UNDERSTANDING THE RIEMANN HYPOTHESIS AND BASIC THEORY OF UNIVALENT FUNCTIONS

A THESIS

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DECLARATION

I declare that the topic "*UNDERSTANDING THE RIEMANN HYPOTHESIS AND BASIC THEORY OF UNIVALENT FUNCTIONS*" for my M.Sc. degree has not been submitted in any other institution or university for the award of any other degree or diploma.

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THESIS CERTIFICATE

This is to certify that the project report entitled **UNDERSTANDING THE RIEMANN HYPOTHESIS AND BASIC THEORY OF UNIVALENT FUNCTIONS** submitted by **Sueet Millon Sahoo** to the National Institute of Technology Rourkela, Orissa for the partial fulfilment of requirements for the degree of master of science in Mathematics is a bonafide record of review work carried out by her under my supervision and guidance. The contents of this project, in full or in parts, have not been submitted to any other institute or university for the award of any degree or diploma.

(Bappaditya Bhowmik)

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ABSTRACT

In this thesis, we study the following topics in complex analysis:-

(1) Riemann Mapping theorem.

(2) Riemann's zeta function.

(3) Basic univalent function theory.

We also study the famous unsolved problem, the *Riemann Hypothesis* during the course and establish a relation between *Riemann zeta* function and number theory through *Euler's theorem.* Lastly, we focus on some basic univalent function theory, which leads us to understand the *Bieberbach conjecture.*

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NOTATION

English Symbols

$\mathbb{C}$	the complex plane.
$\mathbb{D}$	the unit disk $\{z \in \mathbb{C} : \lvert z \rvert < 1\}$.
$\overline{B}(a, R)$	the closed ball center at a and radius R.
$\mathbb{H}(G)$	set of analytic functions in G.
$A \subset B$	A is a proper subset of B.
$\mathcal{S}$	class of normalized analytic univalent functions

CHAPTER 1

INTRODUCTION

Complex variable is a subject which has something for all mathematician. In addition, to having application to other parts of analysis, it can rightly claim to be an ancestor of many areas of mathematics. Actually, in this thesis, we plan to focuss on some topics in complex analysis and the theory of *univalent functions* . The theory of univalent functions is well-studied subject, branch around the turn of the century and yet it remains an active field of current research. One of the major problems of the field was *Bieberbach Conjecture*, dating from the year 1916. For many years, this famous problem has stood as a challenge and has inspired the development of ingenious methods which now form the backbone of the entire subject. This conjecture is now settled by Louis de Branges in the year 1984 (compare [**1**]). But there are still many open problems in the theory of univalent functions that continue to attract mathematicians of recent times. We require some preliminary knowledge on various topics in complex-function theory, so that we start understanding the theory of univalent functions.

In Chapter 1, we plan to study some standard results on classical Complex analysis. In Chapter 2, we discuss compactness and convergence in the family of analytic functions. This will help us to understand the proof of the celebrated *Riemann Mapping Theorem.* We also focus on understanding the popular open problem till date – the *Riemann Hypothesis.* In order to do so, we study *infinite product*, *Weierstrass factorization theorem*, the *Gamma function* and the *Riemann zeta function.* This is the content in Chapter 3. In Chapter 4, we study the basic univalent function theory leading to understand the *Bieberbach Conjecture.*

CHAPTER 2

REVIEW OF SOME TOPICS IN COMPLEX ANALYSIS

In this chapter, we wish to revise some important results from Complex function theory. We start with the *Open Mapping theorem* and *Maximum principle*. We also focus on *Schwarz's lemma* and an extension of this lemma, called *Schwarz-Pick lemma*. We also study the following basic results: *Argument principle* , *Rouche's theorem*, *Hurwitz's theorem*, *normal families*, *Montel's theorem*.

Theorem 2.1 (Open Mapping theorem)**.** *Let G be a region and suppose f is a non constant analytic function on G. Then for any open set U in G; $f(U)$ is open.*

Theorem 2.2 (The Maximum principle)**.** *Let $\Omega \subset \mathbb{C}$ and suppose α is in the interior of Ω. We can therefore, choose a positive number ξ such that $B(\alpha, \xi) \subset \Omega$, it readily follows that there is a point ξ in Ω with $|\xi| > |\alpha|$ i.e if α is a point in Ω with $|\xi| > |\alpha|$ for each ξ in the set Ω then α belongs to $\partial\Omega$.*

Theorem 2.3 (Maximum Modulus theorem)**.** *If f is analytic in a region G and a is a point in G with $|f(a)| \geq |f(z)|$ $\forall z$ in G then f must be a constant function.*

Theorem 2.4 (Schwarz's lemma)**.** *Let $\mathbb{D} = \{z : |z| < 1\}$ and suppose f is analytic on $\mathbb{D}$ with*

(a) $|f(z)| \leq 1$ for z in $\mathbb{D}$.

(b) $f(0) = 0$.

Then $|f^{'}(0)| \leq 1$ and $|f(z)| \leq |z|$ $\forall z \in \mathbb{D}$. Moreover if $|f^{'}(0)| = 1$ or $|f(z)| = |z|$ for some $z \neq 0$ then there is a constant c, $|c| < 1$ such that $f(w) = cw$ $\forall w$ in $\mathbb{D}$.

Proof. Define $g : \mathbb{D} \to \mathbb{C}$ by

$$g(z) = \frac{f(z)}{z} \Rightarrow f^{'}(0) = g(0)\ for\ z \neq 0,$$

then g is analytic in $\mathbb{D}$. According to Maximum Modulus theorem for $|z| \leq r$ and $0 < r < 1$, we have $|g(z)| = \dfrac{|f(z)|}{|z|} \leq r^{-1}$, ($\because$ $|f(z)| \leq 1\ \forall z \in \mathbb{D}$). As r approaches to 1, so we have $|f(z)| \leq |z|\ \forall z \in \mathbb{D}$ and $|f^{'}(0)| = |g(0)| \leq 1$. If $|f(z)| \leq |z|$ for some z in $\mathbb{D}$, $z = 0$ or $|f^{'}(0)| = 1$, then $|g|$ assumes its maximum value inside $\mathbb{D}$. Thus again applying maximum modulus theorem, $|g(z)| \equiv c$ for some constant c with $c = 1$, since $|g(z)| = \dfrac{|f(z)|}{|z|} = c$, so we have$f(z) = cz\ \forall z \in \mathbb{D}$. □

Proposition 2.5. *If $|a| < 1$ then Φ_a is a one-one map of $\mathbb{D} = \{z : |z| < 1\}$ onto itself, the inverse of Φ_a is Φ_{-a}. Furthermore Φ_a maps $\partial\mathbb{D}$ onto $\partial\mathbb{D}$, $\Phi_a(a) = 0$, $\Phi_a^{'}(0) = 1 - |a|^2$, and $\Phi_a^{'}(a) = (1 - |a|^2)^{-1}$.*

Proof. Given that $|a| < 1$. Define the Möbious transformation

$$\Phi_a = \frac{z - a}{1 - \bar{a}z}.$$

Φ_a is analytic in $|z| < |a|^{-1}$, since $1 - \bar{a}z \neq 0$. It is easy to see that,

$$\begin{aligned}
\Phi_a(\Phi_{-a}(z)) = \Phi_a\left(\frac{z + a}{1 + \bar{a}z}\right) &= \frac{\left(\dfrac{z + a}{1 + \bar{a}z}\right) - a}{1 - \bar{a}\left(\dfrac{z + a}{1 + \bar{a}z}\right)} \\
&= \frac{\dfrac{z + a - a - az\bar{a}}{1 + \bar{a}z}}{\dfrac{1 + \bar{a}z - \bar{a}z - \bar{a}a}{1 + \bar{a}z}} = \frac{z(1 - \bar{a}a)}{(1 - \bar{a}a)} = z,
\end{aligned}$$

$$\begin{aligned}
\Phi_{-a}(\Phi_a(z)) = \Phi_{-a}\left(\frac{z - a}{1 - \bar{a}z}\right) &= \frac{\left(\dfrac{z - a}{1 - \bar{a}z}\right) + a}{1 + \bar{a}\left(\dfrac{z - a}{1 - \bar{a}z}\right)} \\
&= \frac{\dfrac{z - a + a - az\bar{a}}{1 - \bar{a}z}}{\dfrac{1 - \bar{a}z + \bar{a}z - \bar{a}a}{1 - \bar{a}z}} = \frac{z(1 - \bar{a}a)}{(1 - \bar{a}a)} = z.
\end{aligned}$$

So, we have $\mathit{\Phi}_{-a}(\mathit{\Phi}_a(z))$=$\mathit{\Phi}_a(\mathit{\Phi}_{-a}(z))$. Hence $\phi_\alpha : \mathbb{D} \to \mathbb{D}$ is one-one and onto. Let θ be a real number; then

$$\Phi_a(e^{i\theta}) = \frac{e^{i\theta} - a}{1 - \bar{a}e^{i\theta}}.$$

This says that $\phi_a(\partial\mathbb{D})$ =$\partial\mathbb{D}$. So $\phi_a : \partial\mathbb{D} \to \partial\mathbb{D}$. It is easy to see that $\phi_a(a) = 0$. Now

$$\Phi_a(z) = \frac{z - a}{1 - \bar{a}z} \Rightarrow \Phi_a^{'}(z) = \frac{1 - \bar{a}a}{1 - z\bar{a}^2}$$

So we have, $\Phi_a^{'}(0) = 1 - |a|^2$ and

$$\Phi_a^{'}(a) = \frac{1 - a\bar{a}}{|1 - a\bar{a}|^2} = \frac{1}{1 - a\bar{a}} = \frac{1}{1 - |a|^2} = (1 - |a|^2)^{-1}.$$

This completes the proof. □

Theorem 2.6 (Schwarz-Pick lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Then for a in $\mathbb{D}$,*

$$|f^{'}(a)| \leq \frac{1 - (|f(a)|)^2}{1 - (|a|)^2}.$$

.

Proof. Suppose f is analytic on $\mathbb{D}$ with $|f(z)| \leq 1$. Suppose let a in $\mathbb{D}$ s.t $|a| < 1$ and $f(a) = \alpha$. let $g = \phi_\alpha \circ f \circ \phi_{-a}$. Then g maps $\mathbb{D}$ into $\mathbb{D}$. Here,

$$g(0) = \phi_\alpha(f(\phi_{-a}(0))) = \phi_\alpha(f(a)) = \phi_\alpha(\alpha) = \frac{\alpha - \alpha}{1 - \alpha\bar{\alpha}} = 0.$$

Now we can apply Schwarz's Lemma. Now $|g^{'}(0)| \leq 1$ and we have

$$\Phi_{-a}(z) = \frac{z + a}{1 + \bar{a}z} \Rightarrow \Phi_{-a}^{'}(z) == \frac{1 + z\bar{a} - \bar{a}z - \bar{a}a}{(1 + z\bar{a})^2}$$

So that $\Rightarrow \Phi'_{-a}(0) = 1 - \bar{a}^2$ and $\phi_{-a}(0) = a$. Now $g(z) = \phi_\alpha \circ f \circ \phi_{-a}(z)$. So on differentiating $g(z)$, we get, $g'(z)) = ((\phi_\alpha \circ f)' \circ \phi_{-a}(z))\phi'_{-a}(z)$. So that

$$\begin{aligned}
g'(0) &= ((\phi_\alpha \circ f)' \circ \phi_{-a}(0))\phi'_{-a}(0) = (\phi_\alpha \circ f)'(a)(1 - |a|^2) \\
&= (\phi'_\alpha \circ f)(a)(f'(a))(1 - |a|^2) = \phi'_\alpha(f(a))f'(a)(1 - |a|^2) \\
&= \phi'_\alpha(\alpha)f'(a)(1 - |a|^2) = \frac{1}{1 - |\alpha^2|}f'(a)(1 - |a|^2) \\
&= \frac{1 - |a|^2}{1 - |\alpha|^2}f'(a).
\end{aligned}$$

