“VIBRATION AND STABILITY OF LAMINATED COMPOSITE DOUBLY CURVED SHELLS BY HIGHER ORDER SHEAR DEFORMATION THEORY”

A THESIS SUBMITTED IN PARTIAL FULFILMENT

OF THE REQUIREMENTS FOR THE DEGREE OF

Master of Technology

in

Structural Engineering

By

NEMI SHARAN

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DEPARTMENT OF CIVIL ENGINEERING

NATIONAL INSTITUTE OF TECHNOLOGY, ROURKELA

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CERTIFICATE

This is to certify that the thesis entitled **"VIBRATION AND STABILITY OF LAMINATED COMPOSITE DOUBLY CURVED SHELLS BY A HIGHER ORDER SHEAR DEFORMATION THEORY"** submitted by **Mr. NEMI SHARAN** in partial fulfillment of the requirements for the award of **Master of Technology** Degree in **Civil Engineering** with specialization in **Structural Engineering** at the National Institute of Technology, Rourkela (Deemed University) is an authentic work carried out by him under my supervision and guidance.

To the best of my knowledge, the matter embodied in the thesis has not been submitted to any other University/ Institute for the award of any degree or diploma.

Date:

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CONTENTS

ABSTRACT

The present study deals with a higher order shear deformation theory of laminated shells as suggested by Reddy and Liu. The theory is based on a displacement field in which the displacements of the middle surface are expanded as cubic functions of the thickness coordinate, and the transverse displacement is assumed to be constant through the thickness. This displacement field leads to the parabolic distribution of the transverse shear stresses (and zero transverse normal strain) and therefore no shear correction factors are used. The theory is also based on the assumption that the thickness to radius ratio of shell is small compared to unity and hence negligible.

The governing equations are derived in orthogonal curvilinear coordinates. These equations are then reduced to those of doubly curved shell. All the quantities are suitably non-dimensionalised. The Navier solution has been used which gives rise to a generalized eigenvalue problem in matrix formulation. The natural frequencies for vibration and buckling loads of laminated orthotropic doubly curved shells and panels with simply supported ends are obtained.

The eigenvalues, and hence the frequency parameters are calculated by using a standard computer program. To check the derivation and computer program, the frequencies in HZ for different layer are compares with earlier results.

The lowest value of frequency parameter and buckling load are computed for the laminated composite doubly curved shell. The effects of various parameters such as number of layers, aspect ratio, modular ratio, etc on the above are studied.

Frequency also increases as number of layers of the shell increases for symmetric cross-ply layout. But when there is unsymmetrical cross-ply layout, then frequency decreases. With the increasing of modular ratio, non-dimensional frequency is also increasing.

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LIST OF NOTATIONS

The principal symbols used in this thesis are presented for easy reference. A symbol is used for different meaning depending on the context and defined in the text as they occur.

English

Notation	Description
A_1 , A_2	Lames Parameter or Surface Metrics
A_{ij} , B_{ij} , D_{ij} E_{ij}, F_{ij}, H_{ij}	Laminate Stiffnesses
$\underset{,}{C}$ $\underline{\ }C$	The matrix notation described in appendix
dS	The distance between points (α, β, z) and $(\alpha + d\alpha, \beta + d\beta, z + dz)$
dV	Volume of the shell element
dt	Time derivative
E_1 , E_2	Longitudinal and transverse elastic moduli respectively
G_{12} , G_{13} , G_{23}	In plane and Transverse shear moduli
h	Total thickness of the shell
h_k	Distance from the reference surface to the layer interface
$k^2_{11,}$ k^2_{22}	The shear correction factors.
k_1 , k_2 , k_6	Quantities whose expression are given in equation [11.b]
$k_1^0, k_2^0, k_6^0, k_1^2$ $k_2^2, k_6^2, k_4^1, k_5^1$	Quantities whose expressions are given in equation [11.b]

L_1 , L_2 , L_3	lames coefficient
n_1 and n_2	axial load applied in α and β direction
a ,b	size of the spherical shell
N_i, M_i, P_i, Q_1 Q_2, K_1, K_2	Stress resultant and stress couples
$M, \overline{M}$	Matrix notation described in appendix
m	Axial half wave number
n	Circumferential wave number
$Q_{ij}^{(k)}$	Reduced stiffness matrix of the constituent layer
R	Radius of the reference surface of spherical shell
R_1 , R_2	Principal radii of curvature
T	Kinetic energy
U, V, W, Φ_1, Φ_2	Amplitude of displacement
U_s	Strain energy
$\overline{U}, \overline{V}, \overline{W}, \overline{\Phi}_1, \overline{\Phi}_2$	Non dimensionalised amplitude of displacement
u, v, w	Displacement component at the reference surface
$\overline{u}$, $\overline{v}$, $\overline{w}$	Non dimensionalised displacement component at any point in the shell
$I_i, \overline{I}_i, \overline{I}_i'$	Inertia terms defined by equation
{X}	Column matrix of amplitude of vibration or eigenvector.

$\{\overline{X}\}$	Non dimensionalised form of column matrix

Greek

Notation	**Description**
α , β , z	Shell coordinates
ε_I	Strain components
$\varepsilon_1^{\ 0}, \varepsilon_2^{\ 0}, \varepsilon_4^{\ 0}$ $\varepsilon_5^{\ 0}, \varepsilon_6^{\ 0}$	Quantities whose expressions are given in equation[11.b]
λ_m, λ_n	Non dimensionalised axial and circumferential wave parameter
ρ	Material density
σ_i	Stress components
ϕ_1, ϕ_2	Rotation at z = 0 of normals to the mid-surface with respect to α and β axes
ω	Natural circular frequency parameter

CHAPTER 1

INTRODUCTION

CHAPTER 1

INTRODUCTION

1.1 GENERAL:

An increasing number of structural designs, especially in the aerospace, automobile, and petrochemical industries are extensively utilizing fiber composite laminated plates and shells as structural elements. The laminated orthotropic shell belongs to the composite shell category. One of the important factors in the analysis of the layered shells is its individual layer properties, which may be anisotropic, orthotropic or isotropic.

A shell is a curved, thin walled structure. Two important classes of shells are plates (shells which are flat when un-deformed) and membranes (shells whose walls offer no resistance to bending). Shells may be made of a single homogeneous or anisotropic material or may be made of layers of different materials.

The primary function of a shell may be to transfer loads from one of its edges to another, to support a surface load, to provide a covering, to contain a fluid, to please the eye or a combination of these.

A thin shell is defined as a shell with a thickness which is small compared to its other dimensions and in which deformations are not large compared to thickness. A primary difference between a shell structure and a plate structure is that, in the unstressed state, the shell structure has curvature as opposed to plates. Membrane action in a shell is primarily caused by in-plane forces (plane stress), though there may be secondary forces resulting from flexural deformations. Where a flat plate acts similar to a beam with bending and shear stresses, shells are analogous to a cable which resists loads through tensile stresses. The ideal thin shell must be capable of developing both tension and compression.

At the present time, many of the existing methods of analysis for multilayered laminated orthotropic plates and shells are direct extensions of those developed earlier for homogeneous isotropic and orthotropic plates and shells. In the classical lamination theory (CLT), the well known Kirchhoff-love hypothesis is assumed to be verified. The range of applicability of the C.L.T solution has been well established for laminated flat plates by Pagano (1969, 1970). These analyses have indicated that the hypothesis of no deformable normal, while acceptable for isotropic plates and shells, is often quite unacceptable for multilayered anisotropic plates and shells with very large ratio of Young's modulus to shear modulus, even if they are relatively thin.

A Mindlin-type first-order transverse shear theory deformation theory (S.D.T) has been first developed by Dong and Tso (1972) for multilayered anisotropic shells. The present study deals with a higher-order but simple shear deformation theory of laminated shells as proposed by Reddy and Liu [24] for plates and shallow shells. The theory is based on a displacement field in which the displacements of the middle surface are expanded as cubic functions of the thickness coordinate, and the transverse displacement is assumed to be constant through the thickness. The latter assumption is equivalent to neglecting the stretching of the normal to the middle surface of the shell. This displacement field leads to the parabolic distribution of the transverse shear stresses (and zero transverse normal strain), and therefore no shear correction factors are used. The governing equations are derived in curvilinear orthogonal coordinates. These equations are then reduced to doubly curved shell. The free vibration and stability analysis of the doubly curved shell is carried out by computing lowest value of frequencies and buckling loads for various shell parameters.

1.2 APPLICATION OF SHELLS

Thin shell structures are light weight constructions using shell elements. These elements are typically curved and are assembled to large structures. Typical applications are fuselages of aero planes, boat hulls and roof structures in some buildings.

Shell structures are mainly used in industrial applications such as automobile, civil, aerospace and petrochemical engineering. Various types of shells are used in civil field such as conoid, hyperbolic paraboloid and elliptical paraboloid shell. All are used for roofing to cover large column-free areas.

Laminated composites are such type of material which has high strength to weight and strength to stiffness ratios. The mechanical properties of the laminated composites depend on the degree of orthotropy of the layers, ratio of the transverse shear modulus to the in-plane shear modulus and stacking sequence of laminates. Many of the classical theories developed for thin elastic shells are based on the Love-Kirchhoff assumptions in which the normal to the mid-plane before deformation is considered to be normal and straight after the deformation.

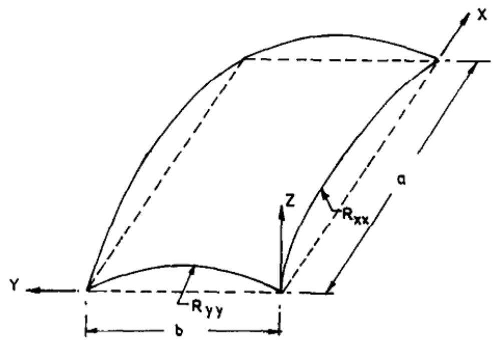

Figure:1.2.1 Elliptical Paraboloidal Shell

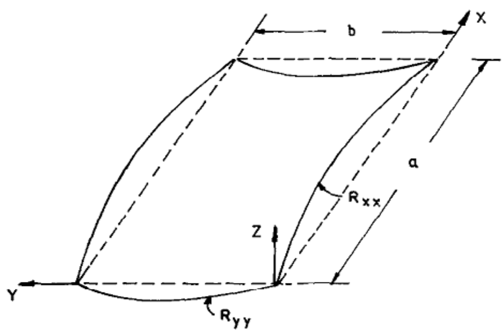

Figure:1.2.2 Hyperbolic Paraboloidal Shell

1.3 VIBRATION OF COMPOSITE SHELLS

Shell vibration was first studied by Germaine in 1821.Aron (1874) gave a set of five equations which reduced to plate equations when curvatures were zero. Logical extensions of the beam and plate equations for both transverse and in-plane motion were introduced by Love (1863-1940) in 1888. Love's equations brought the basic development of the theory of vibration of continuous structures, which have a thickness that is much less than any length or surface dimensions, to a satisfying end.

CHAPTER 2

LITERATURE REVIEW

CHAPTER 2

LITERATURE REVIEW

2.1 INTRODUCTION

The increasing use of high performance fibre reinforced laminated shell structures in aerospace and other applications are well documented. A reliable prediction of the response of these laminated shells or doubly-curved panels must account for transverse shear deformation. Since the matrix material is of relatively low shearing stiffness as compared to the fibers, polymeric composite shell type structures are highly prone to transverse shear related fatigue failures. Additionally, a solution to the problem of deformation of laminated shells of finite dimensions must satisfy the prescribed boundary conditions, which introduce additional complexities into the analysis.

Most of the investigations use

(a) The classical lamination theory (CLT) or the first-order shear deformation theory (FSDT). (A detailed review of CLT-based analyses is available in, e.g., Bert and Reddy [6], and Chaudhuri et al. [8], while that relating to their FSDT counterparts is available in, e.g., Abu Arja and Chaudhuri [1], Chaudhuri and Abu Arja [9], and Librescu et al. [17].)

(b) Higher-order shear deformation theory(HSDT)

Leissa and Kadi (1971) worked on the effect of curvature upon the vibration frequencies of shallow shells. For this purpose, the shell was chosen to have a rectangular boundary supported by shear diaphragms and yielded exact, closed form solutions for convenient comparison in the linear, small deflection regime and, at the same time, represented a situation which could readily occur in practical application. He extended his analysis into non linear, large deflection regime by assuming mode shapes satisfying the non-linear field equations of motion and compatibility approximately by means of the Galerkin procedure.

Bhimaraddi (1983) worked on the free, undamped vibration of an isotropic circular cylindrical shell with higher order displacement model, giving rise to a more realistic parabolic variation of transverse shear strains. In this work in-plane inertia, rotatory inertia, and shear deformation effects on the dynamic response of cylindrical shells was studied.

Bhimaraddi (1991) studied the free vibration analysis of homogeneous and laminated doubly curved shells on rectangular planform and made of an orthotropic material using the three- dimensional elasticity equations. A solution was obtained utilizing the asumption that the ratio of the shell thickness to its middle surface radius is negligible as compared to unity. However, it was shown that by dividing the shell thickness into layers of smaller thickness and matching the interface displacement and stress continuity conditions, very accurate results could be obtained even for very thick shells.

Touratier (1991) generalized the geometrically linear shear deformation theories for small elastic strains for multilayered axisymmetric shells of general shape without any assumption other than neglecting the transverse normal strain. He used a certain sine shear function and no shear correction factor was used. He presented a new theory and compared with the classical theories for a simply-supported thick laminated cylindrical shell under an internal pressure.

Leissa and Chang (1996) derived a rigorous theory which governs the linearly elastic deformation of shells made of laminated composite materials including shear deformation and rotary inertia effects. The equations derived are applicable for static and dynamic problems for shells of arbitrary curvature, but constant thickness.

Qatu (1999) derived accurate equations of deformation for laminated composite deep, thick shells. These equations include shells with a pre- twist and accurate force and moment resultants which are considerably different from plates. He also derived consistent set of equations of motion, energy functionals and boundary conditions.

Tabiei and Tanov(1999) worked on the finite element formulation of a higher order shear deformation shell element for non linear dynamic analysis with explicit time integration scheme. They combined co rotational approach with velocity strain equations of a general third order theory in the formulation of a four noded quadrilateral element with selectively reduced integration.

