# A BRIEF STUDY ON DYNAMICS OF VISCOELASTIC ROTORS – AN OPERATOR BASED APPROACH

A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

**Master of Technology**
**In**
**MECHANICAL ENGINEERING**

[Specialization: Machine Design and Analysis]

**By**

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Under the supervision of

**Dr. HARAPRASAD ROY**


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# CERTIFICATE

This is to certify that the thesis entitled, " A BRIEF STUDY ON DYNAMICS OF VISCOELASTIC ROTORS – AN OPERATOR BASED APPROACH " submitted by Mr. VIVEK SINGH in partial fulfillment of the requirements for the award of MASTER OF TECHNOLOGY Degree in "MECHANICAL ENGINEERING" with specialization in "MACHINE DESIGN AND ANALYSIS" at the National Institute of Technology, Rourkela (India) is an authentic Work carried out by him under my supervision and guidance.

To the best of my knowledge, the matter embodied in the thesis has not been submitted to any other University / Institute for the award of any Degree or Diploma.

Date:

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I am especially indebted to my parents for their love, sacrifice, and support. They are my first teachers after I came to this world and have set great examples for me about how to live, study and work.

**VIVEK SINGH**

# CONTENTS

# LIST OF FIGURES

# A Brief Study on Dynamics of Viscoelastic Rotors - An Operator Based Approach


## ABSTRACT

Viscoelasticity, as the name implies, is a property that combines elasticity and viscosity or in other words such materials store energy as well as dissipates it to the thermal domain when subjected to dynamic loading and most interesting the storage and loss of energy depends upon the frequency of excitation. Modeling of viscoelastic materials is always difficult whereas modeling the elastic behavior is easy, modelling the energy dissipation mechanism possess difficulty. This work attempts to study the dynamics of a viscoelastic rotor-shaft system considering the effect of internal material damping in the rotor. The rotation of rotors introduces a rotary damping force due to internal material damping, which is well known to cause instability in rotor-shaft systems. Therefore, a reliable model is necessary to represent the rotor internal damping for correct prediction of stability limit of spin speed and unbalance response amplitude of a rotor-shaft system. An efficient modelling technique for viscoelastic material, augmenting thermodynamic field (ATF) has been found in literature.

Here the material constitutive relationship has been represented by a differential time operator. Use of operators enables to consider general linear viscoelastic behaviours, represented in the time domain, for which, in general, instantaneous stress and its derivatives are proportional to instantaneous strain and also its derivatives. The operator may be suitably chosen according to the material model. The constitutive relationships for ATF approach is represented in differential

time operator to obtain the equations of motion of a rotor-shaft system after discretizing the system using beam finite element method. The equations thus developed may easily be used to find the stability limit of spin speed of a rotor-shaft system as well as the time response as a result of unbalance when the rotor-shaft system is subjected to any kind of dynamic forcing function.

In this work dynamic behavior of an aluminium rotor is predicted through viscoelastic modelling of the continuum to take into account the effect of internal material damping. To study the dynamics of an aluminium rotor-shaft system stability limit of spin speed, unbalance response amplitude and time response are used as three indices. It is observed that, the operator based approach is more suitable for finding the equation of motion of a viscoelastic rotor which is used to predicts the dynamic behaviour of that continuum.

# CHAPTER ONE

# Introduction

## 1.1 Background and Importance

**Rotordynamics** is a specialized branch of applied mechanics concerned with the behavior and diagnosis of rotating structures. It is commonly used to analyze the behavior of structures ranging from jet engines and steam turbines to auto engines and computer disk storage. At its most basic level rotordynamics is concerned with one or more mechanical structures (rotors) supported by bearings and influenced by internal phenomena that rotate around a single axis. The supporting structure is called a stator. As the speed of rotation increases the amplitude of vibration often passes through a maximum that is called a critical speed. This amplitude is commonly excited by unbalance of the rotating structure; everyday examples include engine balance and tire balance. If the amplitude of vibration at these critical speeds is excessive catastrophic failure occurs. In addition to this, turbomachinery often develops instabilities which are related to the internal makeup of turbomachinery, and which must be corrected. This is the chief concern of engineers who design large rotors. Texts by Goodwin (1989), Kramer (1993), Lalanne and Ferraris (1998) and Vance (1988) are valuable sources of information on dynamic behaviour of rotor-shaft systems and their analysis.

Unlike the viscoelastic structures (which do not spin) viscoelastic rotors are acted upon by rotating damping force generated by the internal material damping, that tends to destabilize the rotor shaft system by generating a tangential force proportional to the rotor spin speed. Thus a reliable model is necessary to represent the constitutive relationship of a rotor-material by taking into account the internal material damping for understanding the dynamic behaviour of a viscoelastic rotor. Such a model is useful for getting an idea about safe speed ranges of rotation, where the rotor is stable. Expression of the constitutive relationships in the frequency domain, i.e. the stress-strain relationship under a sinusoidal excitation at a fixed frequency is well known from literature, but such constitutive relationships are not useful for analysis of transient response for investigating stability.

This work presents the development of equations of motion of a rotor-shaft-system with a viscoelastic rotor after discretizing the system into finite elements. Subsequently these equations are used to study the dynamics of the rotor-shaft system in terms of stability limit of spin speed and time response of a disc as a result of unbalance. For this, the material constitutive relationship has been represented by a differential time operator. Use of operators enables one to consider general linear viscoelastic behaviours, represented in the time domain by multi-element (3, 4 or higher elements) spring-dashpot models or internal variable models (ATF), for which, in general, instantaneous stress and its derivatives are proportional to instantaneous strain and its derivatives. Again such representation is fairly generic, in a sense that the operator may be suitably chosen according to the material model to obtain the equations of motion of a rotor-shaft system.

## 1.2 Linear viscoelasticity

Viscoelasticity, as the name implies, is a property that combines elasticity and viscosity. A material, which is viscoelastic in nature, thus stores and also dissipates energies and therefore the stress in such materials is not in phase with the strain. For this reason, it is extensively used in various engineering applications for controlling the amplitude of resonant vibrations and modifying wave attenuation and sound transmission properties, increasing structural life through reduction in structural fatigue. Nakra (1998) has reported many such applications.

The classical theory of elasticity states that for sufficiently small strains, the stress in an elastic solid is proportional to the instantaneous strain and is independent of the strain rate. In a viscous fluid, according to the theory of hydrodynamics, the stress is proportional to the instantaneous strain rate and is independent of the strain. Viscoelastic materials exhibit solid and fluid behavior. Such materials include plastics, amorphous polymers, glasses, ceramics, and biomaterials (muscle). Viscoelastic materials are characterized by constant-stress creep and constant-strain relaxation. Their deformation response is determined by both current and past stress states, and conversely, the current stress state is determined by both current and past deformation states. The Stress-Strain Curves for a purely elastic and a viscoelastic material are shown in figure 1.1. Due to loss of energy during loading and unloading time, the stress strain curve for viscoelastic material is elliptic in nature (Thompson and Dahleh (1998)). The area enclosed by the ellipse is a hysteresis loop and shows the amount of energy lost (as heat) in a loading and unloading cycle

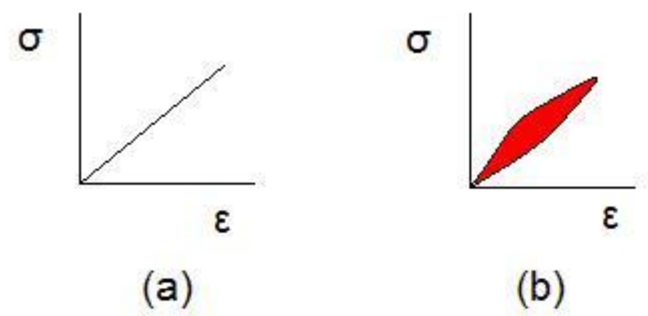


**Figure 1.1 The stress strain curve**

Some phenomena in viscoelastic materials are:

I. If the stress is held constant, the strain increases with time (creep);
II. If the strain is held constant, the stress decreases with time (relaxation);
III. The effective stiffness depends on the rate of application of the load;
IV. If cyclic loading is applied, hysteresis (a phase lag) occurs, leading to dissipation of mechanical energy;
V. Acoustic waves experience attenuation;
VI. Rebound of an object following an impact is less than 100%;
VII. During rolling, frictional resistance occurs.

Viscoelastic behaviour has elastic and viscous components modeled as linear combinations of springs and dashpots, respectively. These models, which include the Maxwell model, the Kelvin-Voigt model, and the Standard Linear Solid Model, are used to predict a material's response under different loading conditions. The Maxwell model of a one dimensional viscoelastic material consists of a linear spring and a linear dashpot connected in series as shown in Figure 1.2a. The Kelvin-Voigt model is consists of a linear spring and a linear damper connected in parallel (Figure 1.2b). More than one spring or linear dampers are included in the model to better approximate material behavior over a broad frequency range. The three-element model of a standard viscoelastic solid is shown in Figure 1.2c. It consists of a linear spring K1 in series with a linear Kelvin element (spring K and dashpot C in parallel). The four-element model is shown in Figure 1.2d. 2-element Voigt model simulates only the creep and 2-element Maxwell's model simulates only the stress relaxation behaviours. Generally 3 or higher element models are capable of depicting both constant-stress creep and constant-strain relaxation behaviours as shown by viscoelastic solids. Bland (1960), Christensen (1982) and Shames and Cozzareli (1992) and many others have given different network comprising linear springs and dashpots to model linear viscoelastic solids.