According to Schwarz's Lemma,

$$|g'(0)| = \frac{1 - |a^2|}{1 - |\alpha|^2}|f'(a)| = \frac{1 - |a|^2}{1 - |f(a)|^2}|f'(a)| \leq 1,$$

$$\Rightarrow |f'(a)| \leq \frac{1 - |f(a)|^2}{1 - |a|^2}. \tag{2.1}$$

Equality occur when $|g'(0)| = 1$ or by virtue of Schwarz's Lemma, when there is a constant c with $|c| = 1$ and

$$f(z) = \phi_{-\alpha}(c\phi_a(z)) \quad \text{for } |z| < 1. \tag{2.2}$$

If $|c| = 1$ and $|a| < 1$ then $f = c\phi_\alpha$. This defines an one-one analytic map of the open unit disk $\mathbb{D}$ onto itself. □

Theorem 2.7. *Let $f : \mathbb{D} \to \mathbb{D}$ be a one-one analytic map of $\mathbb{D}$ onto itself $f(0) = 0$. Then there is a complex number c with $|c| = 1$ such that $f = c\phi_a$.*

Proof. Since f is one-one and onto function then there is an analytic function $g : \mathbb{D} \to \mathbb{D}$ s.t $g(f(z)) = z$ for $|z| < 1$. Applying inequality (2.1) to both f and g, which gives

$$|f'(a)| \leq (1 - |a|^2)^{-1} \text{ and } |g'(0)| \leq 1 - |a|^2.$$

Since $1 = |g'(0)||f'(a)|$, so that we get,$|f'(a)| = (1 - |a|^2)^{-1}$. Applying formula (2.2) we have $f = c\phi_\alpha$ for some c, $|c| = 1$. □

The Argument Principle:

Suppose that f is an analytic function and has a zero of order m at $z = a$. So $f(a) = (z-a)^m(g(z))$ where $g(a) \neq 0$. Hence

$$\begin{aligned} f^{'}(z) &= m(z-a)^{m-1}(g(z)) + (z-a)^m(g^{'}(z)) \\ &= (z-a)^m(g(z))\left(\left(\frac{m}{z-a}\right) + \left(\frac{g^{'}(z)}{g(z)}\right)\right) \\ &= f(z)\left(\left(\frac{m}{z-a}\right) + \left(\frac{g^{'}(z)}{g(z)}\right)\right) \\ \Rightarrow \frac{f^{'}(z)}{f(z)} &= \left(\frac{m}{z-a}\right) + \left(\frac{g^{'}(z)}{g(z)}\right) \end{aligned} \tag{2.3}$$

and $\frac{g^{'}}{g}$ is analytic near $z = a$, since $g(a) \neq 0$. Now suppose that f has a pole of order m at $z = a$; i.e $f(z) = (z-a)^m(g(z))$, where g is analytic and $g(a) \neq 0$.
This gives

$$\frac{f^{'}(z)}{f(z)} = \left(\frac{-m}{z-a}\right) + \left(\frac{g^{'}(z)}{g(z)}\right) \tag{2.4}$$

and again $\frac{g^{'}}{g}$ is analytic near $z = a$.

Definition 2.8 (Meromorphic unction)**.** If G is open and f is a function defined and analytic in G except for poles, then f is a meromorphic function on G.

Theorem 2.9 (Argument Principle)**.** *Let f be meromorphic in G with poles $p_1, p_2, p_3, \cdots p_m$ and zeros $z_1, z_2, z_3, \cdots z_n$ obtained according to multiplicity. If γ is a closed rectifiable curve in G with $\gamma \approx 0$ and not passing through $p_1, p_2, p_3, \cdots p_m, z_1, z_2, z_3, \cdots z_n$ then,*

$$\frac{1}{2\pi i}\int_\gamma \frac{f^{'}(z)}{f(z)} = \sum_{k=1}^{n} n(\gamma; z_k) - \sum_{j=1}^{m} n(\gamma; p_j).$$

Proof. By repeated application of (2.3) and (2.4) we have,

$$\frac{f^{'}(z)}{f(z)} = \sum_{k=1}^{n} \frac{1}{z-a_k} - \sum_{j=1}^{m} \frac{1}{z-p_j} + \frac{g^{'}(z)}{g(z)}$$

where g is analytic and never vanishes in $\mathbb{G}$. According to Cauchy's theorem,

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} &= \int_\gamma \sum_{k=1}^{n} \frac{1}{z-a_k} - \int_\gamma \sum_{j=1}^{m} \frac{1}{z-p_j} + \int_\gamma \frac{g'(z)}{g(z)} \\ &= \sum_{k=1}^{n} n(\gamma; z_k) - \sum_{j=1}^{m} n(\gamma; p_j).\end{aligned}$$

□

Theorem 2.10 (Rouche's Theorem)**.** *Suppose f and g are meromorphic in a neighborhood of $\overline{B}(a;R)$ with no zeros or poles on the circle $\gamma = \{z : |z-a| = R\}$. If z_f, z_g (p_f,p_g) are the number of zeros(poles) of f, g inside γ counted according to their multiplicities and if $|f(z)+g(z)| < |f(z)| + |g(z)|$ on γ then, $Z_f - P_f = Z_g - P_g$.*

Proof. If $\lambda = \dfrac{f(z)}{g(z)}$ and if λ is a positive real number, then this inequality becomes $\lambda + 1 < \lambda + 1$. This is a contradiction, hence the meromorphic function $\dfrac{f}{g}$ maps γ onto $\Omega = \mathbb{C} - [0,\infty)$. If l is a branch of the logarithm on Ω, then $l\left(\dfrac{f(z)}{g(z)}\right)$ is well-defined primitive for $\left(\dfrac{f}{g'}\right)\left(\dfrac{f}{g^{-1}}\right)$ in a neighborhood of γ.Thus

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_\gamma (f/g)'(f/g)^{-1} \\ &= \frac{1}{2\pi i}\int_\gamma \left[\frac{f'}{f} - \frac{g'}{g}\right] \\ &= (Z_f - P_f) - (Z_g - P_g).\end{aligned}$$

So we have $Z_f - P_f = Z_g - P_g$. □

Theorem 2.11 (Hurwitz's Theorem)**.** *Let G be a region and suppose the sequence $\{f_n\}$ in $\mathbb{H}(G)$ converges to f. If f is not identical to zero, $\overline{B}(a;R)$ and $f(z) \neq 0$ for $|z-a| = R$, then there is an integer N such that for $n \geq N$, f and $\{f_n\}$ have the same number of zeros in $B(a;R)$.*

Proof. Let G be a region and $\{f_n\}$ in $\mathbb{H}(G)$ converges to f. Since $f(z) \neq 0$ $\forall |z-a| = R$, let $\delta = \inf\{|f(z)| : |z-a| = R\} > 0$. But $\{f_n\} \to f$ uniformly on $|z| : |z-a| = R$.

So there is an integer N such that if $n \geq N$ and $|z-a| = R$, then $|f(z) - f_n(z)| < \frac{1}{2}\delta < |f(z)| \leq |f(z)| + |f_n(z)|$. According to Rouches theorem f and $\{f_n\}$ have same number of zeros in $B(a;R)$. □

Definition 2.12 (Normal families)**.** A set $\mathbb{F} \subset C(G,\Omega)$ is normal if each sequence in $\mathbb{F}$ has a subsequence which converges to a function f in $C(G,\Omega)$.

Definition 2.13 (Locally bounded)**.** A set $\mathbb{F} \subset \mathbb{H}(G)$ is locally bounded if for each point a in G there are constants M and $r > 0$ such that for all f in $\mathbb{F}$, $|f(z)| \leq M$, for $|z-a| < r$ i.e $\sup\{f(z) : |z-a| < r, f \in \mathbb{F}\} < \infty$.

Definition 2.14 (Equicontinuous at a point)**.** A set $\mathbb{F} \subset C(G,\Omega)$ is equicontinuous at a point $z_0 \in G$ iff for every $\epsilon > 0$ such that for $|z - z_0| < \delta$, $d(f(z), f(z_0)) < \epsilon$ for every f in $\mathbb{F}$.

Definition 2.15 (Equicontinuous over a set)**.** $\mathbb{F}$ is equicontinuous over a set $E \in G$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that for z and z_0 in $\mathbb{F}$ and $|z - z_0| < \delta$, $d(f(z), f(z_0)) < \epsilon$ f $\forall f \in \mathbb{F}$.

Theorem 2.16 (Arzela-Ascoli theorem)**.** *A set $\mathbb{F} \subset C(G,\Omega)$ is normal iff the following two conditions are satisfied:*
(a) For each $z \in G$,$\{f(z) : f \in \mathbb{F}\}$ has compact closure in Ω.
(b) $\mathbb{F}$ is equicontinuous at each point of $\mathbb{G}$.

Theorem 2.17 (Montel's Theorem)**.** *A family $\mathbb{F}$ in $\mathbb{H}(G)$ is normal iff $\mathbb{F}$ is locally bounded.*

Proof. Suppose $\mathbb{F}$ is normal but fails to be locally bounded; then there is a compact set $K \in G$ such that $\sup\{|f(z)| : z \in k, f \in \mathbb{F}\} = \infty$, i.e there is a sequence $\{f_n\}$ in $\mathbb{F}$ such that $\sup\{|f(z)| : z \in k\} \geq n$. Since $\mathbb{F}$ is normal there is a function f in $\mathbb{H}(G)$ and a sequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$. But this gives that $\sup\{|f_{n_k}(z) - f(z)| : z \in k\} \to 0$ as $K \to \infty$. If $|f(z)| \leq M$ for z in K, $n_k \leq \sup\{|f_{n_k}(z) - f(z)| : z \in k\} + M$. Since the right hand side converges to M, so this is a contradiction. So $\mathbb{F}$ is locally bounded.

conversely, suppose $\mathbb{F}$ is locally bounded. Here we use Arzela-Ascoli theorem to show $\mathbb{F}$ is normal. It can be easily shown that the first condition is satisfied. Now only we have to prove the second condition of this theorem, i.e we have to prove $\mathbb{F}$ is equicontinuous at each point of G. Let fix a point $a \in G$ and $\epsilon > 0$, so according to hypothesis $\exists r > 0$ and $M > 0$ such that $\overline{B}(a;r) \subset G$ and $|f(z)| \leq M \; \forall z \in \overline{B}(a;r)$ and $\forall f \in \mathbb{F}$. Let $|z-a| < \frac{1}{2}r$ and $f \in \mathbb{F}$; then using Cauchy's formula with $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$, we get

$$\begin{aligned} |f(a)-f(z)| &\leq \frac{1}{2\pi}\left|\int_\gamma \frac{f(w)(a-z)}{(w-a)(w-z)}dw\right| \\ &\leq \frac{1}{2\pi}|a-z|\left|\int_\gamma \frac{f(w)}{(w-a)(w-z)}dw\right| \qquad (2.5) \end{aligned}$$

At $w = a$

$$\left|\lim_{w\to a} \frac{f(w)(w-a)}{(w-a)(w-z)}\right| = \frac{M}{(1/2)r} = \frac{2M}{r} \qquad (2.6)$$

At $w = z$

$$\left|\lim_{w\to z} \frac{f(w)(w-z)}{(w-a)(w-z)}\right| = \frac{M}{(1/2)r} = \frac{2M}{r} \qquad (2.7)$$

According to Cauchy's formula and from (2.6) and (2.7), we get from (2.5)

$$\int_\gamma \frac{f(w)}{(w-a)(w-z)}dw = 2\pi\left(\frac{2M}{r}+\frac{2M}{r}\right) = \frac{8M\pi}{r} \qquad (2.8)$$

Again from (2.5) and from (2.8) we get,

$$|f(a)-f(z)| = \frac{1}{2\pi}|a-z|\frac{8M\pi}{r} = |a-z|\frac{4M}{r}$$

Let $\delta = \min\{\frac{1}{2r}, \frac{r\epsilon}{4M}\}$. So $|a-z| < \delta$. So $|f(a)-f(z)| < \epsilon \; \forall f \in \mathbb{F}$. Hence the second condition satisfied. Hence it is proved. $\square$

CHAPTER 3

THE RIEMANN MAPPING THEOREM

In this chapter, we put a metric on the family of all holomorphic functions on a fixed domain G, and discuss compactness and convergence in this metric space. We also focus on spaces of meromorphic functions and give a proof of the celebrated *Riemann Mapping theorem.*

Proposition 3.1. *If G is open in $\mathbb{C}$. then there is a sequence $\{K_n\}$ of compact subsets of G such that $G = \bigcup_{i=1}^{n} K_i$. Moreover, the sets $\{K_n\}$ can be chosen to satisfy the following conditions:*

(a) $K_n \subset int(K_n)$.

