A STUDY ON SOLUTION OF DIFFERENTIAL EQUATIONS USING HAAR WAVELET COLLOCATION METHOD

A PROJECT REPORT SUBMITTED IN FULLFILLMENT OF THE REQUIREMENTS OF THE DEGREE OF
MASTER OF SCIENCE

IN
MATHEMATICS

SUBMITTED TO
NATIONAL INSTITUTE OF TECHNOLOGY, ROURKELA

BY
BISHNUPRIYA SAHOO
ROLL NUMBER 410MA2103

UNDER THE SUPERVISION OF
PROF. SANTANU SAHA RAY

DEPARTMENT OF MATHEMATICS
NATIONAL INSTITUTE OF TECHNOLOGY
ROURKELA-769008

NATIONAL INSTITUTE OF TECHNOLOGY
ROURKELA

DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "A study on solution of differential equations using Haar wavelet collocation method" in partial fulfilment for the award of degree of Master of Science, submitted in the Department of Mathematics, National Institute of Technology, Rourkela is an authentic record of my own work carried out under the supervision of Prof. S. Saha Ray. The matter embedded in this thesis has not been submitted by me for the award of any other degree.

Bishnupriya sahoo
Roll No. 410MA2103
Department of Mathematics
NIT, Rourkela

NATIONAL INSTITUTE OF TECHNOLOGY
ROURKELA

CERTIFICATE

This is to certify that the above declaration made by the candidate is correct to the best of my knowledge.

Prof. Santanu Saha Ray
National Institute of Technology
Rourkela-769008
Odisha, India

NATIONAL INSTITUTE OF TECHNOLOGY
ROURKELA

ACKNOWLEDGEMENT

With immense pleasure I wish to express my profound sense of reverence and gratitude to Prof. S. Saha Ray for the encouragement, constructive guidance and thought-provoking discussions throughout the period of my project work, which enabled me to complete the project report so smoothly.

Words at my command are inadequate to convey my profound to my parents whose love, affection and blessings and the support and love of my sisters and friends has inspired me the most.

Bishnupriya sahoo
Roll No. 410MA2103
Department of Mathematics
NIT, Rourkela

ABSTRACT

In this contest of study, problems regarding differential equations are studied when the differential equations: ordinary or partial differential equations have no solution in direct method or it is very difficult to find the required integral.

When this type of problem arises, mainly numerical solution method comes to a picture. From the different numerical methods haar wavelet transform method is one to use it in solving differential equations.

Before coming directly to the solution of differential equations haar wavelet function and its properties are studied. Using the properties of haar wavelet transform a useful term from the differential equation is approximated by the summation of constant multiples of the haar functions which are known functions and easy to handle. Then the other terms of the differential equations are found out by integrating or differentiating the above discussed problem.

Using a logical method the differential equations are solved. And it is observed that the solution gives less error. So this method can be an efficient method.

CONTENTS Page No.

1. **Introduction**:

Numerical Analysis starts with the difficulties in finding the solution of an equation in a direct method or in a theoretical method proposed earlier to find the exact solution. In case of complicacy in finding the solution numerical methods with respectively lesser error are proposed.

We know the methods like Bisection method, Secant method, Newton Raphson method to find the root of an equation. But when the equation includes a differential operator i.e. a differential equation the known numerical solution methods are Picards method, Rungee kutta method etc.

For a better solution i.e. for an approximated solution with lesser error we use various collocation methods. As there is no direct method to solve type of equations like Van-Der-Pol equation and Fisher's equation, hence for the solution, a numerical method can be used.

Here in this contest of study, HAAR Wavelet Transform Method to find the solution of typical differential equations like Van-Der-Pol equation and Fisher's equation. The idea of Wavelets can be summarised as a family of functions constructed from transformation and dilation of a single function called mother wavelet. From various type of continuous and discrete wavelets HAAR Wavelet is the discrete type of wavelet which was 1^{st} proposed and the 1^{st} orthonormal wavelet basis is the Haar basis.

2. THE HAAR WAVELET

Before coming to the term Haar Wavelet let us discuss something about wavelet and wavelet transform. As per Morlet Analysis, signals consist of different features in time and frequency their high frequency components would have shorter time duration than their low frequency components. In order to achieve a good time resolution for high frequency transients and good frequency resolution for low frequency components, Morlet (1982) first introduced the idea of wavelets as a family of functions constructed from translations and dilations of a single function called Mother wavelet and defined by $\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right), a \neq 0, a, b \in \mathcal{R},$ where a is scaling parameter measures degree of compression and b is the translation parameter determines time location of wavelet.

Wavelet transform of $f \in L^2(\mathcal{R})$

$$\mathcal{W}_\psi[f](a,b) = \left(f, \psi_{a,b}\right) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(t)\psi\left(\frac{t-b}{a}\right) dt$$

$\psi_{a,b}(t)$ is called the kernel. If it is $exp(i\omega t)$ then it is known as Fourier transform.

1. Continuous wavelet transform $\mathcal{W}_\psi$ is linear.
2. Wavelet transform is not single transform like Fourier transform, but any transform can be obtained from each of the transform.
3. Inverse wavelet transformation

$$f(t) = C_\psi^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{W}_\psi[f](a,b)\psi_{a,b}(t)(a^{-2}da)\, db$$

$$C_\psi = 2\pi \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(a,b)\right|}{|\omega|} d\omega < \infty \text{ is the admissible condition}$$

.

$\hat{\psi}$ is the Fourier transform of mother wavelet $\psi(t)$

In practical applications, fast numerical algorithms are involved to obtain the solution at discrete points. Continuous wavelet can be computed as discrete grid points. In this basis ψ

can be defined $\psi_{m,n}(t) = a_0^{-\frac{m}{2}}\psi(a_0^{-m}t - nb_0)$ replacing a by a_0^m $(a_0 \neq 0{,}1)$ and b by $nb_0a_0^m$ $(b_0 \neq 0), m, n \in N$

Then the wavelet transform $\bar{f}(m,n) = \mathcal{W}[f](m,n) = \left(f, \psi_{m,n}\right) = \int_{-\infty}^{\infty} f(t)\bar{\psi}_{m,n}(t)dt$

$$\sum_{m,n=-\infty}^{\infty} \bar{f}(m,n)\psi_{m,n}(t) - \text{ Double series or wavelet series.}$$

$\psi_{m,n}(t)$- Discrete wavelet or simple wavelet.