Jayasuriya *et al* (2002) worked on development of isoparametric, finite element for doubly curved anisotropic laminated shells of revolution. Variable-order Legendre polynomials were used in the derivation of interpolation functions. They also developed a user-friendly finite-element code based on the higher-order theory and isoperimetric hierarchical finite-element formulation, and a numerical verification and application of the above model for the study of static and dynamic analysis with mechanical and hydrothermal loadings.

Messina (2003) described the dynamics of freely vibrating multilayered doubly curved shells. The relevant governing differential equations, associated boundary conditions and constitutive equations were derived from one of Reissners mixed variational theorems. Both the governing differential equations and the related boundary conditions were presented in terms of resultant stresses and displacements. In spite of the multi-layer nature of the shell, the theory was developed as if the shell were virtually made of a single layer.

Sudhakar *et al* (2003) formulated a degenerate shell element using higher order shear deformation theory taking the piezoelectric effect in account. An eight-noded element was used to derive global coupled electro elastic behavior of the overall structure.

Amabili (2005) investigated vibrations of doubly curved shallow shells with rectangular base, simply supported at the four edges and subjected to harmonic excitation normal to the surface in the spectral neighborhood of the fundamental mode . In-plane inertia and geometric imperfections were taken into account. The solution was obtained by Lagrangian approach.

Oktem and Chaudhuri (2006) worked on the Fourier series approach to solve a system of five highly coupled linear partial differential equations, generated by the HSDT-based laminated shell analysis, with the C4-type simply supported boundary condition prescribed on two opposite edges, while the remaining two edges are subjected to the SS3-type constraint. The numerical accuracy of the solution was ascertained by studying the convergence characteristics of the deflection and moment of a cross-ply spherical panel, and also by comparison with the available FSDT (first-order shear deformation theory) based analytical solution.

Adam (2006) studied the nonlinear flexural vibrations of shallow shells composed of three thick layers with different shear flexibility, which were symmetrically arranged with respect to the middle surface. He considered shell structures of polygonal platform were hard hinged simply supported (i.e. all in-plane rotations and the bending moment vanish) with the edges fully restrained against displacements in any direction. He formulated kinematic field equations layer wise by first order shear deformation theory. Numerical results of rectangular shallow shells in nonlinear steady-state vibration were presented for various ratios of shell rise to thickness, and non-dimensional load amplitude.

Kulikov and Carrera (2008) worked on higher shell models by new concept of I surfaces inside the shell body. They introduced N (N≥3) I surfaces. The general 3N-parameter shell model was proposed in the framework of the Lagrangian description.

Amabili and Reddy (2009) worked on the use of higher order shear deformation non linear theory for shells of generic shape, taking geometric imperfections into account. They found that results were obtained by keeping non-linear terms of the Von Karman type for amplitudes of about two times the shell thickness.

Tripathy and Suryanarayan (2009) presented vibration and buckling of thin-walled tubular beam shells typical of automotive structures, which are fabricated by joining sheet metal stampings along the two longitudinal edges with periodic spot welds, adhesive bonding, or combination of spot welds and bonding, known as weld bonding. Solutions were obtained for such beam shells of rectangular cross section with two opposite ends simply supported. The beam shell was modeled as an assembly of the constituent walls and Levy-type formulation was used to obtain a series solution for the transverse displacement of each of the walls.

2.2 OBJECTIVE AND SCOPE OF PRESENT INVESTIGATION

In this project, analytical solution of frequency characteristics and buckling loads of laminated composite doubly curved shells will be presented. Higher order shear deformation theory as proposed by Reddy and Liu will be used for the formulation.

The governing equations have been developed. These equations are then reduced to the equations of motion for doubly curved panels and the Navier solution has been obtained for cross-ply laminated composite doubly curved panels. The resulting equations are suitably non-dimensionalized. The Eigen value problem is then solved to obtain the free vibration frequencies and buckling loads.

CHAPTER 3

THEORY AND FORMULATION

CHAPTER-3

THEORY AND FORMULATION

3.1 INTRODUCTION

In the present study, laminated composite doubly curved shells are considered. A number of theories exist for layered composite shells. Many of these theories were developed originally for thin shells, and are based on the Kirchhoff-Love kinematic hypothesis that straight lines normal to the undeformed midsurface remain straight and normal to the middle surface after deformation and undergo no thickness stretching.

The higher order theory is based on a displacement field as proposed by Reddy and Liu in which the displacements of the middle surface are expanded as cubic functions of the thickness coordinate and the transverse displacement is assumed to be constant through the thickness. The additional dependent unknowns introduced with the higher order powers of the thickness coordinate are evaluated in terms of the derivatives of the transverse displacement and the rotation of the normal at the middle surface. This displacement field leads to the parabolic distribution of the transverse shear strains and hence shear stresses and zero transverse normal strain, and therefore no shear correction factors are used. In first order shear deformation theory, shear correction factors are used.

The present study deals with the free vibration and buckling characteristics of thin laminated composite cross-ply doubly curved panels. The displacement components **u, v,** and **w** in the **α, β,** and **z** directions in a laminate element can be expressed in terms of the corresponding mid-plane displacement components **u°, v° , w°,** and the rotations Φ_1 , Φ_2 of the mid-plane normal along **α** and **β** axes. The higher order shear deformation theory is based on a displacement field in which the displacements of the middle surface are expanded as cubic functions of the thickness coordinate and the transverse displacement is assumed to be constant through the thickness. This displacement field leads to parabolic distribution of the transverse shear stresses and zero transverse normal strain and hence no shear correction factors are used.

The governing equations including the effect of shear deformation are presented in orthogonal curvilinear co-ordinates for laminated composite shells. These equations are then reduced to the governing equations for free vibration of laminated composite cross-ply doubly curved shells. The equations are suitably non-dimensionalised. The Navier solution has been used and the generalized eigen value problem so obtained in matrix formulation is solved to obtain the eigen values which are the natural frequencies and the buckling loads.

3.2 BASIC ASSUMPTIONS

A set of simplifying assumptions that provide a reasonable description of the behavior of thin elastic shells is used to derive the equilibrium equations that are consistent with the assumed displacement field.

1. No slippage takes place between the layers.
2. The effect of transverse normal stress on the gross response of the laminate is assumed to be negligible.
3. The line elements of the shell normal to the reference surface do not change their length after deformation.
4. The thickness coordinate of the shell is small compared to the principal radii of curvature (z/R_1, $z/R_2 <<<1$).
5. Normal to the reference surface of the shell before deformation remains straight, but not necessarily normal, after deformation (a relaxed Kirchhoff -Love hypothesis).

3.3 GEOMETRY OF SHELL

Figure 1 shows an element of a doubly curved shell. Here (α, β, z) denote the orthogonal curvilinear coordinates (shell coordinates) such that α and β curves are lines of curvature on the mid surface, z = 0, and z-curves are straight lines perpendicular to the surface, z = 0. For the doubly curved shells discussed here, the lines of principal curvature coincide with the co-ordinate lines. The values of the principal curvature of the middle surface are denoted by K_1 and K_2.

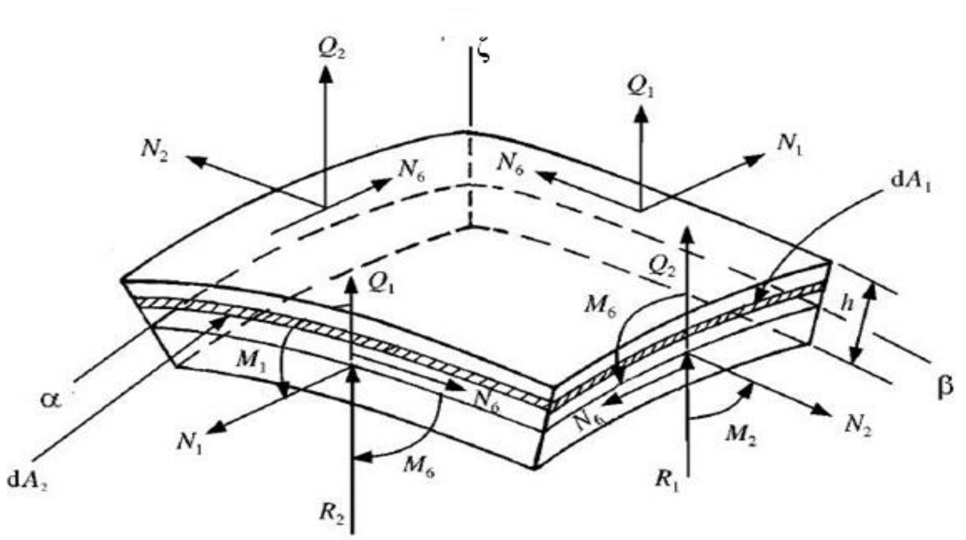

Figure.3.1 Geometry of a laminated shell

The position vector of a point on the middle surface is denoted by r and the position of a point at distance, z, from the middle surface is denoted by R. The distance, ds, between points (α, β, z) and $\left(\alpha + d\alpha, \beta + d\beta, z + dz\right)$ is determined by

$$(ds)^2 = \overline{\mathrm{dr}} \cdot \overline{\mathrm{dr}} \tag{1}$$

$$\overline{dr} = \frac{\partial \bar{r}}{\partial \alpha} d\alpha + \frac{\partial \bar{r}}{\partial \beta} d\beta \tag{2}$$

The magnitude ds of $d\bar{r}$ is given in equation (2), the vectors $\frac{\partial \bar{r}}{\partial \alpha}$ and $\frac{\partial \bar{r}}{\partial \beta}$ are tangent to the and β coordinate lines. Then equation (1) can be modified as

$$(ds)^2 = \frac{\partial \bar{r}}{\partial \alpha}\frac{\partial \bar{r}}{\partial \alpha}(d\alpha)^2 + \frac{\partial \bar{r}}{\partial \beta}\frac{\partial \bar{r}}{\partial \beta}(d\beta)^2 + 2\frac{\partial \bar{r}}{\partial \alpha}\frac{\partial \bar{r}}{\partial \beta} d\alpha . d\beta \tag{3}$$

The following derivation is limited to orthogonal curvilinear coordinates which coincide with the lines of principal curvature of the neutral surface. The third term in equation (3) thus becomes

$$2\frac{\partial\bar{r}}{\partial\alpha}\frac{\partial\bar{r}}{\partial\beta}d\alpha.d\beta = 2\left|\frac{\partial\bar{r}}{\partial\alpha}\right|.\left|\frac{\partial\bar{r}}{\partial\beta}\right|\cos\frac{\pi}{2}d\alpha.d\beta = 0 \qquad (4)$$

Where we define

$$\frac{\partial\bar{r}}{\partial\alpha}\frac{\partial\bar{r}}{\partial\alpha} = \left|\frac{\partial\bar{r}}{\partial\alpha}\right|^2 = A_1^{\ 2} \qquad (5)$$

$$\frac{\partial\bar{r}}{\partial\beta}\frac{\partial\bar{r}}{\partial\beta} = \left|\frac{\partial\bar{r}}{\partial\beta}\right|^2 = A_2^{\ 2}$$

Now the equation (3) becomes

$$(ds)^2 = A_1^{\ 2}(d\alpha)^2 + A_2^{\ 2}(d\beta)^2 \qquad (6)$$

This equation is called the fundamental form and A_1 and A_2 are the fundamental form parameters, Lame parameters, or surface metrics. The distance, dS, between points (α, β, z) and (α + *d*α, β + *d* β, *z* + *dz*) is given by

$$(ds)^2 = d\bar{R}d\bar{R}. = L_1^{\ 2}(d\alpha)^2 + L_2^{\ 2}(d\beta)^2 + L_3^{\ 2}(dz)^2 \qquad (7)$$

In which $d\bar{R} = \frac{\partial\bar{R}}{\partial\alpha}d\alpha + \frac{\partial\bar{R}}{\partial\beta}d\beta + \frac{\partial\bar{R}}{\partial z}dz$ and L_1, L_2, and L_3 are the Lame's coefficients.

$$L_1 = A_1\left[1+\frac{z}{R_1}\right]; L_2 = A_2\left[1+\frac{z}{R_2}\right]; \quad L_3 = 1 \qquad (8)$$

The vectors $\frac{\partial\bar{R}}{\partial\alpha}$ and $\frac{\partial\bar{R}}{\partial\beta}$ are parallel to the vectors $\frac{\partial\bar{r}}{\partial\alpha}$ and $\frac{\partial\bar{r}}{\partial\beta}$.

From the figure the elements of area of the cross section are

$$da_1 = L_1 d\alpha.dz = A_1\left[1+\frac{z}{R_1}\right]d\alpha.dz \qquad (9)$$

$$da_2 = L_2 d\beta.dz = A_2\left[1+\frac{z}{R_2}\right]d\beta.dz$$

The strain displacement equations of a shell are an approximation, within the assumptions made previously, of the strain displacement relations referred to orthogonal curvilinear coordinates. In addition, it is assumed that the transverse displacement, w, does not vary with z.