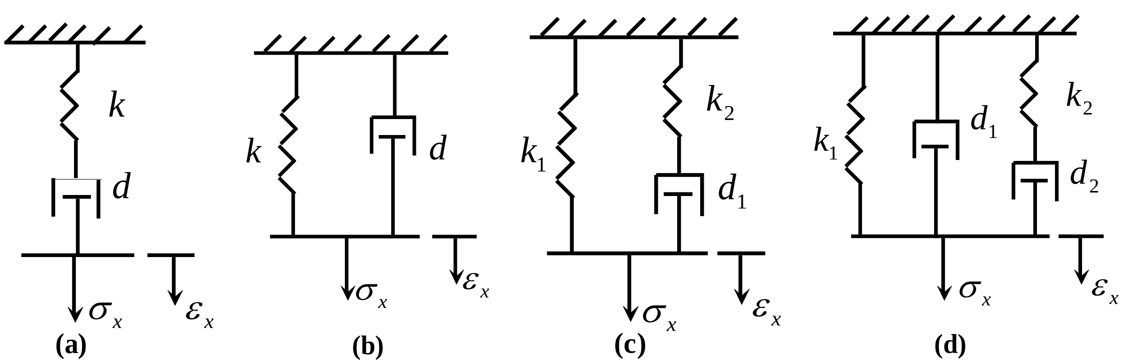


**Figure 1.2 Various spring-dashpot model**

Each model differs in the arrangement of these elements, and all of these viscoelastic models can be equivalently modeled as electrical circuits. In an equivalent electrical circuit, stress is represented by voltage, and the derivative of strain (velocity) by current. The elastic modulus of a spring is analogous to a circuit's capacitance (it stores energy) and the viscosity of a dashpot to a circuit's resistance (it dissipates energy). All materials exhibit some viscoelastic response. In common metals such as steel or aluminium, as well as in quartz, at room temperature and at small strain, the behaviour does not deviate much from linear elasticity. Which may be classified as weakly viscoelastic as the amount of energy dissipated is much less than the energy stored whereas synthetic polymers, wood, and human tissue as well as metals at high temperature display significant viscoelastic effects. Those are generally strongly viscoelastic as energy dissipated is substantially more. In some applications, even a small viscoelastic response can be significant. To be complete, an analysis or design involving such materials must incorporate their viscoelastic behaviour.

## 1.3 Material damping and modelling techniques

Vibration damping is essential to the attainment of performance goals for a variety of advanced engineering systems. In terms of performance, higher damping reduces steady state vibration level at resonance and time needed for vibration to settle. The dynamic analysis of structural components with viscoelastic damping treatment has been subject for the treatment of many years. In common built-up structures that operate in the atmosphere, air damping and joint damping typically dominate system damping. However material damping can also be an important contributor to overall damping in many applications, such as aerospace vehicles, large space structures, etc. Viscoelastic material damping is generally a complex function of frequency, temperature, type of deformation, amplitude and structural geometry.

Damping is the dissipation of mechanical energy and is produced by the some nonconservative forces acting on a given structure. Damping may be classified into external and internal, depending upon the nature of the non-conservative forces acting on the structure. External damping is caused by forces acting on the object, such as damping due to air resistance or Coulomb damping due to friction. Internal damping is caused by physical phenomena intimately linked to the structure of the material. There are various types of models for predicting the material damping. Viscous damping law where the damping force is a function of excitation frequency. For structural materials, under harmonic excitation dissipate energy proportional primarily to the square of excitation amplitude. Lazan (1968) has reported system damping values for different structural materials under different types of deformations. A very good survey has been observer in Rusovici (1999).

The material moduli for linear viscoelastic materials depend on frequency and temperature, but not on strain and stress levels. If harmonic stress and strain states are considered, then the material moduli can be expressed in term of complex number. Where the real part is the material storage modulus and the ratio between imaginary and real part is the loss modulus. The material loss factor is equal to the tangent of the phase angle between the harmonic stress input and the corresponding harmonic strain. Both the loss and storage moduli are frequency and temperature dependent. The frequency and temperature dependence of the complex modulus components must be captured accurately; simple models such as Maxwell or Kelvin model are not able to do that. The Maxwell model behaves like a fluid at low frequencies, while the Kelvin model becomes infinitely stiff at high frequencies. Prediction accuracy increases by adding more springs and dampers. Though the frequency-domain description is simple to use, but faces difficulties when the forcing function contains more than one frequency components (as in the case of periodic excitation) and also for obtaining the transient response (Bert (1973)).

## 1.4 Damping modelling in finite element analysis

The structural response of a system is determined in computational structural dynamics by solving a set of n simultaneous, time-dependent equations. One of the first methods to model damping is to consider proportional damping (Thompson and Dahleh (1998)). In this method, the damping matrix is considered to be proportional to the mass and stiffness matrices through two coefficients (Rayleigh's coefficients).which are calculated directly from the modal damping ratios. The damping ratios are the ratio of actual damping to critical damping for a particular mode of vibration. The drawback of this method is that the damping matrix depends on the arbitrarily determined parameters, and thus has no physical motivation.

Segalman (1987) uses a perturbation approach to model linear viscoelastic structures. Hereditary integrals are used in the equation of motion. Henwood (2002) tried to represent a hysteretic damping matrix by a viscous damping matrix (for values of loss factor 0.4) and extended the work of Crandall (1970) for single-degree-of-freedom to a general structure. The actual energy dissipation mechanism is, however, very complicated. In the internal variable approach the effect of energy dissipation is taken care of by including a dissipation coordinate. Under this approach Bagley and Torvik (1983, 1985) use differential operators of fractional order to model linear frequency dependence of viscoelastic materials using finite elements approach with four empirical model parameters. Time-dependent stresses and strains are related by derivatives of fractional orders. This model connected the molecular theories for uncross linked polymer solids and linear viscoelastic models through fractional calculus. The stiffness matrix of a viscoelastic element was built and the ensuing finite element equations were obtained in the frequency domain. The solutions of the system of equations had to be transformed from the frequency domain back into the time domain. Only loads that have a Laplace transform may be included in subsequent analyses. The Bagley-Torvik approach is only applicable for uncross linked polymer solids of linear viscoelastic materials.

Padovan (1987) based his computational algorithms for finite element analysis of viscoelastic structures on fractional integral-differential operators. Viscoelastic, displacement based, time domain finite element model results, which may be solved for any type of loading. The Padovan model is valid only for viscoelastic materials that may be modeled through fractional calculus.

The model developed by Golla and Hughes (1985) incorporated the hereditary integral form of the viscoelastic constitutive law in a finite element model. The finite element equations are derived in the Laplace domain through the Ritz technique. The time domain equations are obtained from the frequency domain equations by the linear theory of realizations. The method yields a system of second-order matrix differential equations. Internal dissipation coordinates augment the stiffness, damping, and mass matrices. This technique is applicable only for linear viscoelastic materials and may be applied only when the system is initially at equilibrium. McTavish and Hughes (1992, 1993) extended the Golla-Hughes model and formulated the GHM (Golla-Hughes-McTavish) model for linear viscoelastic structures. This new formulation the material modulus is modeled as the sum of mini-oscillators are characterized by a viscous damper, spring and mass. The motion of the mass represents the internal dissipation coordinate. Again, this method leads to a system of second order differential equations of motion in the frequency domain, where the mass, stiffness, and damping matrices are augmented by the internal dissipation coordinates.

Initially, the methodology for modeling frequency-dependent material damping in structural dynamics was motivated by the material science results. Such observation led to the inclusion of augmenting thermodynamic fields (ATF) that could interact with the usual mechanical displacement fields. This method, developed by Lesieutre (1989), had its foundation on the irreversible thermodynamics. ATF method to model frequency dependent material damping of linear viscoelastic structures in a finite element context.

In the first ATF paper introduced by Lesieutre and Mingori (1990), one-dimensional formulation of the ATF model was developed. Lesieutre (1992) further developed the ATF model to include the behavior of high-damping materials and also to demonstrate the use of multiple augmenting fields. The material constitutive equations are derived from the Helmholtz free energy function. The ATF evolution equation is determined from the irreversible thermodynamics assumption that the rate of change ATF is proportional to its deviation from an equilibrium value. The coupled equations of motion are discretized using the method of weighted residuals. The ATF parameters are obtained by iteratively curve fitting or by minimization of the error between the complex modulus to frequency dependent experimental values of storage modulus and loss factor (Roy (2008)). The descriptions of frequency dependent storage modulus and loss factor for different polymeric materials are given by Ferry (1980) and Nashif et. al. (1985) and others. Roy et al. (2008) used the ATF approach to model a viscoelastic continuum of a rotor shaft, to obtain the equations of motion and studied the dynamic behaviour in terms of stability limit of spin speed and unbalance response amplitude.

Lesieutre and Bianchini (1995), Lesieutre et. al. (1996) (approach was developed to extend the ATF method to three-dimensional state) developed a time-domain model of linear Viscoelasticity based on a decomposition of the total displacement into elastic and anelastic parts. They used the motion of anelastic displacement field (ADF) to describe the part of the strain that was not instantaneously proportional to stress. The total and anelastic displacement fields were used to build differential equations for the mass particles and the relaxation of the ADFs respectively, resulting in coupled governing equations that were not explicitly time dependent. The ADF approach is the extension of ATF approach, is more straightforward development in finite element context. The displacement field was made of an elastic component and an anelastic component, where anelastic field is introduced to take in to account the dissipation.

Just like in the ATF model, the ADF parameters are obtained by curve fitting or minimization of errors between ADF complex moduli to the corresponding experimental data (Roy (2008)). The equations of motion are discretized by the weighted residuals method to obtain a finite element model. Multiple ADF may be used to better approximate experimental frequency dependent modulus data for a broad range of frequency.

