NUMERICAL APPRROXIMATION METHODS FOR SOLVING STOCHASTIC DIFFERENTIAL EQUATIONS

A

PROJECT REPORT

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UNDER THE GUIDANCE OF

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DECLARATION

I declare that the topic "Numerical Approximation Methods for Solving Stochastic Differential Equations" for my M.Sc. project under Dr. S. Saha Ray has not been submitted by anyone in any other institution or university for award of any degree.

Place: Prasanta Kumar ojha

Date: Roll No: 410MA2096

CERTIFICATE

This is to certify that the project entitled "Numerical Approximation Methods for Solving Stochastic Differential Equations" submitted by Prasanta Kumar Ojha in partial fulfillment of the requirements for the degree of Master of Science in Mathematics of Department of Mathematics, National Institute of Technology, Rourkela, is an authentic work carried out by him under my supervision.

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CONTENTS

1. Introduction

Stochastic differential equations (SDEs) models play a crucial role in many field of science such as biology, chemistry, climatology, mechanics, physics, economics and finance. In finance, the Black-Scholes stochastic differential equations are used to model the option price. The earliest work on SDEs was description on the Brownian motion done in Einstein's paper "On the motion required by the molecular kinetic theory of heat of small particles suspended in a stationary liquid". The Brownian motion plays an important role in SDEs. In this paper we will study some properties of SDEs and how to solve stochastic differential equation by applying the numerical approximation method.

2. Stochastic process

A stochastic process is a collection of random variables ($X(t); t \in T$), where t is a parameter that runs over an index set T. In general, t is the time parameter and $T \subseteq R$. Each $X(t)$ takes values in some set $S \subseteq R$ called the state space.

3. Wiener process

The most important stochastic process in continuous time is the Wiener process also called Brownian motion (Ref. 4). It is used as building block in many more models. In 1828 the Scottish botanist Robert Brown observed that Pollen grains suspended in water moved in an apparently random way changing direction continuously. This was later explained by the pollen grains being bombarded by water molecules and Brown only contributed to the theory with this name. The precise mathematical explanation was given by American mathematician Norbert wiener in 1923.

A scalar standard Brownian motion or standard Wiener process over $[0, T]$ is a random variable $W(t)$ that depends continuously on $t \in [0, T]$ and satisfies the following three conditions (Ref. 2),

1. $W(0) = 0$ (with probability 1)

2. For $0 \leq s \leq t \leq T$, the increment $W(t) - W(s)$ is normally distributed with mean zero and variance $| t - s |$i.e. $W(t) - W(s) \sim \sqrt{t-s}\ N(0,1)$.

3. For $0 \leq s < t < u < v \leq T$ the increments $W(t) - W(s)$ and $W(v) - W(u)$ are Independent.

We discretized the Brownian motion, by specifying $W(t)$ is discrete at t values. Let us set $\delta t = {}^{T}/_{N}$, where N is a positive integer and W_j denote $W(t_j)$ with $t_j =$

$j\delta$t. Hence from condition 1, we will have $W_0 = 0$ and from condition 2 and 3 we will get $W_j = W_{j-1} + \Delta W_j$, this gives $\Delta W_j = W_j - W_{j-1}$, where , $j = 1,2, \ldots, N$ and $\Delta W_j$are independent random variables of the form $\sqrt{\delta \mathrm{t}}\, N(0,1)$.

