NUMERICAL SOLUTION OF UNCERTAIN SECOND ORDER ORDINARY DIFFERENTIAL EQUATION USING INTERVAL FINITE DIFFERENCE METHOD

A THESIS

Submitted in partial fulfillment of the

requirement of the award of the degree of

Master of Science

In

Mathematics

By

Kshyana prabha Behera

Under the supervision of

Prof. S. Chakraverty

May, 2012

DEPARTMENT OF MATHEMATICS

NATIONAL INSTITUTE OF TECHNOLOGY, ROURKELA-769008

ODISHA, INDIA

NATIONAL INSTITUTE OF TECHNOLOGY, ROURKELA

DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"**NUMERICAL SOLUTION OF UNCERTAIN SECOND ORDER ORDINARY DIFFERENTIAL EQUATION USING INTERVAL FINITE DIFFERENCE METHOD" in partial fulfillment of the requirement for the award of the degree of Master of Science, submitted in the Department of Mathematics, National Institute of Technology, Rourkela is an authentic record of my own work carried out under the supervision of Dr. S. Chakraverty.

The matter embodied in this thesis has not been submitted by me for the award of any other degree.

Date: (KSHYANA PRABHA BEHERA)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Dr. S. Chakraverty
Professor, Department of Mathematics
National Institute of Technology
Rourkela – 769008, Orissa, India

ACKNOWLEDGEMENTS

I wish to express my deep sense of gratitude to my supervisor Dr. S. Chakraverty, Professor, Department of Mathematics, National Institute of Technology, Rourkela for his inspiring guidance and assistance in the preparation of this thesis.

I am grateful to Prof. S. K. Sarangi, Director, National Institute of Technology, Rourkela for providing excellent facilities in the Institute for carrying out research.

I also take the opportunity to acknowledge quite explicitly with gratitude my debt to the Head of the Department Prof. G. K. Panda and all the Professors and Staff, Department of Mathematics, National Institute of Technology, Rourkela for their encouragement during the preparation of this thesis.

I am extremely grateful to my parents, my friends and Smita Tapaswini who are a constant source of inspiration for me.

Kshyana prabha Behera

Roll No-410MA2084

National Institute of Technology

Rourkela – 769008, Orissa
India

ABSTRACT

It is well known that differential equations are in general the backbone of physical systems. The physical systems are modelled usually either by ordinary differential or partial differential equations. Various exact and numerical methods are available to solve different ordinary and partial differential equations. But in actual practice the variables and coefficients in the differential equations are not crisp. As those, are obtained by some experiment or experience. As such the coefficients and the variables may be used in interval or in fuzzy sense. So, we need to solve ordinary and partial differential equations accordingly, that is interval ordinary and interval partial differential equations are to be solved. In the present analysis our target is to use interval computation in the numerical solution of some ordinary differential equations of second order by using interval finite difference method with uncertain analysis.

CONTENTS

CHAPTER-1
INTRODUCTION

In engineering, differential equations are used to model the behaviour of systems such as heat transfer, electric conduction, fluid flow, acoustic wave propagation, and stress distribution. However, sometimes differential equations may not be solved exactly due to the complexity in boundary conditions for geometry of the domain. Many methods, such as Finite Element Method (FEM), Finite Difference Method (FDM), Finite Volume Method (FVM), mesh free methods such as Element Free Galerkin (EFG), Natural Element Method (NEM), and Discrete Element Method (DEM) have been developed to obtain approximate solutions to various differential equations [1].

Finite Difference Method (Pilkey and Wunderlich [1]) is a technique for solving differential equations, in which the differential operator is approximated by a difference operator. In finite difference method the differential equation is approximated by a difference equation whose accuracy depends on the order of the polynomial of the assumed solution.

As mentioned earlier the coefficient of the differential equations may not be crisp. Those may be uncertain and may be given in terms of interval. Now-a-days many different problems (e.g. non linear root finding, solving of linear and non linear systems of equation, linear and global optimization) can be solved by interval methods (see [2], [3]). There are also many different algorithms for finding the solution of initial value problem for ordinary differential equations. The differential equations belong to a large class of important problems in many scientific fields. There are few papers devoted to the interval methods for solving such problems [4]. Related works are reviewed and cited here as follows for a better understanding of the present investigation. Marciniak [5] have applied multistep interval methods for solving the initial value problem. Jankowska and Marciniak [5] developed a new technique of implicit interval Multistep Methods for solving the Initial Value problem. Here, we are considering finite difference method to solve differential equation by taking the variable and parameter in terms of interval.

