Estimation of Parameters of Some Continuous Distribution Functions

Thesis submitted in partial fulfillment of the requirements for the degree of

Master of Science

by

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Certificate

This is to certify that the thesis entitled **"Estimation of parameters of some continuous distribution functions"**, which is being submitted by **Sulagna Mohanty** in the Department of Mathematics, National Institute of Technology, Rourkela, in partial fulfilment for the award of the degree of **Master of Science**, is a record of bonafide review work carried out by her in the Department of Mathematics under my guidance. She has worked as a project student in this Institute for one year. In my opinion the work has reached the standard, fulfilling the requirements of the regulations related to the Master of Science degree. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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Abstract

The thesis addresses the problem of estimation of parameters of some continuous distribution functions. The problem of estimation of parameters of normal and exponential distribution function has been considered. In particular, the maximum likelihood, method of moment, and Bayes estimators has been derived. Further the problem of estimating the parameters of a gamma and weibull distribution function is considered. Similar type of estimators are also derived for this case.

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Chapter 1

Introduction and Motivation

Estimation is one of the major areas of of statistical inference. statistical inference is the process by which conclusions from the sample data is used to draw conclusions about the population from which the sample was selected. The theory of estimation was founded by Prof.R.A.Fisher in a series of fundamental papers round about 1930. Point estimation refers to the process of estimating a parameter from a probability distribution, based on observed data from the distribution. It is one of the core topics in mathematical statistics. The problem of estimation when some parameter is unknown has received considerable attention of statisticians in recent past. Problem of estimation can be found everywhere: in business, in science, as well as in everyday life. In various physical, agricultural and industrial experiments, one comes across situations, where location and scale parameters are to be estimated. For example, in business, a chamber of commerce may want to know the average income of the families in its community, in science, a mineralogist may wish to determine the average iron content of a given core, finally, in everyday life a commuter may want to know how long on the average it will take her drive to work, and a serious gardener may want to know what proportions of certain tulips can be expected to bloom.

If we consider such practical data it is natural that it will follow certain distribution function. In that case we are getting a distribution, and we may be interested in the characteristics of the distribution. So,we need to study the distribution and estimation of its parameters, where the parameter may be unknown.

Suppose we are interested in the quality of production of rice across the country in last ten years. If the collected data follows normal distribution, then by estimating the parameter μ we can have an idea about the average rice production during that period and estimating the parameter σ^2 we can talk about the variability of production of rice in the country.

In Chapter 2, we have discussed some basic results related to the estimation problem. In Chapter 3, the problem of estimation of parameters of exponential and normal distribution is considered and various estimators are obtained. Further, in Chapter 4 the problem is considered for the gamma and weibull distribution function.

Chapter 2

Terminologies and Some Basic Results

In this chapter some definitions and basic results are given which are very much useful for the development of the consequence chapters. Below we start from a very basic concept known as random experiment or statistical experiment.

2.1 Some Definitions

Definition 2.1 (Random experiment) *An experiment in which all outcomes are known in advance, any performance of the experiment that results in an outcome is not known in advance and the experiment can be repeated under identical conditions, is called a random experiment.*

Definition 2.2 (outcome) *The result of a statistical experiment is called an outcome.*

Definition 2.3 (Sample space) *The sample space of a statistical experiment is a pair* (Ω, S)*, where* Ω *is the set of all possible outcomes of an experiment and* S *is the* σ*-field of subsets of* Ω*.*

Definition 2.4 (Event) *An event is a subset of the sample space Ω in which we are interested. Any set $A \in S$ is known as the events.*

Definition 2.5 (Probability measure) *Let (Ω, S) be a sample space. A set function P defined on S is called probability if it satisfies the following conditions,*

(i) $P(A) \geq 0$, $\forall\ A \in S$.

(ii)$P(\Omega) = 1$.

(iii) Let $A_j\ , A_j \in S, j = 1, 2, \ldots$ *be a disjoint sequence of sets.That is* $A_j \bigcap A_k = \emptyset for j \neq k$*.Then*

$$P\Big(\bigcup_{j=1}^{\infty}\Big) = \sum_{j=1}^{\infty} P\Big(A_j\Big)$$

Definition 2.6 (Random variable) *Let (Ω, S) be a sample space. A finite single valued function which maps Ω into $\mathbb{R}$ is called a random variable if the inverse images under X of all Borel sets in $\mathbb{R}$ are events.*

Definition 2.7 (Distribution function) *Let X be an random variable defined on (Ω, S, P). Define a function F on $\mathbb{R}$ by $F(x) = P\{w : X(w) \leq x\}$ for all $x \in \mathbb{R}$. F is nondecreasing, $F(-\infty) = 0$, $F(\infty) = 1$. Then the function F is called the distribution function of the random variable X.*

In this thesis we are suppose to study the continuous distribution function and its parameters, so below we are presenting some results related to this only.

Definition 2.8 (Continuous random variable) *Let X be a random variable defined on (Ω, S, P) with distribution function F. Then X is said to be continuous random variable if F is absolutely continuous that is if there exists a nonnegative function $f(x)$ such that, for every real number x, we have $F(x) = \int_{-\infty}^{x} f(t)\, dt$. The function f is called the probability density function of the random variable X.*

If X is a continuous random variable then we can define its probability density function as below.

Definition 2.9 (Probability density function) *Every nonnegative real valued function f can serve as a probability density function of a continuous random variable X, if $f(x) \geq 0$, and satisfies $\int_{-\infty}^{\infty} f(x)dx = 1$.*

Definition 2.10 (Two dimensional continuous random variable) *A two-dimensional random variable (X, Y) is said to be of continuous type if there exist a nonnegative function $f(.,.)$ such that for every pair $(x, y) \in \mathbb{R}^2$ we have $F(x, y) = \int_{-\infty}^{x}[\int_{-\infty}^{y} f(u, v)dv]du$ where F is the joint distribution function of (X, Y) and the function f is called the joint probability density function of (X, Y).*

Definition 2.11 (Marginal probability density function) *If (X, Y) is a continuous two dimensional random variable with joint PDF $f(x, y)$,then $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy, f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$ are the marginal probability density function of X and Y respectively. Which satisfy $f_X(x) \geq 0, f_Y(y) \geq 0$ and $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$, $\int_{-\infty}^{\infty} f_Y(y)\, dy = 1$.*

Definition 2.12 (Conditional probability density function) *Let (X, Y) be a two-dimensional continuous random variable with joint PDF $f(x, y)$ and marginal PDF of Y at y ,given by $f_Y(y)$,then the conditional PDF of X given $Y = y$ given by $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$, $f_Y(y) > 0$ similarly the conditional PDF of Y given $X = x$ given by $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$, $f_X(x) > 0$.*

Definition 2.13 (Mathematical expectation) *If X is a continuous random variable with probability density function f, we say that the mathematical expectation of X exists and write $E[X] = \int_{-\infty}^{\infty} xf(x)dx$, provided $\int_{-\infty}^{\infty} |x|f(x)dx < \infty$.*

Definition 2.14 (Expected value of function of random variable) *Let X be a continuous random variable with PDF $f(x)$ and g be a Borel measurable function on R such* *that g(X) is a random variable and$E[g(X)]$ exists.Then the expected value of $g(X)$ is given by* $E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx$.