3.4 STRAIN DISPLACEMENT RELATIONS

3.4.1 FIRST ORDER SHEAR DEFORMATION THEORY

According to the first order shear deformation theory, the displacement field is given by:

$$\bar{u} = \left(1+\frac{z}{R_1}\right)u + z\phi_1$$

$$\bar{v} = \left(1+\frac{z}{R_2}\right)v + z\phi_2 \qquad (10)$$

$$\bar{w} = w$$

Here $(\bar{u},\bar{v},\bar{w})$ = the displacement of a point (α, β, z) along the (α, β, z) coordinates; and (u, v, w) = the displacements of a point (α, β, 0). Now substituting equation [10] in the strain displacement relations referred to an orthogonal curvilinear coordinate system we get

$$\varepsilon_1 = \varepsilon_1{}^0 + zk_1$$

$$\varepsilon_2 = \varepsilon_2{}^0 + zk_2$$

$$\varepsilon_4 = \varepsilon_4{}^0 \qquad (11.a)$$

$$\varepsilon_5 = \varepsilon_5{}^0$$

$$\varepsilon_6 = \varepsilon_6{}^0 + zk_6$$

Where,

$$\varepsilon_1^{\;0} = \frac{1}{A_1}.\frac{\partial u}{\partial \alpha} + \frac{w}{R_1};$$

$$\varepsilon_2^{\;0} = \frac{1}{A_2}.\frac{\partial v}{\partial \beta} + \frac{w}{R_2};$$

$$\varepsilon_6^{\;0} = \frac{1}{A_1}.\frac{\partial v}{\partial \alpha} + \frac{1}{A_2}\frac{\partial u}{\partial \beta};$$

$$\varepsilon_4^{\;0} = \phi_2 + \frac{1}{A_2}\frac{\partial w}{\partial \beta} - \frac{u}{R_1}; \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad 11.b$$

$$\varepsilon_5^{\;0} = \phi_1 + \frac{1}{A_1}\frac{\partial w}{\partial \beta} - \frac{v}{R_2};$$

$$k_1 = \frac{1}{A_1}.\frac{\partial \phi_1}{\partial \alpha};$$

$$k_2 = \frac{1}{A_2}.\frac{\partial \phi_2}{\partial \beta};$$

$$k_6 = \frac{1}{A_1}.\frac{\partial \phi_2}{\partial \alpha} + \frac{1}{A_2}\frac{\partial \phi_1}{\partial \beta} + \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{1}{A_1}\frac{\partial v}{\partial \alpha} - \frac{1}{A_2}\frac{\partial u}{\partial \beta}\right);$$

ϕ_1 and ϕ_2 are the rotations of the normals to the reference surface, z = 0, about the α and β coordinate axes, respectively. The displacement field in equation [10] can be used to derive the general theory of laminated shells.

3.4.2 HIGHER ORDER SHEAR DEFORMATION THEORY

The higher order displacement field relations as proposed by Reddy and Liu [24] are

$$\bar{u} = \left(1+\frac{z}{R_1}\right)u + z\phi_1 + z^3\left(\frac{4}{3h^2}\right)\left[-\phi_1 - \frac{1}{A_1}\left(\frac{\partial w}{\partial \alpha}\right)\right]$$

$$\bar{v} = \left(1+\frac{z}{R_2}\right)v + z\phi_2 + z^3\left(\frac{4}{3h^2}\right)\left[-\phi_2 - \frac{1}{A_2}\left(\frac{\partial w}{\partial \beta}\right)\right] \qquad (12)$$

$$\bar{w} = w$$

Here $(\bar{u},\bar{v},\bar{w})$ = the displacement of a point (α, β, z) along the (α, β, z) coordinates; and (u, v, w) = the displacements of a point (α, β, 0). Now substituting equation [12] in strain displacement relations referred to an orthogonal curvilinear coordinate system, we get

$$\varepsilon_1 = \varepsilon_1^{\ 0} + z\left(k_1^{\ 0} + z^2 k_1^{\ 2}\right)$$

$$\varepsilon_2 = \varepsilon_2^{\ 0} + z\left(k_2^{\ 0} + z^2 k_2^{\ 2}\right)$$

$$\varepsilon_4 = \varepsilon_4^{\ 0} + z^2 k_4^{\ 1}$$

$$\varepsilon_5 = \varepsilon_5^{\ 0} + z^2 k_5^{\ 1}$$

$$\varepsilon_6 = \varepsilon_6^{\ 0} + z\left(k_6^{\ 0} + z^2 k_6^{\ 2}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \qquad (13)$$

Where,

$$\varepsilon_1{}^0 = \frac{1}{A_1}.\frac{\partial u}{\partial \alpha} + \frac{w}{R_1};$$

$$\varepsilon_2{}^0 = \frac{1}{A_2}.\frac{\partial v}{\partial \beta} + \frac{w}{R_2};$$

$$\varepsilon_6{}^0 = \frac{1}{A_1}.\frac{\partial v}{\partial \alpha} + \frac{1}{A_2}\frac{\partial u}{\partial \beta};$$

$$\varepsilon_4{}^0 = \phi_2 + \frac{1}{A_2}\frac{\partial w}{\partial \beta};$$

$$\varepsilon_5{}^0 = \phi_1 + \frac{1}{A_1}\frac{\partial w}{\partial \beta};$$

$$k_1{}^0 = \frac{1}{A_1}.\frac{\partial \phi_1}{\partial \alpha};$$

$$k_2{}^0 = \frac{1}{A_2}.\frac{\partial \phi_2}{\partial \beta};$$

$$k_6{}^0 = \frac{1}{A_1}.\frac{\partial \phi_2}{\partial \alpha} + \frac{1}{A_2}\frac{\partial \phi_1}{\partial \beta} + \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{1}{A_1}\frac{\partial v}{\partial \alpha} - \frac{1}{A_2}\frac{\partial u}{\partial \beta}\right);$$

$$k_4{}^1 = -\left(\frac{4}{h^2}\right)\left(\phi_2 + \frac{\partial w}{\partial \beta}\right)$$

$$k_5{}^1 = -\left(\frac{4}{h^2}\right)\left(\phi_1 + \frac{\partial w}{\partial \alpha}\right)$$

$$k_1{}^2 = -\left(\frac{4}{3h^2}\right)\left(\frac{\partial \phi_1}{\partial \alpha} + \frac{\partial^2 w}{\partial \alpha^2}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (14)$$

$$k_2{}^2 = -\left(\frac{4}{3h^2}\right)\left(\frac{\partial \phi_2}{\partial \beta} + \frac{\partial^2 w}{\partial \beta^2}\right)$$

$$k_6{}^2 = -\left(\frac{4}{3h^2}\right)\left(\frac{\partial \phi_2}{\partial \alpha} + \frac{\partial \phi_1}{\partial \beta} + 2\frac{\partial^2 w}{\partial \alpha \partial \beta}\right)$$

Where ϕ_1 and ϕ_2 are the rotations of the normals to the reference surface, z = 0, about α and β coordinate axes, respectively. The stress-strain relations are as given in equation (15).

3.5 STRESS-STRAIN RELATIONS:

The stress-strain relation for the kth orthotropic layer takes the following form:

$$\begin{Bmatrix} \sigma_1^k \\ \sigma_2^k \\ \sigma_4^k \\ \sigma_5^k \\ \sigma_6^k \end{Bmatrix} = \begin{bmatrix} Q_{11}^k & Q_{12}^k & 0 & 0 & Q_{16}^k \\ Q_{12}^k & Q_{22}^k & 0 & 0 & Q_{26}^k \\ 0 & 0 & Q_{44}^k & Q_{45}^k & 0 \\ 0 & 0 & Q_{45}^k & Q_{55}^k & 0 \\ Q_{16}^k & Q_{26}^k & 0 & 0 & Q_{66}^k \end{bmatrix} \begin{Bmatrix} \varepsilon_1^k \\ \varepsilon_2^k \\ \varepsilon_4^k \\ \varepsilon_5^k \\ \varepsilon_6^k \end{Bmatrix} \tag{15}$$

For special orthotropic material, in which the principal axis direction coincides with the axis of the material direction,

$$Q_{16}^K = Q_{26}^K = Q_{45}^K = 0$$

Then,

$$\begin{Bmatrix} \sigma_1^k \\ \sigma_2^k \\ \sigma_4^k \\ \sigma_5^k \\ \sigma_6^k \end{Bmatrix} = \begin{bmatrix} Q_{11}^k & Q_{12}^k & 0 & 0 & 0 \\ Q_{12}^k & Q_{22}^k & 0 & 0 & 0 \\ 0 & 0 & Q_{44}^k & 0 & 0 \\ 0 & 0 & 0 & Q_{55}^k & 0 \\ 0 & 0 & 0 & 0 & Q_{66}^k \end{bmatrix} \begin{Bmatrix} \varepsilon_1^k \\ \varepsilon_2^k \\ \varepsilon_4^k \\ \varepsilon_5^k \\ \varepsilon_6^k \end{Bmatrix} \tag{16}$$

For generalized plane stress condition, the above elastic moduli $\left(Q_{ij}^k\right)$ are related to the engineering constants as follows:

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}$$

$$Q_{12} = \frac{E_1\nu_{21}}{1-\nu_{12}\nu_{21}} = \frac{E_2\nu_{12}}{1-\nu_{12}\nu_{21}} \tag{17}$$

$$Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}$$

$Q_{44} = G_{13}$

$Q_{55} = G_{23}$

$Q_{66} = G_{12}$

$\frac{E_1}{E_2} = \frac{\nu_{12}}{\nu_{21}}$

3.6 GOVERNING EQUATIONS

The governing differential equations, the strain energy due to loads, kinetic energy and formulations of the general problem are derived on the basis of Hamilton's principle.

The equation of motion is obtained by taking a differential element of the shell as shown in Figure 3.1. The figure 3.1 shows an element with internal forces like membrane (N_1, N_2, and N_6), shearing forces (Q_1, and Q_2) and the moment resultants (M_1, M_2 and M_6).

3.6.1 STRAIN ENERGY

The strain energy of a differential shell element can be written as,

$$U = \frac{1}{2}\int_\alpha \int_\beta \int_z [\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_6\varepsilon_6 + \sigma_4\varepsilon_4 + \sigma_5\varepsilon_5].dV \tag{18}$$

dV = Volume of shell element

$$dV = A_1\left(1+\frac{z}{R_1}\right)A_2\left(1+\frac{z}{R_2}\right).d\alpha.d\beta.dz. \tag{19}$$

The variation of strain energy U is given by

$$\delta U = \frac{1}{2}\int_\alpha \int_\beta \int_z [\sigma_1\delta\varepsilon_1 + \sigma_2\delta\varepsilon_2 + \sigma_6\delta\varepsilon_6 + \sigma_4\delta\varepsilon_4 + \sigma_5\delta\varepsilon_5].dV \tag{20}$$

The equation [20] is independent of the material property. Substituting the variation of strain function and dV in equation (20),

$$U=\frac{1}{2}\int_{\alpha}\int_{\beta}\int_{z}[N_1\varepsilon_1{}^0+M_1k_1+N_2\varepsilon_2{}^0+M_2k_2+N_6w_1+M_6\tau_1+N_6w_2+M_6\tau_2+$$

$$Q_1\left(1+\frac{z}{R_1}\right)\varepsilon_4+Q_2\left(1+\frac{z}{R_2}\right)\varepsilon_5]A_1A_2d\alpha d\beta. \tag{21}$$

The variation of strain energy is given as:

$$\delta U=\frac{1}{2}\int_{\alpha}\int_{\beta}\int_{z}[N_1\delta\varepsilon_1{}^0+M_1\delta k_1+N_2\delta\varepsilon_2{}^0+M_2\delta k_2+N_6\delta w_1+M_6\delta\tau_1+N_6\delta w_2+M_6\delta\tau_2+$$

$$Q_1\left(1+\frac{z}{R_1}\right)\delta\varepsilon_4+Q_2\left(1+\frac{z}{R_2}\right)\delta\varepsilon_5]A_1A_2d\alpha d\beta. \tag{22}$$

Substituting for $\varepsilon_1{}^0,\varepsilon_2{}^0,k_1,k_2.....$in equation (22) and integrating the resulting expression, the equation becomes,

$$\delta U=\int_{\alpha}\int_{\beta}\{-\frac{\partial(N_1A_2)}{\partial\alpha}\delta u+N_1\delta v\frac{\partial A_1}{\partial\beta}+\frac{N_1A_1A_2}{R_1}\delta w+M_1\delta\phi_2\frac{\partial A_1}{\partial\beta}-\frac{\partial(M_1A_2)}{\partial\alpha}\delta\phi_1+N_2\delta u\frac{\partial A_2}{\partial\alpha}$$

$$+N_2\frac{A_1A_2}{R_2}\delta w-\frac{\partial(N_2A_1)}{\partial\beta}\delta v+M_2\delta\phi_1\frac{\partial A_2}{\partial\alpha}-\frac{\partial(M_2A_1)}{\partial\beta}\delta\phi_2-N_6\delta u\frac{\partial A_1}{\partial\beta}-\frac{\partial(N_6A_2)}{\partial\alpha}\delta v-M_6\delta\phi_1\frac{\partial A_1}{\partial\beta}$$

$$-\frac{\partial(M_6A_2)}{\partial\alpha}\delta\phi_2-N_6\delta v\frac{\partial A_2}{\partial\alpha}-\frac{\partial(N_6A_1)}{\partial\beta}\delta u-M_6\delta\phi_2\frac{\partial A_2}{\partial\alpha}-\frac{\partial(M_6A_1)}{\partial\beta}\delta\phi_1+Q_1A_1A_2\left[\delta\phi_1-\frac{\delta u}{R_1}\right]-$$

$$\frac{\partial(Q_1A_2)}{\partial\alpha}\delta w+Q_2A_1A_2\left[\delta\phi_1-\frac{\delta u}{R_2}\right]-\frac{\partial(Q_2A_1)}{\partial\beta}\delta w\ \}d\alpha.d\beta \qquad +$$

$$\oint_{\beta}\left[N_1\delta u+N_6\delta v+Q_1\delta w+M_1\delta\phi_1+M_6\delta\phi_2\right]A_2d\beta$$

$$+\oint_{\alpha}\left[N_2\delta v+N_6\delta u+Q_2\delta w+M_2\delta\phi_2+M_6\delta\phi_1\right]A_1d\alpha \tag{23}$$

3.6.2 KINETIC ENERGY

If U be the displacement vector, the kinetic energy of the shell element is given by,

$$T = \frac{1}{2}\int_v \rho \dot{\bar{U}}.\dot{\bar{U}} \quad dV \tag{24}$$

where ρ is the mass density and $\dot{\bar{U}}$ represents differentiation with respect to time.