CHAPTER-2

INTERVAL COMPUTATION

Interval Arithmetic

The interval form of the parameters may be written as

$$[\underline{x}, \overline{x}] = \{x : x \in \Re, \underline{x} \le x \le \overline{x}\}$$

Where $\underline{x}$ is the left value and $\overline{x}$ is the right value of the interval respectively. We define $m = \frac{\underline{x} + \overline{x}}{2}$ as the centre and $w = \overline{x} - \underline{x}$ as the width of the interval $[\underline{x}, \overline{x}]$.

Let $[\underline{x}, \overline{x}]$ and $[\underline{y}, \overline{y}]$ be two elements then the following interval arithmetics are well known [6]

(i) $[\underline{x}, \overline{x}] + [\underline{y}, \overline{y}] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$

(ii) $[\underline{x}, \overline{x}] - [\underline{y}, \overline{y}] = [\underline{x} - \overline{y}, \overline{x} - \underline{y}]$

(iii) $[\underline{x}, \overline{x}] \times [\underline{y}, \overline{y}] = [\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}]$

(iv) $[\underline{x}, \overline{x}] \div [\underline{y}, \overline{y}] = [\underline{x}, \overline{x}] \times \left[\frac{1}{\overline{y}}, \frac{1}{\underline{y}}\right]$,

Provided, $[\underline{y}, \overline{y}] \neq [0, 0]$

CHAPTER 3

TRADITIONAL FINITE DIFFERENCE METHOD

Finite difference method typically involves the following steps:-

- Generate a grid, for example $(x_i, t^{(k)})$ where we want to find an approximate solution.
- Substitute the derivatives in an ODE/PDE or an ODE/PDE system of equations with finite difference schemes. The ODE/PDE then becomes a linear/non linear system of algebraic equations.
- Solve the system of algebraic equations.
- Do the error analysis, both analytically and numerically [7].

Besides, finite difference methods, there are other methods that can be used to solve ODE/PDE such as finite element method, spectral methods etc. Generally finite difference methods are simple to use for problems defined on regular geometrics such as an interval in 1D, a rectangular domain in two space dimensions, and a cubic in three space dimensions.

Below we list three commonly used finite difference formulae to approximate the first order derivative of a function $u(x)$ using the function values only [7].

- The forward finite difference

$$D_+ u(x) = \frac{u(x+h) - u(x)}{h}$$

- The backward finite difference

$$D_0 u(x) = \frac{u(x) - u(x-h)}{h}$$

- The central finite difference

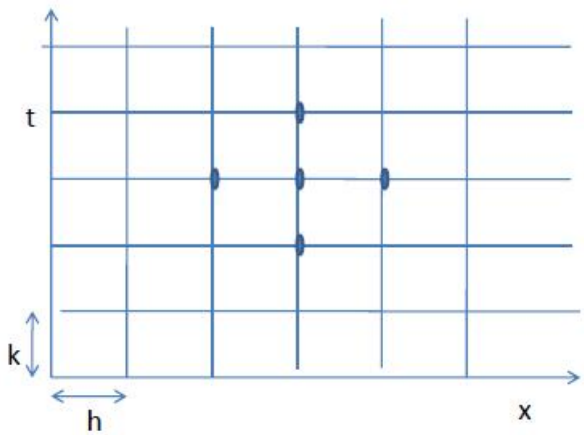
$$D_* u(x) = \frac{u(x+h) - u(x-h)}{2h}$$

Let us consider a differential equation, i.e.

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Assume u is a function of the independent crisp variables x and t.

We subdivide the x-t plane into sets of equal rectangles of sides δx=h, δt=k.