In the recent paper, Roy et. al. (2009) used the both ATF and ADF approaches to model the viscoelastic beam. This method also applied to study the dynamic behaviour of composite beam comprising of several viscoelastic layers. All the models, ATF, ADF and GHM employ additional coordinates to model damping more accurately, whereas the ‘dissipation coordinate’ of GHM is internal to individual elements, while these are continuous in ATF or ADF from element to element. For this advantage ATF/ ADF approaches are used in this work to represent the viscoelastic material behavior.

## 1.5 Important researches on rotor dynamics

The history of rotordynamics is replete with the interplay of theory and practice. W. J. M. Rankine first performed an analysis of a spinning shaft in 1869, but his model was not adequate and he predicted that supercritical speeds could not be attained. In 1895 Dunkerley published an experimental paper describing supercritical speeds. Gustaf de Laval, a Swedish engineer, ran a steam turbine to supercritical speeds in 1889, and Kerr published a paper showing experimental evidence of a second critical speed in 1916.

The Jeffcott rotor (named after Henry Homan Jeffcott), also known as the de Laval rotor in Europe, is a simplified lumped parameter model used to solve these equations. The Jeffcott rotor is a mathematical idealization that may not reflect actual rotor mechanics. Henry Jeffcott was commissioned by the Royal Society of London to resolve the conflict between theory and practice. He published a paper now considered classic in the *Philosophical Magazine* in 1919 in which he confirmed the existence of stable supercritical speeds. August Föppl published much the same conclusions in 1895, but history largely ignored his work.

Between the work of Jeffcott and the start of World War II there was much work in the area of instabilities and modeling techniques culminating in the work of Prohl and Myklestad which led to the Transfer Matrix Method (TMM) for analyzing rotors. The most prevalent method used today for rotordynamics analysis is the Finite Element Method. Nelson (2003) has written extensively on the history of rotordynamics and most of this section is based on his work.

There are many software packages that are capable of solving the rotordynamic system of equations. Rotordynamic specific codes are more versatile for design purposes. These codes make it easy to add bearing coefficients, side loads, and many other items only a rotordynamicist would need. The non-rotordynamic specific codes are full featured FEA solvers, and have many years of development in their solving techniques. The non-rotordynamic specific codes can also be used to calibrate a code designed for rotordynamics.

## 1.6 Internal damping and its effect on rotor dynamics

All types of damping associated to the non-rotating parts of the structure have a usual stabilizing effect. On the other hand damping associated to rotating parts can trigger instability in supercritical ranges. Rotation of rotors introduces a rotary damping force due to internal material damping, which is well known to cause instability in rotor-shaft systems. Thus a reliable model is necessary to represent the rotor internal damping for correct prediction of stability limit of spin speed (SLS) of a rotor-shaft system. Rotary machines, such as motors, compressors and turbines are very common and widely used. Recently the designers of these machines have been required to meet very severe specifications from the demands of high speed operating power or improvements in efficiency and reliability for the design. In such situations, finding some robust and reliable mathematical models, in conjunction with special numerical solution procedures, which enable designers to make an accurate assessment of the relevant parameters, the critical speeds and the dynamic behavior of the system, especially the response of the system to unbalance excitation, is of great importance in order to design for increased speeds of rotation, to optimize weight, to improve reliability, and to reduce maintenance costs.

Modelling the rotor internal damping using the viscous and hysteretic model has been attempted by many researchers [Dimentberg (1961), Tondl (1965), Genta (2005)]. Most of the authors have considered, in general, viscous form of internal damping and used 2-element Voigt model for representing the material behaviour to study the dynamics of rotor-shaft systems. Again Zorzi and Nelson (1977), Ozguven and Ozkan (1984), Ku (1998) developed a finite element model of the rotor material damping by representing its constitutive relationship with a Voigt model (2-element model) where internal material damping force was considered as a superposition of viscous and hysteretic damping forces to take into account the frequency dependent and frequency independent components of energy dissipation per cycle for properly representing the properties of structural materials like steel. In this regard Genta (2004) pointed out the correct interpretation and use of the hysteretic damping model.

However viscous and hysteretic damping models are unsuitable for proper representation of viscoelastic material behaviour, which shows considerable dependence on wide range of excitation frequencies. Not many papers are found to report dynamic simulation of viscoelastic rotors. Grybos (1991) used 3-element material model and studied the dynamics of a viscoelastic rotor. Roy et al. (2008) reported a finite-element approach, where viscoelastic behaviour of a rotor-continuum was represented by ATF (Augmenting Thermodynamic Field). Recently Dutt and Roy (2010) obtained the equation of motion of viscoelastic rotor-shaft system after discretizing the continuum by finite beam element method. The rotor-shaft material was assumed to behave as a linear viscoelastic solid for which the instantaneous stress was obtained by operating the instantaneous strain by a generic linear differential time operator. The advantage of using a generic operator approach is that it may be suitably tailored according to the material constitutive relationship to obtain the equations of motion for a particular material model.

## 1.7 Layout of the present work

As aluminium is normally taken as an elastic material but a careful modelling as done by Lesieutre (1989) and Lesieutre and Mingori (1990) considered aluminium as a viscoelastic material. Viscoelastic modeling of aluminium rotor-shaft is also important from the point of view of predicting the SLS of the rotor-shaft system. Concepts of finite element of rotor shaft-system as given by Zorzi and Nelson (1977) , Nelson and McVaugh (1976) and Rao JS (1996) are extended to take into account the internal material damping using ATF parameters.

In this present reporting, an attempt has been made to study theoretically the stability limit of the spin speed, unbalance vibration response and subsequently time response within the stable zone of operation of a simply supported aluminium rotor-shaft having a central disc made of aluminium. The rotor-shaft material is assumed to behave as a linear viscoelastic solid for which the instantaneous stress is obtained by operating the instantaneous strain by a linear differential time operator. The internal variable approach i.e. ATF is used for modelling the viscoelastic material. The constitutive relationships for ATF approach is represented in differential time operator, where the coefficients of the operator are formed by ATF parameters. The equations of motion of a rotor-shaft system are obtained after discretizing the continuum using finite beam element. So this work is useful for dynamic analysis of viscoelastic rotors under any type of dynamic forcing function.

Based on that review, the objective and scope proposed in this work are as follows.

(a) Finding out the equations of motion and Development Finite Element formulation of the viscoelastic rotor . Operator based constitutive relationship is obtained by using augmenting thermodynamic fields (ATF) approach. The equations of motion is developed by using that constitutive relationship. Finite element method is used to discretised the rotor continuum.

(b) Dynamic behaviour of an aluminium rotor is predicted through viscoelastic modelling of the continuum to take into account the effect of internal material damping. Stability limit of spin speed, unbalance response amplitude and time response are found to study the dynamics.

# CHAPTER TWO

## Viscoelastic Rotor and Its Modelling

# CHAPTER TWO

# Viscoelastic Rotor and Its Modelling

This chapter forms the basis of the entire work as it presents the derivation of the equations of motion of a viscoelastic rotor and Development Finite Element analysis procedure. The material constitutive relationship has been represented by a differential time operator, where the instantaneous stress and its derivatives are proportional to instantaneous strain and also its derivatives. The operator based constitutive relationship plays an important role in developing the equation of motion in the time domain. The finite element method is used to discretised the rotor continuum. The dynamic behaviour of the rotor shaft system includes the stability limit of spin speed, unbalance response amplitude and time response within the stable zone of operation.

## 2.1 Constitutive relationships

The constitutive relationships are obtained from the Helmholtz free energy density function, $\mathcal{H}$ representing a thermodynamic potential, where strain (ε) is an independent variable. The function, $\mathcal{H}$ is defined as [Lesieutre and Mingori (1990)]

$$H = \frac{1}{2}E\varepsilon^2 - \delta\varepsilon\xi + \frac{1}{2}\alpha\xi^2 \tag{1}$$

$$\sigma = \frac{\partial H}{\partial \varepsilon} = E\varepsilon - \delta\xi \tag{2a}$$

$$A = -\frac{\partial \mathcal{H}}{\partial \xi} = \delta\varepsilon - \alpha\xi \qquad 2b)$$

In the preceding equation E is the un-relaxed modulus, σ is the mechanical stress, ξ is the ATF, $A$ is the affinity, α is a material property relating the changes in ***A*** to ξ and δ is the strength of coupling between the mechanical displacement field and the thermodynamic field.

With the assumption [Lesieutre and Mingori(1990)] that $\dot{\xi}$, the rate of change of ξ with time, varies proportionally to its deviation from an equilibrium value for irreversibility of the thermodynamic action, a first order differential equation or relaxation equation is given as

$$\dot{\xi} = SA = -B(\xi - \bar{\xi}) \tag{3}$$

In the preceding equation $B$ is the inverse of relaxation time, $S$ is a constant of proportionality, and $\bar{\xi}$ denotes the values of $\xi$ at equilibrium where $A$ = 0. Putting $A$ = 0 in equations (2b) the values of $\bar{\xi}$, is obtained as $\bar{\xi} = \frac{\delta}{\alpha}\varepsilon$.

