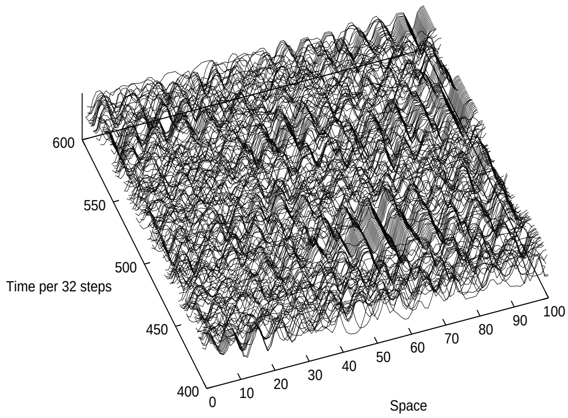

Figure 11: Soliton turbulence

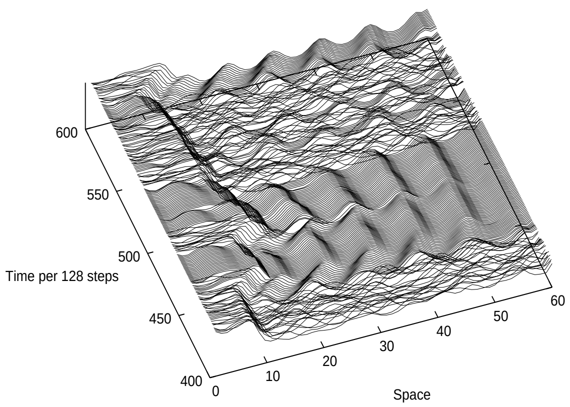

Figure 12: Travelling waves(1)

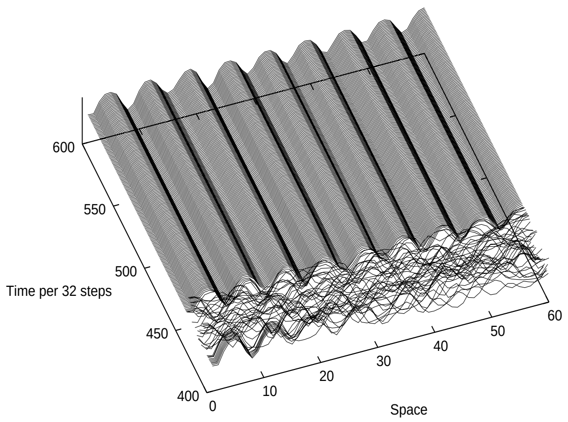

Figure 13: Travelling waves(2)