Definition 2.15 (Conditional expectation) *Let (X, Y) be a two-dimensional random variable defined on a probability space and let h be a Borel measurable function . Assume that $E[h(X)]$ exists. Then the conditional expectation of $h(X)$ given $Y = y$ written as $E[h(X)|Y = y]$ is a random variable that takes the value $E[h(X)|Y = y]$ defined by*

$$E[h(X)|Y] = \int_{-\infty}^{\infty} h(x)f_{X|Y}(x|y)dx.$$

Next we discuss some characteristics of these distribution functions.

Definition 2.16 (Moments) *Moments are parameters associated with the distribution of the random variable X. Let k be a positive integer and c be a constant.If $E[(X - c)^k]$ exists, it is called the moment of k^{th} order about the point c. We will denote $\mu_k = E(X - E(X))^k$.*

Definition 2.17 (Variance) *If $E(X^2)$ exists, the variance is defined by $\sigma^2 = var(x) = E(X - \mu)^2$. The quantity σ is called the standard deviation of X. $\sigma^2 = \mu_2 = E(X^2) - (E(X))^2$.*

Further, we will discuss some terms related to the estimation of parameters of a discrete distribution function.

2.2 Estimation of Parameters

In this thesis we will only discuss the problem of point estimation.

Definition 2.18 (Estimation of parameters) *Suppose $F_\theta(x), \theta \in \Theta$ be a family of distribution functions and θ is taken to be unknown. Here we estimate the unknown parameter θ with the help of samples. We study the theory of point estimation and particularly parametric point estimation.*

Definition 2.19 (Parameter space) *The set of all possible values of the parameters of a distribution function F is called the parameter space. This set is usually denoted by Θ.*

Definition 2.20 (Statistic) *Any function of the random sample $X_1, X_2, \ldots, X_n$ that are being observed say $T(X_1, X_2, \ldots, X_n)$ is called a statistic.*

Definition 2.21 (Estimator) *If a statistic is used to estimate an unknown parameter θ of a distribution, then it is called an estimator and a particular value of the estimator say $T_n(X_1, X_2, \ldots, X_n)$ is called an estimate of θ.*

The process of estimating an unknown parameter is known as estimation.

2.3 Some Characteristics of Estimators

Various statistical properties of estimators can be used to decide which estimator is most appropriate in a given situation.

Definition 2.22 (Unbiasedness) *: A statistic T is an unbiased estimator of of the parameter θ if iff $E(T) = \theta$.*

Example 2.1 *Let X_1, X_2, X_3 be a random sample of size 3 from a Normal population $N(\mu, \sigma^2)$ .The statistic $T = \frac{1}{4}(X_1 + 2X_2 + X_3)$ is an unbiased estimate of μ. Since*

$$E(T) = E([\frac{1}{4}(X_1 + 2X_2 + X_3])$$
$$= \frac{1}{4}(\mu + 2\mu + \mu) = \mu$$

Definition 2.23 (Consistency) *Let $X_1, X_2, \ldots$ be a sequence of iid random variables with common distribution function F_θ ,$\theta \in \Theta$. A sequence of point estimators $T_n(X_1, X_2, \ldots, X_n) = T_n$ will be called consistent for $\psi(\theta)$ if T_n converges to $\psi(\theta)$ in probability that is,*

$$T_n \rightarrow \psi(\theta), \;\; as \; n \rightarrow \infty.$$

Remark 2.1 *If T_n is a consistent estimator of θ and $\psi(\theta)$ is a continuous function of θ,then $\psi(T_n)$ is a consistent estimator of $\psi(\theta)$.*

Remark 2.2 *If T_n is a sequence of consistent estimators such that $E[T_n] \rightarrow \psi(\theta)$ and $Var[T_n] \rightarrow 0$ as $n \rightarrow \infty$, then T_n is a consistent estimator of $\psi(\theta)$.*

Definition 2.24 (Efficiency) *In general there exists more than one consistent estimators.Thus it is necessary to find some criteria to choose between the estimators. Such a criterion which is based on the variances of sampling distributions of estimators is known as efficiency.*

Definition 2.25 (Sufficiency) *An estimator is said to be sufficient for a parameter, if it contains all the information in the sample regarding the parameter. Let $X = (X_1, X_2, \ldots, X_n)$ be a sample from a family of distributions $F_\theta : \theta \in \Theta$. A statistic T is sufficient for θ if and only if the conditional distribution of X given $T = t$, does not depend upon θ.*

Moreover the procedure for checking that the estimator T is sufficient is quite time consuming therefore for determining the sufficient statistics "Factorization Criterion" is used.

Theorem 2.1 (Factorization Criterion) *A statistic $T = t(X)$ is a sufficient statistic for the parameter θ if and only if the joint probability distribution or density of the random sample can be expressed in the form:*

$$f(x_1, x_2, \ldots, x_n; \theta) = g_\theta(t(x)) \times h(x_1, x_2, \ldots, x_n),$$

where $g_\theta(t(x))$ depends on θ and x and $h(x_1, x_2, \ldots, x_n)$ does not depend on θ.

Definition 2.26 (Completeness) *A statistic is said to be complete if the family of distributions of T is is complete. Let $f_\theta(x); \theta \in \Theta$ be a family of pdf's we say the family is complete, if $E_\theta g(X) = 0 \ \ \forall \ \theta \in \Theta$.*

Definition 2.27 (Ancillarity) *A statistic $T(X)$ is said to be ancillary if its distribution does not depend on the underlying model parameter θ.*

2.4 Methods of Estimation

Normally there are two different approaches for obtaining point estimators for parameter is known. Namely classical method and decision theoretic approach. Now we outline some of the most important methods for obtaining estimators. Most commonly used methods under classical estimation are as follows.