Substituting the values of $\bar{\xi}$ in equation (3), the first order relaxation equation is given as

$$\dot{\xi} + B\xi = \frac{B\delta}{\alpha}\varepsilon_x \tag{4}$$

Equation (4) is a first-order linear non-homogeneous differential equation of ξ with respect to time 't'. Equation (4) can be rewritten as

$$\xi = \left(\frac{B\delta}{\alpha}\right)\frac{1}{B+D}\varepsilon_x \tag{5}$$

Where $D=\frac{d}{dt}$, is the first order differential time Operator. Putting values of $\xi$ from equation (5) in equation (2a), the constitutive relationship is rewritten as:

$$\sigma_x = \frac{B\left(E - \dfrac{\delta^2}{\alpha}\right) + ED}{B + D}\varepsilon_x \tag{6}$$

Any standard linear viscoelastic solid can be represented as a combination of multiple linear springs and linear dashpots. The material modulus is a function of operator of time. The general constitutive relation for a linear viscoelastic solid can be expressed as

$$\sigma_x = E(\,)\varepsilon_x \qquad \text{………} \ (7)$$

The instantaneous normal stress $\sigma_x$ is obtained from the equation (7) i.e. by operating the expression of $\varepsilon_x$ by the operator *E* ( ), called the modulus operator. The generic form of modulus operator is expressed in equation (8) below, where *num*(D) and *den*(D) are the numerator and denominator polynomials of differential time operator, D ≡ d/dt. For the special case of linear elastic behaviour, *E* is a constant called the Young's modulus.

The generalized expression of modulus operator for any viscoelastic solid is written as

$$E() = \frac{\mathrm{num(D)}}{\mathrm{den(D)}} = \frac{\sum_{j=0}^{n} a_j \mathrm{D}^j}{\sum_{j=0}^{m} b_j \mathrm{D}^j} \tag{8}$$

Mechanical models for physical representation of viscoelastic behaviour are given by Bland (1960) among others. Expressions of *E* ( ), for 2, 3, and 4-element models for example, as shown in Figure 2.1, are given below.

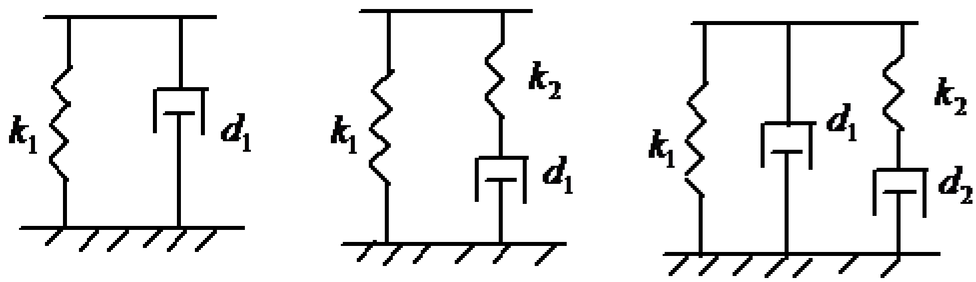


**Figure 2.1: Different Viscoelastic Models**

$$E_2() = a_0 + a_1\mathrm{D}, a_0 = k_1, a_1 = d_1 \tag{9a}$$

$$E_3() = \frac{a_0 + a_1\mathrm{D}}{(b_0 + b_1\mathrm{D})}, a_0 = k_1, a_1 = d_1 + \frac{k_1 d_1}{k_2}, b_0 = 1, b_1 = \frac{d_1}{k_2} \tag{9b}$$

$$E_4() = \frac{a_0 + a_1\mathrm{D} + a_2 D^2}{(b_0 + b_2\mathrm{D})}, a_0 = k_1, a_1 = d_1 + d_2 + \frac{k_1 d_2}{k_2},\ a_2 = \frac{d_1 d_2}{k_2}, b_0 = 1, b_1 = \frac{d_1}{k_2} \tag{9c}$$

For the nomenclature, all springs and dampers directly connected to the ground are called ‘primary’ and those connected in series are called ‘secondary’. Following this, all springs and dashpots with subscript ‘1’ are primary and the ones with subscript ‘2’ are secondary.

The modulus operator for single ATF model

$$E(\ ) = \frac{B\left(E - \frac{\delta^2}{\alpha}\right) + ED}{B + D} \tag{10}$$

The equation (10) represents a three element model. On comparing equation (9b) and equation (10), the polynomial coefficient for numerator and denomerator are given as

$$a_0 = B\left(E - \left(\frac{\delta^2}{\alpha}\right)\right), \qquad a_1 = E, \qquad b_0 = B, \qquad b_1 = 1$$

Using the mechanical analogy (Lesieutre et. al. (1996)) as shown in figure 2.2, which represents the model used in a single ATF approach.

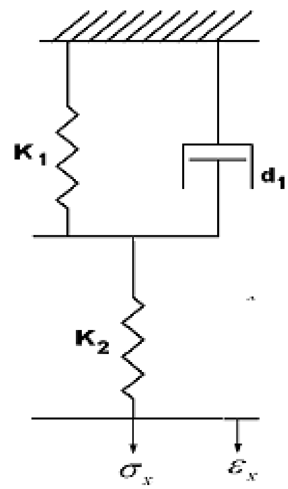


**Figure 2.2: 3-Element model**

Where, $K_2 = E$, $k_1 = \left(\frac{\alpha E}{\delta^2} - 1\right)E$, $d_1 = \frac{\alpha E^2}{B\delta^2}$

## 2.2 Equations of motion

The equations of motion of a rotor-shaft system were found out by Zorzi and Nelson (1977) by considering viscous and hysteretic components of internal material damping. A Voigt element (a 2-element spring-dashpot representation) was used to represent the stress is the sum of two parts one is proportional to strain and another to the strain rate. Thus the representation forms a special case of operator function. For general viscoelastic materials, however, the stress strain relationship is governed by operating an operator, which is a function of D, of which the viscous damping law is a special case (as the modulus operator E( ) for a 2-element Voigt model is obtained by putting $b_1 = 0$ in equation (9b). Hence, in this work the procedure of Zorzi & Nelson (1977) for the viscous damping case has been extended for a general viscoelastic solid by substituting the modulus operate for the 2-element model by modulus operator function for 3-

element models as define above. This process is quite general in a sense that any journal modulus operator may be substituted once the configuration of the spring- dashpot network is decided.

A rotor shaft system has been considered. Figure 2.3 shows the position of a shaft cross section defined by the coordinates of its centre (*v, w*) at a distance *u* from any suitable reference and an element of differential radial thickness *dr* and at a distance *r* (where *r* varies from 0 to $r_0$) subtending an angle d(Ω*t*) (where Ω*t* varies from 0 to 2π) at any instant of time 't'. The cross-section of rotor undergoes two simultaneous rotations due to whirl (Ω) and the spin (ω) respectively radian per second.

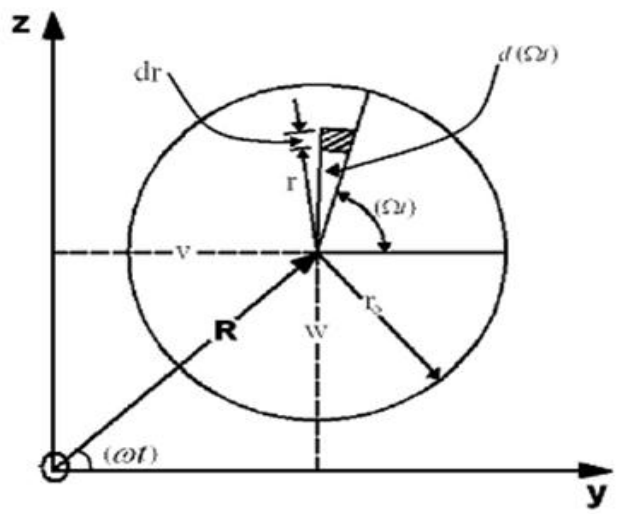


**Figure 2.3 Displaced position of the shaft cross section**

$\sigma_x$, $\varepsilon_x$ denote, respectively, the mechanical stress and strain induced in the element at the instant of time. Zorzi and Nelson (1977) express the mechanical strain in the 'x' direction as

$$\varepsilon_x = -r\cos(\Omega t - \omega t)\frac{\partial^2 R(x,t)}{\partial x^2} \tag{11}$$

Where, R is the displacement of the rotor centre line, ω is the whirl speed.

Zorzi and Nelson (1977) obtained the bending moment expressions after considering a 2-element material model (Voigt model) to represent the constitutive-relationship of the material. The bending moment expressions at any instant of time about the *y* and *z*-axes, $M_{yy}$ and $M_{zz}$ respectively, are expressed as given below.

$$\left.\begin{aligned} M_{yy} &= \int_0^{2\pi}\int_0^{r_0}(w + r\sin(\Omega t))\sigma_x r dr d(\Omega t) \\ M_{zz} &= \int_0^{2\pi}\int_0^{r_0} -(v + r\cos(\Omega t))\sigma_x r dr d(\Omega t) \end{aligned}\right\} \quad \text{............ (12)}$$

The instantaneous bending moments are written next by extending the work by Zorzi and Nelson (1977). Substituting $\sigma_x$ from equation (6) in the bending moment expressions (equations (12)) and utilizing the expressions of $\varepsilon_x$ given in equation (11), the bending moment expressions are rewritten as

$$\begin{aligned} M_{zz} &= \int_0^{2\pi}\int_0^{r_0} -\left(v + r\cos\left(\Omega t\right)\right)\frac{a_0 + a_1 \mathrm{D}}{b_0 + b_1 \mathrm{D}}\left[-r\cos\left(\Omega t - \omega t\right)\frac{\partial^2 R(x,t)}{\partial x^2}\right] r dr d(\Omega t) \\ M_{yy} &= \int_0^{2\pi}\int_0^{r_0}\left(w + r\sin\left(\Omega t\right)\right)\frac{a_0 + a_1 \mathrm{D}}{b_0 + b_1 \mathrm{D}}\left[-r\cos\left(\Omega t - \omega t\right)\frac{\partial^2 R(x,t)}{\partial x^2}\right] r dr d(\Omega t) \end{aligned} \quad \text{.......... (13)}$$