(b) $K \subset G$ *and* K *compact implies* $K \subset k_n$ *for some* n.

(c) *Every component of* $\mathbb{C}_\infty - K_n$ *contains a component of* $\mathbb{C}_\infty - \mathbb{G}$.

Proof. For each positive integer n, let $K_n = \{z : |z| < n\} \bigcap \{z : d(z, \mathbb{C} - G) \geq \frac{1}{n}\}$. Since K_n is bounded and it is intersection of two closed subsets of $\mathbb{C}$. So K_n is compact.Now consider the set $M = \{z : |z| < n+1\} \bigcap \{z : d(z, \mathbb{C} - \mathbb{G}) \geq \frac{1}{n+1}\}$ is open. Hence $K_n \subset M$ and $M \subset K_{n+1}$. So $K_n \subset$ int (K_n). G is an open set, so $G = \bigcup_{n=1}^{\infty} K_n$. Then we can get $G = \bigcup_{n=1}^{\infty}$ int(K_n). If K is compact subset of G, then the set int(K_n) form an open cover of K. So $K \subset K_n$ for some n. Now we wish to prove that every component of $\mathbb{C}_\infty - K_n$ contains a component of $\mathbb{C}_\infty - G$. The unbounded component of $\mathbb{C}_\infty - K_n$ must contain ∞. So the component of $\mathbb{C}_\infty - G$ which contains ∞. Also the unbounded component contains $\{z : |z| > n\}$. So if $\mathbb{D}$ is a bounded component ,it contains a point z with $d(z, \mathbb{C} - G) < \frac{1}{n}$. According to definition this gives a point w in $\mathbb{C} - G$ with

$|z-w| < \dfrac{1}{n}$. But then $z \in B\left(w, \dfrac{1}{n}\right) \subset \mathbb{C}_\infty - K_n$; since disks are connected and z is in the component $\mathbb{D}$ of $\mathbb{C}_\infty - k_n$, $B\left(w, \dfrac{1}{n}\right) \subset \mathbb{D}$. If $\mathbb{D}_1$ is the component of $\mathbb{C}_\infty - \mathbb{D}$ that contains w it follows that $\mathbb{D}_1 \subset \mathbb{D}$. □

Proposition 3.2. $\mathbb{C}(G,\Omega)$ *is a metric space.*

Proof. According to above theorem we have $G = \cup_{n=1}^{\infty} k_n$ where k_n is compact and $k_n \subset int(k_{n+1})$. Define $\rho_n(f,g) = \sup\{d(f(z),g(z)) : z \in k_n\}$ for all functions f, $g \in C(G,\Omega)$.

$$\rho(f,g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{\rho_n(f,g)}{1+\rho_n(f,g)}\right) \tag{3.1}$$

Now, first we have to show that the series in (3.1)is convergent, let $t = \rho_n(f,g)$, then $\frac{t}{1+t} \leq 1$. So the series in (3.1) dominated by the series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$, which is a convergent series . Now we are going to prove that ρ is a metric on $C(G,\Omega)$. it can be easily shown that $\rho(f,g) > 0$, $\rho(f,g) = 0 \Leftrightarrow f = g$, $\rho(f,g) = \rho(g,f)$. Now only we have to establish the triangle inequality condition, i.e to show that $\rho(f,g) \leq \rho(f,h) + \rho(h,g)$. Since $\rho_n(f,g)$ is a metric space, so we have

$$\begin{aligned}
& \rho_n(f,g) \leq \rho_n(f,h) + \rho_n(h,g) \\
\Rightarrow\ & \frac{\rho_n(f,g)}{1+\rho_n(f,g)} \leq \frac{\rho_n(f,h)+\rho_n(h,g)}{1+\rho_n(f,h)+\rho_n(h,g)} \\
\Rightarrow\ & \frac{\rho_n(f,g)}{1+\rho_n(f,g)} \leq \left(\frac{\rho_n(f,h)}{1+\rho_n(f,h)}\right) + \left(\frac{\rho_n(h,g)}{1+\rho_n(h,g)}\right) \\
\Rightarrow\ & \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{\rho_n(f,g)}{1+\rho_n(f,g)}\right) \leq \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{\rho_n(f,h)}{1+\rho_n(f,h)}\right) + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{\rho_n(h,g)}{1+\rho_n(h,g)}\right) \\
\Rightarrow\ & \rho(f,g) \leq \rho(f,h) + \rho(h,g)
\end{aligned}$$

So $C(G,\Omega)$ is a metric space. □

Lemma 3.3. *Let the metric* ρ *be defined as* (3.1). *If* $\epsilon > 0$ *is given then there is a* $\delta > 0$ *and a compact set* $K \subset G$ *such that for* f *and* g *in* $C(G,\Omega)$; $\sup\{d(f(z),g(z)) : z \in K\} < \delta \Rightarrow \rho(f,g) < \epsilon$. *Conversely, if* $\delta > 0$ *and a compact set* K *are given, there is an* $\epsilon > 0$ *such that for* f *and* g *in* $C(G,\Omega)$, $\rho(f,g) < \epsilon \Rightarrow \sup\{d(f(z),g(z)) : z \in K\} < \delta$.

Proof. Now we wish to prove $\sup\{d(f(z),g(z)) : z \in K\} < \delta \Rightarrow \rho(f,g) < \epsilon$. Let $\epsilon > 0$ is fixed and p be a positive number such that $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n < \frac{1}{2}(\epsilon)$. Put $K = K_n$. Choose $\delta > 0$ such that $0 \le t \le \delta$ gives $\frac{t}{1+t} < \frac{1}{2}\epsilon$. Let $f,g \in C(G,\Omega)$ such that $\sup\{d(f(z),g(z)) : z \in K\} < \delta$. Since $K_n \subset K_p$ for $1 \le n \le p$, $0 < \rho_n(f,g) < \delta$.

So

$$\frac{\rho_n(f,g)}{1+\rho_n(f,g)} < \left(\frac{1}{2}\right)\epsilon.$$

Here,

$$\begin{aligned}
\rho(f,g) &= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n\left(\frac{\rho_n(f,g)}{1+\rho_n(f,g)}\right)\\
&= \left(\sum_{n=1}^{p}\left(\frac{1}{2}\right)^n\left(\frac{\rho_n(f,g)}{1+\rho_n(f,g)}\right)\right) + \left(\sum_{n=p+1}^{\infty}\left(\frac{1}{2}\right)^n\left(\frac{\rho_n(f,g)}{1+\rho_n(f,g)}\right)\right)\\
&= \sum_{n=1}^{p}\left(\frac{1}{2}\right)^n\left(\frac{\epsilon}{2}\right) + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Now, we wish to prove that $\rho(f,g) < \epsilon \Rightarrow \sup\{d(f(z),g(z)) : z \in K\} < \delta$. Let K and δ are given, Since $G = \bigcup_{n=1}^{\infty} k_n = \bigcup_{n=1}^{\infty} intK_n$ and K is compact there is an integer $p \ge 1$ such that $K \subset K_p$; this gives $\rho_p(f,g) \ge \sup\{d(f(z),g(z)) : z \in K\}$. Choose $\epsilon > 0$ such that $0 \le s \le 2^p\epsilon$.

$$\Rightarrow \frac{s}{1-s} < \frac{2^p\epsilon}{1-2^p\epsilon} = \delta \Rightarrow \frac{s}{1-s} < \delta$$

$$\Rightarrow 0 \le t \le \delta \Rightarrow \frac{t}{1+t} < \frac{s}{1+s} = 2^p\epsilon$$

So if $\rho_p(f,g) < \epsilon \Rightarrow \dfrac{\rho_p(f,g)}{1+\rho_p(f,g)} < \epsilon$
$\Rightarrow \rho_p(f,g) < \delta \Rightarrow \sup\{d(f(z),g(z)) : z \in K\} < \delta$ □

Proposition 3.4. *(a) A set $\mathbb{O} \subset (C(G,\Omega),\rho)$ is open iff for each f in $\mathbb{O}$ there is a compact set K and a $\delta > 0$ such that $\mathbb{O} \supset \{g : d(f(z),g(z)) < \delta, z \in K\}$.*

(b) A sequence $\{f_n\}$ in $(C(G,\Omega),\rho)$ converges to f iff $\{f_n\}$ converges to f uniformly on all compact subsets of G.

Proof. (a)Let $\mathbb{O}$ is open and $f \in \mathbb{O}$, then for some $\epsilon > 0$, $\{g : \rho(f, g) < \epsilon\} \subset \mathbb{O}$. According to lemma (3.3) there exist $\delta > 0$ and a compact set K such that

$$\{g : d(f(z), g(z)) < \delta; z \in K\} \subset \mathbb{O}.$$

Conversely, if for each $f \in \mathbb{O}$ there is a compact set K and $\delta > 0$ such that

$$\{g : d(f(z), g(z)) < \delta; z \in K\} \subset \mathbb{O},$$

then from the second part of the previous lemma (3.3); we get $\mathbb{O}$ is open.
(b) Given that f_n in $(C(G, \Omega))$ converges to f. Now we have to prove that f_n converges to f uniformly on all compact subsets of G, i.e to prove f_n converges to f $\forall f$ and $\forall\ z \in G$. Let for given $\epsilon > 0$,

$$\begin{aligned} & \rho(f_n, f) < \epsilon \\ \Rightarrow\ & \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{\rho_n(f_n, f)}{1 + \rho_n(f_n, f)}\right) < \epsilon \\ \Rightarrow\ & \rho_n(f_n, f) < \epsilon \\ \Rightarrow\ & \sup\{d(f_n(z), f(z)), z \in K_n\} < \epsilon. \end{aligned}$$

So f_n converges to f $\forall$ f *and* $\forall$ $z \in G$ i.e f_n converges to f uniformly on all compact subsets of G. Conversely, given that f_n converges to f uniformly, $d(f_n, f) < \epsilon \forall f$, then $\sup\{d(f_n, f) : z \in k_n\}$, $\rho_n(f_n, f) < \epsilon \Rightarrow \rho(f_n, f) < \epsilon$. So f_n converges to f. □

Proposition 3.5. *$C(G, \Omega)$ is a complete metric space.*

Proof. Let f_n be a cauchy sequence in $C(G, \Omega)$. If we restrict our domain of the sequence of functions gives a Cauchy sequence f_n to $C(K, \Omega)$ for every compact sets K in G.i.e, for every $\delta > 0$ there is an integer N such that $\sup\{d(f_n(z), f_m(z)) : z \in K\} < \delta$ for $n, m \geq N$. In particular $\{f_n\}$ is a Cauchy sequence in Ω; so there is a point $f(z)$ in Ω such that $f(z) = \lim f_n(z)$. This gives a function $f : G \to \Omega$; it must be shown that f is continuous and $\rho(f_n, f) \to 0$. Let K be compact and fixed $\delta > 0$; choose N so that $\sup\{d(f_n(z), f_m(z)) : z \in K\} < \delta$ should satisfy for $n, m > N$. If z is arbitrary in K but fixed then there is an integer $m > N$ so that $d(f(z), f_m(z)) < \delta$. But then $d(f(z), f_n(z)) < 2\delta$ for all $n \geq N$. Since N does not depend on z, this gives $\sup\{d(f(z), f_n(z)) : z \in K\} \to 0$ as $n \to \infty$. Hence, f_n converges to f uniformly on every compact set in G. In particular converges on all closed balls contained in G. Since uniform limit of a sequence of continuous function is continuous, we see that f is continuous at each point of G. Also $\rho(f_n, f) \to 0$ according to proposition (3.4(b)). □

3.1. Spaces of meromorphic functions

Let G be a region and f is a meromorphic function on G, Let $M(G)$ is the set of all continuous functions on G then consider $M(G)$ as a subset of $C(G, \mathbb{C}_\infty)$. In this section we are going to put a metric d on $\mathbb{C}_\infty$ as follows. Let $z_1, z_2 \in \mathbb{C}$, then

$$d(z_1, z_2) = \frac{2|z_1 - z_2|}{[(1 + |z_1|^2)(1 + |z_2|^2)]^{\frac{1}{2}}};$$

and for each z in $\mathbb{C}$ we define, $d(z, \infty) = \dfrac{2}{(1 + |z|^2)^{\frac{1}{2}}}$.