2.4.1 Method of Moments

Suppose X is a continuous random variable with probability density function (PDF) $f(x; \theta_1, \theta_2, \ldots, \theta_k)$ characterized by k unknown parameters. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from X. Defining the first k sample moments about origin as $m_r^{'} = \frac{1}{n}\sum_{i=1}^{n} X_i^r$, $r = 1, 2, \ldots, k$. The first k population moments about origin are given by $\mu_r^{'} = E(X^r)$, which are in general functions of k unknown parameters. Equating the sample moments and population moments yields k simultaneous equations in k unknowns. $\mu_r^{'} = m_r^{'}$, $r = 1, 2, \ldots, k$. The solutions to the above equations denoted by $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k$ yields the moment estimators of $\theta_1, \theta_2, \ldots, \theta_k$.

2.4.2 Method of Maximum Likelihood Estimation

Suppose $(X_1, X_2, \ldots, X_n)$ be a random vector with PDF $f_\theta(x_1, x_2, \ldots, x_n)$, $\underline{\theta}n \in \Theta$, where $\underline{\theta}$ is a multidimensional vector valued unknown parameter. Then the likelihood function is given by $L(\underline{\theta}; x_1, x_2, \ldots, x_n) = f_{\underline{\theta}}(x_1, x_2, \ldots, x_n)$ which is nothing but a function of unknown parameter $\underline{\theta}$. If $X_1, X_2, \ldots, X_n$ are iid with PDF $f_{\underline{\theta}}(x)$, then the likelihood function is $L(\underline{\theta}; x_1, x_2, \ldots, x_n) = \prod_{i=1}^n f_\theta(x_i)$. The maximum likelihood estimator (MLE) of $\underline{\theta}$ is the value of $\underline{\theta}$ say $\hat{\theta}$ that maximizes the likelihood function $L(\underline{\theta}; x_1, x_2, \ldots, x_n)$. Note that in many cases, the likelihood function can be infinitesimal and it is much easier to deal with the log-likelihood function that is $\log L(\underline{\theta}; x_1, x_2, \ldots, x_n)$. Since log is a monotone function, when likelihood function is maximized, log-likelihood function is also maximized, and vice versa.

Remark 2.3 *Let T be a sufficient statistic for the family of pdf's $f_\theta(x); \theta \in \Theta$. If an MLE of θ exists, it is a function of T.*

Remark 2.4 *If MLE exists then it is the most efficient in the class of such estimators.*

Remark 2.5 (Invariance property) *If T is the MLE of θ and $\psi(\theta)$ is one-to-one function of θ, then $\psi(T)$ is the MLE of $\psi(\theta)$.*

When we estimate the unknown parameter θ of a distribution function $F_\theta(x)$, by an estimator $\delta(X)$ some loss is incurred. Hence we use some loss functions to know the amount of loss incurred as below.

Definition 2.28 (Loss Function) *Loss function represents the loss incurred when the true value of the parameter is θ and we are estimating θ by $\delta(x)$. Throughout the discussion the loss function $L(\theta, \delta(x))$ is taken as nonnegative and real valued in both its arguments. When the correct estimate is chosen the loss becomes zero.Depending on the loss function Bayes estimators are different. Different types of loss functions are discussed below.*

Definition 2.29 (Linear Loss Function) *The linear loss function is defined as*

$$\begin{aligned} L(\theta, \delta(x)) &= c_1\Big(\delta(x) - \theta\Big), \delta(x) \geq \theta \\ &= c_2\Big(\theta - \delta(x)\Big), \delta(x) < \theta \end{aligned}$$

The constants c_1 and c_2 reflect the effect over and under estimating θ.If c_1 and c_2 are functions of θ, the above loss function is called weighted linear loss function.

Definition 2.30 (Absolute Error Loss Function) *The absolute error loss function is defined as*

$$L(\theta, \delta(x)) = |\delta(x) - \theta|.$$

Definition 2.31 (Squared Error Loss Function) *The squared error loss function is defined as*

$$L(\theta, \delta(x)) = k(\delta(x) - \theta)^2.$$

It is also called as quadratic loss function.

Throughout our discussion we have used squared error loss function.

Definition 2.32 (Risk Function) *The average loss of an estimator $\delta(x)$ is known as its risk function and is defined as*

$$R(\theta, \delta) = E[L(\theta, \delta(x))].$$

The goal of an estimation problem is to look for an estimator δ which has uniformly minimum risk for all values of the parameter $\theta \in \Theta$.

2.4.3 Bayes Estimation

In Bayesian Principle the unknown parameter θ which is treated as random variable assumes a probability distribution known as a priori of θ denoted by $\Pi(\theta)$.

To start the estimation of parameters we have the prior information about the unknown parameter θ. Different types of prior are discussed below.

(a) Noninformative Prior A pdf $\Pi(\theta)$ is said to be a noninformative prior if it contains no information about θ. Some simple examples of noninformative priors are $\Pi(\theta) = 1$, $\Pi(\theta) = \frac{1}{\theta}$.

(a) Natural conjugate prior To avoid problem of integration, Statisticians use natural conjugate prior distributions. Usually there is a natural parameter family of distributions such that the posterior distributions also belong to the same family. These priors make the computations much easier. Conjugate priors are usually associated with the exponential family of distributions. Some example of natural conjugate priors are: with sampling from pdf $N(\theta, \sigma^2)$ we take prior distribution $N(\mu, \tau^2)$, the posterior distribution is

$$N\Big(\frac{\sigma^2\mu + x\tau^2}{\sigma^2 + \tau^2}, \frac{\sigma^2\tau^2}{\sigma^2 + \tau^2}\Big)$$

. With sampling distribution Binomial and prior distribution Beta the posterior distribution is Beta.

(b) Jeffreys' invariant prior Jeffreys suggested a general rule for choosing the noninformative prior $\Pi(\theta)$. Where

$$\Pi(\underline{\theta}) \propto \sqrt{I(\underline{\theta})}$$

, where $\underline{\theta}$ vector valued parameter, and

$$I(\underline{\theta}) = -E\Big[\frac{\partial^2 \log f(x|\underline{\theta})}{\partial\theta_i \partial\theta_j}\Big] \tag{2.1}$$

where $I(\underline{\theta})$ is Fisher information matrix.

Definition 2.33 (posterior distribution) *The posterior distribution of θ given $X = x$ is obtained by dividing the joint density of θ and X by the marginal distribution of X. Mathematically*

$$\frac{\Pi(\theta) f_\theta(x)}{\int_\Theta \Pi(\theta) f_\theta(x) d\theta}$$

where Θ is the parameter space.