It may be noted in equation (13) that the operator *E* ( ) is operated exclusively on the terms inside the bracket [ ] containing the expression of strain to give the stress. The operator does not work on other terms, ($v + r\cos(\Omega t)$ and $w + r\sin(\Omega t)$) forming the momentums in the respective planes at any instant of time,‘*t*’. Hence the expressions of momentums are perceived as constants as far as the operator $E$ ( ) is concerned. Following this logic the equation (13) may be rewritten as

$$M_{zz} = \frac{1}{b_0 + b_1 \mathrm{D}} \int_0^{2\pi} \int_0^{r_0} r^2 \left(v + r\cos(\Omega t)\right)[a_0 \cos(\Omega t - \omega t)\frac{\partial^2 R}{\partial x^2} + a_1 \cos(\Omega t - \omega t)\frac{\partial^3 R}{\partial x^2 \partial t}$$
$$- a_1 (\Omega - \omega)\sin(\Omega t - \omega t)\frac{\partial^2 R}{\partial x^2}]dr d(\Omega t)$$

$$M_{yy} = -\frac{1}{b_0 + b_1 \mathrm{D}} \int_0^{2\pi} \int_0^{r_0} r^2 \left(w + r\sin(\Omega t)\right)[a_0 \cos(\Omega t - \omega t)\frac{\partial^2 R}{\partial x^2} + a_1 \cos(\Omega t - \omega t)\frac{\partial^3 R}{\partial x^2 \partial t}$$
$$- a_1 (\Omega - \omega)\sin(\Omega t - \omega t)\frac{\partial^2 R}{\partial x^2}]dr d(\Omega t)$$

After performing the integration

$$M_{zz} = \frac{I}{b_0 + b_1 \mathrm{D}}[a_0 v'' + a_1 \dot{v}'' + \Omega a_1 w'']$$

$$M_{yy} = -\frac{I}{b_0 + b_1 \mathrm{D}}[a_0 w'' + a_1 \dot{w}'' - \Omega a_1 v'']$$

In matrix form

$$\begin{Bmatrix} M_{zz} \\ M_{yy} \end{Bmatrix} = \frac{I}{b_0 + b_1 \mathrm{D}} \left[ \begin{bmatrix} a_0 & a_1\Omega \\ a_1\Omega & -a_0 \end{bmatrix} \begin{Bmatrix} v'' \\ w'' \end{Bmatrix} + \begin{bmatrix} a_1 & 0 \\ 0 & -a_1 \end{bmatrix} \begin{Bmatrix} \dot{v}'' \\ \dot{w}'' \end{Bmatrix} \right] \qquad \text{........... (14)}$$

## 2.3 Finite element formulation

For developing the equations of motion, the rotor-shaft continuum is discretized by using beam finite elements. Due to the motion in *x-y* and *z-x* plane of the rotor, the finite element formulation approximates the mechanical coordinates in two planes simultaneously. Each element contains two nodes at the ends and 4 degrees of freedom, which are the displacements and slopes in x-y and z-x planes, as shown in Figure 2.5.

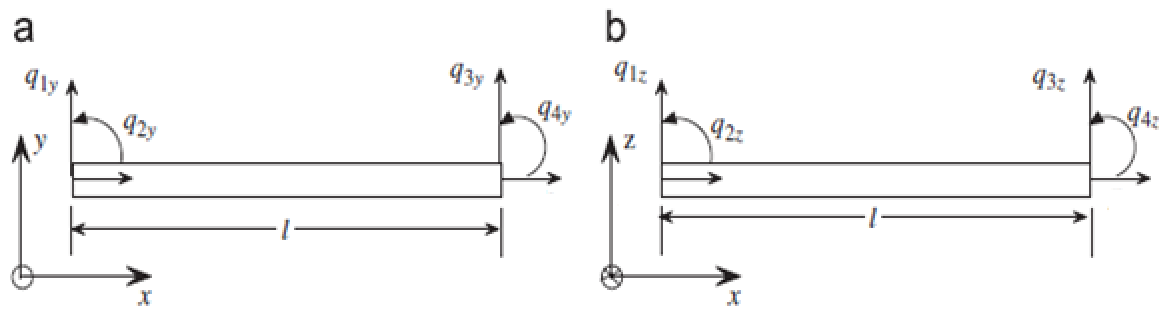


Figure.2.4. Planer beam bending element

The expression of translations and relations with rotations are given as

$$\begin{Bmatrix} v(x,t) \\ w(x,t) \end{Bmatrix} = [\phi(x)]^T \{q(t)\}, \quad \Phi = -\frac{\partial w}{\partial x}, \quad \Gamma = \frac{\partial v}{\partial x} \qquad \text{……….} (15)$$

Where $v$ *and* $w$ denote the deformations along $y$ and $z$ axes and $\Phi$, $\Gamma$ are the rotations about the $y$ and $z$ axes respectively. The Hermite shape function is given as

$$[\phi(x)] = \begin{bmatrix} \{\phi_{xy}(x)\} & \{0\} \\ \{0\} & \{\phi_{zx}(x)\} \end{bmatrix},$$

with subscripts in the elements showing the respective planes.

Assuming that rotor is rotating at a uniform speed (Ω). Using the expressions for differential bending energy as shown in equation (16), along with the expressions of *v(x, t)* and *w(x, t)* and its spatial derivatives from preceding equation (where $\{q\}_{(8x1)}$ are nodal displacement vector (as the rotor continuum is discretized using beam finite elements).

$$dP_B{}^e = \frac{1}{2}\begin{Bmatrix}\Gamma' \\ \Phi'\end{Bmatrix}^T \begin{Bmatrix}M_{zz} \\ M_{yy}\end{Bmatrix} \tag{16}$$

The equations of motion may easily be written using complex coordinates. Expressions of the stiffness, damping and circulatory matrices due to bending are obtained from the strain energy and dissipation function calculated from the expression of bending moments given in the equation (14). Diagonal elements of each coefficient matrix in this equation give rise to a direct matrix (e.g. direct stiffness, direct damping matrix) whereas the cross-diagonal elements give rise cross coupled matrices; reference (Ozguven and Ozkan (1984)) may be seen for details. The expressions of generalized force vectors comprising of forces and moments acting in the 'x-y' and 'z-x' planes i.e. $\left\{[\overline{\mathbf{F}}_{xy}]_{(1x4)}\,[\overline{\mathbf{F}}_{xy}]_{(1x4)}\right\}^T$ . Composition of stiffness, circulatory as well as damping matrices are also shown below.

$$\begin{Bmatrix}\{\overline{\mathrm{F}}_{xy}\}_{(1x4)} \\ \{\overline{\mathrm{F}}_{zx}\}_{(1x4)}\end{Bmatrix} = \frac{I}{(b_0 + b_1\mathrm{D})}\left[a_0[\mathrm{K_b}]_{(8x8)}\{q\}_{(1x8)} + a_1\Omega[\mathrm{K_c}]_{(8x8)}\{q\}_{(1x8)} + a_1[\mathrm{K_b}]_{(8x8)}\{\dot{q}\}_{(1x8)}\right] \quad \text{…….. (17)}$$

The expression of $[K_b]$ and $[K_c]$ are given as

$$[K_b] = \int_0^l I\,[\phi''(x)][\phi''(x)]^T\,dx, \quad [K_c] = \int_0^l I\,[\phi''(x)]\begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix}[\phi''(x)]^T\,dx$$

Operating by the operator *den*(D) = ($b_0$ + $b_1$D) throughout and arranging terms with same orders of differentiation together, the equations of motion of one shaft element are given below. Assuming the rotor is rotating at a uniform speed (Ω).

$$b_1[M]\{\dddot{q}\}+\left(b_0[M]+b_1[G]\right)\{\ddot{q}\}+\left(b_0[G]+a_1[K_b]\right)\{\dot{q}\}$$
$$+\left(a_0[K_b]+a_1\Omega[K_c]\right)\{q\}=\left(b_0+b_1\mathrm{D}\right)\{P\} \qquad \text{......... (18)}$$

Where, ***[M]***$_{(8x8)}$ **=** ***[M***$_{T}$***]***$_{(8x8)}$ **+** ***[M***$_{R}$***]***$_{(8x8)}$ is the inertia matrix, ***[M***$_{T}$***]***$_{(8x8)}$ is the translational mass matrix, ***[M***$_{R}$***]***$_{(8x8)}$ is the rotary inertia matrix, ***[G***$_{T}$***]***$_{(8x8)}$ is the gyroscopic matrix. Effects due to simultaneous action of spin and vibratory motion the rotary inertia and gyroscopic matrix are taken into account. The expressions of translational mass matrix, rotary inertia matrix and gyroscopic matrix are given below after following Rao (1996).

$$\left[M_T\right]=\int_0^l \rho A\phi(x)\phi(x)^T\,dx, \quad \left[M_R\right]=\int_0^l \rho I\phi'(x)\phi'(x)^T\,dx, \quad \left[G\right]=\int_0^l 2\rho I\phi'(x)\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\phi'(x)^T\,dx$$

Where, ρ is the density of the material, *A* is the cross-sectional area. The symbols (•) and (′) stand for single partial differentiation with respect to time, $\frac{\partial}{\partial t}$ and space, $\frac{\partial}{\partial x}$ respectively.

It may be noted that the 3rd order differential equation result in this process. Dutt and Nakra (1992) also obtained a third order differential equation as they modelled the rotor supports of an elastic rotor-shaft using a 4-element spring-dashpot model.