In general $f \in L^2(\mathcal{R})$ can be completely determined by its discrete wavelet transform if the wavelets form a complete system in $L^2(\mathcal{R})$. Otherwise, if the wavelets form an orthonormal basis then they are complete. $\bar{f}(m,n) = \left(f, \psi_{m,n}\right)$

$\Rightarrow f(x) = \sum_{m,n=-\infty}^{\infty}(f, \psi_{m,n}(t))\,\psi_{m,n}(x)$, provided the wavelets form an orthonormal basis.

With the choice $a_0 = 2, b_0 = 1$, there exists a function ψ with good time frequency localisation properties such that $\psi_{m,n}(x) = 2^{-\frac{m}{2}}\psi(2^{-m}x - n)$ constitute an orthonormal basis for $L^2(\mathcal{R})$ is the Haar Wavelet.

Precisely,

$$\psi(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{2} \\ -1, & \frac{1}{2} \leq x < 1 \\ 0, & otherwise \end{cases}$$

and the Haar basis is called the 1st orthonormal basis.

PROPERTIES OF HAAR WAVELET:

1. Haar wavelet is very well localised in the time domain, but not continuous.
2. $\int_0^{\infty} \psi(t)dt = 0$ and $\int_0^{\infty} |\psi(t)|^2 \, dt = 1$
3. Any continuous real function can be approximated by linear combination of Φ(t), Φ(2t), Φ(4t),... , Φ(2^k t),... and their shifted functions.

   This extends the function space where any function can be approximated by continuous functions.
4. Any continuous real function can be approximated by linear combination of the constant functions $\psi(t), \psi(2t), \psi(4t), \ldots, \psi(2^k t)$ ...and their shifted functions.
5. Each two Haar function is orthogonal to each other i.e.

$$\int_{-\infty}^{\infty} 2^{mm1} \psi(2^m t - n) \psi(2^{m1} t - n1) dt = \delta_{m,m1} \delta_{n,n1}$$

6. Wavelet function or scaling function with different scale *m* have a functional relationship Φ(t)= Φ(2t)+ Φ(2t-1) and $\psi(t)$= Φ(2t)- Φ(2t-1).

Practical Application of Haar Wavelet:

Its practical use is in photography or more specifically construction of high resolution camera. We know that if the pixel rate of camera is high then it is a good one. This pixel word comes from the term point. In this case small squares are assumed as points and Haar wavelet is defined in it. The smaller square size gives rise to more resolution and a better quality picture.

Haar Wavelets and Integration of Haar Wavelets:

The Haar Wavelet family for $x \in [0,1)$ is defined by

$$h_i(x) = \begin{cases} 1 & for\ x \in [\alpha, \beta) \\ -1 & for\ x \in [\beta, \gamma) \\ 0 & elesewhere \end{cases}$$

where $\alpha = \frac{k}{m}$, $\beta = \frac{k+0.5}{m}$, $\gamma = \frac{k+1}{m}$

where $m = 2^j, j = 0,1,2, \dots \dots \dots J, k = 0,1, \dots \dots . m-1$ where j indicates the level of wavelet, k denotes translation parameter and J denotes the maximum level of resolution.

The index i in $h_i(x)$ is determined by $i = m + k + 1$ where the minimum values of k and m are 0 and 1 respectively.

The maximum value of $i = 2^{J+1} = 2M$

The introduced notations are as follows

$$p_i(x) = \int_0^x h_i(x')dx' \qquad q_i(x) = \int_0^x p_i(x')dx'$$

In particular when i=1

$$h_1(x) = \begin{cases} 1 & \text{for } x \in [0,1) \\ 0 & \text{elsewhere} \end{cases}$$

$$p_1(x) = \begin{cases} x & \text{for } x \in [0,1) \\ 0 & \text{elsewhere} \end{cases}$$

$$q_1(x) = \begin{cases} \frac{x^2}{2} & \text{for } x \in [0,1) \\ 0 & \text{elsewhere} \end{cases}$$

Else

$$h_i(x) = \begin{cases} 1 & for\ x \in [\alpha, \beta) \\ -1 & for\ x \in [\beta, \gamma) \\ 0 & elesewhere \end{cases}$$

$$p_i(x) = \begin{cases} x - \alpha & for\ x \in [\alpha, \beta) \\ \gamma - x & for\ x \in [\beta, \gamma) \\ 0 & \text{elsewhere} \end{cases}$$

$$q_i(x) = \begin{cases} \frac{(x-\alpha)^2}{2} & \text{for } x \in [\alpha, \beta) \\ \frac{(\alpha-\beta)^2 + (\beta-\gamma)^2 - (\gamma-x)^2}{2} & \text{for } x \in [\beta, \gamma) \\ \frac{(\alpha-\beta)^2 + (\beta-\gamma)^2}{2} & \text{for } x \in [\gamma, 1) \\ 0 & \text{elsewhere} \end{cases}$$

The Method of Solution:

Form the property of the Haar wavelet Transformation, $y''(x)$ which is a function of x can be approximated by the Haar Wavelet function like wise

$$y''(x) = \varphi\left(x, y(x), y'(x)\right) = \sum_{i=1}^{2M} a_i h_i(x)$$

To get the solution $y(x)$, it is difficult to find if the Differential Equation is nonlinear type or complicated to integrate $y''(x) or\ y'(x)$.

But approximating the $y''(x)$ with Haar wavelet function it is quite easier to have $y''(x)$or $y'(x)$ explicitly in terms of x.

i.e.

$$y'(x) = \int y''(x)dx$$

$$= \int \left[\sum_{i=1}^{2M} a_i h_i(x)\right] dx$$

$$= \sum_{i=1}^{2M} \int a_i h_i(x)dx$$

and

$$y(x) = \int y'(x)dx$$

$$= \int \int \left[\sum_{i=1}^{2M} a_i h_i(x)\right] dx\, dx$$

$$= \sum_{i=1}^{2M} \int \int a_i h_i(x)dx$$

3. Methods of Solution of Ordinary Differential Equation (Initial Value Problem):

The Initial Value Problem is in the form $y'' = \emptyset(x, y, y')$,

with initial conditions $y(a) = \alpha, y(a) = \beta$

Case 1: $a \in [0, 1)$

$$y''(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{3.1}$$

$$\Rightarrow y'(x) - y'(a) = \int_a^x \sum_{i=1}^{2M} a_i h_i(x')dx' = \sum_{i=1}^{2M} a_i p_i(x) - \int_0^a \sum_{i=1}^{2M} a_i h_i(x')dx' \tag{3.2}$$

$$\Rightarrow y(x) - y(a) = \int_a^x \sum_{i=1}^{2M} a_i p_i(x')dx' = (x-a)y'(a) + \sum_{i=1}^{2M} a_i q_i(x)$$
$$- \int_0^a \sum_{i=1}^{2M} a_i p_i(x')dx' - (x-a)A, \qquad where\ A = \int_0^a \sum_{i=1}^{2M} a_i h_i(x) \tag{3.3}$$