Definition 2.34 (Bayes Risk) *Bayes risk associated with an estimate δ is defined as the expected value of the risk function $R(\theta, \delta)$ with respect to the prior distribution $\Pi(\theta)$ of θ and is given by,*

$$\begin{aligned} R^*(\theta, \delta) &= E[R(\theta, \delta)] \\ &= \int R(\theta, \delta)\Pi(\theta)d\theta \\ &= \int E[L(\theta, \delta)]\Pi(\theta)d\theta. \end{aligned}$$

Definition 2.35 (Bayes Estimator) *A Bayes estimator is that which minimizes the Bayes risk defined above.Accordingly if δ_o is Bayes estimator of θ with prior distribution $\Pi(\theta)$,then we must have*

$$R^*(\theta, \delta_o) = \inf R^*(\theta, \delta).$$

Theorem 2.2 *The Bayes estimator of a parameter $\theta \in \Theta$ with respect to the qudratic loss function $L(\theta, \delta) = (\theta - \delta)^2$ turns out to be*

$$\delta(x) = E\{\theta | X = x\}.$$

Chapter 3

Estimation of parameters of Normal and Exponential Distribution

In this chapter the problem of estimation of parameters of a normal and exponential distribution is considered. First we will consider the estimation problem for normal distribution.

3.1 Normal Distribution

Method of Moments Estimator: Let $X \sim N(\mu, \sigma^2)$, where μ and σ^2 are unknown. The first two moments about origin are given by

$$\mu_1{}' = E(X) = \mu,$$

$$\mu_2{}' = E(X^2) = \sigma^2 + \mu^2$$

and the sample moments are given by,

$$m_1{}' = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and

$$m_2{}' = \frac{1}{n}\sum_{i=1}^{n} X_i^2.$$

So, using method of moments, we have

$$\begin{aligned} m_1{}' &= \mu_1{}' \\ \Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i &= \mu \\ \Rightarrow \hat{\mu} &= \bar{X}. \end{aligned}$$

Further we have,

$$\begin{aligned} & m_2{}' = \mu_2{}' \\ \Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i^2 &= \sigma^2 + \mu^2 \\ \Rightarrow \bar{X}^2 + \sigma^2 &= \frac{1}{n}\sum_{i=1}^{n} X_i^2. \end{aligned}$$

After solving for σ^2 we get,

$$\begin{aligned} \sigma^2 &= \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \\ &= \frac{S^2}{n}, \quad \text{where} \quad S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2, \\ &= \hat{\sigma}^2. \end{aligned}$$

The method of moments estimator of μ and σ^2 are $\hat{\mu} = \bar{X}$, and $\hat{\sigma}^2 = \frac{S^2}{n}$ respectively.

Maximum Likelihood Estimator: Let $X_1, X_2, \ldots, X_n$ be identically and independently distributed random samples taken from normal distribution $X \sim N(\mu, \sigma^2)$. The pdf of the random variable X is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty; \sigma > 0; \ -\infty < \mu < \infty. \tag{3.1}$$

The likelihood function is given by,

$$\begin{aligned} L(\underline{x}, \mu, \sigma^2) &= \prod_{i=1}^{n}\Big[\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}\Big]. \\ &= \Big(\frac{1}{2\pi\sigma^2}\Big)^{\frac{n}{2}} e^{\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}. \end{aligned}$$

The log likelihood function is given by

$$\log L(\underline{x}, \mu, \sigma^2) = \frac{-n}{2}\log(2\pi) - \frac{-n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2.$$

Then the likelihood equations are given by,

$$\frac{\partial[\log L(\underline{x}, \mu, \sigma^2)]}{\partial\mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu) = 0,$$

and

$$\frac{\partial[\log L(\underline{x}, \mu, \sigma^2)]}{\partial\sigma^2} = \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i - \mu)^2 = 0.$$

The solution of the above equations yield the MLE of μ and σ^2 as,

$$\hat{\mu} = \bar{X}$$

and

$$\hat{\sigma}^2 = \frac{S^2}{n}.$$

The maximum likelihood estimators of μ and σ^2 are

$$\hat{\mu} = \bar{X}$$
$$\hat{\sigma}^2 = \frac{S^2}{n}.$$

Bayes Estimator: Let $X_1, X_2, \ldots, X_n$ be a sample from normal distribution with unknown mean μ and variance $\sigma^2 = 1$. So, the pdf of the random variable is given by

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-(x-\mu)^2/2}.$$

The likelihood function is given by

$$\begin{aligned} L(\underline{x}, \mu) &= \prod_{i=1}^{n}\left[\frac{1}{\sqrt{2\pi}}e^{-\frac{(x_i-\mu)^2}{2}}\right] \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}e^{\frac{-1}{2}\left(\sum_{i=1}^{n}x_i^2 - 2\mu\sum_{i=1}^{n}x_i + \mu^2\right)} \end{aligned}$$

Let the prior PDF of μ be $N(0,1)$. Which is given by

$$g(\mu) = \frac{1}{\sqrt{2\pi}} e^{\frac{-\mu^2}{2}}.$$

The joint PDF of $\underline{X}$ and μ is given by,

$$f(\underline{x}, \mu) = \frac{1}{(2\pi)^{\frac{n+1}{2}}} e^{\frac{-(\sum_{i=1}^{n} x_i^2 - 2\mu n\bar{x} + (n+1)\mu^2)}{2}}.$$

The marginal PDF of $\underline{X}$ is given by,

$$\begin{aligned}
h(\underline{x}) &= \int_{-\infty}^{\infty} f(x, \mu) d\mu \\
&= \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{\frac{n+1}{2}}} e^{\frac{-(\sum_{i=1}^{n} x_i^2 - 2\mu n\bar{x} + (n+1)\mu^2)}{2}} d\mu \\
&= \frac{1}{(2\pi)^{\frac{n+1}{2}}} e^{\frac{-1}{2}\sum_{i=1}^{n} x_i^2} \int_{-\infty}^{\infty} e^{\frac{-(n+1)}{2}[(\mu - \frac{n\bar{x}}{n+1})^2 - (\frac{n\bar{x}}{n+1})^2]} d\mu \\
&= \frac{1}{(2\pi)^{\frac{n+1}{2}}} e^{\frac{-1}{2}\sum_{i=1}^{n} x_i^2} e^{(\frac{n}{2})(\frac{n\bar{x}}{n+1})^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(n+1)}{2}(\mu - \frac{n\bar{x}}{n+1})^2} d\mu \\
&= \frac{1}{(2\pi)^{\frac{n+1}{2}}} e^{\frac{-1}{2}(\sum_{i=1}^{n} x_i^2 - \frac{n^2\bar{x}^2}{n+1})} \frac{1}{(n+1)^{\frac{1}{2}}}.
\end{aligned}$$

Now the posterior PDF is given by,

$$\begin{aligned}
f(\mu|\underline{x}) &= \frac{f(\underline{x}, \mu)}{h(\underline{x})} \\
&= \frac{(n+1)^{\frac{1}{2}}}{\sqrt{2\pi}} e^{\frac{-(n+1)}{2}(\mu - \frac{n\bar{X}}{n+1}))^2}.
\end{aligned}$$

A Bayes estimator of μ with respect to the squared error loss function $L(\mu, \delta) = (\mu - \delta)^2$, is given by,

$$\begin{aligned}
\hat{\mu} &= E(\mu|\underline{x}) \\
&= \int_{-\infty}^{\infty} \mu f(\mu|\underline{x}) d\mu \\
&= \frac{(n+1)^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu e^{-(\frac{n+1}{2})(\mu - \frac{n\bar{X}}{n+1})^2} \\
&= \frac{n\bar{X}}{n+1}.
\end{aligned}$$

3.2 Exponential Distribution(one-parameter)

In this section we will discuss the estimation problem for one-parameter exponential distribution.