$\bar{U} = U_1\bar{n}_1 + U_2\bar{n}_2 + W\bar{n}_1$

$$T = \frac{\rho h}{2}\int_\alpha\int_\beta \left\{ \left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 + \frac{h^2}{12}\left[\left(\frac{\partial \phi_1}{\partial t}\right)^2 + \left(\frac{\partial \phi_2}{\partial t}\right)^2\right]\right\} A_1 A_2 .d\alpha.d\beta \tag{25}$$

The variation of kinetic energy is given as

$$\delta T = \rho h\int_\alpha \int_\beta \left[\dot{u}\,\delta\dot{u} + \dot{v}\,\delta\dot{v} + \dot{w}\,\delta\dot{w} + \frac{h^2}{12}(\dot{\phi}_1\,\delta\dot{\phi}_1 + \dot{\phi}_2\,\delta\dot{\phi}_2)\right] A_1 A_2 .d\alpha.d\beta \tag{26}$$

This equation contains time derivatives of the variations, ie $\delta\dot{u}$ etc. To eliminate these terms, integrating equation (26) by parts (the variations of limits t = t_1 and t = t_2 must vanish) and neglecting the terms $\rho h^3/12\ddot{\phi}_1$ and $\rho h^3/12\ddot{\phi}_2$ which represent rotatory inertia, the above equation reduces to,

$$\int_{t1}^{t_2} \delta T.dt = -\rho h\int_{t1}^{t_2}\int_\alpha \int_\beta \left[\ddot{u}\,\delta u + \ddot{v}\,\delta v + \ddot{w}\,\delta w\right] A_1 A_2 .d\alpha.d\beta.dt \tag{27}$$

If the shell is subjected to both body and surface forces and if q_1,q_2 and q_n are the components of body and surface forces along the parametric lines, then the variation of work done by the external loads are,

$$\begin{aligned}
\delta W_s &= \int_\alpha \int_\beta (q_1\delta u + q_2\delta v - q_n\delta w) A_1.A_2.d\alpha.d\beta \\
\delta W_{e1} &= \oint_\beta \left(\bar{N}_1\delta u + \bar{N}_6\delta v + \bar{Q}_1\delta w + \bar{M}_1\delta\phi_1 + \bar{M}_6\delta\phi_2\right) A_2.d\beta \\
\delta W_{e2} &= \oint_\alpha \left(\bar{N}_6\delta u + \bar{N}_2\delta v + \bar{Q}_2\delta w + \bar{M}_6\delta\phi_1 + \bar{M}_2\delta\phi_2\right) A_1.d\alpha
\end{aligned} \tag{28}$$

3.6.3 HAMILTON'S PRINCIPLE

The equations of equilibrium are derived by applying the dynamic version of the principle of virtual work, which is the Hamilton's Principle.

It states that among the set of all admissible configurations of system, the actual motion makes the quantity $\int_{t1}^{t_2} L.dt$ stationary, provided the configuration is known at the limits t = t_1 and t = t_2. Mathematically this means $\delta \int_{t1}^{t_2} L.dt$

Here, L is called Lagrangian and is equal to L= T - (U –V) (29)

Where, T = Kinetic energy, U= Strain energy, V= potential of all applied loads, δ = Mathematical operation called variation. It is analogous to partial differentiation. It is clear from equation [29] that the Lagrangian consists of kinetic, strain energy and potential of applied loads.

By applying the dynamic version of the principle of virtual work (Hamilton's principle), integrating the displacement gradients by parts in the resulting equation and setting the coefficients of δu, δv, δw, $\delta\varphi_1$ and $\delta\varphi_2$ to zero separately, the following equations of equilibrium are obtained:

$$\partial u : \frac{\partial (A_2 N_2)}{\partial \alpha} + \frac{\partial (A_1 N_6)}{\partial \beta} - N_2 \frac{\partial A_2}{\partial \alpha} + N_6 \frac{\partial A_1}{\partial \beta} + k_1 \left[\frac{\partial (A_2 M_1)}{\partial \alpha} + M_6 \frac{\partial A_1}{\partial \beta} - M_2 \frac{\partial A_2}{\partial \alpha} + \frac{\partial (A_1 M_6)}{\partial \beta} \right] + q_1 A_1 A_2 = \left[\bar{I}_1 \ddot{u} + \bar{I}_2 \ddot{\phi}_1 - \bar{I}_3 \frac{1}{A_2} \frac{\partial \ddot{w}}{\partial \alpha} \right] A_1 A_2$$

(30.a)

$$\partial v : \frac{\partial (A_2 N_6)}{\partial \alpha} + \frac{\partial (A_1 N_2)}{\partial \beta} - N_1 \frac{\partial A_1}{\partial \beta} + N_6 \frac{\partial A_2}{\partial \beta} + k_2 \left[\frac{\partial (A_2 M_6)}{\partial \alpha} - M_1 \frac{\partial A_1}{\partial \beta} + M_2 \frac{\partial A_1}{\partial \beta} + \frac{\partial (A_2 M_6)}{\partial \alpha} \right] + q_2 A_1 A_2 = \left[\bar{I}'_1 \ddot{v} + \overline{I_2}' \ddot{\phi}_2 - \bar{I}'_3 \frac{1}{A_2} \frac{\partial \ddot{w}}{\partial \beta} \right] A_1 A_2$$

(30.b)

$$\partial w : \frac{\partial (A_2 Q_1)}{\partial \alpha} + \frac{\partial (A_1 Q_2)}{\partial \beta} - A_1 A_2 k_1 N_1 - A_1 A_2 k_2 N_2 +$$

$$\frac{4}{3h^2}\left[\frac{\partial}{\partial \alpha}\left(\frac{1}{A_1}\frac{\partial (A_2 P_1)}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{1}{A_2}\frac{\partial (A_2 P_2)}{\partial \beta}\right) + \frac{\partial}{\partial \alpha}\left(\frac{1}{A_1}\frac{\partial (A_1 P_6)}{\partial \beta}\right) + \frac{\partial}{\partial \beta}\left(\frac{1}{A_2}\frac{\partial (A_2 P_6)}{\partial \alpha}\right) - \frac{\partial}{\partial \beta}\left(\frac{P_1}{A_2}\frac{\partial A_1}{\partial \beta}\right) - \frac{\partial}{\partial \alpha}\left(\frac{P_2}{A_1}\frac{\partial A_2}{\partial \alpha}\right) + \frac{\partial}{\partial \alpha}\left(\frac{P_6}{A_1}\frac{\partial A_1}{\partial \beta}\right) + \frac{\partial}{\partial \beta}\left(\frac{P_6}{A_2}\frac{\partial A_2}{\partial \alpha}\right)\right]$$

$$-\frac{4}{h^2}\left[\frac{\partial (A_2 K_1)}{\partial \alpha} + \frac{\partial (A_1 K_2)}{\partial \beta}\right] + q_n A_1 A_2 - n_1 \frac{\partial^2 w}{\partial \alpha^2} - n_2 \frac{\partial^2 w}{\partial \beta^2} = I_1 \ddot{w} A_1 A_2 +$$

$$\left\{\overline{I_3}\frac{\partial (A_2 \ddot{u})}{\partial \alpha} + \bar{I}_5 \frac{\partial (A_2 \ddot{\phi}_1)}{\partial \alpha} + \overline{I_3}\frac{\partial (A_1 \ddot{v})}{\partial \beta} + \bar{I}_5 \frac{\partial (A_1 \ddot{\phi}_2)}{\partial \beta} - \frac{16}{9}\frac{I_7}{h^4}\left[\frac{\partial}{\partial \alpha}\left(\frac{A_2}{A_1}\frac{\partial \ddot{w}}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{A_1}{A_2}\frac{\partial \ddot{w}}{\partial \beta}\right)\right]\right\}$$

(30.c)

$$\partial \phi_1 : \frac{\partial (M_1 A_2)}{\partial \alpha} + \frac{\partial (M_6 A_1)}{\partial \beta} - M_2 \frac{\partial A_2}{\partial \alpha} + M_6 \frac{\partial A_1}{\partial \alpha} + M_6 \frac{\partial A_1}{\partial \beta} - \frac{4}{3h^2}\left[\frac{\partial (A_2 P_1)}{\partial \alpha} + \frac{\partial (A_1 P_6)}{\partial \beta} - P_2 \frac{\partial A_2}{\partial \alpha} + P_6 \frac{\partial A_1}{\partial \beta}\right] -$$

$$Q_1 A_1 A_2 + \frac{4}{h^2}\overline{K}_1 A_1 A_2 = \left[\bar{I}_2 \ddot{u} + \bar{I}_4 \ddot{\phi}_1 - \bar{I}_5 \frac{1}{A_1}\frac{\partial \ddot{w}}{\partial \alpha}\right] A_1 A_2$$

(30d)

$$\partial \phi_2 : \frac{\partial (M_6 A_2)}{\partial \alpha} + \frac{\partial (M_2 A_1)}{\partial \beta} - M_1 \frac{\partial A_2}{\partial \beta} + M_6 \frac{\partial A_2}{\partial \alpha} - \frac{4}{3h^2}\left[\frac{\partial (A_2 P_6)}{\partial \alpha} + \frac{\partial (A_1 P_2)}{\partial \beta} - P_1 \frac{\partial A_1}{\partial \beta} + P_6 \frac{\partial A_2}{\partial \alpha}\right] -$$

$$Q_1 A_1 A_2 + \frac{4}{h^2}\overline{K}_2 A_1 A_2 = \left[\bar{I}_2 \ddot{v} + \bar{I}_4 \ddot{\phi}_2 - \bar{I}_5 \frac{1}{A_2}\frac{\partial \ddot{w}}{\partial \beta}\right] A_1 A_2$$

(30e)

q_1, q_2, q_n can be defined as the transverse loads

The inertias $\bar{I}_1$ and $\bar{I}_1'$ (i=1, 2, 3, 4, 5) are defined by the equations,

$$\bar{I}_1 = I_1 + 2k_1 I_2$$

$$\bar{I}_1' = I_1 + 2k_2 I_2$$

$$\bar{I}_2 = I_2 + k_1 I_3 - \frac{4}{3h^2} I_4 - \frac{4k_1}{3h^2} I_5$$

$$\bar{I}'_2 = I_2 + k_2 I_3 - \frac{4}{3h^2} I_4 - \frac{4k_2}{3h^2} I_5$$

$$\bar{I}_3 = \frac{4}{3h^2} I_4 + \frac{4k_1}{3h^2} I_5$$

$$\bar{I}'_3 = \frac{4}{3h^2} I_4 + \frac{4k_1}{3h^2} I_5 \qquad (31)$$

$$\bar{I}_4 = I_3 - \frac{8}{3h^2} I_5 + \frac{16}{9h^2} I_7$$

$$\bar{I}_5 = \frac{4}{3h^2} I_5 + \frac{16k_1}{9h^4} I_7$$

where $$\left(I_1, I_1, I_1, I_1, I_1, I_1\right) = \sum_{k=1}^{n} \int_{h_k}^{h_{k+1}} \rho^k \left(1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^6\right) d\zeta \qquad (32)$$

Where $\boldsymbol{\rho}^{(k)}$ is the density of the material of the k[th] layer.

3.7 STRESS RESULTANTS AND STRESS COUPLES

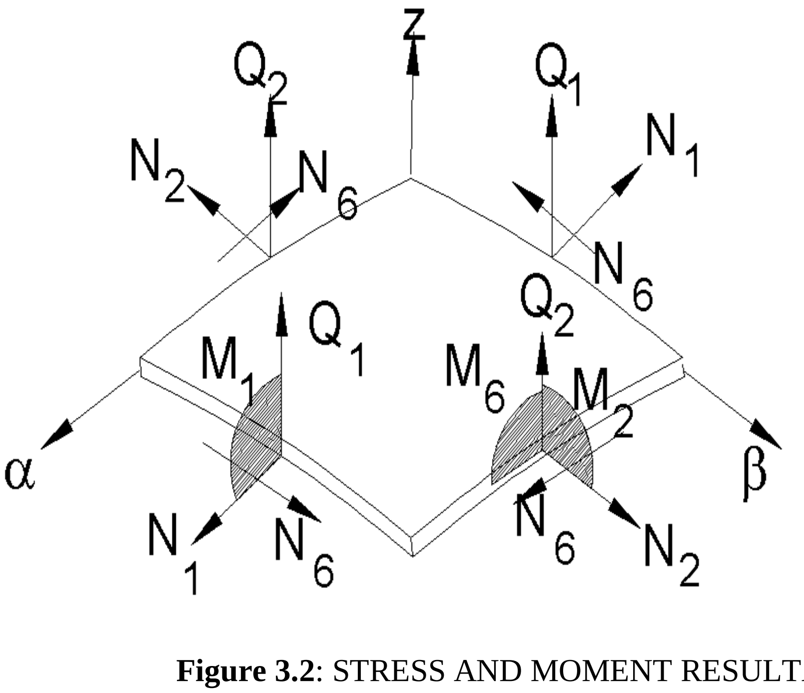

Figure 3.2: STRESS AND MOMENT RESULTANTS

Let N_1 be the tensile force, measured per unit length along β coordinate line, on a cross section perpendicular to α coordinate line. Then the total tensile force on the differential element in the α direction is

$$N_1.A_2 = \int_{-h/2}^{h/2} \sigma_1 da_2 dz \tag{33}$$

In which h = the thickness of the shell (z = -h/2 and z = h/2 denote the bottom and top surfaces of the shell) and da_2 is the area of cross section. Using equation (9), the remaining stress resultants per unit length are given by:

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_{12} \\ N_{21} \\ Q_1 \\ Q_2 \\ M_1 \\ M_2 \\ M_{12} \\ M_{21} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_1\left(1+\frac{z}{R_2}\right) \\ \sigma_2\left(1+\frac{z}{R_1}\right) \\ \sigma_6\left(1+\frac{z}{R_2}\right) \\ \sigma_5\left(1+\frac{z}{R_2}\right) \\ \sigma_4\left(1+\frac{z}{R_1}\right) \\ z\sigma_1\left(1+\frac{z}{R_2}\right) \\ z\sigma_2\left(1+\frac{z}{R_1}\right) \\ z\sigma_6\left(1+\frac{z}{R_2}\right) \\ z\sigma_6\left(1+\frac{z}{R_1}\right) \end{Bmatrix} dz \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 34$$

In contrast to the plate theory (where $1/R_1 = 0, 1/R_2 = 0$), the shear stress resultants, N_{12} and N_{21}, and the twisting moments, M_{12} and M_{21}, are, in general, not equal. For shallow shells the terms z/R_1 and z/R_2 can be neglected in comparison with unity. Hence $N_{12} = N_{21} = N_6$ and $M_{12} = M_{21} = M_6$.