In particular when a=0,

$$y'(x) = y'(0) + \sum_{i=1}^{2M} a_i p_i(x) \tag{3.4}$$

$$y(x) = y(0) + xy'(0) + \sum_{i=1}^{2M} a_i q_i(x) \tag{3.5}$$

To get $y(x)$ we have to 1st find the unknowns a_i's by solving $2M$ system of equations i.e. $y'' = \emptyset(x, y, y')$ at x_j's which are the collocation points,

$$\sum_{i=1}^{2M} a_i h_i(x_j) =$$
$$\emptyset\left(x_j, (x_j - a)y'(a) + \sum_{i=1}^{2M} a_i q_i(x_j) - \int_0^a \sum_{i=1}^{2M} a_i p_i(x'_j)dx' - (x_j - a)A, y'(a) + \sum_{i=1}^{2M} a_i p_i(x) - \int_0^a \sum_{i=1}^{2M} a_i h_i(x'_j)dx'\right)$$

Case 2: $a \in (-\infty, 0) \cup [1, \infty)$

We can find $y(x)\ and\ y'(x)$ by integrating $y''(x)$ from $a\ to\ x$ using interval summation rule of integration.

Example: 3.1

Solution of Van Der Pol Equation

$$u'' + u' + u + u^2u' = -\sin t - \sin t \, . \cos^2 t$$

with initial condition $u(0) = 1, u'(0) = 0$

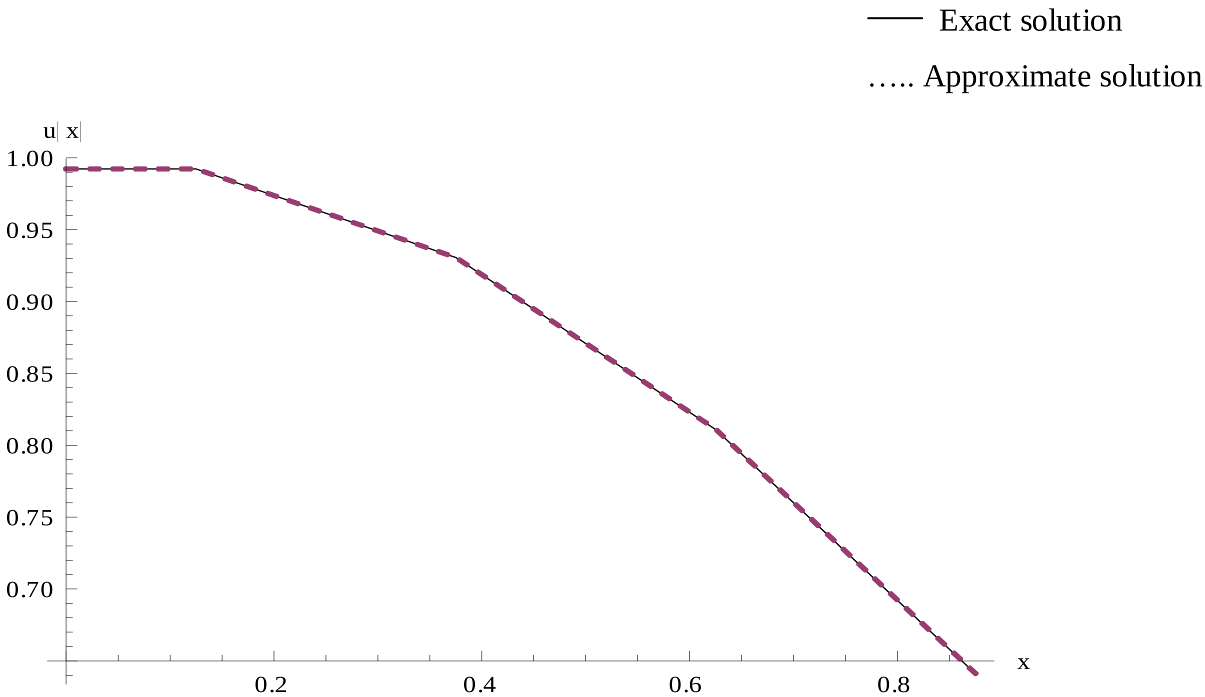
The exact solution is

$$u = \ cos \ t$$

Table 3.1

The comparison of exact solution and Haar Solution of Van-Der-Pol Equation:

x	*Exact Solution*	*Haar Solution*	*Error*
0	1	1	0
0.125	0.992198	0.99224	0.0000425132
0.375	0.930508	0.930632	0.000124536
0.625	0.810963	0.81117	0.000206634
0.875	0.640997	0.641235	0.000238164



[Fig 3.1 Comparison of exact solution and Haar solution of Van-Der-Pol Equation]

4. Methods of Solution of Ordinary Differential Equation (Boundary Value Problem):

Case 1: $y''(x) = \emptyset(x, y, y')$ with $y'(0) = \alpha_1, y'(1) = \beta_1$ (4.1.1)

$$y''(x) = \sum_{i=1}^{2M} a_i h_i(x) \quad (4.1.2)$$

$$\int_0^x y''(x)dx = \int_0^x \sum_{i=1}^{2M} a_i h_i(x)\, dx$$

$$y'(x) - \alpha_1 = \sum_{i=1}^{2M} a_i p_i(x) \quad (4.1.3)$$

$$\int_x^1 y''(x)dx = \int_x^1 \sum_{i=1}^{2M} a_i h_i(x)\, dx = \sum_{i=1}^{2M} \left[\int_0^1 a_i h_i(x)dx - \int_0^x a_i h_i(x)dx \right]$$

$$\beta_1 - y'(x) = a_1 - \sum_{i=1}^{2M} a_i p_i(x) \quad (4.1.4)$$

From (4.1.3) and (4.1.4) we have $a_1 = \alpha_1 - \beta_1$

Hence the corresponding approximations are

$$y''(x) = (\alpha_1 - \beta_1)h_1(x) + \sum_{i=2}^{2M} a_i h_i(x) \quad (4.1.5)$$

$$y'(x) = \alpha_1 + (\alpha_1 - \beta_1)p_1(x) + \sum_{i=2}^{2M} a_i p_i(x) \quad (4.1.6)$$

$$y(x) = y(0) + \alpha_1 x + (\alpha_1 - \beta_1)q_1(x) + \sum_{i=2}^{2M} a_i q_i(x) \quad (4.1.7)$$

$$(\alpha_1 - \beta_1)h_1(x_j) + \sum_{i=2}^{2M} a_i h_i(x_j)$$
$$= \emptyset\left(x_{j,} y(0) + \alpha_1 x_j + (\alpha_1 - \beta_1)q_1(x_j) + \sum_{i=2}^{2M} a_i q_i(x)\, \alpha_1 + (\alpha_1 - \beta_1)p_1(x_j) + \sum_{i=2}^{2M} a_i p_i(x_j) \right)$$