Method of Moments Estimator: Let X be a random variable which has probability density function $f(x) = \frac{1}{\beta}e^{-\frac{x}{\beta}}$, $\beta > 0$ and $x > 0$.

Let $X_1, X_2, \ldots, X_n$ are identically and independently distributed random samples taken from X. The first moment about origin is given by

$$\mu_1{}' = E(X) = \beta.$$

The first sample moment about origin is given by

$$m_1{}' = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

So, using method of moments we have,

$$m_1{}' = \mu_1{}'.$$

Further simplifying we have,

$$\begin{aligned} \beta &= \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X} \\ &= \hat{\beta}, \quad \text{say.} \end{aligned}$$

So, the method of moment estimator of β is $\bar{X}$.

Maximum Likelihood Estimator:

Let $X_1, X_2, \ldots, X_n$ be identically and independently distributed random samples taken from one-parameter Exponential distribution $Ex(\beta)$. So, the pdf of the rv X is given by

$$f(x; \beta) = \frac{1}{\beta}e^{-x/\beta}, \quad x > 0, \quad \beta > 0. \tag{3.2}$$

The likelihood function is given by,

$$\begin{aligned} L(\underline{x};\beta) &= \prod_{i=1}^{n} \frac{e^{\frac{-x_i}{\beta}}}{\beta} \\ &= \frac{1}{\beta^n} e^{\frac{-1}{\beta}\sum_{i=1}^{n} x_i} \end{aligned}$$

The log likelihood function is given by

$$\begin{aligned} \log L(\underline{x};\beta) &= \log(\frac{1}{\beta^n} e^{\frac{-1}{\beta}\sum_{i=1}^{n} x_i}) \\ &= -n\log\beta - \frac{1}{\beta}\sum_{i=1}^{n} x_i. \end{aligned}$$

The likelihood equation is given by,

$$\begin{aligned} \frac{\partial}{\partial\beta}\log L(\underline{x};\beta) = \frac{-n}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n} x_i &= 0. \\ \Rightarrow \frac{1}{\beta}(-n + \frac{1}{\beta}\sum_{i=1}^{n} x_i) &= 0. \end{aligned}$$

Solving for β we get,

$$\hat{\beta} = \bar{X}.$$

Therefore the maximum likelihood estimator of β is $\bar{X}$.

Bayes Estimator: Let $X_1, X_2, \ldots, X_n$ be identically and independently distributed random samples taken from one-parameter Exponential distribution $Ex(\beta)$. So, the pdf of the rv X is given by

$$f(x;\beta) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \quad x > 0, \quad \beta > 0. \tag{3.3}$$

The likelihood function is given by,

$$\begin{aligned} L(\underline{x};\beta) &= \prod_{i=1}^{n} \frac{e^{\frac{-x_i}{\beta}}}{\beta} \\ &= \frac{1}{\beta^n} e^{\frac{-1}{\beta}\sum_{i=1}^{n} x_i} \\ &= \frac{1}{\beta^n} e^{\frac{-S}{\beta}}, \quad \text{where} \quad S = \sum_{i=1}^{n} x_i. \end{aligned}$$

Considering the inverted gamma prior, the prior pdf of β is given by,

$$g(\beta|a,b) = \frac{a^b}{\Gamma(b)} \frac{e^{\frac{-a}{\beta}}}{\beta^{b+1}}, \ a > 0, \ b > 0, \ \beta > 0.$$

The joint pdf of X and β is given by,

$$f(\underline{x}, \beta) = \frac{a^b}{\Gamma(b)} \frac{e^{\frac{-1}{\beta}(S+a)}}{\beta^{n+b+1}}, \ a > 0, \ b > 0, \ \beta \ > 0.$$

The marginal pdf of X is given by,

$$\begin{aligned} h(\underline{x}) &= \int_0^\infty f(\underline{x}, \beta) d\beta \\ &= \frac{a^b}{\Gamma(b)} \int_0^\infty \frac{e^{\frac{-1}{\beta}(S+a)}}{\beta^{n+b+1}} d\beta, \\ &= \frac{a^b}{\Gamma(b)} \int_0^\infty e^{-t} \frac{t^{n+b+1}}{(S+a)^{n+b}} dt, \quad (\text{put} \ \ t = \frac{S+a}{\beta}), \\ &= \frac{a^b \Gamma(n+b)}{\Gamma(b)(S+a)^{n+b}}. \end{aligned}$$

Now the posterior PDF is given by,

$$f(\beta|\underline{x}) = \frac{(S+a)^{n+b} e^{\frac{-(S+a)}{\beta}}}{\beta^{n+b+1}\Gamma(n+b)}.$$

A Bayes estimator of β with respect to the squared error loss function $L(\beta, \delta) = (\beta - \delta)^2$, is given by,

$$\begin{aligned} \hat{\beta} &= E(\beta|\underline{x}) \\ &= \int_0^\infty \beta f(\beta|\underline{x}) d\beta \\ &= \int_0^\infty \beta \frac{(S+a)^{n+b} e^{\frac{-(S+a)}{\beta}}}{\beta^{n+b+1}\Gamma(n+b)} d\beta \\ &= \frac{n\bar{X} + a}{n+b-1}. \end{aligned}$$

3.3 Exponential Distribution(two-parameter)

In this section we will discuss the estimation problem for two-parameter exponential distribution.