The shell under consideration is composed of finite number of orthotropic layers of uniform thickness, as shown in Figure 2. In view of assumption 1, the stress resultant in equation [34] can be expressed as

$$\left(N_{i,}M_i\right)=\sum_{k=1}^{n}\int_{h_{k-1}}^{hk}\sigma_i{}^k\left(1,z\right).dz..................\left(i=1,2,6\right)$$

$$Q_i=\sum_{k=1}^{n}\int_{h_{k-1}}^{hk}\sigma_i{}^k.dz..................\left(i=4,5\right) \quad .... \quad (35)$$

In which n = the number of layers in the shell; h_k and h_{k-1} is the top and bottom z coordinates of the k^{th} lamina.

Substitution of equation [11] and [15] into equation [35] leads to the following expression for the stress resultants and stress couples in the first order shear deformation theory:

$$\begin{aligned} N_i &= A_{ij}.\varepsilon_j{}^0+B_{ij}.K_j{}^0; \\ M_i &= B_{ij}\varepsilon_j{}^0+D_{ij}K_j{}^0; \\ Q_1 &= A_{45}.\varepsilon_4{}^0+A_{55}\varepsilon_5{}^0; \\ Q_2 &= A_{44}.\varepsilon_4{}^0+A_{45}\varepsilon_5{}^0; \end{aligned} \quad .... \quad (36)$$

Here A_{ij}, B_{ij} and D_{ij} denote the extensional, flexural-extensional coupling, and flexural stiffness. They may be defined as:

$$\left(A_{ij},B_{ij},D_{ij}\right)=\sum_{k=1}^{n}\int_{h_{k-i}}^{h_k}Q_{ij}{}^k(1,z,z^2).dz. \quad (37)$$

For i, j = 1, 2, 4, 5, 6.

h_k and h_{k-1} are the distances measured as shown in figure -1

According to the higher order theory, similarly,

$$\left(N_{i,}M_i,P_i\right)=\sum_{k=1}^{n}\int_{h_{k-1}}^{hk}\sigma_i{}^k\left(1,z,z^3\right).dz..................\left(i=1,2,6\right)$$

$$\left(Q_1,K_1\right)=\sum_{k=1}^{n}\int_{h_{k-1}}^{hk}\sigma_4{}^k\left(1,z^2\right).dz \quad .... \quad (38)$$

$$(Q_2, K_2) = \sum_{k=1}^{n} \int_{h_{k-1}}^{hk} \sigma_5{}^k (1, z^2) dz$$

In which n = the number of layers in the shell; h_k and h_{k-1} is the top and bottom z coordinates of the kth lamina.

Substituting of equation [13] and [15] into equation [38] leads to the following expression for the stress resultants and stress couples

$$\begin{aligned}
N_i &= A_{ij}.\varepsilon_j{}^0 + B_{ij}.k_j{}^0 + E_{ij}k_j{}^2; \\
M_i &= B_{ij}\varepsilon_j{}^0 + D_{ij}k_j{}^0 + F_{ij}k_j{}^2; \\
P_i &= E_{ij}\varepsilon_j{}^0 + F_{ij}k_j{}^0 + H_{ij}k_j{}^2; \qquad (i, j) = 1,2,6 \\
Q_1 &= A_{5j}.\varepsilon_j{}^0 + D_{5j}k_j{}^1; \\
Q_2 &= A_{4j}.\varepsilon_j{}^0 + D_{4j}k_j{}^1; \\
K_1 &= D_{4j}\varepsilon_j{}^0 + F_{4j}k_j{}^1; \\
K_2 &= D_{5j}\varepsilon_j{}^0 + F_{5j}k_j{}^1; \qquad (j = 4,5)
\end{aligned} \qquad \text{.........................} \qquad (39)$$

where *Aij*, *Bij* , etc. are the laminate stiffnesses expressed as

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \sum_{k=1}^{n} \int_{h_{k-i}}^{h_k} Q_{ij}{}^k (1, z, z^2, z^3, z^4, z^5).dz \quad . \qquad (40)$$

The strain displacement relations (13) are substituted in the equations for the stress resultants and stress couples given in equation (39). Since the solution for the equations of motion is done by using the Navier solution, therefore such a solution exists only for specially antisymmetric cross ply laminate for which the following laminate stiffnesses are zero.

$$A_{16} = A_{26} = B_{16} = B_{26} = D_{16} = D_{26} = A_{45} = 0$$

The expression for the stress resultants and stress couples so obtained are then substituted into the equation of motion (30). The equation of motion in terms of the displacements for doubly curved shells hence reduces to

$$
\begin{aligned}
&(A_{11}+2B_{11}k_1+D_{11}k_1^2)\frac{\partial^2 u}{\partial \alpha^2}+(A_{66}+2B_{66}k_1+D_{66}k_1^2)\frac{\partial^2 u}{\partial \beta^2}+\\
&\left[A_{12}+B_{12}k_2+A_{66}+B_{66}k_2+B_{12}k_1+D_{12}k_2k_1+k_1B_{66}+D_{66}k_2k_1\right]\frac{\partial^2 v}{\partial\alpha\partial\beta}+\\
&\left[A_{11}k_1+A_{12}k_2+B_{11}k_1^2+B_{12}k_2k_1\right]\frac{\partial w}{\partial\alpha}-\frac{4}{3h^2}E_{11}\frac{\partial^3 w}{\partial\alpha^3}-\frac{4}{3h^2}F_{11}k_1\frac{\partial^3 w}{\partial\alpha^3}-\frac{4}{3h^2}\left[E_{12}+2E_{66}+F_{12}k_1+2F_{66}\right]\frac{\partial^3 w}{\partial\alpha\partial\beta^2}\\
&+\left[B_{11}-\frac{4}{3h^2}E_{11}+D_{11}k_1-\frac{4}{3h^2}F_{11}k_1\right]\frac{\partial^2\phi_1}{\partial\alpha^2}+\left[B_{66}-\frac{4}{3h^2}E_{66}+D_{66}k_1-\frac{4}{3h^2}F_{66}\right]\frac{\partial^2\phi_1}{\partial\beta^2}+\\
&\left[B_{12}-\frac{4}{3h^2}E_{12}+B_{66}-\frac{4}{3h^2}E_{66}+D_{12}k_1-\frac{4}{3h^2}F_{12}k_1+D_{66}k_1-\frac{4}{3h^2}F_{66}k_1\right]\frac{\partial^2\phi_2}{\partial\alpha\partial\beta}=\\
&I_1\ddot u+\bar I_2\ddot\phi_1-\bar I_3\frac{\partial\ddot w}{\partial\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&\left[A_{66}+B_{66}k_1+B_{21}k_2+D_{21}k_2k_1+k_2B_{66}+D_{66}k_2k_1+B_{21}k_1+A_{21}\right]\frac{\partial^2 u}{\partial\alpha\partial\beta}\\
&+\left[A_{66}+2B_{66}k_2+D_{66}k_2^2\right]\frac{\partial^2 v}{\partial\alpha^2}+\left[A_{22}+2B_{22}k_2+D_{22}k_2^2\right]\frac{\partial^2 v}{\partial\beta^2}+\\
&\left[A_{21}k_1+A_{22}k_2+k_1k_2B_{21}+k_2^2B_{22}\right]\frac{\partial w}{\partial\beta}-\frac{4}{3h^2}E_{22}\frac{\partial^3 w}{\partial\beta^3}-\frac{4}{3h^2}F_{22}\frac{\partial^3 w}{\partial\beta^3}\\
&-\frac{4}{3h^2}\left[2E_{66}+E_{21}+F_{66}+F_{21}\right]\frac{\partial^3 w}{\partial\alpha^2\partial\beta}\\
&+\left[B_{66}-\frac{4}{3h^2}E_{66}\right]\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}+B_{21}\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}-\frac{4}{3h^2}E_{21}\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}+\\
&D_{66}k_2\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}-\frac{4}{3h^2}F_{66}k_2\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}+D_{21}k_2\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}-\frac{4}{3h^2}F_{21}k_2\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}+\\
&\left[B_{66}-\frac{4}{3h^2}E_{66}-\frac{4}{3h^2}F_{66}k_2+D_{66}k_2\right]\frac{\partial^2\phi_2}{\partial\alpha^2}+\left[B_{22}-\frac{4}{3h^2}E_{22}-\frac{4}{3h^2}F_{22}k_2+D_{22}k_2\right]\frac{\partial^2\phi_2}{\partial\beta^2}\\
&=I_1\ddot v+\bar I_2\ddot\phi_2-\bar I'_3\frac{\partial\ddot w}{\partial\beta}
\end{aligned}
$$

$$-\left[k_1\left(A_{11}+B_{11}k_1\right)+k_2\left(A_{21}+B_{21}k_1\right)\right]\frac{\partial u}{\partial\alpha}-\left[k_1\left(A_{11}k_1+A_{12}k_2\right)+k_2\left(A_{21}k_1+A_{22}k_2\right)\right]w+$$

$$-\left[k_1\left(A_{12}+B_{12}k_2\right)+k_2\left(A_{22}+B_{22}k_2\right)\right]\frac{\partial v}{\partial\beta}+$$

$$-\left[k_1\left(B_{11}-\frac{4}{3h^2}E_{11}\right)+k_2\left(\frac{4}{3h^2}E_{21}k_2+B_{21}\right)+\left(A_{55}-\frac{4}{h^2}D_{55}\right)\right]\frac{\partial\phi_1}{\partial\alpha}$$

$$-\left[k_1\left(B_{12}-\frac{4}{3h^2}E_{12}\right)+k_2\left(-\frac{4}{3h^2}E_{22}+B_{22}\right)+\left(A_{44}-\frac{4}{h^2}D_{44}\right)\right]\frac{\partial\phi_2}{\partial\beta}$$

$$+\left[\frac{4}{3h^2}E_{11}k_1+\frac{4}{3h^2}k_2E_{21}+\left(A_{55}-\frac{4}{h^2}D_{55}\right)\right]\frac{\partial^2 w}{\partial\alpha^2}+$$

$$\left[\frac{4}{3h^2}k_1E_{12}+\frac{4}{3h^2}k_2E_{22}+\left(k_2E_{21}+k_2E_{22}\right)+\left(A_{44}-\frac{4}{h^2}D_{44}\right)\right]\frac{\partial^2 w}{\partial\beta^2}+$$

$$\frac{4}{3h^2}\left[\begin{array}{l}\left(E_{11}+F_{11}k_1\right)\frac{\partial^3 u}{\partial\alpha^3}+\left(k_1+k_2\right)\frac{\partial^2 w}{\partial\alpha^2}+\left(F_{11}-\frac{4}{3h^2}H_{11}\right)\frac{\partial^3\phi_1}{\partial\alpha^3}+\left(E_{12}+F_{12}k_2+2E_{66}\right)\frac{\partial^3 v}{\partial\alpha^2\partial\beta}+\\ F_{12}\frac{\partial^3\phi_2}{\partial\alpha^2\partial\beta}-\frac{4}{3h^2}H_{11}\frac{\partial^4 w}{\partial\alpha^4}+\left(E_{12}+F_{21}k_2+F_{22}k_2+E_{22}\right)\frac{\partial^3 v}{\partial\beta^3}+\\ \left(F_{21}-\frac{4}{3h^2}H_{21}+F_{22}-\frac{4}{3h^2}H_{22}\right)\frac{\partial^3\phi_2}{\partial\beta^3}-\\ \left(\frac{4}{3h^2}H_{21}-\frac{4}{3h^2}H_{22}\right)\frac{\partial^4 w}{\partial\beta^4}+2E_{66}\frac{\partial^3 u}{\partial\alpha\partial\beta^2}+\\ k_2F_{66}\frac{\partial^3 v}{\partial\alpha\partial\beta^2}+\left(F_{66}-\frac{4}{3h^2}H_{66}\right)\frac{\partial^3\phi_2}{\partial\alpha\partial\beta^2}+H_{66}\frac{\partial^4 w}{\partial\alpha^2\partial\beta^2}\end{array}\right]$$

$$=-I_1\ddot{w}-\left[\frac{4}{3h^2}\right]^2 I_7\left[\frac{\partial^2\ddot{w}}{\partial\alpha^2}+\frac{\partial^2\ddot{w}}{\partial\beta^2}\right]+\left[\bar{I}'_3\frac{\partial\ddot{v}}{\partial\beta}\right]+\left[\bar{I}_5\frac{\partial\ddot{\phi}_1}{\partial\alpha}+\bar{I}_5\frac{\partial\ddot{\phi}_2}{\partial\beta}\right]$$

$$\left[B_{11}+D_{11}k_1-\frac{4}{3h^2}E_{11}-\frac{4}{3h^2}F_{11}k_1\right]\frac{\partial^2 u}{\partial\alpha^2}+\left[B_{11}k_1+B_{12}k_2-\frac{4}{3h^2}E_{11}k_1-\frac{4}{3h^2}E_{12}k_2-A_{55}+\frac{4}{3h^2}D_{55}\right]\frac{\partial w}{\partial\alpha}$$

$$+\left[B_{12}+D_{12}k_2+B_{66}+D_{66}k_2-\frac{4}{3h^2}E_{12}-\frac{4}{3h^2}F_{12}k_2+E_{66}+F_{66}k_2\right]\frac{\partial^2 v}{\partial\alpha\partial\beta}+$$

$$\left[D_{11}-\frac{8}{3h^2}F_{11}-\frac{16}{9h^4}H_{11}\right]\frac{\partial^2\phi_1}{\partial\alpha^2}$$

$$+\left[D_{12}-\frac{4}{3h^2}F_{12}+D_{66}-\frac{4}{3h^2}F_{66}-\frac{4}{3h^2}F_{12}+\frac{16}{9h^4}H_{12}-\frac{4}{3h^2}F_{66}+\frac{16}{9h^4}H_{66}\right]\frac{\partial^2\phi_2}{\partial\alpha\partial\beta}$$

$$-\frac{4}{3h^2}F_{11}\frac{\partial^3 w}{\partial\beta^3}-\frac{16}{9h^4}H_{11}\frac{\partial^3 w}{\partial\beta^3}-\left[\frac{4}{3h^2}F_{12}-\frac{4}{3h^2}F_{66}+\frac{16}{9h^4}H_{12}\right]\frac{\partial^3 w}{\partial\beta^2\partial\alpha}$$

$$+\left[B_{66}+D_{66}k_1-\frac{4}{3h^2}E_{66}-\frac{4}{3h^2}F_{66}k_1\right]\frac{\partial^2 u}{\partial\beta^2}+$$