4. **Stochastic differential equations**

A stochastic differential equations (SDEs) is an object of the following type

$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$

$X_0 = x$.

where a, b: $[0,\infty) \times R \rightarrow R$. The value ‘$x$’ is called the initial condition. A solution of this equation is a stochastic process, $X(t)$ satisfying

$X_t = X_0 + \int_o^t a(s, X_s)\, ds + \int_o^t b(s, X_s)\, dW_s.$

5. Itô's Integral

The wiener process is nowhere differentiable, thus we cannot apply the general calculus rules. Itô's calculus named after Kiyoshi Itô a Japanese mathematician extends the rules and methods from calculus to stochastic process. His famous book 'stochastic processes' investigate the independent increments, stationary process, Markov process and some theory of diffusion process. Now we will define Itô's integral,

Let X_t and $Y_t = g(t, X_t)$ are two stochastic processes. Here $g(t, X_t)$ is a twice differentiable function on $[0, T] \times R$. Then

$dX_t = u\, dt + v\, dW_t$ (5.1) and

$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)\, dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)(dX_t)^2$ (5.2)

where for computing $(dX_t)^2$ we use the following properties

$$(dt)^2 = 0$$

$$dtdW_t = dW_t dt = 0$$

$$(dW_t)^2 = dWdW = dt$$

Then, $(dX_t)^2 = (u\, dt + v\, dW_t)(u\, dt + v\, dW_t)$

$$= uv(dt)^2 + uvdtdW_t + vudW_t dt + v^2 dW_t dW_t = v^2 dt\ .$$

Therefore, eq. (5.2) can be written as,

$dY_t = [\frac{\partial g}{\partial t}(t, X_t) + \frac{\partial g}{\partial x}(t, X_t)u + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)\, v^2]\, dt + \frac{\partial g}{\partial x}(t, X_t)\, v\, W_t$ (5.3)

Hence eq. (5.3) can be written in integral form as,

$$g(t,X_t)v^2 = g(0,X_0) + \int_0^t [\frac{\partial g}{\partial s}(s,X_s) + \frac{\partial g}{\partial x}(s,X_s)u(s) +$$

$$\frac{1}{2}\frac{\partial^2 g}{\partial x^2}(s,X_s)v^2(s)]\, ds + \int_0^t \frac{\partial g}{\partial x}\, (s,X_s)\, v(s)\, dW_s. \quad (5.4)$$

Some properties of Itô integral

1. **(Integration by parts)**

Suppose $f(t,\omega)$ is continuous and of bounded variation with respect to $s \in [0,t]$. Then

$$\int_0^t f(s)dW_s = f(t) - \int_0^t W_s\, df_s.$$

2. **(Itô product formula)**

If X_t,Y_t are the stochastic processes, then

$$d(\,X_t Y_t) = \,X_t dY_t + Y_t dX_t + dX_t dY_t$$

3. **(Stochastic chain rule)**

In the deterministic case, if $dX/dt = f(X)$ then, for any smooth function V, the chain rule say that

$$\frac{dV\big(X(t)\big)}{dt} = \frac{dV\big(X(t)\big)}{dX}\frac{d\big(X(t)\big)}{dt} = \left\{\frac{dV\big(X(t)\big)}{dX}\right\} f(X(t))$$

Now, suppose that X satisfies the Itô stochastic differential equation, then

$$dV = (dV/dX)\, dX$$

So $dV\big(X(t)\big) = \frac{dV(X(t))}{dX}\Big(f\big(X(t)\big)dt + g\big(X(t)\big)dW_t\Big)$

From Itô's result we will get the correct formulation

$$dV\big(X(t)\big) = \frac{dV\big(X(t)\big)}{dX}dX + \frac{1}{2}g(X(t))^2\frac{d^2V(X(t))}{dX^2}dt$$

6. Example for Itô Integral

Using Itô's integral we shall show that

$$\int_0^t W_s^3 \, dW_s = \frac{1}{4} W_t^4 - \frac{3}{2} \int_0^t W_s^2.$$

Let us consider $g(t,x) = \frac{1}{4} x^4$ and $X_t = W_t$ then $Y_t = g(t, X_t) = \frac{1}{4} W_t^4$, $\frac{\partial g}{\partial x} = x^3$ and $\frac{\partial^2 g}{\partial x^2} = 3\, x^2$.