Corollary 3.6. $M(G) \cup \{\infty\}$ *is a complete metric space.*

Corollary 3.7. $\mathbb{H}(G) \cup \{\infty\}$ *is closed in* $C(G, \mathbb{C}_\infty)$.

3.2. The Riemann Mapping Theorem

The theorem was stated by *Bernhand Riemann* in 1851 in his Ph.D. thesis. According to Riemann Mapping theorem, any two proper simply connected domains in the plane are homeomorphic. Even though class of continuous functions are vastly larger than the class of conformal maps, it is not easy to construct a one-to-one function onto the disc, knowing only that the domains are simply connected.

Theorem 3.8. *Let G be a simply connected region which is not the whole plane and $a \in G$. Then there is a unique analytic function $f : G \to \mathbb{C}$ having the properties:*

$$(3.2) \qquad \begin{aligned} &(a) \quad f(a) = 0 \ \textit{and} \ f^{'}(a) > 0 \\ &(b) \quad f \ \textit{is one} - \textit{one}. \\ &(c) \quad f(G) = \{z : |z| < 1\} \end{aligned}$$

Proof. We have to show that (1)Uniqueness of f having the properties in (3.2).
(2) Existence of f. (1) Uniqueness: Let g be another analytic function which is defined by $g : G \to \mathbb{C}$, which satisfies all the conditions of (3.2), then $g(a) = 0$ and $g^{'}(a) > 0$ implies that $a = g^{-1}(0)$ $(f \circ g^{-1})(0) = f(a) = 0$. According to theorem (2.7), $(f \circ g^{-1})$ is a one-one map then there is constant c with $|c| = 1$ and $(f \circ g^{-1})(z) = cz \forall z$ then $f(z) = cg(z)$ since $f^{'}(a) > 0$, so $cg^{'}(a) > 0$ and also we have $g^{'}(a) > 0$. So c must be 1. So finally we have $f = g$, i.e, f is unique. (2) Existence: Consider the family of functions $\mathbb{F}$ of all analytic functions f having properties (a) and (b) from (3.2) and satisfying $|f(z)| < 1$ for z in G. Now only we have to choose a member of $\mathbb{F}$ having property (c) of the equation (3.2). Suppose $\{K_n\}$ is a sequence of compact subsets of G such that $G = \bigcup_{n=1}^{\infty} K_n$ and $a \in K_n$ for each n, then $\{f(K_n)\}$ is a sequence of compact subsets of $\mathbb{D}$ where $\mathbb{D} = \{z : |z| < 1\}$.As n becomes larger, $\{f(K_n)\}$ becomes larger and larger and tries to fill out the disc ID. i.e, $\mathbb{D} = \bigcup_{n=1}^{\infty} f(K_n)$. In a simply connected region every non vanishing analytic function has an analytic square root. So to prove existence of f, we have to prove the following lemma.

Lemma 3.9. *Let G be a region which is not the whole plane and such that every non vanishing analytic function on G has an analytic square root. If $a \in G$, then there is an* *analytic function f on G such that*

$$(a) \quad f(a) = 0 \text{ and } f^{'}(a) > 0$$

$$(b) \quad f \text{ is } one - one.$$

$$(c) \quad f(G) = \mathbb{D} = \{z : |z| < 1\}$$

Proof. Define $\mathbb{F} = \{f \in \mathbb{H}(G) : f \text{ is } one - one, a \in Gf(a) = 0, f^{'}(a) > 0, f(G) \subset \mathbb{D}\}$. Since $f(G) \subset \mathbb{D}$, $\sup\{|f(z)| : z \in G\} \leq 1$ for $f \in \mathbb{F}$. According to Montel's theorem $\mathbb{F}$ is normal if it is non-empty, i.e, we have to prove (i) $\mathbb{F} \neq \phi$ and (ii) $\mathbb{F}^{-} = \mathbb{F} \cup \{0\}$. First assume that equation (i) and (ii) hold. Consider the function $f \to f^{'}(a)$ of $\mathbb{H}(G) \to \mathbb{C}$. This is a continuous function. Since $\mathbb{F}^{-}$ is compact, then there exist $f \in \mathbb{F}^{-}$ with $f^{'}(a) \geq g^{'}(a) \forall g \in \mathbb{F}$. Because $\mathbb{F} \neq \phi$ then $f \in \mathbb{F}$. Now only we have to show that $f(G) = \mathbb{D}$ i.e, we have to show that (a)$f(G) \subset \mathbb{D}$ and (b)$\mathbb{D} \subset f(G)$. To prove equation $f(G) = \mathbb{D}$; let$w$ does not belongs to $f(G)$. Then the function $\dfrac{f(z) - w}{1 - \bar{w}f(z)}$ is analytic in G and never vanishes. By hypothesis there is an analytic function $h : G \to \mathbb{C}$ such that $[h(z)]^2 = \dfrac{f(z) - w}{1 - \bar{w}f(z)}$. Since the Möbius transformation $T(\xi) = \dfrac{\xi - w}{1 - \bar{w}\xi}$ maps $\mathbb{D}$ onto $\mathbb{D}, h(G) \subset \mathbb{D}$. Define $g : G \to \mathbb{C}$ by

$$g(z) = \left(\frac{|h^{'}(a)|}{h^{'}(a)}\right)\left(\frac{h(z) - h(a)}{1 - \overline{h(a)}h(z)}\right).$$

Here, it is easy to see that $g(a) = 0$, and

$$g^{'}(z) = \left(\frac{|h^{'}(a)|}{h^{'}(a)}\right)\left(\frac{h^{'}(z)(1 - h(a)\overline{h(a)})}{[1 - \overline{h(a)}h(z)]^2}\right).$$

Hence

$$g^{'}(a) = \left(\frac{|h^{'}(a)|}{h^{'}(a)}\right)\left(\frac{h^{'}(a)(1 - h(a)\overline{h(a)})}{[1 - \overline{h(a)}h(a)]^2}\right) = \frac{|h^{'}(a)|}{1 - |h(a)|^2} > 0.$$

Now, $|h(a)|^2 = |-w| = |w|$ and

$$h(z)^2 = \frac{f(z - w)}{1 - \bar{w}f(z)}.$$

On differentiation of the function h yields,

$$\begin{aligned} 2h(z)h^{'}(z) &= \frac{(1-\bar{w}f(z))f^{'}(z)-(f(z)-w)(-\bar{w}f^{'}(z))}{(1-\bar{w}f(z))^2} \\ &= \frac{f^{'}(z)-\bar{w}f(z)f^{'}(z)+f(z)f^{'}(z)\bar{w}-w\bar{w}f^{'}(z)}{(1-\bar{w}f(z))^2} \\ &= \frac{f^{'}(z)(1-|w|^2)}{(1-\bar{w}f(z))^2}. \end{aligned}$$

Hence,

$$2h(a)h^{'}(a) = \frac{f^{'}(a)(1-|w|^2)}{(1-\bar{w}f(a))^2} = f^{'}(a)(1-|w|^2)\ (\because\ f(a)=0).$$

and as a result,

$$h^{'}(a) = \frac{f^{'}(a)(1-|w|^2)}{2h(a)}.$$

Now

$$\begin{aligned} g^{'}(a) &= \frac{|h^{'}(a)|}{1-|h(a)|^2} > 0 \\ \Rightarrow g^{'}(a) &= \frac{f^{'}(a)(1-|w|^2)}{2h(a)(1-|w|)} = \frac{f^{'}(a)(1-|w|^2)}{2\sqrt{|w|}(1-|w|)} = \frac{f^{'}(a)(1+|w|)}{2\sqrt{|w|}} > f^{'}(a). \end{aligned}$$

This gives that g is in $\mathbb{F}$ and contradicts the choice of f. So $w \in f(G)$. So, $\mathbb{D} \subset f(G)$. Again $f(G) \subset \mathbb{D}$. So we get $f(G) = \mathbb{D}$. Now we are going to prove the equations (i) and (ii). Since $G \neq \mathbb{C}$, let $b \in \mathbb{C} - G$ and let g be an analytic function on G such that $[g(z)]^2 = z - b$. If z_1 and z_2 are points in G, and $g(z_1) = \pm g(z_2)$, then it follows that $z_1 = z_2$. In particular g is one to one . According to Open mapping theorem there is a $r > 0$ such that $B(a, R) \subset g(G)$. So there is a point z in G such that $g(z) \in B(-g(a); r)$ then $r > |g(z) + g(a)| = |-g(z) - g(a)|$. Since $B(a, R) \subset g(G)$, so there is a w in G with $g(w) = -g(z)$; but $B(a, R) \subset g(G)$ shows that $w = z$ which gives $g(z) = 0$. But then $z - b = [g(b)]^2 = 0$ implies that b is in G, a contradiction. Hence $g(G) \cap \{\xi : |\xi + g(a)| < r\} = \phi$. Let U be the disk $\{\xi : |\xi + g(a)| < r\} = B(-g(a); r)$. There is a Möbius transformation T such that $T(\mathbb{C}_\infty - U^-) = \mathbb{D}$. Let $g_1 = T \circ g$; then $g_1(G) \subset \mathbb{D}$. If $\alpha = g(a)$, then let $g_2(z) = \phi_\alpha \circ g_1(z)$; so we will have that $g_2(G) \subset \mathbb{D}$ and g_2 is analytic, but we also have that $g_2(a) = 0$. Now there is a complex number c with $|c| = 1$, such that $g_3(z) = cg_2(z)$ has positive derivative at $z = a$ and is therefore in $\mathbb{F}$.

Here f is $g_3 = c(\phi_\alpha \circ T \circ g)$. So from this we conclude that the set $\mathbb{F}$ is nonempty. Suppose $\{f_n\}$ is a sequence in $\mathbb{F}$ and $f_n \to f$ in $H(G)$. Clearly $f(a) = 0$ and since $f_n'(a) \to f^{'}(a)$, then it follows that $f^{'}(a) \geq 0$. Let z_1 be an arbitrary element of G and put $\xi = f(z_1)$; let $\xi_n = f_n(z_1)$. Again let $z_2 \in G$, $z_1 \neq z_2$ and let K be a closed disk centered at z_2 such that $z_1 \notin K$. Then $f_n(z) \to \xi_n$ never vanishes on disk K. Since f_n is one- one. But $f_n(z) - \xi n \to f(z) - \xi$ uniformly on K. According to Hurwitz's theorem gives that $f(z) - \xi$ never vanishes on K or $f(z) \equiv \xi$. If $f(z) \equiv \xi$ on K, then f is the constant function ξ throughout G; since $f(a) = o$ we have that $f(z) \equiv 0$. Otherwise we get that $f(z_1) \neq f(z_2)$ for $z_1 \neq z_2$; that is f is one-one. But if f is one-one then $f^{'}$ can never vanish. So $f^{'}(a) > 0$ and f is in $\mathbb{F}$. This proves the equation (ii) completely, which proves the existence of f in $\mathbb{F}$. □

Now from this above lemma we conclude that f exists with properties in equation (3.2). This completes the proof of the Riemann mapping theorem. □

CHAPTER 4

RIEMANN HYPOTHESIS

In this Chapter, our main aim is to introduce the most well-known open problem the *Reimann Hypothesis*. For this purpose we study the *Reimann zeta function*. In order to do so we concentrate in the following topics: *infinite product, Weierstrass factorization theorem, Factorization of sine function*. We also focus on *Gamma function, Bohr-mollerup theorem* . At last we give a link between *Zeta function* and number theory by introducing *Euler's theorem*.

Let G be a open connected set. Let $\{a_k\}$ be a sequence in G which has no limit point in G. Consider an integer sequence $\{m_k\}$. Actually the question is, "is there an analytic function f on G and such that the only zeros of f are at the points a_k, with the multiplicity of the zero at a_k equal to m_k. To answer this question we start with the following definition.

Definition 4.1 (Infinite product)**.** If $\{z_n\}$ is a sequence of complex number and if $z = \lim \prod_{k=1}^{n} z_k$ exists, then z is the number in $\{z_n\}$ and it is denoted by $z = \prod_{n=1}^{\infty} z_n$.