$$\left[D_{66}-\frac{8}{3h^2}F_{66}+\frac{16}{9h^4}H_{66}\right]\frac{\partial^2\phi_1}{\partial\beta^2}+\frac{16}{9h^4}H_{66}\frac{\partial^2 w}{\partial\alpha\ \partial\beta}-\left[A_{55}-\frac{4}{3h^2}D_{55}\right]\phi_1$$

$$+\frac{4}{h^2}\left[D_{55}-\frac{4}{3h^2}F_{55}\right]\phi_1+\frac{4}{h^2}\left[D_{55}-\frac{4}{3h^2}F_{55}\right]\frac{\partial w}{\partial\alpha}$$

$$=I_2\ddot{u}+\bar{I}_4\ddot{\phi}_1-\bar{I}_5\frac{\partial\ddot{w}}{\partial\alpha}$$

$$\left[B_{66}+D_{66}k_2-\frac{4}{3h^2}E_{66}-\frac{4}{3h^2}F_{66}k_2\right]\frac{\partial^2 v}{\partial\alpha^2}+\left[B_{66}+k_1D_{66}+B_{21}+k_1D_{21}-\frac{4}{3h^2}E_{66}-\frac{4}{3h^2}F_{66}k_1\right]\frac{\partial^2 u}{\partial\alpha\partial\beta}+$$
$$\left[D_{66}-\frac{4}{3h^2}F_{66}+D_{21}-\frac{4}{3h^2}F_{21}-\frac{4}{3h^2}F_{66}-\frac{16}{9h^4}H_{66}\right]\frac{\partial^2\phi_1}{\partial\alpha\partial\beta}+\left[D_{66}-\frac{4}{3h^2}F_{66}-\frac{4}{3h^2}F_{66}+\frac{16}{9h^4}H_{66}\right]\frac{\partial^2\phi_2}{\partial\alpha^2}$$
$$-\left[\frac{8}{3h^2}F_{66}+\frac{4}{3h^2}F_{21}+\frac{4}{3h^2}H_{66}\right]\frac{\partial^3 w}{\partial\alpha^2\partial\beta}+\left[B_{22}+D_{22}k_2-\frac{4}{3h^2}\left(E_{21}+F_{21}k_2+F_{22}k_2+E_{22}\right)\right]\frac{\partial^2 v}{\partial\beta^2}+$$
$$\left[B_{22}k_2+B_{21}k_1-\frac{4}{3h^2}\left(E_{21}k_2+E_{22}k_2\right)\right]\frac{\partial w}{\partial\beta}+\left[D_{22}-\frac{4}{3h^2}F_{22}-\frac{4}{3h^2}\left(F_{21}-\frac{4}{3h^2}H_{21}+F_{22}-\frac{4}{3h^2}H_{22}\right)\right]\frac{\partial^2\phi_2}{\partial\beta^2}+$$
$$\left[-\frac{4}{3h^2}F_{22}+\frac{16}{9h^4}H_{21}+\frac{16}{9h^4}H_{22}\right]\frac{\partial^3 w}{\partial\beta^3}-\left[A_{44}-\frac{4}{h^2}D_{44}+\frac{4}{h^2}\left(D_{44}-\frac{4}{h^2}F_{44}\right)\right]\phi_2-$$
$$\left[A_{44}-\frac{4}{h^2}D_{44}+\frac{4}{h^2}\left(D_{44}-\frac{4}{h^2}F_{44}\right)\right]\frac{\partial w}{\partial\beta}$$

$$=\bar{I}_2\ddot{v}+\bar{I}_4\ddot{\phi}_2-\bar{I}_5\frac{\partial\ddot{w}}{\partial\beta}$$

3.8 BOUNDARY CONDITIONS

Up to now, the analysis has been general without reference to the boundary conditions. For reasons of simplicity, only simply supported boundary conditions are considered along all edges for the shell. The boundary conditions for the simply supported doubly curved shell are obtained as given below

$$N_1 = 0,\ v = 0,\ w = 0,$$

Following the Navier solution procedure, the following solution form which satisfies the boundary conditions in the above equation is assumed:

$$
\begin{aligned}
u &= U \cos \lambda_m \alpha . \sin \lambda_n \beta \quad e^{i\omega t} \\
v &= V \sin \lambda_m \alpha . \cos \lambda_n \beta \quad e^{i\omega t} \\
\omega &= W \sin \lambda_m \alpha . \sin \lambda_n \beta \quad e^{i\omega t} \\
\phi_1 &= \Phi_1 \cos \lambda_m \alpha . \sin \lambda_n \beta \quad e^{i\omega t} \\
\phi_2 &= \Phi_2 \sin \lambda_m \alpha . \cos \lambda_n \beta \quad e^{i\omega t}
\end{aligned}
\qquad \text{..............} \qquad (42)
$$

where , $\lambda_m = \dfrac{m\pi}{a}$, $\lambda_n = \dfrac{n\pi}{b}$ and U, V, W, Φ_1 and Φ_2 are the maximum amplitudes, m and n are known as the axial half wave number and circumferential wave number respectively. Introducing the expressions [42] into the governing equations of motion in terms of displacements [41], the following equation in matrix form is obtained, which is a general eigen value problem.

$$[C]\{X\} = \omega^2 [M]\{X\} \qquad (43)$$

$$[C]\{X\} = \lambda [B]\{X\} \qquad (43.a)$$

Where,

ω^2 is the eigenvalue

λ is the critical buckling load

{X} is a column matrix of amplitude of vibration or eigenvector.

[C], [M] and [B] are 5 x 5 matrices.

For convenience, the elements of the above matrices are suitably non-dimensionalised as follows

$$
\begin{aligned}
U &= \overline{U}h \\
V &= \overline{V}h \\
W &= \overline{W}h \\
\Phi_1 &= \overline{\Phi}_1 \\
\Phi_2 &= \overline{\Phi}_2 \\
A_{ij} &= \overline{A}_{ij}Q_2h \qquad E_{ij} = \overline{E}_{ij}Q_2h^4 \\
B_{ij} &= \overline{B}_{ij}Q_2h^2 \qquad F_{ij} = \overline{F}_{ij}Q_2h^5 \\
D_{ij} &= \overline{D}_{ij}Q_2h^3 \qquad H_{ij} = \overline{H}_{ij}Q_2h^7
\end{aligned}
\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (44)
$$

(- (bar) on top indicates non-dimensionalised quantities)

And

$$
\begin{aligned}
I_1 &= I_1'\rho^1 h \\
I_2 &= I_2'\rho^1 h^2 \\
I_3 &= I_3'\rho^1 h^3 \\
I_4 &= I_4'\rho^1 h^4 \\
I_5 &= I_5'\rho^1 h^5 \\
I_7 &= I_7'\rho^1 h^7
\end{aligned}
\qquad (45)
$$

$\left(I_1', I_2', I_3', \ldots\ldots\ldots\right.$ are the non-dimensionalised quantities

Also

$$
\begin{aligned}
\overline{I}_1 &= I_1'\rho^1 h = \tilde{I}_1\rho^1 h \\
\overline{I}_2 &= \left[I_2' - \frac{4}{3}I_4'\right]\rho^1 h^2 = \tilde{I}_2\rho^1 h^2 \\
\overline{I}_3 &= \left[I_3' - \frac{8}{3}I_5' + \frac{16}{9}I_7'\right]\rho^1 h^3 = \tilde{I}_3\rho^1 h^3 \\
\overline{I}_5 &= \left[I_5' - \frac{4}{3}I_7'\right]\rho^1 h^5 = \tilde{I}_5\rho^1 h^5
\end{aligned}
\qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (46)
$$

After non–dimensionalising the terms, the equation [43] in matrix form can be written as given below

$$\left[\overline{H}\right]\left\{\overline{X}\right\}=\overline{\omega}^{2}\left\{\overline{X}\right\} \tag{47}$$

$$\left[\overline{H}\right]\left\{\overline{X}\right\}=\overline{\lambda}\left\{\overline{X}\right\} \tag{47.a}$$

Where,

$$\overline{\omega}^{2}=\frac{\rho b^{2}\omega^{2}}{Q_{2}}$$

$$\lambda=n_{1}\frac{a^{4}}{b^{2}h^{2}(1-\nu_{12}\nu_{21})} \tag{48}$$

A non-trivial solution for the column matrix $\left\{\overline{X}\right\}$ will give the required eigenvalues, which are the values of the square of the frequency parameter $\overline{\omega}$ in the present case. The lowest value of $\overline{\omega}$ is of particular interest. λ will give the critical buckling load.

CHAPTER 4

RESULTS& DISCUSSION

CHAPTER 4

NUMERICAL RESULTS AND DISCUSSIONS

4.1 INTRODUCTION

The frequency parameters and buckling loads are calculated by using a computer program for laminated composite doubly curved shells. The results obtained using the present theory are compared to earlier results and are tabulated. The numerical values of the lowest value of frequency parameter and non-dimensional buckling load are presented for various shell parameters in this chapter to study the effect of number of layers, the orientation of layers and the 'a/h' ratio (side by thickness ratio).

SOLUTION OF EIGEN VALUE PROBLEM AND COMPUTER PROGRAM

A standard subroutine in the computer program to find the eigenvalue of matrices has been used, which consists of root power method of iteration with Wielandt's deflection technique. The program is called RTPM, which is capable of finding the required number of roots in descending order. The change of the sign of the determinant value is checked for values of one percent on either side of the root to verify the convergence. The RTPM program gives the highest value of eigenvalue first, so if the $\left[\overline{M}\right]$matrix is taken as $\left[\overline{C}\right]^{-1}\left[\overline{H}\right]$, then the highest value of $\left(\frac{1}{\overline{\omega}^2}\right)$is obtained, that is the lowest value of $\overline{\omega}^2$ and $\overline{\omega}$ which is of particular interest. These programs are written in FORTRAN language.

4.2 The validation of the formulation and comparison of results

Using the formulation developed above, numerical studies are carried out. The lowest value of the frequencies has been calculated at first for two layer, three layer and four layer laminated composite spherical shells for various values of R/a and a/h by higher order shear deformation theory. These results are compared with earlier available results. This also serves to check on the validity of the present theory, as the results are mostly agreeable. Table 4. 1 and Table 4.2 shows the comparison of present results and those of J .N. Reddy [24] for the non-dimensional frequency parameter $\bar{\omega}^2 = \frac{Q_{22}a^2\omega^2}{b^2h^2E_2}$ of [0/90], [0/90/0] and [0/90/90/0] simply supported spherical shell. The geometrical and material properties used are

$\bar{E}_{11} = 25\bar{E}_{22}, \quad \bar{G}_{12} = \bar{G}_{13} = 0.5\bar{E}_{22}, \quad \bar{G}_{23} = 0.2\bar{E}_{22}, \quad \bar{\nu}_{12} = 0.25, \quad \rho = 1$

For all the panels, a/b =1 and $R_1 = R_2 = R$.

Table 4.1: Comparison of lowest non-dimensional frequency for a simply supported laminated composite spherical shell (a/h=100)

R/a	**Theory**	$0^o/90^o$			$0^o/90^o/0^o$			$0^o/90^o/90^o/0^o$		
		[24]	**Present value**	**Difference**	**[24]**	**Present value**	**Difference**	**[24]**	**Present value**	**Difference**
5	HSDT	28.840	28.886	0.046	31.020	30.927	-0.093	31.100	31.026	-0.074
10	HSDT	16.710	16.811	0.101	20.350	20.245	-0.105	20.380	20.300	-0.08
20	HSDT	11.840	11.911	0.071	16.620	16.511	-0.109	16.63	16.540	-0.09
50	HSDT	10.060	10.230	0.17	15.420	15.291	-0.129	15.42	15.320	-0.1
100	HSDT	9.784	9.964	0.18	15.240	15.110	-0.13	15.23	15.140	-0.09
Plate	HSDT	9.688	9.687	-0.001	15.170	15.171	0.001	15.170	15.169	-0.001

Table 4.2: Comparison of lowest non-dimensional frequency for a simply supported laminated composite spherical Shell (a/h=10)

R/a	Theory	$0^o/90^o$			$0^o/90^o/0^o$			$0^o/90^o/90^o/0^o$		
		[24]	**Present value**	**Difference**	**[24]**	**Present value**	**Difference**	**[24]**	**Present value**	**Difference**
5	HSDT	9.337	9.332	-0.005	12.060	12.110	0.05	12.040	12.649	0.609
10	HSDT	9.068	9.095	0.027	11.860	11.948	0.088	11.840	12.498	0.658
20	HSDT	8.999	9.035	0.036	11.810	11.907	0.097	11.790	12.460	0.67
50	HSDT	8.980	9.019	0.039	11.790	11.896	0.106	11.780	12.450	0.67
100	HSDT	8.977	9.017	0.04	11.790	11.894	0.104	11.780	12.448	0.668
Plate	HSDT	8.976	8.877	-0.099	11.790	11.890	0.100	11.780	12.445	0.665

As can be seen from the Table 4.1 and Table 4.2, the values are quite agreeable.

The validation of the formulation for buckling load is similarly done by comparing the results with that of Librescu *et al* [17]. It can be seen from Table 4.3 that the results are in agreement.

Table 4.3: Comparison of non-dimensional buckling load for a simply supported laminated composite doubly curved shell (a/h=10)

$\overline{E}_{11} = 40\overline{E}_{22}, \quad \overline{G}_{12} = \overline{G}_{13} = 0.6\overline{E}_{22}, \quad \overline{G}_{23} = 0.5\overline{E}_{22}, \quad \overline{\nu}_{12} = 0.25, \quad \rho = 1$

Curvature	Present work	Theory	
		FSDT [17]	HSDT [17]
R_x/a=5, R_y/a=5	12.530	12.214	12.431
R_x/a=10, R_y/a=5	12.236	11.822	12.007
R_x/a=10, R_y/a=20	11.861	11.479	11.697
R_x/a=20, R_y/a=20	11.797	11.406	11.610
Plate	11.753	11.353	11.555

4.3 NUMERICAL RESULTS:

The governing equations are solved for doubly curved shells (spherical, parabolic, and elliptical paraboloid shells). Analysis is carried out for simply supported boundary condition. The material properties for the parametric study are assumed as:

$\overline{E_{11}} = 25\times10^9, \quad \overline{E_{22}} = 01\times10^9, \quad \overline{G_{12}} = 0.5\times10^9, \quad \overline{G_{13}} = 0.5\times10^9$

$\overline{G_{23}} = 0.2\times10^9 \quad \nu_{12} = 0.25 \quad \frac{a}{h} = 100$

The geometry for various shells unless otherwise specified is

For spherical shell: $\frac{R_1}{a} = 5, \ \frac{R_2}{a} = 5$

For hyperbolic paraboloid shell: $\frac{R_1}{a} = -5, \ \frac{R_2}{a} = 5$

For elliptical paraboloidal shell: $\frac{R_1}{a} = 5, \ \frac{R_2}{a} = 7.5$

4.3.1 VIBRATION ANALYSIS

The first study is done to investigate the variation in R/a ratio on the non-dimensional frequency parameter.