Using Itô's integral, we obtained that

$$dY_t = W_t^3 \, dW_t + \frac{3}{2} W_t^2 \, (dW_t^2)^2$$

$$= W_t^3 \, dW_t + \frac{3}{2} W_t^2 dt$$

Its integral form is

$$\frac{1}{4} W_t^4 = \frac{1}{4} W_0^4 + \int_0^t W_s^3 \, \mathrm{d}W_s + \frac{3}{2} \int_0^t W_s^2 \mathrm{d}W_s$$

$$\int_0^t W_s^3 \mathrm{d}W_s = \frac{1}{4} W_t^4 - \frac{3}{2} \int_0^t W_s^2 \mathrm{d}W_s \quad \text{since } W_0 = 0.$$

SDEs play an important role in the following historical problem in finance called Black- Scholes diffusion equation.

$$\begin{cases} \mathrm{dX} = \mu \, \mathrm{X} \, \mathrm{dt} + \sigma \, \mathrm{X} \, \mathrm{d}W_t \\ X_0 = 0. \end{cases} \tag{6.1}$$

where μ and σ are constants.

The solutions of the Black- Scholes diffusion equation (Ref. 3) is of the form

$$X(t) = X_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma \, W_t}$$

For getting the solution, let us consider

$X = f(t, Y) = X_0 e^Y$

where $Y = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t$. Then by Itô's formula

$$dX = X_0 e^Y dY + \frac{1}{2} e^Y dY dY \qquad (6.2)$$

where $dY = \left(\mu - \frac{1}{2}\sigma^2\right) dt + dW_t$. Now $dYdY = \sigma^2 dt$. So eq. (6.1) can be written as

$$\begin{aligned} dX &= X_0 e^Y \left(\mu - \frac{1}{2}\sigma^2\right) dt + X_0 e^Y \sigma \, \mathrm{d}W_t + \frac{1}{2}\sigma^2 e^Y \, dt \\ &= \mu X_0 e^Y \, dt - \frac{1}{2} X_0 e^Y \sigma^2 \, dt + X_0 e^Y \sigma \, \mathrm{d}W_t + \frac{1}{2}\sigma^2 e^Y \, dt \\ &= \mu X_0 e^Y \, dt + X_0 e^Y \sigma \, \mathrm{d}W_t \\ &= \mu \, X dt + \sigma \, X dW_t. \text{ since } X = X_0 e^Y. \end{aligned}$$

7. Strong and weak convergence for SDEs

7.1 Strong Convergence

We say that a general time discrete approximation Y^δ with maximum step size δ converges strongly to X_T at time T if

$$\lim_{\delta \to 0} E(|X_T - Y^\delta (T)|) = 0$$

7.2 Weak Convergence

A discrete-time approximation $w_{\delta t}$ with step-size δt is said to convergence weakly to the solution $X(T)$ if

$$\lim_{\delta \to 0} E\{f(w_{\delta t}(T))\} = E\{f(X(T))\}$$

8. Convergence Orders

We shall say that a time discrete approximation Y^{δ} convergence strongly with order $\gamma > 0$ at time T if there exist a positive constant C, which does not depend on δ, a $\delta_0 > 0$ such that

$$\epsilon(\delta) = E(|X_T - Y^{\delta}(T)|) \leq C\delta^{\gamma}$$

for each $\delta \in (0,\delta_0)$.

Similarly we shall say that a time discrete approximation Y^{δ} converges weakly with order $\beta > 0$ to X at time as $\delta \to 0$ if for each $g \in C_P^{2(\beta+1)}(R^d, R)$ there exists a positive constant C, which does not depend on δ, and a finite $\delta_0 > 0$ such that

$$\left|E(g(X_T)) - E\left(g\left(Y^{\delta}(T)\right)\right)\right| \leq C\delta^{\beta}$$

for each $\delta \in (0,\delta_0)$

9. Euler- Maruyama method

The Euler- Maruyama method (EM method) is applied to approximate the numerical solution of SDEs. It is a simple generalization of the Euler method for ordinary differential equations to stochastic differential equations. It is named after Leonhard Euler and Gisiro Maruyama.