Solving above system of equations for unknowns y (0) and a_i, at $i \neq 1$, approximate solution $y(x)$ in (4.1.7) can be found out.

Case 2: $$y''(x) = \emptyset(x, y, y') \ \ with\, y(0) = \alpha_2, y(1) = \beta_2 \tag{4.2.1}$$

$$y''(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{4.2.2}$$

$$\int_0^x y''(x)dx = \int_0^x \sum_{i=1}^{2M} a_i h_i(x)\, dx$$

$$y'(x) = y'(0) + \sum_{i=1}^{2M} a_i p_i(x) \tag{4.2.3}$$

$$y(x) = \alpha_2 + xy'(0) + \sum_{i=1}^{2M} a_i q_i(x) \tag{4.2.4}$$

$$y'(0) = \beta_2 - \alpha_2 - \sum_{i=1}^{2M} a_i C_i(x) \tag{4.2.4}$$

$$y(x) = \alpha_2 + x\left[\beta_2 - \alpha_2 - \sum_{i=1}^{2M} a_i C_i(x)\right] + \sum_{i=1}^{2M} a_i q_i(x) \tag{4.2.5}$$

$$y'(x) = \beta_2 - \alpha_2 - \sum_{i=1}^{2M} a_i C_i(x) + \sum_{i=1}^{2M} a_i p_i(x) \tag{4.2.6}$$

Now

$$\sum_{i=1}^{2M} a_i h_i(x_j) = \emptyset\left(x_{j,}\alpha_2 + x_j\left[\beta_2 - \alpha_2 - \sum_{i=1}^{2M} a_i C_i(x_j)\right] + \sum_{i=1}^{2M} a_i q_i(x_j), \beta_2 - \alpha_2 - \sum_{i=1}^{2M} a_i C_i(x_j) + \sum_{i=1}^{2M} a_i p_i(x_j)\right)$$

Solving above system of equations for unknowns a_i, approximate solution $y(x)$ in (4.2.5) can be found out.

Case 3: $y''(x) = \emptyset(x, y, y')$ with $y'(0) = \alpha_3, y(1) = \beta_3$ (4.3.1)

$$y''(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{4.3.2}$$

$$\int_0^x y''(x)dx = \int_0^x \sum_{i=1}^{2M} a_i h_i(x)\, dx$$

$$y'(x) = \alpha_3 + \sum_{i=1}^{2M} a_i p_i(x) \tag{4.3.3}$$

$$\int_x^1 y'(x)dx = \int_x^1 \left[\alpha_3 + \sum_{i=1}^{2M} a_i p_i(x)\right] dx = \alpha_3(1-x) + \sum_{i=1}^{2M}\left[\int_0^1 a_i p_i(x)dx - \int_0^x a_i p_i(x)dx\right]$$

$$y(x) = \beta_3 - \alpha_3(1-x) - \sum_{i=1}^{2M} a_i C_i + \sum_{i=1}^{2M} a_i q_i(x) \tag{4.3.4}$$

$$\sum_{i=1}^{2M} a_i h_i(x_j) = \emptyset\left(x_{j,}\beta_3 - \alpha_3(1-x_j) - \sum_{i=1}^{2M} a_i C_i + \sum_{i=1}^{2M} a_i q_i(x_j), \alpha_3 + \sum_{i=1}^{2M} a_i p_i(x_j)\right)$$

Solving above system of equations for unknowns a_i, approximate solution $y(x)$ in (4.3.4) can be found out.

Case 4: $y''(x) = \emptyset(x, y, y')\ \ with\ y(0) = \alpha_4, y'(0) = \beta_4$ (4.4.1)

$$y''(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{4.4.2}$$

$$\int_x^1 y''(x)dx = \int_x^1 \sum_{i=1}^{2M} a_i h_i(x)\, dx$$

$$y'(x) = \beta_4 + \sum_{i=1}^{2M}\left[\int_0^1 a_i h_i(x)dx - \int_0^x a_i h_i(x)dx\right]$$

$$y'(x) = \beta_4 - a_1 + \sum_{i=1}^{2M} a_i p_i(x) \tag{4.4.3}$$

$$y(x) = \alpha_4 + (\beta_4 - a_1)x + \sum_{i=1}^{2M} a_i q_i(x) \tag{4.4.4}$$

$$\sum_{i=1}^{2M} a_i h_i(x_j) = \emptyset\left(x_{j,}\alpha_4 + (\beta_4 - a_1)x_j + \sum_{i=1}^{2M} a_i q_i(x_j)\,, \beta_4 - a_1 + \sum_{i=1}^{2M} a_i p_i(x_j) \right)$$

Solving above system of equations for unknowns a_i, approximate solution $y(x)$ in (4.4.4) can be found out.