The above differential equation may be written as,

$$\Rightarrow \frac{u(t_{i,j+1}) - u(t_{i,j})}{k} = \frac{a^2}{h^2}\left[u(x_{i+1,j}) - 2u(x_{i,j}) + u(x_{i-1,j})\right]$$

CHAPTER-4

Numerical Examples for Traditional Finite Difference Method

Example: 1 in this example, we will find the deflection of simply supported beam at $x = 50''$ as shown in Fig. 1. For the calculation we use step size $\Delta x = 25''$, Tension applied (T)= 7200 Ibs, E=Young's modulus of elasticity of the beam(psi)= 30 Msi, Second moment of areal(in^4) (I)= 120 in^4, Uniform loading intensity (Ibs/in) (q)= 5400 Ibs, length of beam(in) (L)= 75 in. [8]

$$\frac{d^2y}{dx^2} - \frac{Ty}{EI} = \frac{qx(L-x)}{2EI} \tag{1}$$

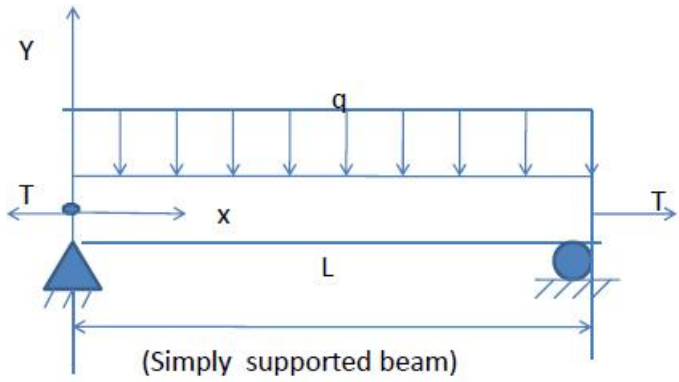


Fig. 1 Simply supported beam [8]

Substituting the given values in equation (1) we get

$$\frac{d^2y}{dx^2} - 2\times10^{-6}\,y = 7.5\times10^{-7}\,x(75-x) \tag{2}$$

Now, approximating the derivatives $\frac{d^2y}{dx^2}$ at node 'i' by the central divided difference method (shown in Fig 2).

Fig. 2 Illustration of finite difference nodes using central divided difference method [8]

$$\frac{d^2y}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2} \tag{3}$$

We can rewrite the equation as

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2} - \frac{7200y_i}{(32\times10^6)(120)} = \frac{(5400)x_i(75 - x_i)}{2(30\times10^6)(120)} \tag{4}$$

Since, Δx=25, we have 4 nodes as given

The locations of the 4 nodes are

$$x_0 = 0, \mathrm{x}_1 = 25, \mathrm{x}_2 = 50, \mathrm{x}_3 = 75$$

Writing the equation at each node, we get four simultaneous equations with four unknowns as below

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.0016 & -0.003202 & 0.0016 & 0 \\ 0 & 0.0016 & -0.003202 & 0.0016 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 9.375\times10^{-4} \\ 9.375\times10^{-4} \\ 0 \end{bmatrix}$$

Solving the above matrix equation we get

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5852 \\ -0.5852 \\ 0 \end{bmatrix}$$

The complete exact solution is given by

$$y = 0.375x^2 - 28.125x + 3.75\times10^5 + c_1e^{0.0014142x} + c_2e^{-0.0014142x} \tag{5}$$

Applying the given boundary condition we have

$$c_1 = -1.77557286\times10^5$$

$$c_2 = -1.97442714\times10^5$$

Substituting these values in equation (5) we get the solution as

$$y = 0.375x^2 - 28.125x + 3.75\times10^5 - 1.77557286\times10^5 e^{0.0014142x} - 1.97442714\times10^5 e^{-0.0014142x}$$

Example: 2

This problem has been taken from [8].

Here we take the case of a pressure vessel to check its ability to withstand pressure. For a thick pressure vessel of inner radius a and outer radius b, the differential equation for the radial displacement u of a point along the thickness is given by [8]

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \qquad (6)$$

The inner radius a=5″ and the outer radius b=8″, here are taken as given in [8] and the material of the pressure vessel is ASTMA36 steel. The yield strength of this type of steel is 36ksi.Two strain gages that are bounded tangentially at the inner and the outer radius measure normal tangential strain as

$\epsilon_{t|r=a} = 0.00076$

$\epsilon_{t|r=b} = 0.000375$

At the maximum needed pressure, since the radial displacement and tangential strain are related by

$$\in_t = \frac{u}{r}$$

So we have

$$u|_{r=a} = 0.00076\times5 = 0.0038''$$
$$u|_{r=b} = 0.000375\times8 = 0.0030''$$

Now let us divide the radial thickness of the pressure vessel into 6 equidistance nodes, and we will find the radial displacement profile as given in [8].