We will use Itô's integral to construct EM method. Now let us set $\delta t = {}^{T}/_{L}$ for some integer L and let $\tau_j = j\delta t$ for $j = 0\ldots L$. Let $X(\tau_j)$ is abbreviated as X_j setting successively $t = \tau_{j+1}$and $t = \tau_j$ in eq. (5.1), we will obtain

$$X(\tau_{j+1}) = X_0 + \int_0^{\tau_{j+1}} a\,(s, X_s)\ ds + \int_0^{\tau_{j+1}} b\,(s, X_s)\, dW_s \tag{9.1}$$

$$X\,(\tau_j) = X_0 + \int_0^{\tau_j} a\,(s, X_s)\, \mathrm{ds} + \int_0^{\tau_j} b\,(s, X_s)\, dW_s \tag{9.2}$$

If we subtract from eq. (9.1) to eq. (9.2), we will obtain

$$X\,(\tau_{j+1}) = \mathrm{X}\,(\tau_j) + \int_{\tau_j}^{\tau_{j+1}} a\,(s, X_s)\ ds + \int_{\tau_j}^{\tau_{j+1}} b\,(s, X_s)\, dW_s \tag{9.3}$$

For the first integral we will use the conventional deterministic quadrature rule

$$\int_{\tau_j}^{\tau_{j+1}} a\,(s, X_s)\ ds \approx (\tau_{j+1} - \tau_j)\, a\,(s, X_s) = \delta \mathrm{t}\, a\left(\tau_j, X_{\tau_j}\right) \tag{9.4}$$

and for the second integral we will use Itô's formula

$$\int_{\tau_j}^{\tau_{j+1}} b\,(s, X_s)\, dW_s \approx b(\tau_j, X_{\tau_j})\,(W(\tau_{j+1}) - W\,(\tau_j)) \tag{9.5}$$

Combining eq. (9.4) and eq. (9.5) we will get the desired EM method.

The initial condition is $X_0 = X(0)$

$$X_{j+1} = X_j + \delta \mathrm{t}\, a(s, X_s) + b\,(s, X_s)\,(W_{j+1} - W_j) \tag{9.6}$$

Equation (9.6) can be written as

$X_{j+1} = X_j + \delta \mathrm{t}\, a(s, X_s) + b\,(s, X_s)\, \Delta W_j$.

where $\Delta W_j = (W_{j+1} - W_j)$

We have defined $N(0,1)$ to be the standard random variable. Each random number ΔW_i is computed as $\Delta W_i = \sqrt{\delta t}\,\eta$, where η is chosen from $N(0,1)$.

Euler-Maruyama scheme is strongly convergent with order $\frac{1}{2}$ and weakly convergent with order 1.

10. Milstein's method

The Milstein's method, named after Grigori N. Milstein a Russian mathematician, is a technique for the approximate numerical solution of SDEs.

Let τ_j, τ_{j+1}be two consecutive points in the time discretization $[0, T]$, from EM method we have

$$X\,(\tau_{j+1}) = X\,(\tau_j) + \int_{\tau_j}^{\tau_{j+1}} a\,(s, X_s)\;ds + \int_{\tau_j}^{\tau_{j+1}} b\,(s, X_s)\;dW_s.$$

We will apply Itô's formula to the expression $a(s, X_s)$ and $b(s, X_s)$, which are the coefficient in our SDEs.

$$X(\tau_{j+1}) = X(\tau_j) + \int_{\tau_j}^{\tau_{j+1}}\Big(a + \int_{\tau_j}^{s} a'\,a + \frac{1}{2}\,a''b^2\Big)\,du + \int_{\tau_j}^{s}(a'\,b - dW_s)\;ds + \int_{\tau_j}^{\tau_{j+1}}\Big(b + \int_{\tau_j}^{s} b'\,a + \frac{1}{2}\,b''\,b^2\,\Big)\,du + \int_{\tau_j}^{s} b'\,b\;dW_u)\;dW_s \qquad (10.1)$$