Case 5: $$y''(x) = \emptyset(x, y, y') \;\; with\, y(0) = y(1), y'(0) = y'(1) \tag{4.5.1}$$

$$y''(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{4.5.2}$$

$$\int_0^1 y''(x)dx = \int_0^1 \sum_{i=1}^{2M} a_i h_i(x)\, dx$$

$$y'(1) - y'(0) = a_1 \quad \Rightarrow \qquad a_1 = 0 \tag{4.5.3}$$

$$y'(x) = y'(0) + \sum_{i=2}^{2M} a_i p_i(x) \tag{4.5.4}$$

$$0 = \int_0^1 y'(x)dx = \int_0^1 \left[y'(0) + \sum_{i=2}^{2M} a_i p_i(x) \right] dx = y'(0) + \sum_{i=2}^{2M} a_i C_i$$

$$\Rightarrow y'(0) = -\sum_{i=2}^{2M} a_i C_i \tag{4.5.5}$$

$$\int_0^x y'(x)dx = \int_0^x \left[y'(0) + \sum_{i=2}^{2M} a_i p_i(x) \right] dx = \int_0^x \left[-\sum_{i=2}^{2M} a_i C_i + \sum_{i=2}^{2M} a_i p_i(x) \right] dx$$

$$\Rightarrow y(x) = y(0) + \sum_{i=2}^{2M} a_i [q_i(x) - xC_i] \tag{4.5.6}$$

$$\sum_{i=2}^{2M} a_i h_i(x_j) = \emptyset\left(x_{j,} y(0) + \sum_{i=2}^{2M} a_i [q_i(x) - xC_i]\,, + \sum_{i=2}^{2M} [a_i p_i(x) - a_i C_i] \right)$$

Solving above system of equations for unknowns a_i, approximate solution $y(x)$ in (4.5.6) can be found out.

Example 4.1

$$y'' = 2y^3 \quad \text{with boundary conditions } y'(0) = -1, y'(1) = -\frac{1}{4}$$

The exact solution is

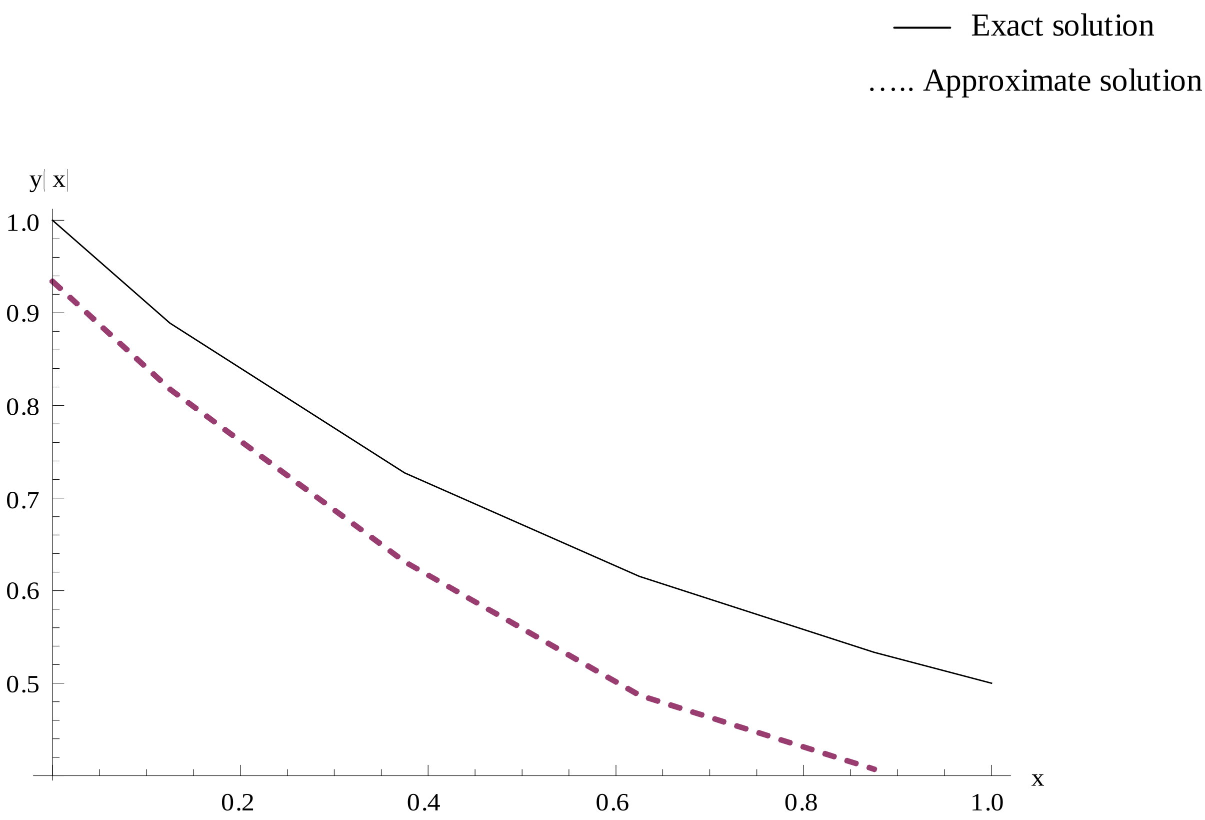
$$y = \frac{1}{1+x}$$

This is a BVP of case 2 discussed above.

Table 4.1

The comparison of exact solution and Haar Solution of $y'' = 2y^3$

x	1/(1+*x*)	*Haar y*(*x*)	*Error*
0	1	0.934012	0.0659877
0.125	0.888889	0.81755	0.0713384
0.375	0.727273	0.631248	0.0960249
0.625	0.615385	0.486979	0.128406
0.875	0.533333	0.407181	0.126152



[Fig 4.1 Comparison of exact solution and Haar solution of $y'' = 2y^3$]

Example 4.2

$$y'' + y' + y + y^2 y' + \sin x + \sin x\, cos^2 x = 0, y(0) = 1, y(1) = \cos 1$$