Proposition 4.2. *Let* $\mathrm{Re\,z_n} > 0 \forall \mathrm{n} \geq 1$*. Then* $\prod_{n=1}^{\infty} z_n$ *converges to a non-zero number iff the series* $\sum_{n=1}^{\infty} \log z_n$ *converges.*

Proof. Let $p_n = (z_1 z_2 z_3 \cdots z_n)$, $z = re^{i\theta}$, $-\pi < \theta < \pi$ and $l(p_n) = \log|p_n| + i\theta_n$ where $\theta - \pi < \theta_n < \theta + \pi$. If $S_n = \log z_1 + \log z_2 + \log z_3 + \cdots + \log z_n$, then $\exp(S_n) = \log(z_1 z_2 z_3 \cdots z_n) = z_1 z_2 z_3 \cdots z_n = \prod_{k=1}^{n} z_k = p_n$. So that $s_n = l(p_n) + 2\pi i k_n$ for some integer k_n. Suppose $p_n \to z$, then it is easy to see that $S_n - S_{n-1} = \log z_n \to 0$ and also $l(p_n) - l(p_{n-1}) \to 0$, hence $k_n - k_{n-1} \to 0$ as $n \to \infty$. Since each k_n is an integer this gives that there is an n_0 and a k such that $k_m = k_n = k$ for $m, n \geq n_0$. So $s_n \to l(z) + 2\pi i k$; i.e the series $\sum_{n=1}^{\infty} z_n$ converges. Conversely, suppose $\sum_{n=1}^{\infty} z_n$ converges. if $s_n = \sum_{k=1}^{n} z_k$ and $s_n \to s$ then $\exp(s_n) \to \exp s$. But $\exp s_n = \prod_{k=1}^{n} z_k$ so that $\prod_{n=1}^{\infty} z_n$ is convergent to $z = e^s \neq 0$. $\square$

Consider the power series expansion of

$$\log(1+z) = \sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots ,$$

which has radius of convergence one. If $|z| < 1$, then

$$\begin{aligned}
\left|1 - \frac{\log(1+z)}{z}\right| &= \left|1 - \frac{z - \frac{z^2}{2} + \frac{z^3}{3} - ...}{z}\right| \\
&= \left|1 - \left(1 - \frac{z}{2} + \cdots\right)\right| = \left|\frac{z}{2} - \frac{z^2}{3} + \cdots\right| \\
&\leq \left|\frac{z}{2} - \frac{z^2}{2} + \cdots\right| = \frac{1}{2}|z - z^2 + z^3 - \cdots| \\
&\leq \frac{1}{2}(|z| + |z^2| + |z^3| + \cdots) \ (\because |z| < 1) \\
&= \frac{1}{2}\left(\frac{|z|}{1-|z|}\right).
\end{aligned}$$

If we further require $|z| < \frac{1}{2}$ then,

$$\left|1 - \frac{\log(1+z)}{z}\right| \leq \frac{1}{2}.$$

So for $|z| < \frac{1}{2}$,

$$\frac{1}{2}|z| \leq \log(1+z) \leq \frac{3}{2}|z|.$$

As a result,

$$1 - \log(1+z) \Rightarrow 1 + \frac{|\log(1+z)|}{|z|} \leq \frac{1}{2} \Rightarrow \frac{\log|1+z|}{|z|} \leq \frac{1}{2}.$$

Proposition 4.3. *Let* $\operatorname{Re} z_n > -1$*; then the series* $\sum \log(1+z_n)$ *converges absolutely iff the series* $\sum z_n$ *converges absolutely.*

Proof. Given that $\operatorname{Re} z_n > -1$. If $\sum |z_n|$ converges, then $z_n \to 0$; so let $|z_n| < \frac{1}{2}$. We know that,

$$\frac{1}{2}|z| \leq |\log(1+z)| \leq \frac{3}{2}|z| \tag{4.1}$$

So $\sum \log(1+z_n)$ converges. Conversely, $\sum |\log(1+z_n)|$ converges, then $|z_n| < \frac{1}{2}$ for sufficiently large n. From equation (4.1) we get; that $\sum |z_n|$ converges. $\square$

Definition 4.4 (Absolutely Converges of an infinite Product)**.** If $\operatorname{Re} \mathrm{z_n} > 0$ for all n then the infinite product $\prod z_n$ is said to converge absolutely if the series $\sum \log z_n$ converges absolutely.

Corollary 4.5. *If* $\operatorname{Re} z_n$*, then the product* $\prod z_n$ *converges absolutely iff the series* $\sum(z_n - 1)$ *converges absolutely.*

Proof.

$$\begin{aligned} & \sum (z_n - 1) \text{ converges absolutely} \\ \Leftrightarrow\ & \sum \log(1 + z_n - 1) = \sum \log(z_n) \text{ converges absolutely (According to Proposition(4.3))} \\ \Leftrightarrow\ & \prod_{n=1}^{\infty} z_n \ \text{ converges to a nonzero. (According to Proposition(4.2))} \end{aligned}$$

$\square$

Lemma 4.6. *Let X be a set and let $f_1, f_2, f_3, \cdots$ be a functions from X into $\mathbb{C}$ such that $f_n(x) \to f(x)$ uniformly for X. If there is a constant a such that* $\operatorname{Re} \mathrm{f(x)} \leq \mathrm{a}$ *for all $x \in X$, then* $\exp f_n(x) \to \exp(f(x))$ *uniformly for x in X.*

Proof. We wish to prove that $|\exp f_n(x) - \exp f(x)| \leq \epsilon$. Given that $f_n(x) \to f(x)$ uniformly for X. If $\epsilon > 0$ is given then choose $\delta > 0$ such that $|e^z - 1| < \epsilon e^{-a}$ whenever $|z| < \delta$. Now choose n_0 such that $|f_n(x) - f(x)| < \delta$ for all x is X whenever $n \geq n_0$. Thus,

$$\epsilon e^{-a} > |\exp[f_n(x) - f(x)] - 1| = \left| \frac{\exp f_n(x)}{\exp f(x)} - 1 \right| \Rightarrow |\exp f_n(x) - \exp f(x)| \leq \epsilon e^{-a} |\exp f(x)| \leq \epsilon.$$

□

Lemma 4.7. *Let (X, d) be a compact metric space and let g_n be a sequence of continuous functions from X into $\mathbb{C}$ such that $\sum g_n(x)$ converges absolutely and uniformly for x in X. Then the product $f(x) = \prod_{n=1}^{\infty}(1 + g_n(x))$ converges absolutely and uniformly for x in X. Also there is an integer n_0 such that $f(x) = 0$ iff $g_n(x) = -1$ for some n, $1 \leq n \leq n_0$.*

Proof. Let (X, d) be a compact metric space and let $\{g_n\}$ be a sequence of continuous functions from X into $\mathbb{C}$. Given that $\{g_n(x)\}$ converges uniformly for x in X there is an integer n_0 such that $|g_n(x)| < \frac{1}{2} \forall x \in X$ and $n \leq n_0$. Here $\mathrm{Re}\,[1 + \mathrm{g_n(x)}] > 0$. So $|\log(1 + g_n(x))| \leq \frac{3}{2}|g_n(x)| \forall n > n_0$ and x in X. Thus $h(x) = \sum_{n=n_0+1}^{\infty} \log(1 + g_n(x))$ converges uniformly for x in X. Since h is a continuous function and X is compact, then it follows that h must be bounded. In particular, there is a constant a such that $\mathrm{Re\,h(x) < a}$ for all x in X. So $\exp h(x) = \prod_{n=n_0+1}^{\infty}(1 + g_n(x))$. According to lemma (4.6) we have $\exp h(x)$ converges. So, $f(x) = [(1 + g_1(x)).....(1 + g_{n_0}(x))] \exp h(x)$. We know that $\exp h(x)$ is never be zero. So if $f(x) = 0$ for any x in X. So it must be $g_n(x) = -1$ for some n, $1 \leq n \leq n_0$. □

Theorem 4.8. *Let G be a region in $\mathbb{C}$ and let $\{f_n\}$ be a sequence in $\mathbb{H}(G)$ such that no f_n is identically zero. If $\sum[f_n(z) - 1]$ converges absolutely and uniformly on compact subsets of $\mathbb{G}$, then $\prod_{n=1}^{\infty} f_n(z)$ converges in $\mathbb{H}(G)$ to an analytic function $f(z)$. If a is a zero of only at a finite number of the functions f_n, and the multiplicity of the zero of f at a is the sum of the multiplicities of the zeros of the functions f_n at a.*

Proof. Given that $\sum[f_n(z) - 1]$ converges uniformly and absolutely on compact subsets of G. So according to lemma (4.7) $f(z) = \prod f_n(z)$ converges uniformly and absolutely on compact subsets of G. That is, the infinite product converges in $\mathbb{H}(G)$. Again suppose $f(a) = 0$ and let $r > 0$ be chosen such that $\overline{B}(a; r) \subset \mathbb{G}$. By hypothesis, $\sum[f_n(z) - 1]$ converges uniformly on $\overline{B}(a; r)$. According to lemma (4.7), $f(z) = f_1(z) \cdots f_2(z) g(z)$

where g does not vanish in $\overline{B}(a;r)$. So the multiplicities of the zero of f at a is the sum of the multiplicities of the zeros of the functions f_n at a. $\square$

Definition 4.9. An elementary function is one of the following functions $E_p(z)$ for $p = 0, 1, 2, \cdots$.

$$\begin{aligned} E_0(z) &= 1 - z \\ E_p(z) &= (1-z)\exp(z + \frac{z^2}{2} + \frac{z^3}{3} + ... + \frac{z^p}{p}), p \geq 1 \end{aligned} \tag{4.2}$$

The function $E_p\left(\dfrac{z}{a}\right)$ has a simple zero at $z = a$ and no other zero. Also if b is a point in $\mathbb{C} - G$, then $E_p\left(\dfrac{a-b}{z-b}\right)$ has a simple zero at $z = a$ and is analytic in G.

Lemma 4.10. *If* $|z| \leq 1$ *and* $p \geq 0$ *then* $|1 - E_p(z)| \leq |z|^{p+1}$.

Proof. Let $p \geq 1$, $E_p(z) = 1 + \sum\limits_{k=1}^{\infty} a_k z^k$ be the power series expansion about $z = 0$. By differentiating both the side of the equation (4.2) we get,

$$\begin{aligned} E_p'(z) &= -\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) + (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right)(1 + z + \cdots + z^{p-1}) \\ &= \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right)((1-z)(1 + z + \cdots + z^{p-1}) - 1) \\ &= \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right)\left(\frac{1-z^p}{1-z}(1-z) - 1\right) \\ &= \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right)(1 - z^p - 1) \\ &= -z^p \exp\left(z + \frac{z^2}{2} + ...\frac{z^p}{p}\right). \end{aligned}$$

Now $a_1 = a_2 = \cdots = a_p = 0$, the co-efficient of the expansion of $\exp\left(z + \frac{z^2}{2} + ...\frac{z^p}{p}\right)$ are all positive, $a_k \leq 0$ for $k \geq p+1$. Thus $|a_k| = -a_k$ for $k \geq p+1$; this gives,

$$0 = E_p(1) = 1 + \sum_{k=p+1}^{\infty} a_k \; or \; \sum_{k=p+1}^{\infty} |a_k| = -\sum_{k=p+1}^{\infty} a_k = 1.$$

Hence, for $|z| \leq 1$,

$$|E_p(z) - 1| = |\sum_{k=p+1}^{\infty} a_k z^k| = |z|^{p+1} \sum_{k=p+1}^{\infty} a_k z^{k-p-1}| \leq |z|^{p+1}| \sum_{k=p+1}^{\infty} a_k| = |z|^{p+1}.$$

So we have, $|E_p(z) - 1| \leq |z|^{p+1}$. $\square$

Theorem 4.11. *Let* $\{a_k\}$ *be a sequence in* $\mathbb{C}$ *such that* $\lim |a_n| = \infty$ *and* $a_n = 0$ *for all* $n \geq 1$.*(This is not a sequence of distinct points; but by hypothesis, no point is repeated on infinite number of times.)If* $\{p_n\}$ *be any sequence of integers such that*