By applying Euler-Maruyama method to eq. (10.1), we obtain

$$X(\tau_{j+1}) = X(\tau_j) + \int_{\tau_j}^{\tau_{j+1}} a\;ds + \int_{\tau_j}^{\tau_{j+1}}\Big(b + \int_{\tau_j}^{s} b'b\;dW_u\,\Big)\;dW_s \approx X(\tau_j) + a\,\delta t + b\,\Delta W_i + \int_{\tau_j}^{\tau_{j+1}}\int_{\tau_j}^{s} b'\,b\;dW_u\;dW_s \qquad (10.2)$$

We approximate the eq. (10.1) by

$$\int_{\tau_j}^{\tau_{j+1}}\int_{\tau_j}^{s} b'\,b\;dW_u\;dW_s = b'b\int_{\tau_j}^{\tau_{j+1}}\int_{\tau_j}^{s} dW_u dW_s,$$

Then we obtain,

$b'b \int_{\tau_j}^{\tau_{j+1}} \int_{\tau_j}^{s} dW_u dW_s = \frac{1}{2} b'b \, ((\Delta W_i)^2 - \delta t)$, substituting this result in the previous approximation, we will get the desired Milstein's method.

The initial condition is $X_0 = x$,

$X_{j+1} = X_j + a\delta t + b\Delta W_j + \frac{1}{2} b'b \, ((\Delta W_i)^2 - \delta t)$.

Milstein's scheme is strongly and weakly convergent with order 1.

11. Strong order 1.5 Taylor method

The strong order 1.5 strong Taylor scheme can be obtained by adding more terms from Ito -Taylor expansion to the Milstein scheme (Ref. 1). The strong order 1.5 Ito -Taylor scheme is

$$X_{i+1} = X_i + a\delta t + b\Delta W_i + \frac{1}{2} bb_x \left(\Delta W_i^2 - \delta t \right) + a_x b \Delta Z_i + \frac{1}{2} \left(aa_x + \frac{1}{2} b^2 a_{xx} \right) (\delta t)^2 + \left(ab_x + \frac{1}{2} b^2 b_{xx} \right) (\Delta W_i \delta t - \Delta Z_i) + \frac{1}{2} b \, (bb_{xx} + b_x^2) \left(\frac{1}{3} \Delta W_i^2 - \delta t \right) \Delta W_i$$

the initial condition is $X_0 = x$.

Where the partial derivatives with respect to x are denoted by subscripts and the additional random variables ΔZ_i is normally distributed with mean zero and variance $E \, (\Delta Z_i^2) = \frac{1}{3} \, (\delta t)^3$ and correlated with covariance

$E (\Delta Z_i \Delta W_i) = \frac{1}{2} \, (\delta t)^2$, where ΔZ_i can be generated as

$$\Delta Z_i = \frac{1}{2} \, \delta t \, (\Delta W_i + \Delta V_i / \sqrt{3})$$

where ΔV_i is chosen independently from $\sqrt{\delta t} \; N \, (0, 1)$.

12. Applications for SDEs

By using EM-method, Milstein's method and Strong order 1.5 Taylor methods we can obtain the solution of given SDEs

$\frac{dX}{dt} = Y$

$\frac{dY}{dt} = X - X^3 + 0.2\, X\, \frac{dW}{dt}$

with initial conditions $X_0 = 0, t \in [0, 10]$ and $h = 0.001$.