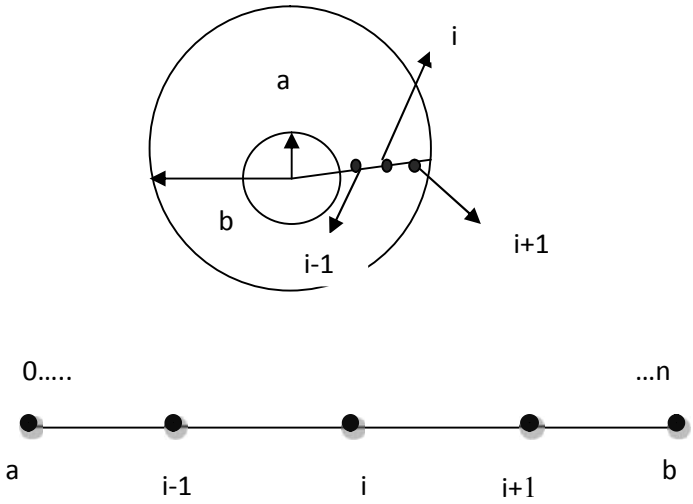


Fig. 3 Nodes along the radial direction [8]

We have

$$\frac{d^2u}{dr^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta r)^2} \tag{7}$$

$$\frac{du}{dr} \approx \frac{u_{i+1} - u_i}{\Delta r} \tag{8}$$

Substituting these approximations from equation (2) and equation (3) in equation (1) we get

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta r)^2} + \frac{1}{r_i}\frac{u_{i+1} - u_i}{\Delta r} - \frac{u_i}{r_i^2} = 0 \tag{9}$$

$$\left[\frac{1}{(\Delta r)^2} + \frac{1}{r_i \Delta r}\right] u_{i+1} + \left[\frac{-2}{(\Delta r)^2} - \frac{1}{r_i \Delta r} - \frac{1}{r_i^2}\right] u_i + \frac{1}{(\Delta r)^2} u_{i-1} = 0 \tag{10}$$

Finally for 6 nodes we have the matrix equation as

Solution of the above matrix equation may be obtained as

$u_0 = 0.0038''$
$u_1 = 0.0035''$
$u_2 = 0.0033''$
$u_3 = 0.0032''$
$u_4 = 0.0031''$
$u_5 = 0.0030''$

CHAPTER-5

NUMERICAL EXAMPLES OF INTERVAL FINITE DIFFERENCE METHOD

The interval finite difference method has been discussed by the following examples only.

Example-3

The same problem has been taken as mentioned in Example1**,** but here with some interval parameters. In this problem T, q, L and I are crisp and E has been taken as interval. The data are as below

$$T = 7200Ibs,\ q = 5400Ibs/in,\ L = 75in,\ E = [29,31]Msi,\ and\ I = 120in^2$$

Substituting the above values in the given equation (1), we get

$$\Rightarrow \frac{d^2[\underline{y},\overline{y}]}{dx^2} - \frac{[60\underline{y},60\overline{y}]}{[29\times10^6,31\times10^6]} = \frac{22.5x(75-x)}{[29\times10^6,31\times10^6]} \tag{11}$$

It may be noted that y now will be in interval viz,

$$y = [\underline{y},\overline{y}]$$

We have used here the upper and lower variable in two second order ordinary differential equation and discretise the ordinary differential equation as below

$$\frac{d^2\underline{y}}{dx^2} = \frac{\underline{y}_{i+1} - 2\underline{y}_i + \underline{y}_{i-1}}{(\Delta x)^2} \tag{12}$$

$$\frac{d^2\overline{y}}{dx^2} = \frac{\overline{y}_{i+1} - 2\overline{y}_i + \overline{y}_{i-1}}{(\Delta x)^2} \tag{13}$$

now from equation (11) and equation (12) we have

$$\frac{\underline{y}_{i+1} - 2\underline{y}_i + \underline{y}_{i-1}}{(\Delta x)^2} - \frac{60\underline{y}_i}{29\times10^6} = \frac{22.5x_i(75-x_i)}{29\times10^6}$$

Since, Δx=25, we have 4 nodes as given earlier

Writing the equation for each node, we get the following

Node-1: From the simply supported boundary condition at x=0, we obtain

$$\underline{y}_1 = 0$$

Node-2: for node-2 the equation is

$$0.0016\underline{y}_3 - 0.003202\underline{y}_2 + 0.0016\underline{y}_1 = 9.698\times10^{-4}$$