Again equation (18) may be further combined and rewritten as

$$[\mathcal{A}]\{\dot{\mathcal{X}}\}+[\mathcal{B}]\{\mathcal{X}\}=\{\mathcal{P}\} \qquad (19)$$

$$[\mathcal{A}]=\begin{bmatrix}[A_3] & [0] & [0]\\ [0] & [A_3] & [0]\\ [0] & [0] & [A_3]\end{bmatrix},\ [\mathcal{B}]=\begin{bmatrix}[0] & -[A_3] & [0]\\ [0] & [0] & -[A_3]\\ [A_0] & [A_1] & [A_2]\end{bmatrix},\ \{\mathcal{X}\}=\begin{Bmatrix}\{q\}\\ \{\dot{q}\}\\ \{\ddot{q}\}\end{Bmatrix},\ \{\mathcal{P}\}=(b_0+b_1\mathrm{D})\begin{Bmatrix}\{0\}\\ \{0\}\\ \{P\}\end{Bmatrix}$$

# CHAPTER THREE

# Results and Discussion

## 3.1. The Rotor-Shaft System

A rotor shaft system as shown in figure 3.1, is made of aluminium ($E$ = 7.13e10 Pa, $\rho$ = 2750 Kg/m$^3$), has been considered. The rotor shaft (Length = 0.75 m, Diameter = 0.05 m) is mounted at the ends in bearings considered as simply supported ends. The aluminium disc (Disc diameter = 0.15 m, Disc thickness = 0.03 m) is put centrally and has an unbalance, $U$ = 10 gm-mm. Following Lesieutre and Mingori (1990) the ATF parameters of aluminium are $B$ = 8000 sec$^{-1}$, $\alpha$ = 8000 Pa, $\delta$ = 4.7766e6 Pa.

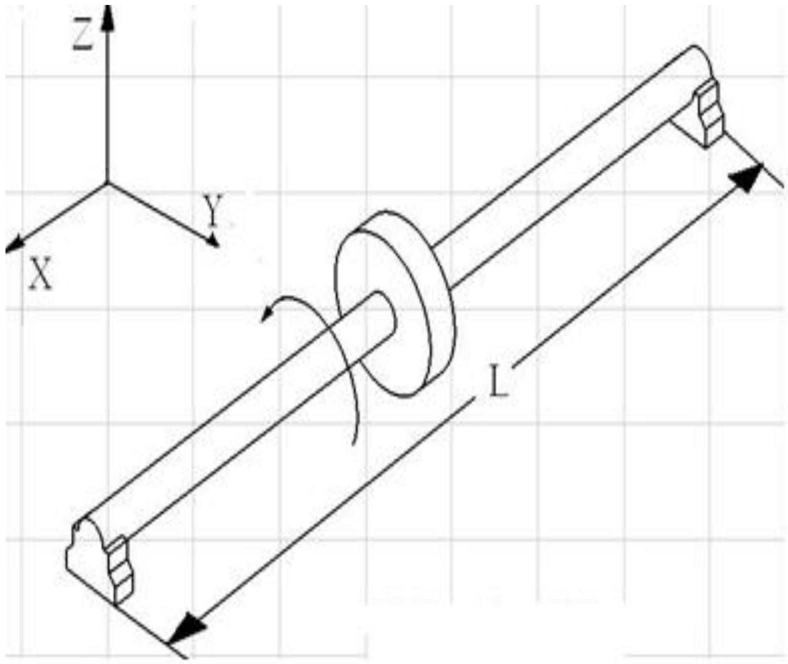


Figure 3.1 Schematic diagram of Rotor

## 3.2. Stability limit of spin speed and Unbalance response amplitude

Spin-speed for stability of the aluminium rotor–shaft system (Figure 3.2) has been found out from the sign of the maximum real part of the eigenvalues. The figure shows a plot of the maximum real parts for various spin speed. SLS corresponds to the spin speed when the maximum value of the real part of all the eigenvalues touches the zero line. Steady state synchronous Unbalance Response Amplitude (UBR) of the disc is plotted in figure 3.3 within the respective stable speed zones of operation. In certain speed the response has a maximum amplitude, which indicate the resonance of the system.

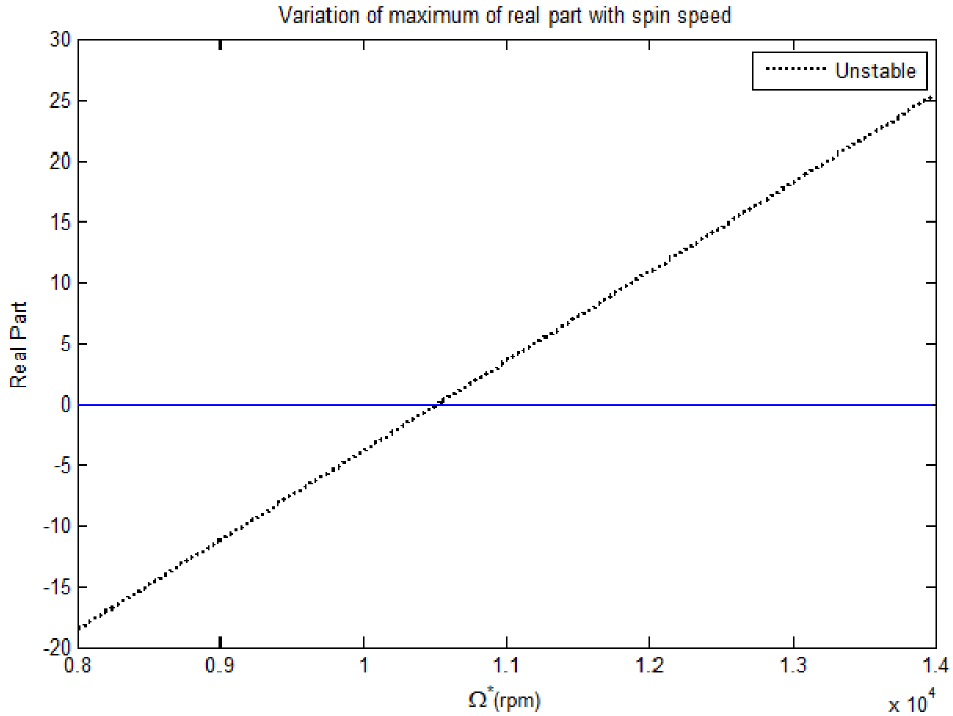


**Figure: 3.2 Variation of maximum real parts vs. spin-speed**

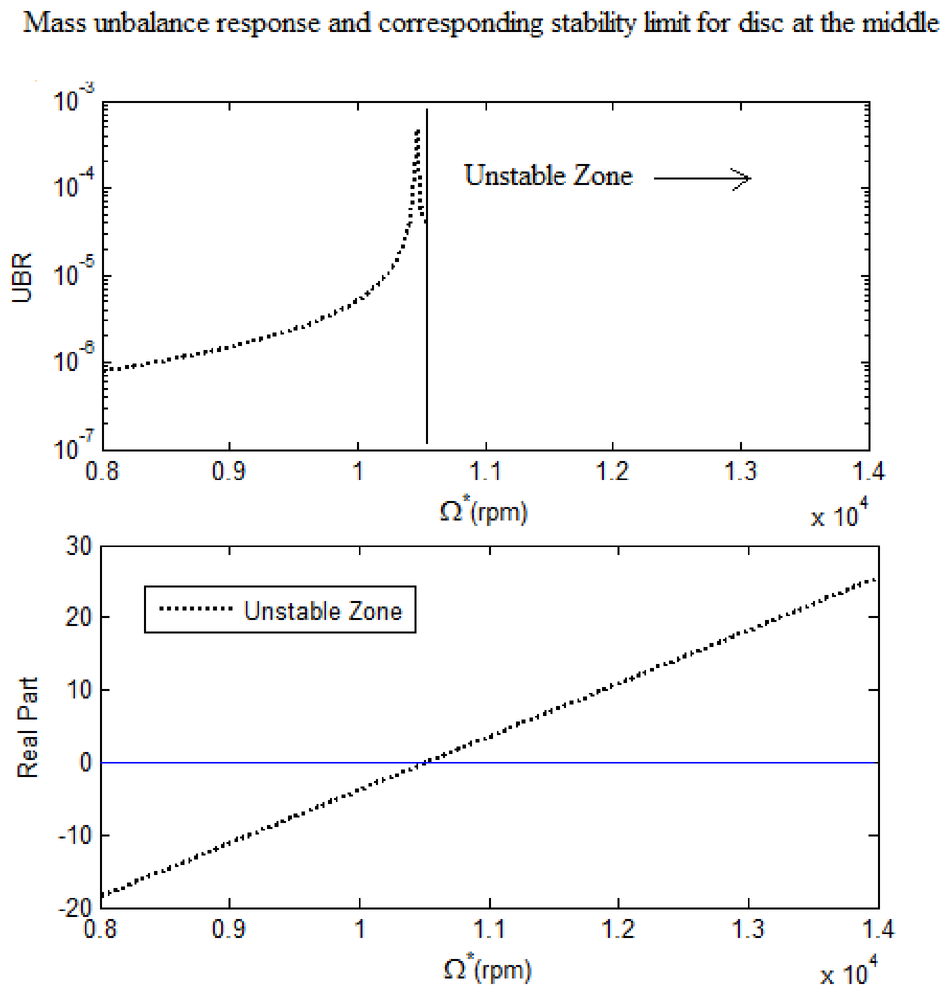


**Figure: 3.3 Mass UBR and corresponding stability for disc at the middle.**

Figure 3.4 shows the unbalance response amplitude at different disc positions when the rotor is divided into 10 elements having 11 nodes. It is observed at $6^{th}$ node (when disc is at the middle) the amplitude of unbalance response is maximum and further shifting disc towards the left and right from the middle the UBR amplitude decreases despite of changing the unstable zone (beyond spin speed of 1.05e+004 rpm). So, figure 3.4 shows that by changing disc position only UBR amplitude changes without changing unstable zone.