The three numerical schemes applied to the present problem is given below,

Euler-Maruyama Scheme:

$$X_{i+1} = 2X_i - X_{i-1} + h^2((X_i - X_i^3)h + 0.2X_i\sqrt{h}\,\eta)$$

Milstein Scheme:

$$X_{i+1} = 2X_i - X_{i-1} + h^2((X_i - X_i^3)h + 0.2X_i\sqrt{h}\,\eta + \frac{1}{2}0.04X_i(\,h\,\eta^2 - h))$$

Strong order 1.5 Taylor Scheme:

$$X_{i+1} = 2X_i - X_{i-1} + h^2((X_i - X_i^3)\,h + 0.2X_i\sqrt{h}\,\eta + \tfrac{1}{2}0.04X_i(h\,\eta^2 - h) - 0.6X_i^3\,\Delta Z + \tfrac{1}{2}\left((X_i - X_i^3)(1 - 3X_i^2) + \tfrac{1}{2}0.04X_i^2(-6X_i)\right)h^2 + \left((X_i - X_i^3)0.2\right)\left(\sqrt{h}\,\eta h - \Delta Z\right) + \tfrac{1}{2}0.2X_i0.04\left(\tfrac{1}{3}\,h\,\eta^2 - h\right)\sqrt{h}\,\eta)$$

where $\Delta Z = \frac{1}{2}(\,h^{\frac{3}{2}}\eta + \frac{h^{\frac{3}{2}}\eta}{\sqrt{3}})$

The numerical approximation solution for given SDEs is cited in Table-1 as follow,

Table-1: Comparison Result for the present problem using the three numerical methods

t	Euler-Maruyama Scheme	Milstein Scheme	Strong order 1.5 Taylor Scheme
0	0	0	0
2	1.99495	1.99492	1.99495
4	3.92231	3.92217	3.92305
6	5.54695	5.54651	5.55711
8	6.43582	6.43496	6.48756
10	6.23568	6.23448	6.37289