Table 4.4 shows the variation of R/a ratio on the non-dimensional frequency parameters of the hyperbolic paraboloid shell.

Table 4.4: Variation in non-dimensional frequency parameter with change in curvature ratio (R/a) for laminated composite hyperbolic paraboloidal shell

$\frac{R_1}{a}$	$\frac{R_2}{a}$	0/90			0/90/90/0		
		a/h=200	a/h=100	a/h=10	a/h=200	a/h=100	a/h=10
-1	1	45.590	32.826	12.582	12.352	12.330	10.166
-2	2	19.494	15.377	9.402	14.354	14.328	11.826
-3	3	13.533	11.798	9.028	14.764	14.737	12.167
-4	4	11.567	10.721	8.958	14.910	14.883	12.269
-5	5	10.774	10.308	8.945	14.979	14.952	12.346
-10	10	9.990	9.919	8.966	15.071	15.044	12.422

Here it is seen that as R/a ratio increases, then non-dimensional frequencies are decreasing.

The study is extended to elliptical paraboloidal shells by increasing the R/a ratio, keeping the R_1/R_2 ratio constant, as given in Table 4.5.

Table 4.5: Variation in non-dimensional frequency parameter with change in curvature ratio (R/a) for laminated composite elliptical paraboloidal shell

R_1/a	R_2/a	a/h =100			a/h=10		
		0/90	0/90/90/0	0/90/0/90/0	0/90	0/90/90/0	0/90/0/90/0
1	1.5	107.105	107.763	107.634	12.928	15.095	15.093
2	3.0	56.682	57.788	57.730	10.270	13.216	13.180
3	4.5	38.821	40.438	40.398	9.613	12.800	12.755
4	6.0	29.924	31.996	31.978	9.363	12.648	12.599
5	7.5	24.692	27.175	27.166	9.242	12.577	12.526
10	15	15.034	18.855	18.868	9.076	12.480	12.427

In case of elliptical paraboloidal shell, with the increase of R/a ratio, non-dimensional frequency decreases. With the increase of thickness parameter (a/h), i.e., as thickness decreases, then frequency parameter is also increasing.

Table 4.6 shows the variation of modular ratio on the non-dimensional frequency parameter. Four layer cross-ply [0/90/90/0] and a/h = 100 value is chosen for the study.

Table 4.6: Variation in non-dimensional frequency parameter with modular ratio for laminated composite doubly curved shells of [0/90/90/0] and a/h=100

$\frac{\overline{E}_{11}}{\overline{E}_{22}}$	Hyperbolic paraboloidal shell	Elliptical paraboloidal shell	Spherical shell
	R_1/a= -5,R_2/a=5	R_1/a= 5,R_2/a=7.5	R_1/a=R_2/a=5
25	14.952	27.175	31.026
40	18.432	29.465	33.131
60	22.220	32.101	35.538

With the increasing modular ratio, it is seen that the non-dimensional frequency parameter is also increasing. But hyperbolic paraboloidal shell has less non-dimensional frequency; hence it is more stable than the other two doubly curved shells.

Table 4.7 shows the variation of the non-dimensional frequency with number of layers for all shell geometries.

Table 4.7: Variation in non-dimensional frequency parameter with number of layers for laminated composite doubly curved shells a/h=100

Shell	Curvature	0/90	0/90/0	0/90/90/0	0/90/0/90/0
Spherical	$R_1/a=R_2/a=5$	28.886	30.928	31.026	31.009
Elliptical	$R_1/a=5, R_2/a=7.5$	24.692	27.090	27.175	27.166
Hyper paraboloid	$R_1/a=-5, R_2/a=5$	10.308	14.924	14.952	14.977

Above table shows that as number of layers changes, non-dimensional frequencies generally increase.

Some of the above results are also presented in graphical form as shown in Figures 4.1-4.3.

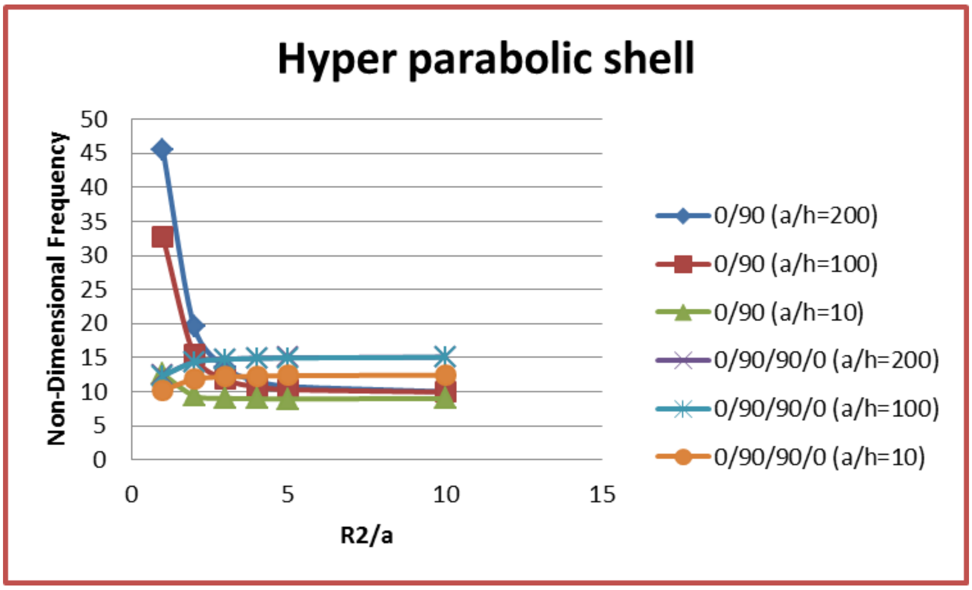

Figure 4.1: Variation in non-dimensional frequency with R_2/a of hyperbolic paraboloid shells for various a/h ratios

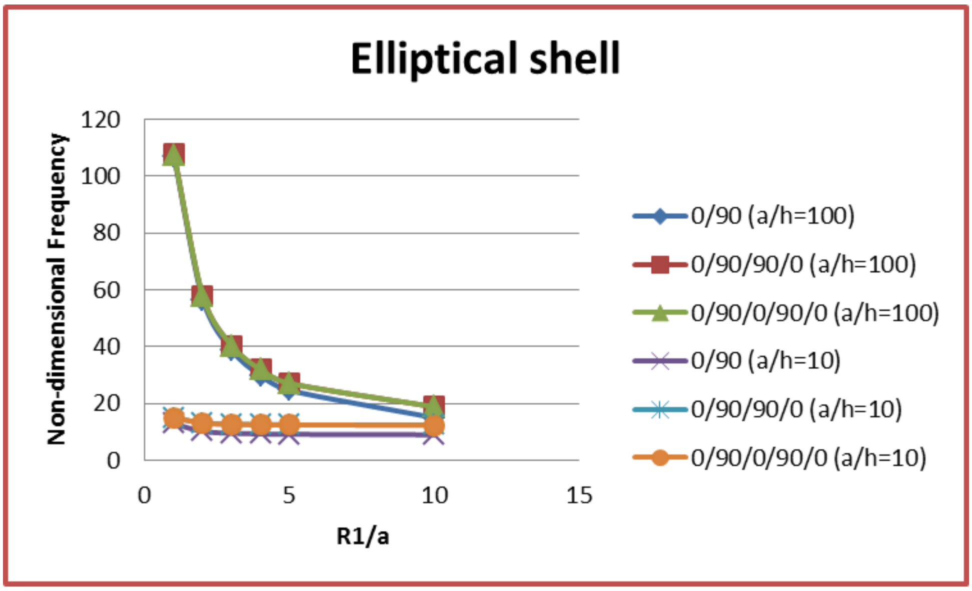

Figure 4.2: Variation of non-dimensional frequency with R_1/a of elliptical paraboloid shell for various a/h ratios

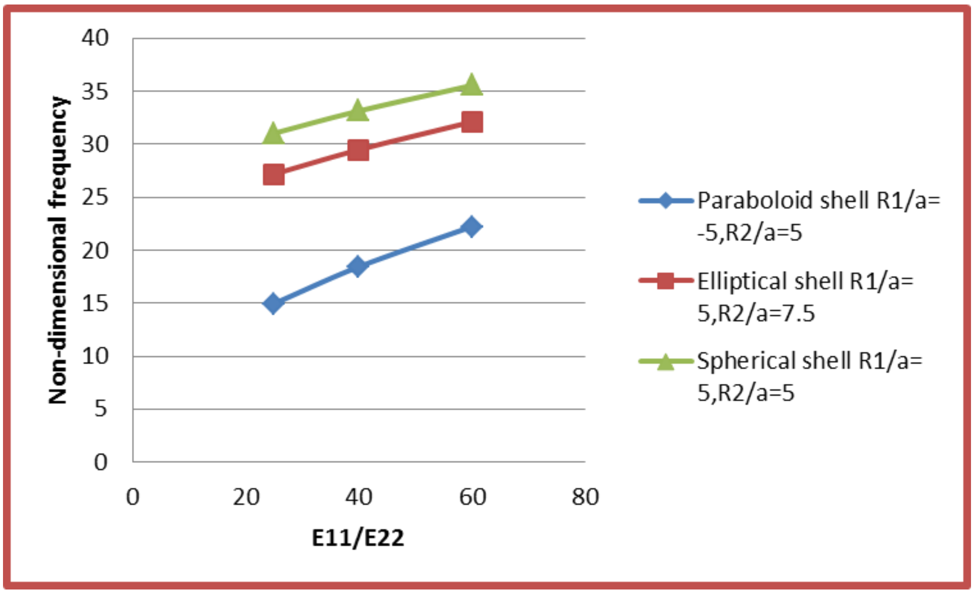

Figure 4.3: Variation of non-dimensional frequency of cross-ply spherical shell, hyperbolic and elliptical paraboloid shell with modular ratio.

4.3.2 STABILITY ANALYSIS

For determination of non-dimensional buckling load, the geometrical and material properties used are

$$\overline{E}_{11} = 25\overline{E}_{22}, \quad \overline{G}_{12} = \overline{G}_{13} = 0.5\overline{E}_{22}, \quad \overline{G}_{23} = 0.2\overline{E}_{22}, \quad \overline{\nu}_{12} = 0.25, \quad \rho = 1$$

Table 4.8 shows the variation in non-dimensional buckling load with R/a ratio for a hyperbolic paraboloidal shell. It is observed that here as R/a ratio increases, non-dimensional buckling load decreases.

Table 4.8: Variation in non-dimensional buckling load with curvature ratio (R/a) for laminated composite hyperbolic paraboloidal shell

$\frac{R_1}{a}$	$\frac{R_2}{a}$	0/90			0/90/90/0		
		a/h=200	a/h=100	a/h=10	a/h=200	a/h=100	a/h=10
-1	1	254.088	132.165	20.560	18.594	18.529	12.720
-2	2	40.545	25.285	9.925	21.935	21.859	15.040
-3	3	19.003	14.468	8.828	22.584	22.505	15.490
-4	4	13.745	11.822	8.563	22.813	22.734	15.650
-5	5	11.869	10.876	8.471	22.919	22.840	15.724
-10	10	10.138	10.001	8.398	23.061	22.982	15.822

Similar study is done next for the laminated composite elliptical paraboloid shell in Table 4.9 and the same variation is observed.

Table 4.9: Variation in non-dimensional buckling load with curvature ratio (R/a) for laminated composite elliptical paraboloid shell

R_1/a	R_2/a	a/h=100			a/h=10		
		0/90	0/90/90/0	0/90/0/90/0	0/90	0/90/90/0	0/90/0/90/0
1	1.5	1310.338	1327.928	1324.926	19.186	26.322	26.301
2	3.0	335.768	349.220	348.530	11.169	18.449	18.338
3	4.5	154.832	168.000	167.737	9.644	17.006	16.877
4	6.0	91.445	104.575	104.461	9.106	16.502	16.367
5	7.5	62.092	75.218	75.173	8.856	16.269	16.131
10	15	22.935	36.076	36.123	8.521	15.959	15.816

Table 4.10 shows the variation of modular ratio on the non-dimensional buckling load for all shell geometries. Four layer cross-ply was taken with a/h ratio of 100. The non-dimensional buckling load is found to increase with increase in modular ratio for all shell geometries.

Table 4.10: Variation in non-dimensional buckling load with modular ratio for laminated composite doubly curved shells [0/90/90/0] and a/h=100.

$\frac{\overline{E}_{11}}{\overline{E}_{22}}$	Hyperbolic paraboloid shell	Elliptical paraboloid shell	Spherical shell
	R_1/a= -5,R_2/a=5	R_1/a= 5,R_2/a=7.5	R_1/a=R_2/a=5
25	98.235	75.218	22.840
40	112.059	88.452	34.710
60	128.961	105.006	50.442

Table 4.11 shows the variation of the non-dimensional frequency with number of layers for all shell geometries.

Table 4.11: Variation in non-dimensional buckling load with number of layers for laminated composite doubly curved shells a/h=100

Shell	Curvature	0/90	0/90/0	0/90/90/0	0/90/0/90/0
Spherical	$R_1/a=R_2/a=5$	85.161	97.606	98.235	98.126
Elliptical	$R_1/a=5,R_2/a=7.5$	62.092	74.748	75.218	75.173
Hyper paraboloid	$R_1/a= -5,R_2/a=5$	10.876	22.756	22.840	22.918

Above table shows that with increase in number of layers, the non-dimensional buckling load generally increases.