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|} \right)^{p_n+1} < \infty \ \forall \quad r > 0, \tag{4.3}$$

then $f(z) = \prod_{n=1}^{\infty} E_{p_n}(z/a_n)$ *converges in* $\mathbb{H}(\mathbb{C})$*.The function,* f *is an entire function with zero only at the points* a_n*. If* z_0 *occurs in the sequence* $\{a_n\}$ *exactly* m *times, then* f *has a zero at* $z = z_0$ *of multiplicity* m*. Furthermore, if* $p_n = n - 1$*, then the equation* (4.3) *will be satisfied.*

Proof. Suppose there are integers p_n such that $\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|} \right)^{p_n+1} < \infty$ is satisfied. According to previous lemma $|1 - E_{p_n}(z/a_n)| \leq \left| \frac{z}{a_n} \right|^{p_n+1} \leq \left(\frac{r}{|a_n|} \right)^{p_n+1}$, whenever $|z| \leq r$ and $r \leq |a_n|$. For a fixed $r > 0$ there is an integer N such that $|a_n| \geq r \forall n \geq N$ ($\because \lim |a_n| = \infty$). Thus for each $r > 0$ the series $\sum |1 - E_{p_n}((z/a_n))|$ is dominated by the convergent series. So $\sum [1 - E_{p_n}(z/p_n)]$ converges absolutely in $H(\mathbb{C})$. The infinite product $\prod_{n=1}^{\infty} Ep_n(z/p_n)$ converges in $\mathbb{H}(\mathbb{C})$. Now we have to show that $\{p_n\}$ can be found so that (4.3) holds for all r is a trivial matter. For any r there is an integer N such that $|a_n| > 2r$ for all $n \leq N$. This gives that $\left(\frac{r}{|a_n|} \right) < \frac{1}{2}$ for all $n \geq N$. So if $p_n = n - 1 \forall n$, the tail end of the series is dominated by $\sum (\frac{1}{2})^n$, thus the series $\sum \left(\frac{r}{|a_n|} \right)^{p_n+1} < \infty$. $\square$

Theorem 4.12 (Weierstrass Factorization theorem)**.** *Let* f *be an analytic function and let* $\{a_n\}$ *be the non-zero. Zeros of* f *repeated according to multiplicity; suppose* f *has a zero at* $z = 0$ *of order* $m \geq 0$*. Then there is an entire function* g *and a sequence of integers* p_n *such that*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n} \right).$$

Proof. According to be the preceding theorem integer $\{p_n\}$ can be chosen such that

$$h(z) = z^n \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{p_n}\right)$$

has the same zeros as f with the same multiplicities. It follows that $\dfrac{f(z)}{g(z)}$ has removable singularities at $z = 0, a_1, a_2, \cdots$. Thus $\dfrac{f}{h}$ is an entire function and furthermore, has no zeros since $\mathbb{C}$ is simply connected domain there is an entire function g such that $\dfrac{f(z)}{h(z)} = e^{g(z)}$. □

Application of Weierstrass Factorization theorem :

The main aim of the Weierstrass Factorization theorem is to show that every analytic function can be factored.Here we discuss this fact with an example. Here, we show the factorization of the analytic function $\sin \pi z$. Let γ be the rectangle path $[n+\frac{1}{2}+ni, -n-\frac{1}{2}+ni, -n-\frac{1}{2}-ni, n+\frac{1}{2}-ni, n+\frac{1}{2}+ni]$ and wish to calculate $\int_\gamma \pi(z^2-a^2)^{-1}\cot \pi z dz$ for $a \neq$ an integer, we also show that $\lim_{n\to\infty}\int_\gamma \pi(z^2-a^2)^{-1}\cot \pi z dz = 0$, and $\pi \cot \pi z = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{a^2-n^2}$ for $a \neq$ an integer. We wish to prove the following three things:

$$(1) \quad \int_\gamma \pi(z^2-a^2)^{-1}\cot \pi z dz \quad \text{for} \quad a \neq \text{ an integer,}$$

$$(2) \quad \lim_{n\to\infty}\int_\gamma \pi(z^2-a^2)^{-1}\cot \pi z dz = 0,$$

$$(3) \quad \pi \cot \pi a = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{a^2-n^2}.$$

Now consider the integral $\int_\gamma \pi(z^2-a^2)^{-1}\cot \pi z dz$. Here $f(z) = \pi(z^2-a^2)^{-1}\cot \pi z$ attain its poles at $z = \pm a, 0, \pm 1, \pm 2, \cdots \pm n$. We calculate,

$$\begin{aligned}\operatorname{Res}\{f(z);0\} &= \lim_{z\to 0}\frac{z\pi\cos\pi z}{(z^2-a^2)\sin\pi z} = \lim_{z\to 0}\left(\frac{z\cos\pi z}{\sin\pi z}\right)\left(\frac{-\pi}{a^2}\right)\\ &= \lim_{z\to 0}\left(\frac{\cos\pi z - z\pi\sin\pi z}{\pi\cos\pi z}\right)\left(\frac{-\pi}{a^2}\right) = \frac{1}{a^2},\end{aligned}$$

$$\operatorname{Res}\{f(z);\pm n\} = \frac{1}{n^2-a^2} \text{ and } \operatorname{Res}\{f(z);a\} = \lim_{z\to a}\frac{(z-a)\pi}{(z-a)(z+a)}\cot\pi z = \frac{\pi}{2a}\cot\pi a,$$

$$\operatorname{Res}\{f(z);-a\} = \lim_{z\to -a}\frac{(z+a)\pi}{(z-a)(z+a)}\cot\pi z = \frac{\pi}{2a}\cot\pi a.$$

As $|\cot \pi z| \leq 2$ in γ, we see that

$$(4.4) \qquad I_n := \int_\gamma \left(\frac{\pi}{z^2 - a^2}\right) \cot \pi z = 2\pi z\left(\frac{-1}{a^2} + \sum_{j=1}^{n} \left(\frac{2}{j^2 - a^2}\right) + \left(\frac{\pi}{a} \cot \pi z\right)\right).$$

Now, $|z^2 - a^2| > |z|^2 - |a|^2 > n^2 - a^2 \Rightarrow \dfrac{1}{|z^2 - a^2|} < \dfrac{1}{n^2 - a^2}$. Hence,

$$|I_n| \leq 4n\left(\frac{1}{n^2 - a^2}\right).$$

So we have the following result, $\lim_{n\to\infty} \int_\gamma \pi(z^2 - a^2)^{-1} \cot \pi z dz = 0$, which establish (2). Now from (4.4), we get:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{n^2 - a^2}\right) + \left(\frac{\pi}{a} \cot \pi z\right) - \frac{1}{a} = 0$$

$$\Rightarrow \frac{\pi}{a} \cot \pi z = \frac{1}{a^2} + \sum_{n=1}^{\infty} \left(\frac{2}{n^2 - a^2}\right)$$

$$(4.5) \qquad \Rightarrow \pi \cot \pi z = \frac{1}{a} + \sum_{n=1}^{\infty} \left(\frac{2}{n^2 - a^2}\right).$$

Now consider the trigonometric function $f(z) = \sin \pi z$, which is an entire function.

$$(4.6) \qquad f(z) = \sin \pi z \Rightarrow f^{'}(z) = \pi \cos \pi z \Rightarrow \frac{f^{'}(z)}{f(z)} = \pi \cot \pi z.$$

Now the zeros of $\sin \pi z = \dfrac{1}{2i}(e^{i\pi z} - e^{-i\pi z})$ are precisely the integers; moreover, each zero is simple. Since $\sum_{n=-\infty}^{\infty} \left(\dfrac{r}{n}\right)^2 < \infty$ for all $r > 0$. According to Weierstrass Factorization theorem, choose $p_n = 1 \ \forall \ n$. Thus we have the following expression:

$$(4.7) \qquad \sin \pi z = [\exp g(z)] z \prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n}$$

So

$$\begin{aligned} \prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n} &= \left(\prod_{n=-1}^{-\infty} \left(1 - \frac{z}{n}\right) e^{z/n}\right)\left(\prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n}\right) \\ &\Rightarrow \left((1 - z)e^{-z+z}\right)\left((1 - \frac{z}{2})e^{-z/3+z/3}\right) \cdots \\ &= \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n} \end{aligned}$$

So

$$\sin \pi z = [\exp g(z)] z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \tag{4.8}$$

where $g(z)$ is an entire function. Now differentiating the equation (4.7) then we get;

$$\frac{f^{'}(z)}{f(z)} = g^{'}(z) + \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}. \tag{4.9}$$

So, now comparing the equation (4.5), (4.6) and (4.9); we get

$$\pi \cot \pi z = g^{'}(z) + \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \frac{f^{'}(z)}{f(z)} = \pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

So, we have $g^{'}(z) = 0 \Rightarrow g(z) = a$ where a is constant. So from equation (4.9) we get

$$\begin{aligned} \sin \pi z &= e^a z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \\ \Rightarrow \frac{\sin \pi z}{\pi z} &= \frac{e^a}{\pi} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right). \end{aligned}$$

As $z \to 0$ $\dfrac{\sin \pi z}{\pi z} \to 1$ and $\prod_{n=1}^{\infty} \left(1 - \dfrac{z^2}{n^2}\right) \to 1$. So $\dfrac{e^a}{\pi} = 1$ as $z \to 0$ implies that $e^a = \pi$, so, $\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \dfrac{z^2}{n^2}\right)$ and the converges uniformly over compact subsets of $\mathbb{C}$.

4.1. The Gamma function

Definition 4.13. The gamma function, $\Gamma(z)$ is the meromorphic function on $\mathbb{C}$ with simple poles at $z = 0, \pm 1, \pm 2, \pm 3, \cdots$ defined by

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n},$$

where γ is a constant chosen so that $\Gamma(1) = 1$.

Now first thing we have to show the existence of γ. Let $z = 1$ and substitute in $\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1} e^{z/n}$, then we get; $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-1} e^{1/n}$ is a finite number,

$$c = \prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-1} e^{1/n}$$

which is clearly positive. Let $\gamma = \log c$; it follows that with this choice of γ,i.e

$$\gamma = \log c = \log\left(\prod_{n=1}^{\infty}(1+\frac{1}{n})^{-1}e^{-1/n}\right) = \sum\left(1+\frac{1}{n}\right)^{-1} e^{-1/n} = \sum e^{-1/n}\left(\frac{n+1}{n}\right)^{-1}.$$

For $z = 1$, $\Gamma(1) = e^{-\gamma}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-1} = 1$. So $\Gamma(1) = 1$. The constant γ is called Euler's constant and it satisfies

$$e^{\gamma} = \prod_{n=1}^{\infty}(1+\frac{1}{n})^{-1}e^{-1/n}. \tag{4.10}$$

Here in this equation both the sides involve only positive real numbers, so we can take both the sides log then we get; $\log(e^{\gamma}) = \log\left(\prod_{k=1}^{\infty}(1+\frac{1}{k})^{-1}e^{-1/k}\right)$

$$\begin{aligned}
\Rightarrow \gamma &= \sum_{k=1}^{\infty}\log\left(\left(1+\frac{1}{k}\right)^{-1} e^{1/k}\right) = \sum_{k=1}^{\infty}\left[\frac{1}{k} - \log(k+1) + \log k\right] \\
&= \lim_{n\to\infty}\sum\left[\frac{1}{k} - \log(k+1) + \log k\right] \\
&= \lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) - \log(n+1)\right] \\
&= \lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) - \log n - \log\left(\frac{n+1}{n}\right)\right] \\
&= \lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) - \log n\right].
\end{aligned}$$

From the definition of $\Gamma(z)$ is defined as

$$\begin{aligned}
\Gamma(z) &= \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1} e^{z/n} = \frac{e^{-\gamma z}}{z}\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)^{-1} e^{k/n} \\
&= \frac{e^{-\gamma z}}{z}\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{ke^{z/k}}{z+k}\right) = \lim_{n\to\infty}\frac{e^{-\gamma z n!}}{z(z+1)....(z+n)}\exp\left(z(1+\frac{1}{2}+....+\frac{1}{n}))\right)
\end{aligned}$$

However $e^{-\gamma z}\exp\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)z\right] = n^{z}\exp\left[z\left(-\gamma+1+\frac{1}{2}+....+\frac{1}{n}-\log n\right)\right]$.