Method of Moments Estimators: Let X be a random variable which has probability density function $f(x;\alpha,\beta) = \frac{1}{\beta}e^{-(x-\alpha)/\beta}$; $\alpha < x < \infty$, $-\infty < \alpha < \infty$, $\beta > 0$. Let $X_1, X_2, \ldots, X_n$ are identically and independently distributed random samples taken from X. The first two moments about origin are given by

$$\mu_1{}' = E(X) = \alpha + \beta.$$

$$\mu_2{}' = E(X^2) = 2\beta(\alpha+\beta) + \alpha^2.$$

and the sample moments are given by,

$$m_1{}' = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

and

$$m_2{}' = \frac{1}{n}\sum_{i=1}^{n} X_i^2.$$

So, using method of moments, we have

$$\begin{aligned} m_1{}' &= \mu_1{}' \\ \Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i &= \alpha + \beta = \bar{X} \\ \Rightarrow \hat{\alpha} &= \bar{X} - \hat{\beta}. \end{aligned}$$

and

$$\begin{aligned} m_2{}' &= \mu_2{}' \\ \Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i^2 &= 2\beta(\alpha+\beta) + \alpha^2. \end{aligned}$$

Solving for β we have

$$\hat{\beta} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

So, the method of moment estimators of α and β are obtained as

$$\hat{\alpha} = \bar{X} - \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2,$$

and

$$\hat{\beta} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

respectively.

Maximum Likelihood Estimator: Let $X_1, X_2, \ldots, X_n$ be identically and independently distributed random samples taken from two-parameter Exponential distribution $Ex(\alpha, \beta)$. So, the pdf of the random variable X is given by

$$f(x; \alpha, \beta) = \frac{1}{\beta}e^{-(x-\alpha)/\beta};\ \alpha < x < \infty,\ \ -\infty < \alpha < \infty,\ \beta > 0. \tag{3.4}$$

The likelihood function is given by,

$$L(\underline{x}; \alpha, \beta) = \frac{1}{\beta^n}e^{-\frac{1}{\beta}\sum_{i=1}^{n}(x_i-\alpha)}.$$

The loglikelihood function is given by

$$\begin{aligned} \log L(\underline{x}; \alpha, \beta) &= \log(\frac{1}{\beta^n}e^{-\frac{1}{\beta}\sum_{i=1}^{n}(x_i-\alpha)}) \\ &= -n\log\beta - \frac{1}{\beta}\sum_{i=1}^{n}(x_i - \alpha). \end{aligned}$$

The likelihood equations are given by,

$$\begin{aligned} \frac{\partial}{\partial\alpha}(\log L(\underline{x}; \alpha, \beta)) &= 0 \\ \Rightarrow \hat{\alpha} = \min(X_1, X_2, \ldots, X_n) &= X_{(1)}. \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial\beta}(\log L(\underline{x}; \alpha, \beta) &= 0 \\ \Rightarrow \frac{-n}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n}(x_i - \alpha) &= 0. \end{aligned}$$

After some simplification we get,

$$\hat{\beta} = \frac{1}{n}\sum_{i=1}^{n}(X_i - X_{(1)}).$$

Therefore the maximum likelihood estimator of α and β are $X_{(1)}$ and $\frac{1}{n}\sum_{i=1}^{n}(X_i - X_{(1)})$. respectively.

Bayes Estimator: Let $X_1, X_2, \ldots, X_n$ be identically and independently distributed random samples taken from two-parameter Exponential distribution $Ex(\alpha, \beta)$.

So, the pdf of the random variable X is given by,

$$f(x;\alpha,\beta) = \frac{1}{\beta}e^{-(x-\alpha)/\beta};\ \alpha < x < \infty,\ \ -\infty < \alpha < \infty,\ \beta > 0. \tag{3.5}$$

The likelihood function is given by,

$$\begin{aligned} L(\underline{x};\alpha,\beta) &= \prod_{i=1}^{n}\frac{1}{\beta}e^{-(x-\alpha)/\beta} \\ &= \frac{1}{\beta^n}e^{-\frac{1}{\beta}\sum_{i=1}^{n}(x_i-\alpha)} \\ &= \frac{1}{\beta^n}e^{\frac{-1}{\beta}\{S+n(x_{(1)}-\alpha)\}}. \end{aligned}$$

where $X_{(1)}$ is the first order statistic in the sample. Here $\underline{X} = (X_1, X_2, \ldots, X_n)$ and $S = \sum_{i=1}^{n}(x_i - x_{(1)})$.

The joint pdf is given by,

$$\begin{aligned} \Pi(\underline{x};\alpha,\beta) &= L(\underline{x};\alpha,\beta)g(\alpha,\beta) \\ &= \frac{1}{\beta^{n+1}}e^{\frac{-1}{\beta}\{S+n(x_{(1)}-\alpha)\}}. \end{aligned}$$

The marginal pdf of α is given by,

$$\begin{aligned} \Pi_1(\alpha|\underline{x}) &= \int_0^\infty \Pi(\underline{x};\alpha,\beta)d\beta \\ &= \frac{k}{[S+n(x_{(1)}-\alpha)]^n}\ ,\quad -\infty < \alpha < x_{(1)}, \end{aligned}$$

where

$$k^{-1} = \int_{-\infty}^{x_{(1)}} \frac{d\alpha}{[S + n(x_{(1)} - \alpha)]^n}.$$

Let

$$S + n(x_{(1)} - \alpha) = V,$$

then

$$d\alpha = -dV/n.$$

Finally we get,

$$k^{-1} = \frac{1}{n(n-1)S^{n-1}}.$$

Substituting k in $\Pi_1(\alpha|\underline{x})$ we have

$$\Pi_1(\alpha|\underline{x}) = \frac{n(n-1)S^{n-1}}{[S + n(x_{(1)} - \alpha)]^n}.$$

Now the Bayes estimator of α with respect to the squared error loss function is given by,

$$\begin{aligned} \hat{\alpha} &= E(\alpha|\underline{x}) \\ &= \int_{-\infty}^{x_{(1)}} \alpha \frac{n(n-1)S^{n-1}}{[S + n(x_{(1)} - \alpha)]^n} d\alpha. \end{aligned}$$

After certain calculations we get,

$$\hat{\alpha} = X_{(1)} - \frac{S}{n(n-2)}.$$

The marginal pdf of β is given by,

$$\begin{aligned} \Pi_2(\beta|\underline{x}) &= \int_{-\infty}^{x_{(1)}} \Pi(\underline{x}; \alpha, \beta) d\alpha \\ &= \frac{S^{n-1}}{\Gamma(n-1)\beta^n} e^{-S/\beta}. \end{aligned}$$

The Bayes estimator of β with respect to the squared error loss function is given by,

$$\begin{aligned} \hat{\beta} &= E(\beta|\underline{x}) \\ &= \frac{S}{n-2}\ , \ n > 2. \end{aligned}$$

Chapter 4

Estimation of Parameters of Some Non-Normal Distribution Functions

In this chapter we will take up a problem of estimating parameter of non-normal distribution functions, such as gamma and weibull distribution functions.

4.1 Gamma Distribution

First we consider the problem for a two parameter gamma distribution function.