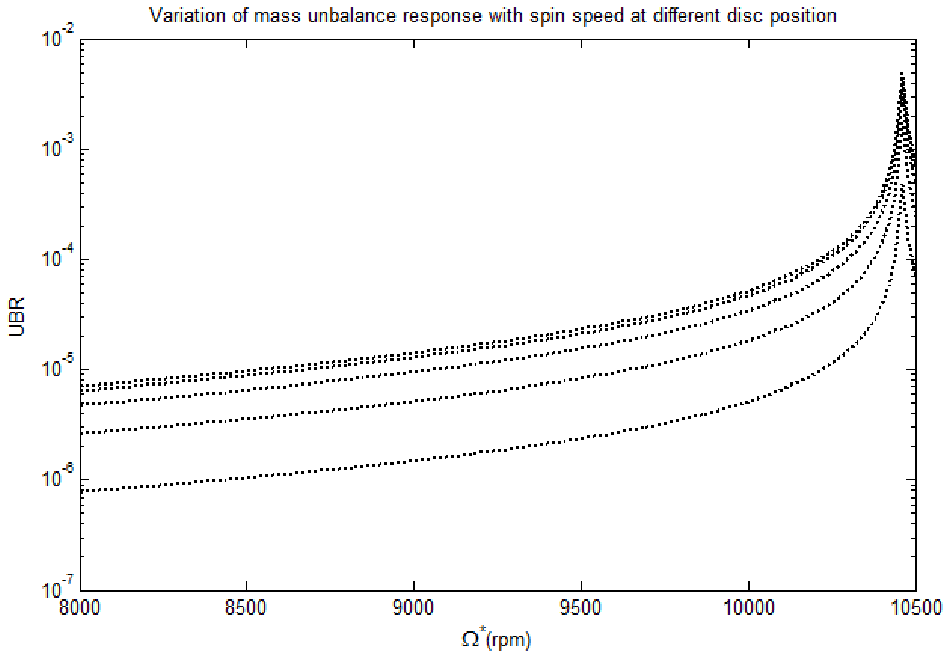


**Figure 3.4 Unbalance Response Amplitude at different disc position.**

# 3.3 Time response of the disc

In this section the theoretical results of time response both for transient and steady state excitation have been analyzed. In transient analysis the rotor is perturbed by an initial velocity of 0.38 m/s at the disc node, whereas in the case of steady state analysis, the rotor is excited due to unbalance.

Figure 3.5 and figure 3.6 shows the UBR amplitude for different time span in stable zone, both in Y and Z direction when the rotor is perturbed by initial velocity at the centre of the rotor. As it is a case of transient excitation, after certain instant both the transient response in Y and Z direction decreases and reaches at zero. It is due to the nature of the viscoelastic material of the rotor, because after certain instant the energy dissipates and vibration reduces.

Also, figure 3.4 and figure 3.5 shows the, the UBR amplitude for different time span for steady state in stable zone, both in Y and Z direction when the rotor is excited, due to unbalance. As it is a case of steady state excitation, the response amplitude is constant throughout the time span.

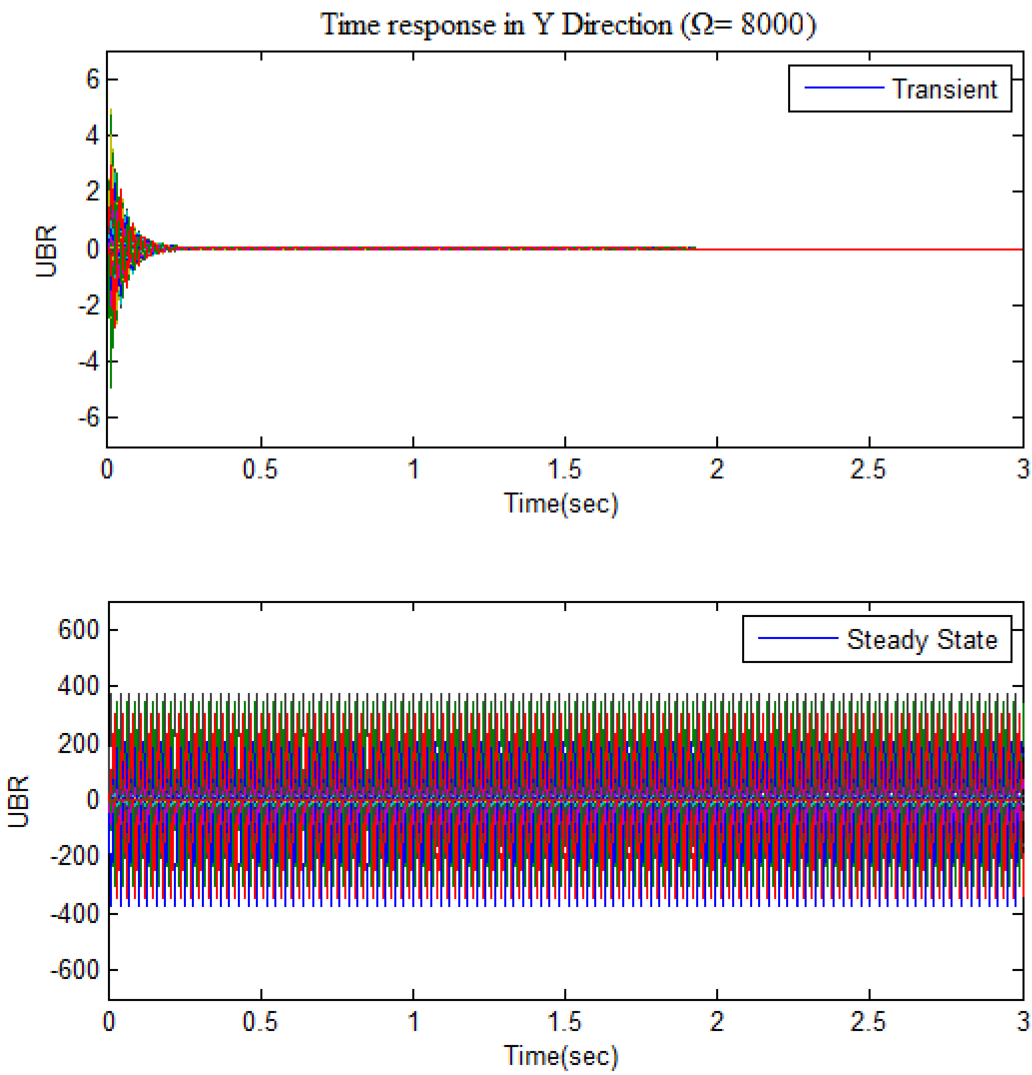


**Figure: 3.5 Time response of rotor in Stable Zone (Y- Direction).**

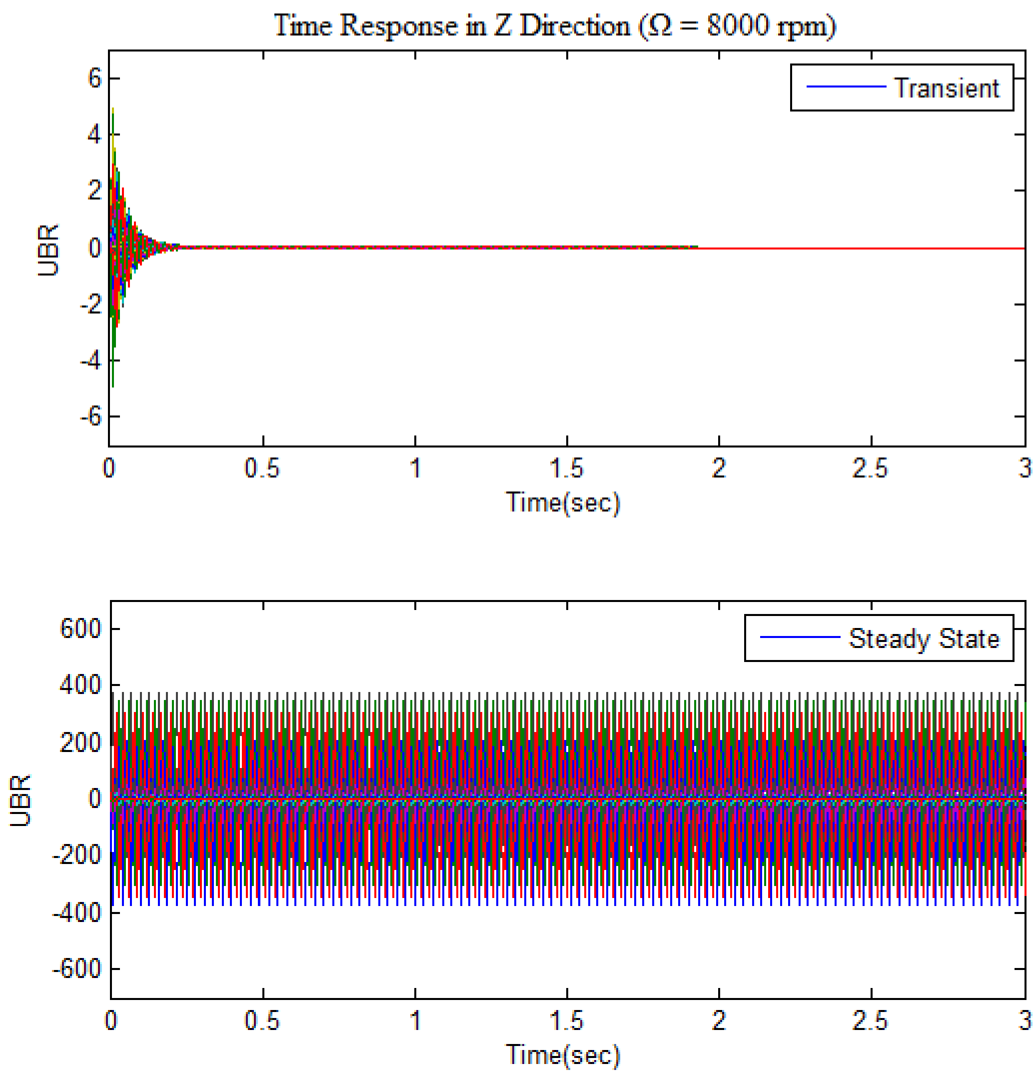


**Figure: 3.6   Time response of rotor in Stable Zone (Z- Direction).**

Figure 3.7 and figure 3.8 shows time response of rotor in unstable zone. When the rotor is allowed to rotate above SLS, the response amplitude monotonically increases with time. So beyond this speed the system becomes unstable and does not reach any steady state. It is the uncontrolled state of vibration, after certain time the failure of rotor takes place. So it is advisable to rotate much below that speed.