The exact solution is

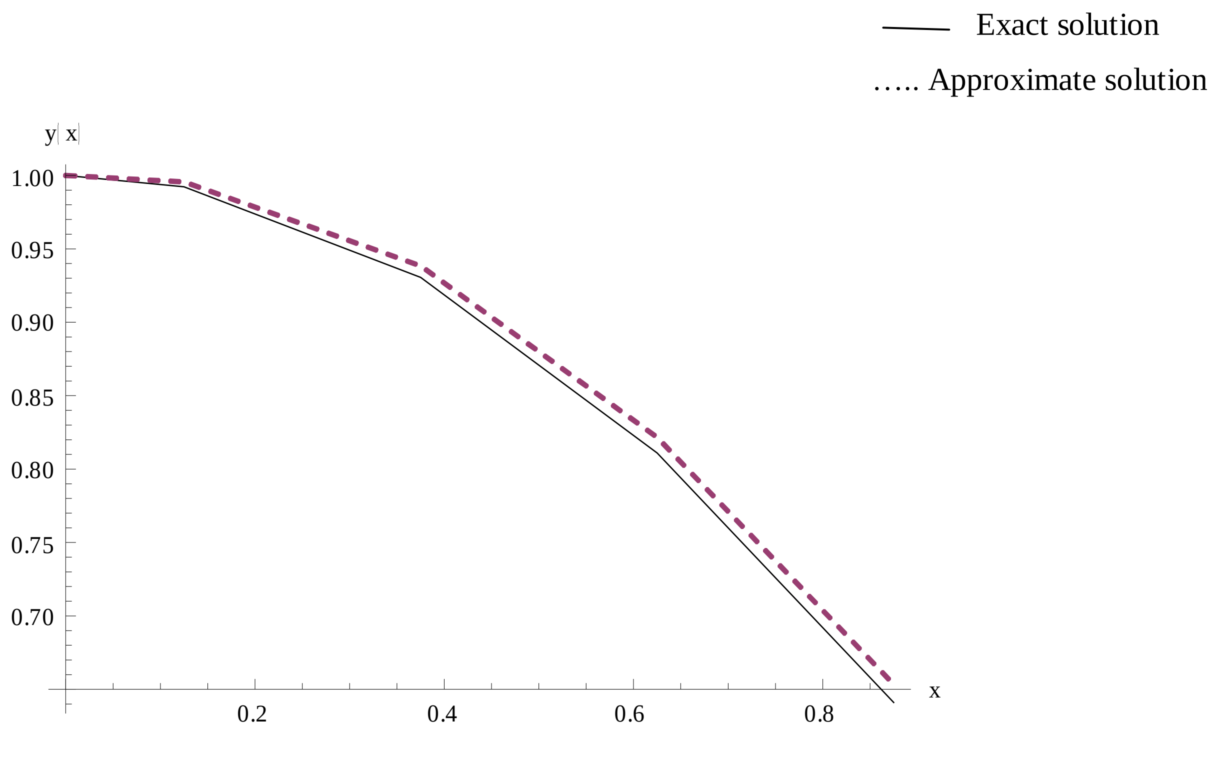
$$y = \cos x$$

This is a Boundary Value Problem of Case 2 discussed above.

Table 4.1

The comparison of exact solution and Haar Solution

x	*Cos x*	*Haar y*(*x*)	*Error*
0	1	1	0
0.125	0.992198	0.995543	0.00334484
0.375	0.930508	0.938405	0.00789711
0.625	0.810963	0.821593	0.0106297
0.875	0.640997	0.653346	0.123487



[Fig 4.2 Comparison of exact solution and Haar solution]

5. **Method to solve Partial Differential Equation**:

Example 5.1 (Solution Method of Sine-Gordon Equation)

Let us assume the Partial Differential Equation with two independent variables. For example let it be Sine-Gordon equation i.e.

$$\frac{1}{L^2}\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = \sin u$$

where $u = u(x,t)$ is a function of *x* and *t* , (*x*,*t*) are discrete points in the form (x_j, t_s) where

$$x_j = \frac{j-0.5}{2M}, \qquad t_{s+1} = t_s + \Delta t$$

Keeping in view with the initial and boundary conditions we have to approximate

$$u_{xxt} = \sum_{i=1}^{2M} a_i h_i(x)$$

The following steps show the clear view of logic behind the approximated solution method.

Let us consider, the Partial Differential Equation

$$\emptyset\left(\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, u\right) = 0 \qquad (5.1.1)$$

where $u = u(x,t)$ with the initial and boundary conditions

$$u(0,t) = \varphi(t), u^{'}(0,t) = \omega(t)$$

$$u(x,0) = f(x), \dot{u}(x,0) = g(x)$$

Step1: We approximate

$$\ddot{u}'' = \sum_{i=1}^{2M} a_s(i) h_i(x) \qquad t \in [t_s, t_{s+1}], \quad x \in [0,1] \qquad (5.1.2)$$

Step 2: We integrate above once and twice w.r.t. x from 0 to x and w.r.t. t from t_s to t we get

$$\dot{u}''(x,t) = (t - t_s)\sum_{i=1}^{2M} a_s(i) h_i(x) + \dot{u}''(x, t_s)$$

$$u''(x,t) = \frac{1}{2}(t-t_s)^2 \sum_{i=1}^{2M} a_s(i) h_i(x) + (t-t_s)\dot{u}''(x,t_s) + u''(x,t_s)$$

$$\ddot{u}(x,t) = \sum_{i=1}^{2M} a_s(i) q_i(x) + x\ddot{u}'^{(0,t)} + \ddot{u}(0,t)$$

$$\dot{u}(x,t) = (t-t_s)\sum_{i=1}^{2M} a_s(i) q_i(x) + \dot{u}(x,t_s) + x\left[\dot{u}'(0,t) - \dot{u}'(0,t_s)\right] + \dot{u}(0,t) - \dot{u}(0,t_s)$$

$$u(x,t) = \frac{1}{2}(t-t_s)^2 \sum_{i=1}^{2M} a_s(i) q_i(x) + u(x,t_s) + x[u'(0,t) - u'(0,t_s) - (t-t_s)\dot{u}'(0,t_s)] + u(0,t) - u(0,t_s) - (t-t_s)\dot{u}(0,t_s) \qquad (5.1.3)$$

Step 3: The approximated values (5.1.3) in step 2 are substituted in equation (5.1.1) in step 1.

At t=t_{s+1} solving the system of equations (5.1.1) generated by *2M* collocation points for unknowns $a_s(i)$'s, the approximate solution u(x,t) in (5.1.3) can be found out.

Clearly these $a_s(i)$'s are only valid for the range $t \in [t_s, t_{s+1}]$

Step 4: Initially t_s=0 and $t_{s+1} = t_s + \Delta t$
After 1st iteration $u(x, t_{s+1}), \dot{u}(x, t_{s+1})$ are obtained which are treated as the boundary condition and initial condition instead of $u(0,t) = \varphi(t), u'(0,t) = \omega(t)$ and $u(x, t_{s+2})$ can be found out following the previous steps.

Proceeding likewise we will have the set of solutions at different $x \in [0,1]$ and t. Hence we get a two dimensional solution treating each iteration as one dimensional problem.