CHAPTER 5

CONCLUSION

CHAPTER 5

5.1 CONCLUSION:

The governing equations including the effect of shear deformations have been presented in orthogonal curvilinear coordinates for laminated orthotropic doubly curved shells. The theory is based on a displacement field as proposed by Reddy and Liu [24] in which the displacements of the middle surface are expanded as cubic functions of the thickness coordinate and transverse displacement is assumed to be constant through the thickness. This displacement field leads to the parabolic distribution of the transverse shear strain and hence shear stresses and therefore no shear correction factors are used.

The governing equations of motion are derived by integrating the displacement gradients by parts and setting the coefficients of $\partial u, \partial v, \partial w, \partial \phi_1$ and $\partial \phi_2$ to zero separately. In the present study, for first two equations the moment terms are also considered.

The governing equations are then specialized for a doubly curved shell. These equations have been solved for simply supported doubly covered shells and the associated eigenvalue problem has been solved by means of a computer program. Lowest frequencies and buckling loads have been considered throughout.

The results of the present theory have been compared with the earlier results available by Reddy and Liu [24]. Results are compared and found that it matches that of Reddy and Liu for spherical shells. The study is extended to vibration of hyperbolic and elliptical paraboloid shells and influence of various parameters like aspect ratio, number of layers, modular ratio,etc on the same are studied. The non-dimensional buckling load for axial compression in one direction is also studied for various shell geometries and variation in other parameters.

Following conclusions were observed and are summarized again below:

1) With the increasing modular ratio, non-dimensional frequency is also increasing. But hyperbolic paraboloid shell has less non-dimensional frequency as compared to elliptical paraboloid shell and spherical shell.
2) With the increase of curvature ratio R/a frequency decreases.
3) With the increase of a/h ratio, frequency also increases.

4) Increase of thickness parameter (a/h) ratio then non-dimensional frequency of the doubly curved shell increases.
5) It is also observed that as number of layers of the shell increases, buckling load increases.
6) In comparison among spherical, hyperbolic paraboloid shell and ellipitical paraboloid shell, the hyperbolic paraboloidal shell has least non-dimensional buckling load.
7) In case of both shell geometries, with the increase of curvature ratio(R/a) buckling load decreases.
8) As a/h ratio increases, the non-dimensional buckling load also increases.
9) As number of layers increases, in general the non-dimensional frequency and the non-dimensional buckling load increases.

Thus by the above study it can be seen that by suitably changing the orientation of the layers or the number of layers, the properties of the laminated composite doubly curved shells can be tailored to suit the particular needs.

5.2 SCOPE FOR FUTURE WORK

1) HSDT can be used to study the vibration and buckling characteristics of thick laminated composite shells.
2) HSDT is more important for laminated composite plates and shells because of greater shear effect
3) Incorporation of finite element techniques to take care of higher shear deformation theory instead of analytical method used in this work.

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APPENDIX

The non-dimensionalised coefficients of the [C_{ij}] and [M_{ij}] matrices of Higher order shear deformation theory are given below,

$$\overline{C}(1,1) = \left[\overline{A}_{11} + 2KH\overline{B}_{11} + K^2H^2\overline{D}_{11}\right]\frac{\overline{\lambda}^2{}_m}{S^2} + \left[\overline{A}_{66} + 2KH\overline{B}_{66} + K^2H^2\overline{D}_{66}\right]\overline{\lambda}^2{}_n$$

$$\overline{C}(1,2) = \left[\left(\overline{A}_{12} + \overline{A}_{66}\right) + H\left(\overline{B}_{12} + \overline{B}_{66}\right) + KH\left(\overline{B}_{12} + \overline{B}_{66}\right) + KH^2\left(\overline{D}_{12} + \overline{D}_{66}\right)\right]\frac{\overline{\lambda}_m\overline{\lambda}_n}{S}$$

$$\overline{C}(1,3) = \left[\left(KH\overline{A}_{11} + H\overline{A}_{12} + K^2H^2\overline{B}_{11} + KH^2\overline{B}_{11}\right)\frac{\overline{\lambda}_m J}{SH}\right] + \frac{4H}{3SJ}\left[\left(\overline{E}_{12} + 2\overline{E}_{66}\right) + KH\left(\overline{F}_{12} + 2\overline{F}_{66}\right)\right]\overline{\lambda}_m\overline{\lambda}^2{}_n +$$

$$\left[\frac{4}{3}\left(\overline{E}_{11} + KH\overline{F}_{11}\right)\right]\frac{\overline{\lambda}^3{}_m H}{S^3 J}$$

$$\overline{C}(1,4) = \left[\left(\overline{B}_{11} - \frac{4}{3}\overline{E}_{11} + KH\overline{D}_{11} - \frac{4}{3}KH\overline{F}_{11}\right)\frac{\overline{\lambda}^2{}_m}{S^2} + \left(\overline{B}_{66} - \frac{4}{3}\overline{E}_{66} + KH\overline{D}_{66} - \frac{4}{3}KH\overline{F}_{66}\right)\overline{\lambda}^2{}_n\right]$$

$$\overline{C}(1,5) = \left[\left(\overline{B}_{12} + \overline{B}_{66}\right) - \frac{4}{3}\left(\overline{E}_{12} + \overline{E}_{66}\right) + KH\left(\overline{D}_{12} + \overline{D}_{66}\right) - \frac{4}{3}KH\left(\overline{F}_{12} + \overline{F}_{66}\right)\right]\frac{\overline{\lambda}_m\overline{\lambda}_n}{S}$$

$$\overline{C}(2,1) = \left[\left(\overline{A}_{66} + \overline{A}_{21}\right) + KH\left(\overline{B}_{21} + \overline{B}_{66}\right) + H\left(\overline{B}_{21} + \overline{B}_{66}\right) + KH^2\left(\overline{D}_{12} + \overline{D}_{66}\right)\right]\frac{\overline{\lambda}_m\overline{\lambda}_n}{S}$$

$$\overline{C}(2,2) = \left[\left(\overline{A}_{66} + 2H\overline{B}_{66} + H^2\overline{D}_{66}\right)\frac{\overline{\lambda}^2{}_m}{S^2} + \left(\overline{A}_{22} + 2H\overline{B}_{22} + H^2\overline{D}_{22}\right)\overline{\lambda}^2{}_n\right]$$

$$\overline{C}(2,3) = -\left[\begin{array}{l}\left(KH\overline{A}_{21} + H\overline{A}_{22} + KH^2\overline{B}_{21} + H^2\overline{B}_{22}\right)\frac{\overline{\lambda}_n J}{H} + \frac{4}{3}\left(2\overline{E}_{66} + \overline{E}_{21} + 2H\overline{F}_{66} + H\overline{F}_{21}\right)\frac{\overline{\lambda}^2{}_m\overline{\lambda}_n H}{S^2 J} + \\ \frac{4\overline{\lambda}^3{}_n H}{3J}\left(\overline{E}_{22} + \overline{F}_{22}\right)\end{array}\right]$$

$$\overline{C}(2,4) = \left[\left(\overline{B}_{21} + \overline{B}_{66}\right) + H\left(\overline{D}_{21} + \overline{D}_{66}\right) - \frac{4}{3}\left(\overline{E}_{21} + \overline{E}_{66}\right) + \frac{4}{3}H\left(\overline{F}_{21} + \overline{F}_{66}\right)\right]\frac{\overline{\lambda}_m\overline{\lambda}_n}{S}$$

$$\overline{C}(2,5) = \left[\left(\overline{B}_{66} + H\overline{D}_{66} - \frac{4}{3}\overline{E}_{66} - \frac{4}{3}H\overline{F}_{66}\right)\frac{\overline{\lambda}^2{}_m}{S^2} + \left(\overline{B}_{22} + H\overline{D}_{22} - \frac{4}{3}\overline{E}_{22} - \frac{4}{3}H\overline{F}_{22}\right)\overline{\lambda}^2{}_n\right]$$

$$\overline{C}(3,3) = \left[\begin{array}{l}\left(K^2H\overline{A}_{11} + K\overline{A}_{12} + K\overline{A}_{21} + \overline{A}_{22}\right)J^2 + \left(\overline{A}_{11} - 8\overline{D}_{55} + \frac{8}{3}KH\overline{E}_{11} + \frac{4}{3}H\overline{E}_{12} + \frac{4}{3}H\overline{E}_{21} + 16\overline{F}_{55}\right)\frac{\overline{\lambda}^2{}_m}{S^2} \\ + \left(\overline{A}_{44} - 8\overline{D}_{44} + \frac{4}{3}KH\overline{E}_{21} + \frac{8}{3}H\overline{E}_{22} + \frac{4}{3}KH\overline{E}_{12} + 16\overline{F}_{44}\right)\overline{\lambda}^2{}_n + \frac{16}{9S^4J^2}\overline{H}_{11}H^2\overline{\lambda}^4{}_m + \frac{16}{9J^2}\overline{H}_{11}H^2\overline{\lambda}^4{}_n + \\ \frac{16H^2}{9S^2J^2}\left(\overline{H}_{12} + \overline{H}_{21} + 4\overline{H}_{66}\right)\overline{\lambda}^2{}_m\overline{\lambda}^2{}_n\end{array}\right]$$

$$\overline{C}(3,4)=\left[\begin{array}{l}\left(\overline{A}_{55}-8\overline{D}_{55}+16\overline{F}_{55}-KH\overline{B}_{11}+\frac{4}{3}KH\overline{E}_{11}-H\overline{B}_{21}+H\overline{E}_{21}\right)\frac{\overline{\lambda}_m J}{SH}-\frac{4}{3}\left(\overline{F}_{11}-\frac{4}{3}\overline{H}_{11}\right)\frac{\overline{\lambda}^3{}_m H}{S^3 J}-\\ \frac{4}{3}\left(\overline{F}_{21}-\frac{4}{3}\overline{H}_{21}+2\overline{F}_{66}-\frac{8}{3}\overline{H}_{66}\right)\overline{\lambda}{}_m\overline{\lambda}^2{}_n\frac{H}{SJ}\end{array}\right]$$

$$\overline{C}(3,5)=\left[\begin{array}{l}\left(\overline{A}_{44}-8\overline{D}_{44}+16\overline{F}_{44}-KH\overline{B}_{12}+\frac{4}{3}KH\overline{E}_{12}-H\overline{B}_{22}+\frac{4}{3}H\overline{F}_{22}\right)\frac{\overline{\lambda}_n J}{H}-\frac{4}{3}\left(\overline{F}_{12}-\frac{4}{3}\overline{H}_{12}+2\overline{F}_{66}-\frac{8}{3}\overline{H}_{66}\right)\\ \frac{\overline{\lambda}^2{}_m\overline{\lambda}{}_n H}{S^2 J}-\frac{4}{3}\left(\overline{F}_{22}-\frac{4}{3}\overline{H}_{22}\right)\overline{\lambda}^3{}_n\frac{H}{J}\end{array}\right]$$

$$\overline{C}(4,4)=\left[\left(\overline{D}_{11}-\frac{4}{3}\left(2\overline{F}_{11}-\frac{4}{3}\overline{H}_{11}\right)\right)\frac{\overline{\lambda}^2{}_m}{S^2}+\left(\overline{D}_{66}-\frac{4}{3}\left(2\overline{F}_{66}-\frac{4}{3}\overline{H}_{66}\right)\right)\overline{\lambda}^2{}_n+\left(\overline{A}_{55}-8\overline{D}_{55}+16\overline{F}_{55}\right)\frac{J^2}{H^2}\right]$$

$$\overline{C}(4,5)=\left[\left(\overline{D}_{12}+\overline{D}_{66}-\frac{4}{3}\left(2\overline{F}_{12}+2\overline{F}_{66}\right)+\frac{16}{9}\left(\overline{H}_{12}+\overline{H}_{66}\right)\right)\frac{\overline{\lambda}_m\overline{\lambda}_n}{S}\right]$$

$$\overline{C}(1,3)=\overline{C}(3,1)$$

$$\overline{C}(2,3)=\overline{C}(3,2)$$

$$\overline{C}(1,4)=\overline{C}(4,1)$$

$$\overline{C}(1,5)=\overline{C}(5,1)$$

$$\overline{C}(2,4)=\overline{C}(4,2)$$

$$\overline{C}(2,5)=\overline{C}(5,2)$$

$$\overline{C}(3,5)=\overline{C}(5,3)$$

$$\overline{C}(5,4)=\overline{C}(4,5)$$

$$M(1,1) = -(I_1' + 2K_1 h I_2')$$

$$M(1.2) = M(2,1) = 0$$

$$M(1,3) = M(3,1) = (\frac{4}{3} I_4' + \frac{4}{3} K_1 h I_5') \frac{\bar{\lambda}_m}{a} h$$

$$M(1,4) = M(4,1) = -(I_2' + K_1 h I_3' - \frac{4}{3} I_4' - \frac{4}{3} K_1 h I_5')$$

$$M(1,5) = M(5,1) = 0$$

$$M(2,2) = -(I_1' + 2K_2 h I_2')$$

$$M(2,3) = M(3,2) = (\frac{4}{3} I_4' + \frac{4}{3} K_2 h I_5') \frac{\bar{\lambda}_n}{b} h$$

$$M(2,4) = M(4,2) = 0$$

$$M(2,5) = -(I_2' + K_2 h I_3' - \frac{4}{3} I_4' - \frac{4}{3} K_2 h I_5')$$

$$M(3,3) = -\frac{16}{9} I_7' (\bar{\lambda}_m^2 (\frac{h}{a})^2 + \bar{\lambda}_n^2 (\frac{h}{b})^2) - I_1'$$

$$M(3,4) = M(4,3) = (\frac{4}{3} I_5' - \frac{16}{9} I_7') \bar{\lambda}_m \frac{h}{a}$$

$$M(3,5) = M(5,3) = (\frac{4}{3} I_5' - \frac{16}{9} I_7') \bar{\lambda}_n \frac{h}{b}$$

$$M(4,4) = M(5,5) = -(I_3' - \frac{8}{3} I_5' + \frac{16}{9} I_7')$$

$$B(3,3) = \bar{\lambda}^2$$