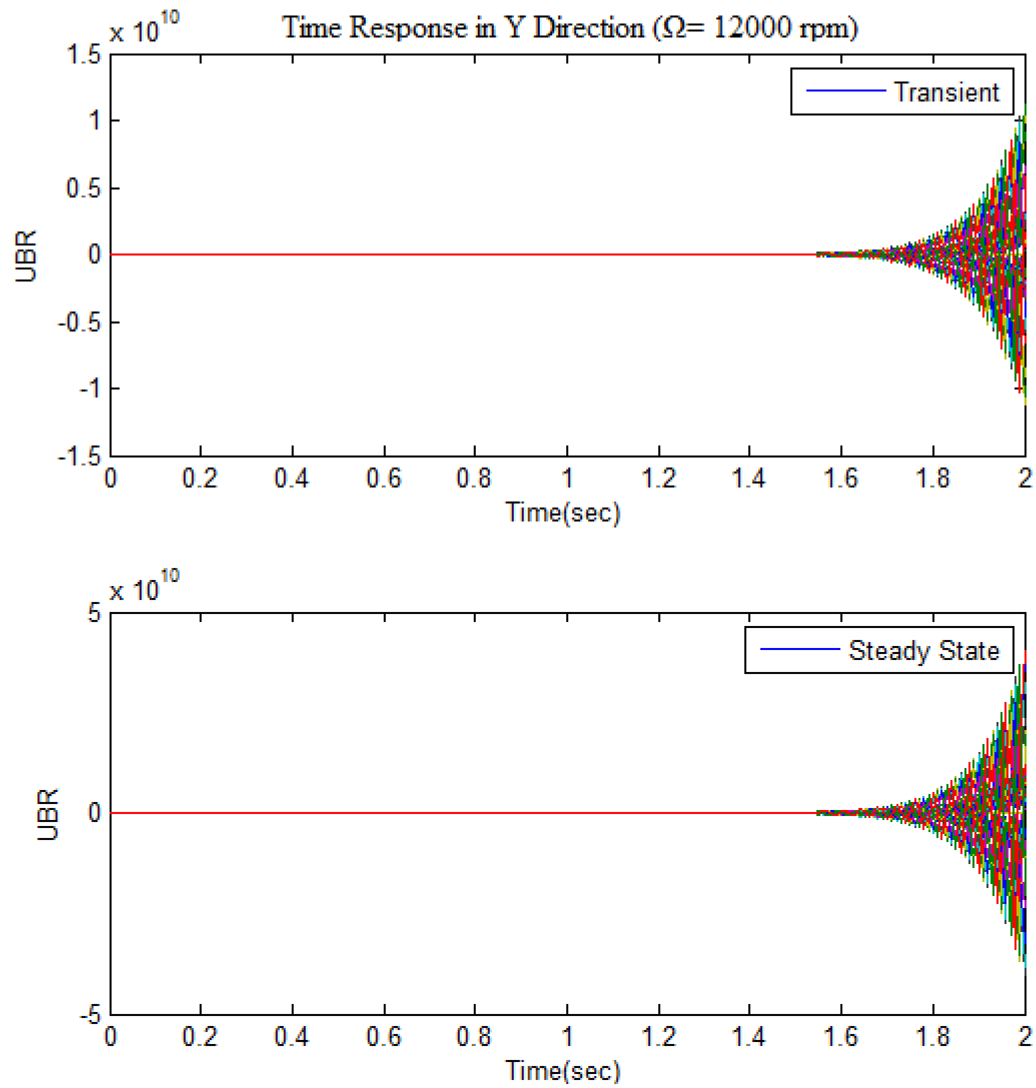


**Figure 3.7: Time response of rotor in Unstable Zone (Y- Direction)**

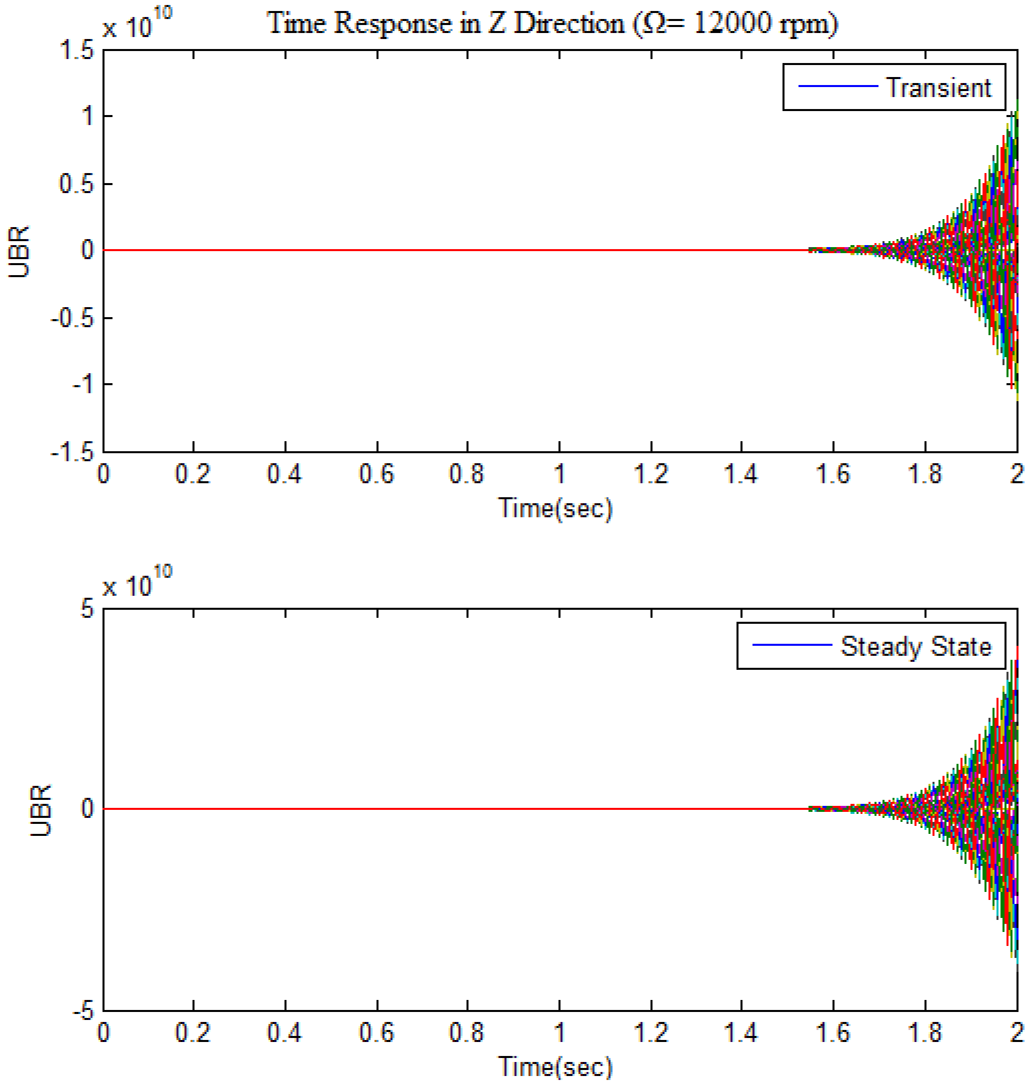


**Figure3.8: Time response of rotor in Unstable Zone (Z- Direction).**

# CHAPTER FOUR

# Conclusions and scope for future work

## 4.1 Conclusions

This project work has given the equations of motion of a rotor-shaft system having a viscoelastic rotor. The linear viscoelastic rotor-material behaviour is represented in the time domain where the instantaneous stress is obtained by operating the instantaneous strain. The mechanical analogy i.e. rheological model is sometime difficult to represent for all viscoelastic materials. The operator may be suitably chosen according to the material model. The formulation has been found very useful to generate equations motion by discretizing the rotor continuum into finite beam elements and study the dynamic behaviour of rotor-shaft systems in terms of stability limit of the spin speed as well as unbalance response of the disc. Temporal variation of disc response has also been plotted as a further verification of stability of the rotor-shaft system. So this work is useful for dynamic analysis of viscoelastic rotors under any type of dynamic forcing function.

## 4.2 Scope for future work

This study has given birth to different other possibilities which may be taken up as future research activities in this area.

1) In the present work ATF approach is represented in differential time operator based approach to obtain the equations of motion of a rotor-shaft system is more suitable for finding the equation of motion of a viscoelastic rotor which is used to predicts the dynamic behaviour of the continuum.

2) In this work 3-element mechanical model is used for physical representation of viscoelastic behaviour for simplicity. Here, 4-element model can also be used for better prediction of stability limit speed (SLS) and unbalance response (UBR) of the rotor-shaft system.

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