Node-3: for node-3 the equation is

$$0.0016\underline{y}_4 - 0.003202\underline{y}_3 + 0.0016\underline{y}_2 = 9.698\times10^{-4}$$

Node-4: From the simply supported boundary condition at x=75, we obtain

$$\underline{y}_4 = 0$$

Similarly equation (11) and equation (12) we have

$$\frac{\overline{y}_{i+1} - 2\overline{y}_i + \overline{y}_{i-1}}{(\Delta x)^2} - \frac{60\overline{y}_i}{31\times10^6} = \frac{22.5x_i(75-x_i)}{31\times10^6}$$

In the same way we can have the following again

Node-1: from the simply supported boundary condition at x=0, we obtain

$$\overline{y}_1 = 0$$

Node-2: for node-2 the equation is

$$0.0016\overline{y}_3 - 0.003202\overline{y}_2 + 0.0016\overline{y}_1 = 9.072\times10^{-4}$$

Node-3: for node-3 the equation is

$$0.0016\overline{y}_4 - 0.003202\overline{y}_3 + 0.0016\overline{y}_2 = 9.072\times10^{-4}$$

Node-4: from the simply supported boundary condition at x=75 we obtain

$$\overline{y_4} = 0$$

Thus, we have four simultaneous equations with four unknowns for lower bound of the deflection and can be written in matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.0016 & -0.003202 & 0.0016 & 0 \\ 0 & 0.0016 & -0.003202 & 0.0016 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \underline{y}_3 \\ \underline{y}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 9.698\times10^{-4} \\ 9.698\times10^{-4} \\ 0 \end{bmatrix}$$

Solving this matrix equation, we get

$$\begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \underline{y}_3 \\ \underline{y}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5979 \\ -0.5979 \\ 0 \end{bmatrix}$$

Similarly, we have four simultaneous equations with four unknowns for upper bound of the deflection and can be written in matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.0016 & -0.003201 & 0.0016 & 0 \\ 0 & 0.0016 & -0.003201 & 0.0016 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{y}_1 \\ \overline{y}_2 \\ \overline{y}_3 \\ \overline{y}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 9.072\times10^{-4} \\ 9.072\times10^{-4} \\ 0 \end{bmatrix}$$

Solution of the above gives

$$\begin{bmatrix} \overline{y}_1 \\ \overline{y}_2 \\ \overline{y}_3 \\ \overline{y}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5599 \\ -0.5599 \\ 0 \end{bmatrix}$$

Finally the complete solution in interval form may be written as

$$y_1 = [\underline{y}_1, \quad \overline{y}_1] = [0, \quad 0]$$
$$y_2 = [\underline{y}_2, \quad \overline{y}_2] = [-0.5979, \quad -0.5599]$$
$$y_3 = [\underline{y}_3, \quad \overline{y}_3] = [-0.5979, \quad -0.5599]$$
$$y_4 = [\underline{y}_4, \quad \overline{y}_4] = [0, \quad 0].$$

EXAMPLE-4

Next, we take the example 2 in the interval form

The differential equation is given by

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \tag{14}$$

Let, the radial displacement u is given now in interval form as

$$u|_{r=a} = [0.0037, 0.0039]$$
$$u|_{r=b} = [0.0029, 0.0031]$$

As done in previous example we will have two ordinary differential equation for upper and lower bounds of u. Consequently after bit calculation we arrive at the following matrix equation for lower value of u.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2.7778 & -5.8851 & 3.0754 & 0 & 0 & 0 \\ 0 & 2.7778 & -5.8504 & 3.0466 & 0 & 0 \\ 0 & 0 & 2.7778 & -5.8223 & 3.0229 & 0 \\ 0 & 0 & 0 & 2.7778 & -5.7990 & 3.0030 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{u}_0 \\ \underline{u}_1 \\ \underline{u}_2 \\ \underline{u}_3 \\ \underline{u}_4 \\ \underline{u}_5 \end{bmatrix} = \begin{bmatrix} 0.0037 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.0029 \end{bmatrix}$$