Example 5.2 (Numerical Solution Method of Fisher's Equation)

Fisher's Equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 6u(1-u) \qquad (5.2.1)$$

With initial and boundary conditions

$$u(x,0) = \frac{1}{(1+e^x)^2}$$
$$u(0,t) = \frac{1}{(1+e^{-5t})^2}$$
$$u'(0,t) = \frac{-2e^{-5t}}{(1+e^{-5t})^3}$$
$$\dot{u}(0,t) = \frac{10e^{-5t}}{(1+e^{-5t})^3}$$
$$\dot{u}'(0,t) = \frac{-30e^{-10t}}{(1+e^{-5t})^4} + \frac{10e^{-5t}}{(1+e^{-5t})^3} \qquad (5.2.2)$$

Let us approximate

$$\dot{u}''(x,t) = \sum_{i=1}^{2M} a_s(i) h_i(x)$$

$$u''(x,t) = u''(x,t_s) + (t - t_s) \sum_{i=1}^{2M} a_s(i) h_i(x)$$

$$u'(x,t) = u'(0,t) + (t - t_s) \sum_{i=1}^{2M} a_s(i) p_i(x) + u'(x,t_s) - u'(0,t_s)$$

$$u(x,t) = (t - t_s) \sum_{i=1}^{2M} a_s(i) q_i(x) + u(x,t_s) - u(0,t_s) + x[u'(0,t) - u'(0,t_s)] + u(0,t)$$

$$\dot{u}(x,t) = \sum_{i=1}^{2M} a_s(i) q_i(x) + x\dot{u}'(0,t) + \dot{u}(0,t)$$

(5.2.3)

So,

$$\begin{aligned}\sum_{i=1}^{2M} a_s(i) q_i(x) + x\dot{u}'(0,t) + \dot{u}(0,t) \\ &= u''(x,t_s) + (t - t_s) \sum_{i=1}^{2M} a_s(i) h_i(x) \\ &+ 6\left[(t - t_s) \sum_{i=1}^{2M} a_s(i) q_i(x) + u(x,t_s) - u(0,t_s) + x[u'(0,t) - u'(0,t_s)]\right. \\ &\left. + u(0,t)\right]\left[1 - (t - t_s) \sum_{i=1}^{2M} a_s(i) q_i(x) + u(x,t_s) + u(0,t_s) - x\left[u'(0,t) - u'(0,t_s)\right]\right. \\ &\left. - u(0,t)\right] \qquad (5.2.4)\end{aligned}$$

At $t=t_{s+1}$ solving the system of equations (5.2.4) generated by *2M* collocation points for unknowns $a_s(i)$'s, the approximate solution $u(x,t)$ in (5.2.3) can be found out when $x \in [0,1], and\ t \in [t_s, t_{s+1}]$.

The exact solution of the given Fisher's equation is

$$u(x,t) = \frac{1}{(1 + e^{x-5t})^2}$$

The solution is brought out by using MATHEMATICA.The comparison of approximate Haar solution and exact solution is cited in the tables below.

Table 5.2.1

t	*x*=0		
	u(*x*,*t*)	*Exact*	*Error*
0.0	0.25	0.25	0
0.1	0.387456	0.387456	0
0.2	0.534447	0.534447	0
0.3	0.668428	0.668428	0
0.4	0.77580	0.77580	0
0.5	0.854038	0.854038	0
0.6	0.907397	0.907397	0
0.7	0.942235	0.942235	0
0.8	0.964351	0.964351	0
0.9	0.978147	0.978147	0
1.0	0.986659	0.986659	0

Table 5.2.2

t	*x*=0.125		
	u(*x*,*t*)	*exact*	*Error*
0.0	0.219765	0.219765	0
0.1	0.331601	0.351254	0.0196532
0.2	0.477158	0.498133	0.0209742
0.3	0.618858	0.637102	0.0182438
0.4	0.73778	0.751751	0.0139708
0.5	0.827364	0.837044	0.00967978
0.6	0.889889	0.896045	0.00615616
0.7	0.931341	0.934923	0.0035815
0.8	0.957919	0.959749	0.00183003
0.9	0.974601	0.975291	0.000690132
1.0	0.984935	0.984903	0.0000317345

Table 5.2.3

t	x=0.375		
	$u(x,t)$	*exact*	*Error*
0.0	0.16592	0.16592	0
0.1	0.121348	0.282183	0.160836
0.2	0.241005	0.424263	0.183258
0.3	0.40585	0.569897	0.164047
0.4	0.569473	0.698033	0.12856
0.5	0.707203	0.798002	0.0907986
0.6	0.810899	0.869469	0.0585698
0.7	0.883233	0.917596	0.034363
0.8	0.931182	0.948759	0.0175776
0.9	0.961928	0.96844	0.00651222
1.0	0.981231	0.980677	0.000553802

Table 5.2.4

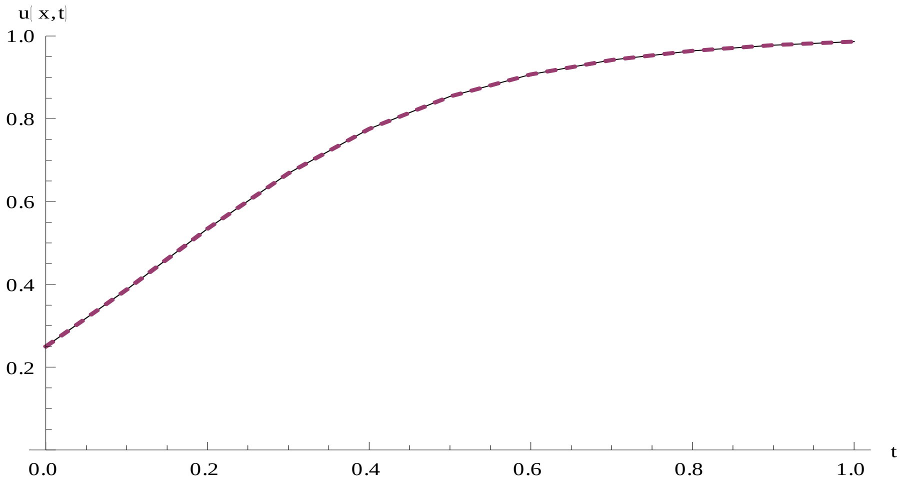
t	x=0.625		
	$u(x,t)$	*exact*	*Error*
0.0	0.121553	0.121553	0
0.1	-0.102173	0.219765	0.321938
0.2	-0.0989373	0.351254	0.450191
0.3	0.054176	0.498133	0.443956
0.4	0.261209	0.637102	0.375893
0.5	0.468066	0.751751	0.283685
0.6	0.643754	0.837044	0.193289
0.7	0.777303	0.896045	0.118742
0.8	0.871258	0.934923	0.0636648
0.9	0.933956	0.959749	0.0257925
1.0	0.974354	0.975291	0.00093736