Method of Moments Estimator: Let $X_1, X_2, \ldots, X_n$ be a random sample from $Gamma(\alpha, \beta)$ distribution where both the parameters are unknown. The problem is to get the estimators for both α and β.

The first two moments about origin are given by

$$\mu_1{}' = E(X) = \alpha\beta$$

$$\mu_2{}' = E(X^2) = \alpha(\alpha+1)\beta^2.$$

The the sample moments are obtained as,

$$m_1{}' = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and

$$m_2{}' = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

So, using method of moments, we have

$$\begin{aligned} m_1{}' &= \mu_1{}' \\ \Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i &= \alpha\beta \\ \Rightarrow \hat{\alpha} &= \frac{\bar{X}}{\beta}, \end{aligned}$$

and

$$\begin{aligned} m_2{}' &= \mu_2{}' \\ \Rightarrow \frac{1}{n}\sum_{i=1}^{n} x_i^2 = \alpha(\alpha+1)\beta^2 &= \frac{\bar{X}}{\beta}(\frac{\bar{X}}{\beta}+1)\beta^2. \end{aligned}$$

For solving β we have,

$$\begin{aligned} \Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i^2 &= \bar{X}(\bar{X}+\beta) \\ \Rightarrow \beta\bar{X} &= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\bar{X})^2. \end{aligned}$$

Finally we get,

$$\hat{\beta} = \frac{S^2}{n\bar{X}}, \quad \text{where} \quad S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2.$$

The method of moments estimators of α and β are

$$\hat{\alpha} = \frac{\bar{X}}{\hat{\beta}}$$

$$\hat{\beta} = \frac{S^2}{n\bar{X}}.$$

Maximum Likelihood Estimator: Let $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables taken from Gamma distribution. The pdf is given by,

$$f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta};\ 0 < x < \infty;\ \alpha,\ \beta > 0.$$

The likelihood function is given by

$$\begin{aligned} L(\underline{x};\alpha,\beta) &= \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x_i/\beta} \\ &= \frac{1}{(\Gamma(\alpha))^n \beta^{n\alpha}} \prod_{i=1}^{n} x_i^{\alpha-1} e^{-\sum_{i=1}^{n} x_i/\beta}. \end{aligned}$$

The log likelihood function is given by

$$\log(L(\underline{x};\alpha,\beta)) = -n\log\Gamma(\alpha) - n\alpha\log\beta + (\alpha-1)\sum_{i=1}^{n}\log x_i - \frac{1}{\beta}\sum_{i=1}^{n} x_i.$$

The likelihood equations are given by,

$$\frac{\partial}{\partial\alpha}(\log(L) = -n\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - n\log\beta + \sum_{i=1}^{n}\log x_i = 0. \tag{4.1}$$

Further we have,

$$\frac{\partial}{\partial\beta}(\log(L) = -n\frac{\alpha}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n} x_i = 0. \tag{4.2}$$

Solving we get,

$$\hat{\beta} = \frac{\bar{X}}{\alpha}.$$

Now substituting the value of β in equation (4.1) we get,

$$\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \log\alpha = \frac{1}{n}\sum_{i=1}^{n}\log x_i - \log(\bar{X}).$$

The above equation is to be solved for $\hat{\alpha}$, In this case the equation is not easily solvable. So we may use some numerical methods like Newton Raphson method for getting the value of α. Then substitute back to get the value of β.

Bayes Estimator: Let $X_1, X_2, \ldots, X_n$ be a sample which follows Gamma distribution. Then, the pdf of the random variable X is given by,

$$f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}; \quad 0 < y < \infty; \; \alpha, \; \beta > 0.$$

Let the corresponding conjugate prior for β is,

$$g(\beta) = \frac{1}{\beta^5} e^{\frac{-1}{\beta}}, \quad \beta > 0.$$

The joint pdf of X and β is given by,

$$f(x,\beta) = \frac{1}{\beta^5\Gamma(\alpha)} x^{\alpha-1} e^{-(x+1)/\beta}.$$

Now the marginal PDF is given by ,

$$\begin{aligned} h(x) &= \int_0^\infty \frac{1}{\beta^5\Gamma(\alpha)} x^{\alpha-1} e^{-(x+1)/\beta} d\beta \\ &= \frac{x^{\alpha-1}\Gamma(\alpha+4)}{(x+1)^{\alpha+4}\Gamma(\alpha)}. \end{aligned}$$

The posterior PDF is given by the conditional PDF that is

$$f(\beta|\underline{x}) = \frac{f(x,\beta)}{h(x)} = \frac{(x+1)^{\alpha+4} e^{-(x+1)/\beta}}{\Gamma(\alpha+4)\beta^{\alpha+5}}. \tag{4.3}$$

Now with respect to squared error loss function the Bayes estimator of β is given by,

$$\begin{aligned} E(\beta|\underline{x}) &= \int_0^\infty \beta \frac{f(x,\beta)}{h(x)} d\beta && (4.4) \\ &= \int_0^\infty \beta \frac{(x+1)^{\alpha+4} e^{-(x+1)/\beta}}{\Gamma(\alpha+4)\beta^{\alpha+5}} d\beta \\ &= \frac{x+1}{\alpha+3} \\ &= \hat{\beta}, \quad \text{say.} && (4.5) \end{aligned}$$

4.2 Weibull Distribution

In this section we consider the two-parameter weibull distribution function. Here we are interested in estimating the unknown parameters.

Method of Moments Estimator: Let $Y_1, Y_2, \ldots, Y_n$ be a sample from Weibull(α, β) distribution. The first two moments about origin are given by

$$\mu_1{}' = E(Y) = \beta^{\frac{1}{\alpha}}\Gamma(1 + \frac{1}{\alpha}),$$

and

$$\mu_2{}' = E(Y^2) = \beta^{\frac{2}{\alpha}}\Gamma(1 + \frac{2}{\alpha}).$$

The sample moments are given by,

$$m_1{}' = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

and

$$m_1{}' = \frac{1}{n}\sum_{i=1}^{n} Y_i^2.$$

So,using method of moments, we have

$$\begin{aligned} m_1{}' &= \mu_1{}' \\ \Rightarrow \frac{1}{n}\sum_{i=1}^{n} Y_i &= \beta^{\frac{1}{\alpha}}\Gamma(1 + \frac{1}{\alpha}) \end{aligned}$$

and

$$\begin{aligned} m_2{}' &= \mu_2{}' \\ \Rightarrow \frac{1}{n}\sum_{i=1}^{n} Y_i^2 &= \beta^{\frac{2}{\alpha}}\Gamma(1 + \frac{2}{\alpha}). \end{aligned}$$

Solving for α and β we get the method of moment estimators.