Definition 4.14. For $z \neq 0, -1, \cdots$

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdot(z+n)} \tag{4.11}$$

The formula of the Gamma function yields a simple derivation of the functional equation satisfied by the gamma function.

Functional equation for Riemann zeta function

For $z \neq 0, -1, \cdots$, $\Gamma(z+1) = z\Gamma(z)$. To obtain this important equation substitute $z+1$ for in equation (4.11); that gives

$$\begin{aligned} \Gamma(z+1) &= \lim_{n\to\infty} \frac{n!n^{z+1}}{(z+1)(z+2)\cdots(z+n+1)} \\ &= z\lim_{n\to\infty} \left[\frac{n!n^z}{z(z+1)(z+2)\cdots(z+n)}\right]\left[\frac{n}{z+n+1}\right] = z\Gamma(z) \end{aligned}$$

So we have,

$$\Gamma(z+n) = z(z+1)\cdots(z+n+1). \tag{4.12}$$

So for a nonnegative integer and $z \neq 0, -1, \cdots$. In particular setting $z = 1$ gives that

$$\Gamma(n+1) = n! \tag{4.13}$$

Here Γ has simple poles at $z = 0, -1, \cdots$ we wish to find poles the poles, $\mathrm{Res}\,(\Gamma; -\mathrm{n}) = \lim_{\mathrm{z}\to-\mathrm{n}}(\mathrm{z}+\mathrm{n})\Gamma(\mathrm{z})$ for each non-negative integer n, we have,

$$(z+n)\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)(z+2)...(z+n+1)}$$

$$\Rightarrow \lim_{z\to-n}(z+n)\Gamma(z)n = \lim_{n\to\infty} \frac{\Gamma(z+n+1)}{z(z+1)(z+2)...(z+n+1)} = \frac{\Gamma(z+1)}{(-n)(-n+1).....(-1)}$$

$$\mathrm{Res}\,(\Gamma; -\mathrm{n}) = \frac{(-1)^{\mathrm{n}}}{\mathrm{n}!}, \mathrm{n} \geq 0 \tag{4.14}$$

Theorem 4.15. *Let f be a function defined on $(0, \infty)$ such that $f(x) > 0$ for all $x > 0$. Suppose f has the following properties:*

(a) $\log f(x)$ *is a convex function;*

(b) $f(x+1) = xf(x)\forall x$;

(c) $f(1) = 1$.

Then $f(x) = \Gamma(x)$ *for all* x.

Proof. Here f be a function defined on $(0;\infty)$ such that $f(x) > 0$ for all $x > 0$. f has the following properties (a) $\log f(x)$ is a convex function and (b) $f(x+1) = x(x)\forall x$.

$$f(x+n) = x(x+1).......(x+n-1)f(x) \tag{4.15}$$

for nonnegative integer n.So if $f(x) = \Gamma(x)$ for $0 < x \leq 1$, this equation will give that f and Γ are everywhere identical. Let $0 < x \leq 1$ and let n be an integer larger than 2.

$$\frac{\log f(n-1) - \log f(n)}{(n-1)-n} \leq \frac{\log f(x+n) - \log f(n)}{(x+n)-n} \leq \frac{\log f(n+1) - \log f(n)}{(n+1)-n}.$$

Since equation (4.15) holds, we have that $f(m) = (m-1)!$ for every integer $m \geq 1$.

$$x\log(n-1) \leq \log f(x+n) - \log(n-1)! \leq x \log n!$$

Adding $\log(n-1)!$ to each side of this inequality and applying the exponential gives

$$\begin{aligned}
(n-1)^x \leq \frac{f(x+n)}{(n-1)!} &\leq n^x \\
\Rightarrow (n-1)^x(n-1)! &\leq f(x+n) \leq n^x(n-1)! \\
\Rightarrow (n-1)^x(n-1)! &\leq x(x+1)....(x+n-1)f(x) \leq n^x(n-1)! \\
\Rightarrow \frac{(n-1)^x(n-1)!}{x(x+1)....(x+n-1)} &= \frac{n^x(n-1)!}{x(x+1)....(x+n-1)} \\
&= \frac{n^x n!}{x(x+1)....(x+n-1)}\left(\frac{x+n}{n}\right).
\end{aligned}$$

Since the term in the middle of this sandwich $f(x)$ does not involve the integer n and since the inequality holds for all integers $n \geq 2$, we may vary the integers on the left and right hand side independently of one another and preserve the inequality. In particular, $n+1$ may be substituted for n on the left while allowing the right hand side to remain unchanged. This gives

$$\frac{n^x n!}{x(x+1)....(x+n-1)} \leq f(x) \leq \frac{n^x n!}{x(x+1)....(x+n-1)}\left(\frac{x+n}{n}\right) \forall\, n \geq 2 \text{ and } x \in [0,1].$$

Now take the limits as $n \to \infty$. Since $\lim \dfrac{x+n}{n} = 1$. According to Gauss's formula implies that $\Gamma(x) = f(x)$ for $0 < x \leq 1$. The result now follows by applying $f(x+n)$ formula and Functional equation. $\square$

Lemma 4.16. *Let $S = \{z : a \leq \operatorname{Re} z \leq A\}$ where $0 < a < A < \infty$.*
(a)For every $\epsilon > 0$ there exist $\delta > 0$ such that $\forall z \in S$, $\left|\int_\alpha^\beta e^{-t}t^{z-1}\right| < \epsilon$, whenever $0 < \alpha < \beta < \delta$.
(b) For every $\epsilon > 0$ there is a number k such that for all z in S, $\left|\int_\alpha^\beta e^{-t}t^{z-1}\right| < \epsilon$, whenever $\beta > \alpha > k$.

Proof. (a) If $0 < t \leq 1$, so $\log t < 0$. If $z \in S$, then $(\mathrm{Re\,z}-1)\log \mathrm{t} \leq (\mathrm{a}-1) \leq (\mathrm{a}-1)\log \mathrm{t}$. Since $e^{-t} \leq 1$, so we have $|e^{-t}t^{z-1}| \leq t^{\mathrm{Re\,z}-1} \leq t^{a-1}$. If $0 < \alpha < \beta < 1$, then

$$\left|\int_\alpha^\beta t^{z-1}dt\right| \leq \int_\alpha^\beta t^{a-1}dt = \frac{1}{a}(\beta^a - \alpha^a)\forall z \in S.$$

If $\epsilon > 0$, then we can choose δ, $0 < \delta < 1$ such that $a^{-1}(\beta^a - \alpha^a) < \epsilon$, for $|\alpha - \beta| < \delta$. So, $\left|\int_\alpha^\beta e^{-t}t^{z-1}\right| < \epsilon$,
(b) Let $z \in S$ and $t \geq 1$, $|t^{z-1}| \geq t^{A-1}$. Since $t^{A-1}\exp(-\frac{-1}{2}t)$ is continuous on $[1,\infty)$ and converges to zero as $t \to \infty$, there exist a constant c such that $t^{A-1}\exp(\frac{-1}{2}t) \leq c\forall t \geq 1$

$$|e^{-t}t^{z-1}| \leq ce^{\frac{-1}{2}t}\forall z \in S,\ t \geq 1.$$

If $\beta > \alpha > 1$, then

$$\left|\int_\alpha^\beta e^{-t}t^{z-1}dt\right| \leq c\int_\alpha^\beta e^{\frac{-1}{2}t} = 2c\left(e^{\frac{-1}{2}\alpha} - e^{\frac{-1}{2}\beta}\right)$$

Again any $\epsilon > 0$ there exist a number $k > 1$ such that $|2c\left(e^{\frac{-1}{2}\alpha} - e^{\frac{-1}{2}\beta}\right)| < \epsilon$ whenever $\alpha, \beta > k$. $\square$

Proposition 4.17. *If $\mathbb{G} = \{z : \operatorname{Re} z > 0\}$ and*

$$f_n(z) = \int_n^{\frac{1}{n}} e^{-t}t^{z-1}dt \ for\ n \geq 1$$

and z in $\mathbb{G}$, then each f_n is analytic on $\mathbb{G}$ and the sequence is convergent in $H(\mathbb{G})$.

Proof. Think of f_n as the integral of $\phi(t,z) = e^{-t}t^{z-1}$ along the straight line segment $[\frac{1}{2}, n]$. and f_n is analytic. Now if k is a compact subset of $\mathbb{G}$ there are positive real numbers a and A such that $k \subset \{z : a \leq \mathrm{Re\,z} \leq \mathrm{A}\}$. Since

$$f_m(z) - f_n(z) = \int_{\frac{1}{m}}^{\frac{1}{n}} e^{-t}t^{z-1}dt + \int_n^m e^{-t}t^{z-1}dt for m > n.$$

From the previous lemma (4.16) imply that f_n is a cauchy sequence in $\mathbb{H}(G)$. But $\mathbb{H}(G)$ is complete. So that f_n must converge. □

Lemma 4.18. *(a)* $\left(1 + \frac{z}{n}\right)^n$ *converges to* e^z *in* $\mathbb{H}(\mathbb{C})$.
(b) If $t \geq 0$, *then* $\left(1 - \frac{t}{n}\right)^n \leq e^{-t}\ \forall n \geq t$.

Proof. Let K be a compact subset of the plane. Then $|z| < n \forall z \in K$ and n sufficiently large. It suffices to show that $\lim_{n\to\infty} n \log\left(1 + \frac{z}{n}\right) = z$ uniformly for z in K. We know that

$$\log(1+w) = \sum_{k=1}^{\infty} (-1)^{k-1}\frac{w^k}{k} for |w| < 1.$$

Let $n > |z| \forall z \in K$; if z is any point in K, then

$$(4.16) \qquad \begin{aligned} n\log(1+\frac{z}{n}) &= z - (\frac{1}{2})(z^2 n) + (\frac{1}{3})(z^3 n^2) - \cdots \\ \Rightarrow n\log(1+\frac{z}{n}) - z &= z\left[\frac{-1}{2}(\frac{z}{n}) + \frac{1}{3}(\frac{z}{n})^2 - \cdots\right] \\ \Rightarrow |n\log\left(1+\frac{z}{n} - z\right)| &\leq |z|\sum_{k=2}^{\infty}\frac{1}{k}|\frac{z}{n}|^{n-1} \leq |z|\sum_{k=1}^{\infty}|\frac{z}{n}|^{n-1} = \frac{|z|^2}{n}\frac{1}{1-|z/n|} \leq \frac{R^2}{n-R}, \end{aligned}$$

where $R \geq |z|\ \forall\ z \in K$. If $n \to \infty$, then this difference goes to zero uniformly for z in K. So $\left(1 + \frac{z}{n}\right)^n$ converges to e^z in $\mathbb{H}(\mathbb{C})$. (b) Now let $t \geq 0$. Substitute $-t$ for z in (4.16) where $t \leq n$,then we have

$$\begin{aligned} & n\log\left(1 - \frac{t}{n}\right) + t = -t\sum_{\infty}^{k=1}\frac{1}{k}\left(\frac{t}{n}\right)^{x-1} \leq 0 \\ \Rightarrow\ & n\log\left(1 - \frac{t}{n}\right) \leq -t \\ \Rightarrow\ & \log\left(1 - \frac{t}{n}\right)^n \leq -t \Rightarrow \left(1 - \frac{t}{n}\right)^n \leq e^{-t} \end{aligned}$$

Hence $\left\{\left(1-\frac{t}{n}\right)^n\right\}$ is convergent. □

Theorem 4.19. *If* $\operatorname{Re} z > 0$, *then* $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$

Proof. Fix $x > 1$ and let $\epsilon > 0$. According to lemma (4.16)(b), we can chose $k > 0$ such that

$$\int_k^r e^{-t}t^{z-1}dt < \frac{\epsilon}{4} \tag{4.17}$$

whenever $r > k$. Let n be any integer grater than k and let f_n be the function defined as

$$f_n = \int_{\frac{1}{n}}^n e^{-t}t^{z-1}dt.$$

We see that

$$\begin{aligned}
f_n - \int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}dt &= \int_{\frac{1}{n}}^n e^{-t}t^{z-1}dt - \int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}dt \\
&= -\int_0^{\frac{1}{n}} \left(1-\frac{t}{n}\right)^n t^{x-1}dt - \int_{\frac{1}{n}}^n \left(1-\frac{t}{n}\right)^n t^{x-1}dt \\
&+ \int_{\frac{1}{n}}^n \left(1-\frac{t}{n}\right)^n t^{x-1}dt \\
&= -\int_0^{\frac{1}{n}} \left(1-\frac{t}{n}\right)^n dt + \int_{\frac{1}{n}}^n \left[e^{-t} - \left(1-\frac{t}{n}\right)^n\right] t^{x-1}dt.
\end{aligned}$$