The results for three numerical schemes are cited by Figs. 1-3 as follows,

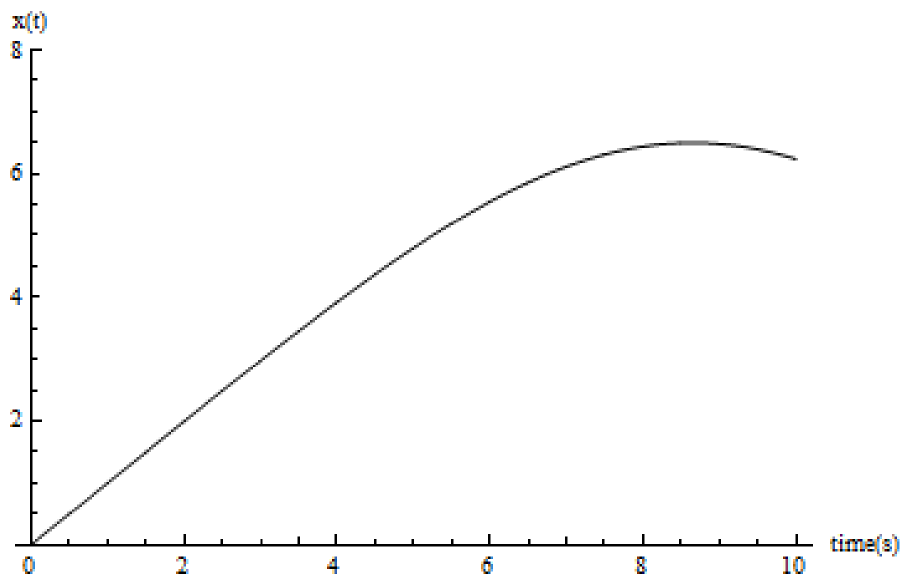

Fig.1. Solution of given SDE by using Euler- Maruyama scheme

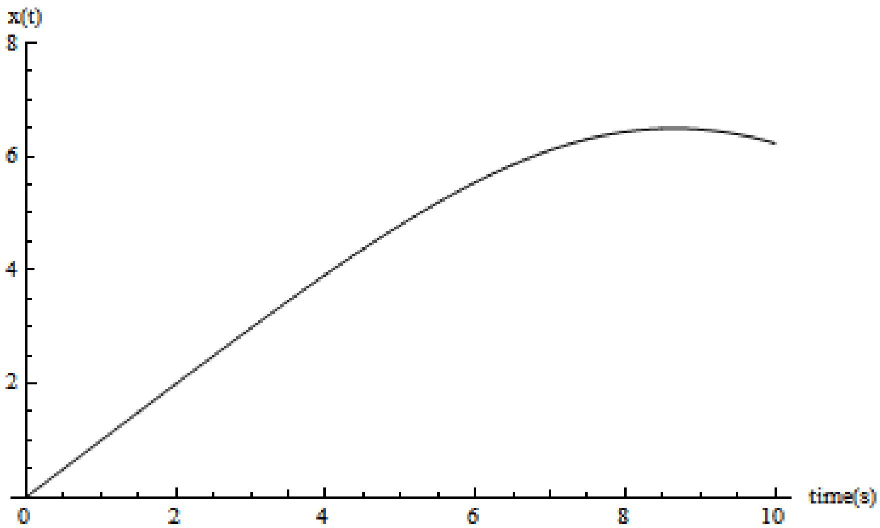

Fig.2. Solution of given SDE by using Milstein scheme

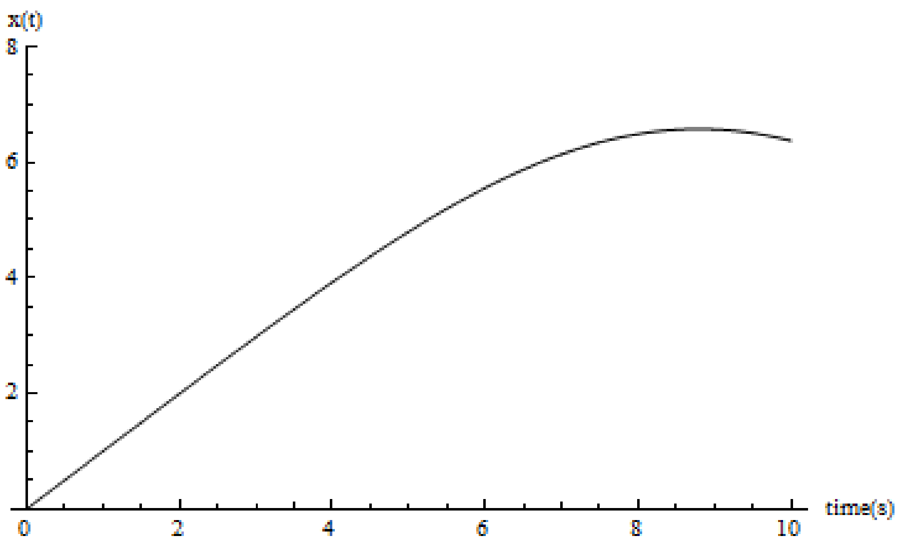

Fig.3. Solution of given SDE by using strong order 1.5 Taylor scheme

11. Conclusion

Numerical methods for the solution of stochastic differential equations are essential for analysis of random phenomenon. In the above problem we obtained the solutions by using three numerical approximations viz. Euler-Maruyama methods, Milstein's Method and Strong order 1.5 Taylor method very accurately. The numerical methods are very easy to use and efficient to calculate the solution for stochastic differential equations. There is a good agreement for numerical computations between these three methods (Euler-Maruyama methods, Milstein's Method and Strong order 1.5 Taylor method). The numerical results as cited in the Table-1 are shown with less round off errors. The numerical solutions are plotted by Fig.1-3 using a package $Mathematica$ ® 7. For solving any kind of SDEs these numerical methods are more simple and efficient to calculate.

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