Table 5.2.5

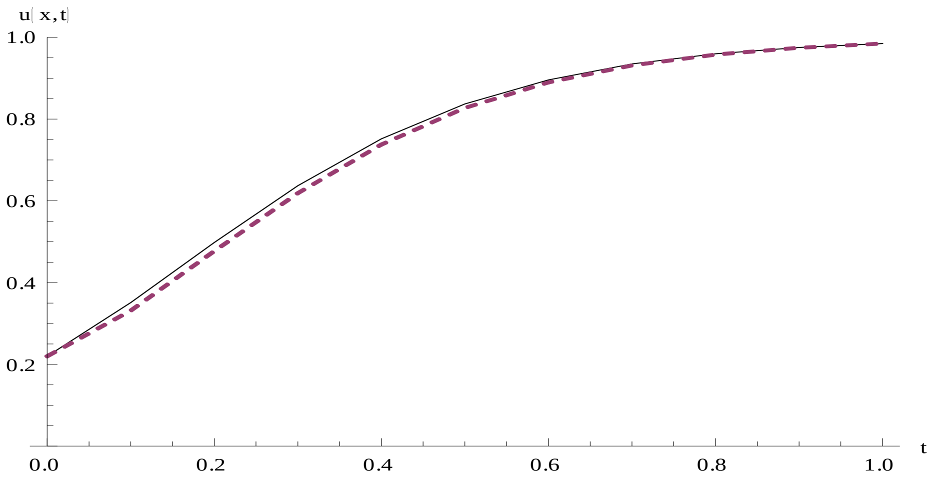
t	x=0.875		
	$u(x,t)$	*Exact*	*Error*
0.0	0.0865624	0.0865624	0
0.1	-0.176485	0.16592	0.342406
0.2	-0.361698	0.282183	0.643882
0.3	-0.330027	0.424263	0.75429
0.4	-0.158322	0.569897	0.728218
0.5	0.082333	0.698033	0.6157
0.6	0.335699	0.798002	0.462303
0.7	0.560501	0.869469	0.308968
0.8	0.737147	0.917596	0.180449
0.9	0.864426	0.948759	0.0843328
1.0	0.950774	0.96844	0.0176659

—— Exact solution

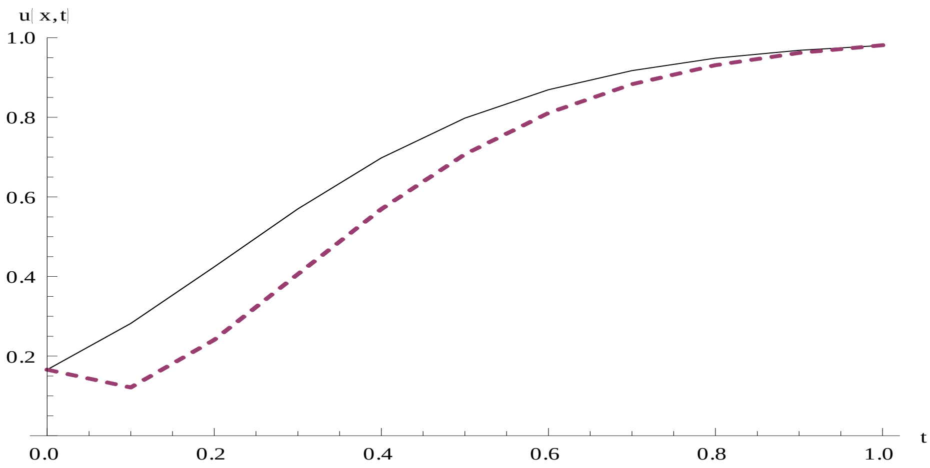
….. Approximate solution



[Fig 5.2.1 Plot of exact and approximate solution of Fisher's equation at x=0]



[Fig 5.2.2 Plot of exact and approximate solution of Fisher's equation at x=0.125]



[Fig 5.2.3 Plot of exact and approximate solution of Fisher's equation at x=0.375]

6. Conclusion

1. Haar Wavelet transform method is best shoot for the initial value problems, boundary value problems as well as the partial differential equations with less error.
2. It is easy to get solution in two dimensions so it is better over all other methods.
3. It is observed that if the level of resolution is more i.e. if the collocation points are more then we can get a better solution with lesser error.
4. Like Fisher's equation or Sine Gordon equation we can solve all non linear type critical partial differential equations.
5. One of the drawbacks is that to find the constants we solve linear or nonlinear equations which may impossible without mathematical software.
6. Simply availability and fast convergence of the Haar wavelets provide a solid foundation for highly nonlinear problems of differential equations.
7. These discussed method with far less degrees of freedom and small computation time provides better solution.
8. It can be concluded that this method is quite suitable, accurate, and efficient in comparison to other classical methods.

REFERENCES

1. S. Gopalkrishnan, Mira Mitra, "Wavelet Methods for Dynamical Problems," CRC Press, 2010.
2. L. Debnath, "Wavelet Transform and their Applications," Birkhauser Boston, 2002.
3. S. Saha Ray, "On Haar wavelet operational matrix of general order and its application for the numerical solution of fractional Bagley Torvik equation," Applied Mathematics and Computation (Elsevier), **218**, 9, pp. 5239-5248, 2011.
4. S. Islam, I. Aziz, B. Sarler, "Numerical Solution of second order boundary value problems by collocation method with Haar Wavelets," Mathematical and Computer Modeling (Elsevier), **52**, pp. 1577-1590, 2010.
5. U. Lepik, "Numerical Solution of evolution equations by the Haar wavelet method", Applied Mathematics and Computation (Elsevier), **185**, pp. 695-704, 2006.
6. U. Lepik, "Application of Haar wavelet transform to solving integral and differential equations," Applied Mathematics and Computation (Elsevier), **57**, 1, pp. 28-46, 2007.
7. U. Lepik, "Numerical solution of differential equations using Haar wavelets," Mathematics and Computers in Simulation(Elsevier), **68**, pp. 127-143, 2005
8. U. Lepik, "Haar wavelet method for solving stiff differential equations," Mathematical modeling and analysis (Taylor and Francis), **14**, 4, pp. 467-481, 2009.
9. U. Lepik, "Haar Wavelet method for solving higher order differential equations," International Journal of Mathematics and Computation (CESER Publications), **1**, N08, pp. 84-94, 2008.
10. C. F. Chen, C. H. Hsiao, "Haar Wavelet Method for solving Lumped and distributed parameter systems," IEEE Proceeding Control Theory Appl., **144**, 1, pp. 87-94, 1997.