Maximum Likelihood Estimator: Let $Y_1, Y_2, \ldots, Y_n$ are independent random variables which follows Weibull distribution. The pdf of the random variable is given by,

$$f(y;\alpha,\beta) = \frac{\alpha}{\beta} y^{\alpha-1} e^{-y^{\alpha}/\beta};\ y > 0,\ \alpha,\ \beta > 0.$$

The likelihood function is given by

$$\begin{aligned} L(\underline{y};\alpha,\beta) &= \prod_{i=1}^{n} \frac{\alpha}{\beta} y_i^{\alpha-1} e^{-y_i^{\alpha}/\beta}, \\ &= \frac{\alpha^n}{\beta^n} \prod_{i=1}^{n} y_i^{\alpha-1} e^{-\sum_{i=1}^{n} y_i^{\alpha}/\beta}. \end{aligned}$$

Here $\underline{y} = (y_1, y_2, \ldots, y_n)$ represents random sample of values of Y. The log likelihood function is given by

$$\begin{aligned} \log L(\underline{y};\alpha,\beta) &= n\log\alpha - n\log\beta + \sum_{i=1}^{n} \log y_i^{\alpha-1} - \frac{1}{\beta}\sum_{i=1}^{n} y_i^{\alpha} \\ &= n\Big(\log\alpha - \log\beta\Big) + (\alpha-1)\sum_{i=1}^{n} \log y_i - \frac{1}{\beta}\sum_{i=1}^{n} y_i^{\alpha}. \end{aligned}$$

The Likelihood equations are given by,

$$\frac{\partial}{\partial\alpha}(\log(L(\underline{y};\alpha,\beta)) = \frac{n}{\alpha} + \sum_{i=1}^{n} \log y_i - \frac{\alpha}{\beta}\sum_{i=1}^{n} y_i^{\alpha-1} = 0 \qquad (4.6)$$

and

$$\frac{\partial}{\partial\beta}(\log(L(\underline{y};\alpha,\beta)) = \frac{-n}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n} y_i^{\alpha} = 0. \qquad (4.7)$$

The above two equations can be solved numerically to get the value of MLEs.

Bayes estimator: Let $Y_1, Y_2, \ldots, Y_n$ be a sample which follows Weibull distribution. Then, the pdf of the random variable Y is given by,

$$f(y;\alpha,\beta) = \frac{\alpha}{\beta} y^{\alpha-1} e^{-y^{\alpha}/\beta};\ y > 0,\ \alpha,\beta\ > 0 \qquad (4.8)$$

The likelihood function is given by,

$$L(\underline{y};\alpha,\beta) = \frac{\alpha^n}{\beta^n}\lambda^{\alpha-1}e^{-\sum_{i=1}^n y_i^\alpha/\beta}$$

where $\lambda = \prod_{i=1}^n y_i$. Now considering the Jeffreys' prior

$$h(\alpha,\beta) = \frac{1}{\alpha\beta},$$

the joint distribution is given by,

$$\begin{aligned}\Pi(\underline{y};\alpha,\beta) &= L(y;\alpha,\beta)h(\alpha,\beta)\\ &= k\frac{\alpha^{n-1}}{\beta^{n+1}}\lambda^{\alpha-1}e^{-\sum_{i=1}^n y_i^\alpha/\beta},\end{aligned}$$

where

$$\begin{aligned}k^{-1} &= \int_0^\infty\int_0^\infty \Pi(y;\alpha,\beta)d\beta d\alpha\\ &= \int_0^\infty \frac{\alpha^{n-1}\lambda^{\alpha-1}}{(\sum_{i=1}^n y_i^\alpha)^n}\Gamma(n)d\alpha.\end{aligned}$$

Thus

$$\Pi(\underline{y};\alpha,\beta) = \frac{\alpha^{n-1}\lambda^{\alpha-1}e^{-\sum_{i=1}^n y_i^\alpha/\beta}}{\Gamma(n)\int_0^\infty \frac{\alpha^{n-1}\lambda^{\alpha-1}}{(\sum_{i=1}^n y_i^\alpha)^n})}d\alpha. \tag{4.9}$$

The posterior of α and β are given by,

$$\Pi_1(\alpha|\underline{y}) = \frac{\frac{\alpha^{n-1}\lambda^{\alpha-1}}{(\sum_{i=1}^n y_i^\alpha)}}{\int_0^\infty \frac{\alpha^{n-1}\lambda^{\alpha-1}}{(\sum_{i=1}^n y_i^\alpha)^n}d\alpha} \tag{4.10}$$

and

$$\Pi_2(\beta|\underline{y}) = \frac{\int_0^\infty \lambda^{\alpha-1}\alpha^{n-1}e^{-\sum_{i=1}^n y_i^\alpha/\beta}d\alpha}{\Gamma(n)\beta^{(n+1)}\int_0^\infty \frac{\alpha^{n-1}\lambda^{\alpha-1}}{(\sum_{i=1}^n y_i^\alpha)^n}d\alpha} \tag{4.11}$$

respectively. From equation (4.10) and (4.11) the Bayes estimators for α and β are given by,

$$\hat{\alpha} = \int_0^\infty \frac{\frac{\alpha^n\lambda^{\alpha-1}}{(\sum_{i=1}^n y_i^\alpha)^n}d\alpha}{\int_0^\infty \frac{\alpha^{n-1}\lambda^{\alpha-1}}{(\sum_{i=1}^n y_i^\alpha)^n}d\alpha} \tag{4.12}$$

and

$$\hat{\beta} = \int_0^\infty \frac{\frac{\alpha^{n-1}\lambda^{\alpha-1}}{(\sum_{i=1}^n y_i^\alpha)^{n-1}} d\alpha}{(n-1)\int_0^\infty \frac{\alpha^{n-1}\lambda^{\alpha-1}}{(\sum_{i=1}^n y_i^\alpha)^n} d\alpha} \tag{4.13}$$

respectively. The above are difficult to evaluate. One may use some numerical integration methods for the evaluation these integrals.

Conclusions and Scope of Future Works

In this project work, at first I have studied the basic requirements for the development of subsequent chapters. I have learned some techniques for estimating parameters of a distribution function such as maximum likelihood, method of moments and the Bayesian approach. In particular the problem has been considered for the case of normal, exponential, gamma and weibull distribution. I have also studied some characteristics of the estimators. The work of the thesis can be extended in various directions. Some of the future works to be carried out are listed below.

- Numerical methods can be adopted to find the closed form of some estimators which are obtained in Chapter 4.
- The risk of different estimators can be compared numerically also graphically.
- Bayes estimator with respect to different loss functions can be obtained.

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