Solution of the above gives,

$\underline{u}_0 = 0.0037$
$\underline{u}_1 = 0.0034$
$\underline{u}_2 = 0.0033$
$\underline{u}_3 = 0.0031$
$\underline{u}_4 = 0.0030$
$\underline{u}_5 = 0.0029$

In a similar way we may write the matrix equation for upper value of u as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2.7778 & -5.8851 & 3.0754 & 0 & 0 & 0 \\ 0 & 2.7778 & -5.8504 & 3.0466 & 0 & 0 \\ 0 & 0 & 2.7778 & -5.8223 & 3.0229 & 0 \\ 0 & 0 & 0 & 2.7778 & -5.7990 & 3.0030 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{u}_0 \\ \overline{u}_1 \\ \overline{u}_2 \\ \overline{u}_3 \\ \overline{u}_4 \\ \underline{u}_5 \end{bmatrix} = \begin{bmatrix} 0.0039 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.0031 \end{bmatrix}$$

Corresponding solutions may be obtained for upper value of u as

$\overline{u}_0 = 0.0039''$
$\overline{u}_1 = 0.0036''$
$\overline{u}_2 = 0.0034''$
$\overline{u}_3 = 0.0033''$
$\overline{u}_4 = 0.0032''$
$\overline{u}_5 = 0.0031''$

Finally we may write the interval solution of the problem as

$u_0 = [\underline{u}_0, \overline{u}_0] = [0.0037, 0.0039]$
$u_1 = [\underline{u}_1, \overline{u}_1] = [0.0034, 0.0036]$
$u_2 = [\underline{u}_2, \overline{u}_2] = [0.0033, 0.0035]$
$u_3 = [\underline{u}_3, \overline{u}_3] = [0.0031, 0.0033]$
$u_4 = [\underline{u}_4, \overline{u}_4] = [0.0030, 0.0032]$
$u_5 = [\underline{u}_5, \overline{u}_5] = [0.0029, 0.0031]$.

CHAPTER-6

CONCLUSIONS

We have successfully applied interval finite difference method for the solution of second order ordinary differential equation. In this study the related ordinary differential equation has been written as a combination of two ordinary differential equations containing the upper and lower dependent variables. Then these are solved separately and the interval results obtained gives the bound of the exact solution (of crisp problem).

References

[1] Uncertainties in the solutions to Boundary element Method: An interval approach by Bartlomiej Franciszek Zalewskt, Department of civil engineering, CASE WESTERN RESERVE UNIVERSITY, august (2008).

[2] Jaulin L., Kieffer M., Didrit O., Watter E, Applied interval analysis. Springer-Verlag, London (2001).

[3] Hammer R, Hocks M., Kulisch U., Ratz D, Numerical Toolbox for verified computing I. Basic Numerical problems, Springer-verlag, Berlin (1993).

[4] Malgorzata A. Jankowska, Poznan University of Technology, Institute of Applied Mechanices, Piotrowo 3, 60-965 Poznan, Poland.

[5] Jankowska, M., Marciniak, A: Implicit interval Multistep Methods for solving the Initial Value problem. Computational methods in silence and Technology 8(1) 17-30, (2001).

[6] Alefeld, G. And Herzberer, J., Introduction to Interval Computations, Academic Press, New York, NY, (1983).

[7] Zhilin Li, Finite Difference Methods Basics, center for Research Computation and department of Mathematics North Carolina State University, Raleigh, NC 27695.

[8] Autar Kaw, Cuong Nguyen, Luke Snyder, Finite Difference Methods of solving Ordinary differential equations, (Text book notes of Finite Difference Method of solving Ordinary Differential Equations), May 9,(2012).

[9] Bhat R. B, Chakraverty S, Numerical Analysis in Engineering, alpha science international ltd (2004).

[10] Samsundaram, Ordinary Differential Equations, A first course, narosa, (2002).

[11] M. D. Raisinghania, Ordinary and Partial Differential Equations, S. Chand Publishing (2010).

[12] Moore, R.E: Interval analysis. Prentice-Hall, Englewood cliffs, NJ (1966).

[13] Markov S. M., in directed Interval arithmetic and its applications, Journal of Universal Computer Science. 1, 514-526, (1995).

[14] Moore R. E., Blerbaun F., Methods and applications of interval analysis, SIAM, (1979).

[15] F. B. Hildebrand, Introduction to Numerical Analysis, Mc Graw Hill, Inc, New York, (1974).