Now by lemma (4.18(b)) and lemma (4.16(a)) we have the following result,

$$\int_0^{\frac{1}{n}} \left(1-\frac{t}{n}\right)^n t^{x-1}dt \le \int_0^{\frac{1}{n}} e^{-t}t^{x-1}dt \le \frac{\epsilon}{4}$$

for sufficiently large n. Part(a) of the preceding lemma gives

$$\left|\left(1-\frac{t}{n}\right)^n - e^{-t}\right| \le \frac{\epsilon}{4MK}$$

for $t$in $[0,k]$ where $M = \int_0^k t^{x-1}dt$

$$\left|\int_{\frac{1}{n}}^k \left[e^{-t} - \left(1-\frac{t}{n}\right)^n t^{x-1}dt\right]\right| \le \frac{\epsilon}{4}$$

using the lemma (4.18)(b) and (4.17) we get;

$$\left|\int_k^n \left[e^{-t} - \left(1-\frac{t}{n}\right)^n t^{x-1}dt\right]\right| \le 2\int_k^n e^{-t}t^{x-1}dt \le \frac{\epsilon}{2}$$

for $n > k$. If we combine these inequalities, we get

$$\left| f_n(x) - \int_0^n \left(1 - \frac{t}{n}\right)^n t^{x-1} dt \right| \le \frac{\epsilon}{2} - \frac{\epsilon}{4} = \frac{\epsilon}{2}$$

for n sufficiently large, that is

$$\begin{aligned} 0 &= \lim \left[f_n(x) - \int_0^n \left(1 - \frac{t}{n}\right)^n t^{x-1} dt \right] \\ &= \lim \left[f_n(x) - \frac{n! n^x}{x(x+1)......(x+n)} \right] \\ &= f(x) - \Gamma(x) \Rightarrow f(x) = \Gamma(x). \end{aligned}$$

□

4.2. Reimann zeta function

Definition 4.20. The Reimann Zeta function is defined for $\operatorname{Re} z > 1$ by the equation

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}.$$

Now we have $n^{-z}\Gamma(z) = \int_0^\infty e^{-nt} t^{z-1} dt$. If $\operatorname{Re} z > 1$ and we sum this equation over all positive n, then

$$\zeta(z)\Gamma(z) = \sum_{n=1}^{\infty} n^{-z}\Gamma(z) = \sum_{n=1}^{\infty} \int_0^\infty e^{-nt} t^{z-1} dt.$$

Now our aim is to show that this infinite sum can be taken inside the integral sign. for this we want to prove these following results.

Lemma 4.21. *(a) Let* $S = \{z : \operatorname{Re z} \ge \mathrm{a}\}$ *where* $a > 1$*. If* $\epsilon > 0$ *then there is a number* δ*,* $0 < \delta < 1$*, such that for all* z *in* S*.*

$$\left| \int_\alpha^\beta (e^t - 1)^{-1} t^{z-1} dt \right| < \epsilon,$$

whenever $\delta > \beta > \alpha$*.*

(b) Let $S = \{z : \operatorname{Re z} \le \mathrm{A}\}$ *where* $-\infty < A < \infty$*. If* $\epsilon > 0$*, then there is a number* $k > 1$

such that $\forall z \in S$.

$$\left| \int_{\alpha}^{\beta} (e^t - 1) t^{z-1} dt \right| < \epsilon,$$

whenever $\beta > \alpha > k$.

Proof. (a) Let $S = \{z : \mathrm{Re\,z} \geq \mathrm{a}\}$ where $a > 1$. Since $e^t - 1 \geq t \forall t \geq 0$. we have that for $0 < t \leq 1$ and $z \in S$

$$\left| (e^t - 1) t^{z-1} \right| \leq t^{a-1} \left[\frac{1}{e^t - 1} \leq \frac{1}{t} \right].$$

Since $a > 1$ the integral $\int_0^1 t^{a-2} dt$ is finite. So that δ can be found such that

$$\left| \int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{z-1} dt \right| < \epsilon.$$

.

(b) If $t \geq 1$ and $z \in S$, then

$$\left| (e^t - 1) t^{z-1} \right| \leq (e^t - 1)^{-1} t^{A-1} \leq c e^{\frac{1}{2}t} (e^t - 1)^{-1}.$$

Since $e^{\frac{1}{2}t}(e^t - 1)^{-1}$ is integrable on $[1, \infty)$ the required number k can be found. □

Corollary 4.22. *(a) If* $S = \{z : a \leq \mathrm{Re\,z} \leq \mathrm{A}\}$ *where* $1 < a < \infty$, *then the integral*

$$\left| \int_0^{\infty} (e^t - 1)^{-1} t^{z-1} dt \right|$$

converges uniformly on S.

(b) If $S = \{z : \mathrm{Re\,z} \leq \mathrm{A}\}$ *where* $-\infty < A\infty$, *then the integral*

$$\left| \int_1^{\infty} (e^t - 1)^{-1} t^{z-1} dt \right|$$

converges uniformly on S.

Proposition 4.23. *For* $\mathrm{Re\,z} > 1$,

$$\zeta(z)\Gamma(z) = \int_0^{\infty} (e^t - 1)^{-1} t^{z-1} dt$$

Proof. According to above corollary this integral is analytic function in the region $\{z : \mathrm{Re}\, z > 1\}$. Thus, it suffices to show that $\zeta(z)\Gamma(z)$ equals this integral for $z = x > 1$. From the previous lemma we have there exist $\alpha, \beta, 0 < \alpha < \beta < \infty$ such that

$$\int_0^\alpha (e^t - 1)^{-1} t^{x-1} dt < \frac{\epsilon}{4}. \int_\beta^\infty (e^t - 1)^{-1} t^{x-1} dt < \frac{\epsilon}{4}$$

Since $\sum_{k=1}^{n} e^{-kt} \leq \sum_{k=1}^{\infty} e^{-kt} = (e^t - 1)^{-1}$ for $\forall n \geq 1$, we have,

$$\sum_{n=1}^{\infty} \int_0^\alpha (e^t - 1)^{-1} t^{x-1} dt < \frac{\epsilon}{4} \text{ and } \sum_{n=1}^{\infty} \int_\beta^\infty (e^t - 1)^{-1} t^{x-1} dt < \frac{\epsilon}{4}.$$

Now

$$\begin{aligned}
&\left|\zeta(z)\Gamma(z) - \int_0^\infty (e^t - 1)^{-1} t^{x-1} dt\right| \\
= &\left|\sum_{n=1}^{\infty} \int_0^\infty e^{-nt} t^{x-1} dt - \int_0^\infty (e^t - 1)^{-1} t^{x-1} dt\right| \\
= &\left[\sum_{n=1}^{\infty} \left[\int_0^\alpha e^{-nt} t^{x-1} dt + \int_\alpha^\beta e^{-nt} t^{x-1} dt + \int_\beta^\infty e^{-nt} t^{x-1} dt\right]\right] \\
- &\left[\int_0^\alpha (e^t - 1)^{-1} t^{x-1} dt + \int_\alpha^\beta (e^t - 1)^{-1} t^{x-1} dt + \int_\beta^\infty (e^t - 1)^{-1} t^{x-1} dt\right] \\
= &\frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \left|\sum_{n=1}^{\infty} \int_\alpha^\beta e^{-nt} t^{x-1} dt - \int_\alpha^\beta (e^t - 1)^{-1} t^{x-1} dt\right|
\end{aligned}$$

But $\sum_{n=1}^{\infty} e^{-nt}$ converges to $(e^t - 1)^{-1}$ uniformly on $[\alpha, \beta]$; So that the right hand side is exactly ϵ. ☐

Actually we wish to use the above proposition to extend the domain of definition of ζ to $\{z : \mathrm{Re}\, z > -1\}$. Consider the Laurent series expansion of $(e^z - 1)^{-1}$is as follows

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} a_n z^n \tag{4.18}$$

for some constants $a_1, a_2, \cdots$. Thus $(e^t - 1)^{-1} - t^{-1}$ remains bounded in a neighborhood of $t = 0$ of $t = 0$, which implies that

$$\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) dt$$

converges uniformly on compact subsets of the right half plane $\{z : \mathrm{Re\,z} > 0\}$ and therefore represents an analytic function there. Hence

$$\zeta(z)\Gamma(z) = \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) t^{z-1} dt + (z-1)^{-1} + \int_1^\infty \frac{t^{z-1}}{e^t - 1} dt \tag{4.19}$$

and each of these summands, except $(z-1)^{-1}$ analytic in the right half plane. Thus one may define $\zeta(z)$ for $\mathrm{Re\,z} > 0$ setting it equal to $[\Gamma(z)]^{-1}$ times the right hand side of (4.19). In this manner ζ is meromorphic in the right half plane with a simple pole at $z = 1$ ($\sum_{n=1}^\infty n^{-z}$ diverges) whose residue is 1.

Now suppose $0 < \mathrm{Re\,z} < 1$; then $(z-1)^{-1} = -\int_1^\infty t^{z-2} dt$.

$$\begin{aligned}
\zeta(z)\Gamma(z) &= \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) t^{z-1} + \int_1^\infty \frac{t^{z-1}}{e^t - 1} - t^{z-2} dt \\
\int_1^\infty \frac{t^{z-1}}{e^t - 1} - t^{z-2} dt &= \int_1^\infty \frac{t^{z-1} - (e^t - 1) t^{z-1}}{e^t - 1} dt \\
&= \int_1^\infty \frac{t^{z-1} - e^t t^{z-2} + t^{z-2}}{e^t - 1} dt
\end{aligned}$$

Again considering the Laurent expansion of $(e^z - 1)^{-1}$; we see that $(e^z - 1)^{-1} - t^{-1} + \frac{1}{2} \leq ct$ for some constant c and all t in the unit interval $[0, 1]$. Thus the integral

$$\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) t^{z-1} dt$$

is uniformly convergent on compact subsets of $\{z : \mathrm{Re\,z} > -1\}$. Also, since

$$\lim_{t \to \infty} t \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) = 1$$

there is a constant c such that

$$\left(\frac{1}{e^t - 1} - \frac{1}{t}\right) \leq \frac{c}{t}, t \geq 1.$$

This gives that the integral

$$\int_1^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) t^{z-1} dt$$

converges uniformly on compact subsets of $\{z : \mathrm{Re\,z} < 1\}$ using these last two integrals with equation gives

$$\zeta(z)\Gamma(z) = \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) t^{z-1} dt - \frac{1}{2z} + \int_1^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) t^{z-1} dt \tag{4.20}$$

for $0 < \operatorname{Re} z < 1$. But since both integrals converge in the strip $-1 < \operatorname{Re} z < 1$ (4.20) can be used to define $\zeta(z)$ in $\{z : -1 < \operatorname{Re z} < 1\}$, Since the term $(2z)^{-1}$ appears on the right hand side of Equation (4.20) will have ζ have a pole at $z = 0$? the answer in no. As we want to define $\zeta(z)$, we must divide (4.20) by $\Gamma(z)$. When this happens the term in equation becomes $[2z\Gamma(z)]^{-1} = [2\Gamma(z+1)]^{-1}$ which is analytic at $z = 0$. Thus, if ζ is so defined in the strip $\{z : -1 < \operatorname{Re z} < 1\}$ it is analytic there. If this is combined with (4.20) $\zeta(z)$ is defined for $\operatorname{Re z} > -1$ with a simple pole at $z = 1$. Now if $-1 < \operatorname{Re z} < 0$, then

$$\int_1^\infty t^{z-1}dt = \frac{-1}{2}$$

We insert the above integral in (4.22), which yields

$$\zeta(z)\Gamma(z) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) t^{z-1}dt, -1 < \operatorname{Re} z < 0.$$

We see that

$$\cot(1/2it) = \frac{2}{it} - 4it \sum_{n=1}^\infty \frac{1}{t^2 + 4n^2\pi^2} \quad for\ t \neq 0$$

as

$$\frac{1}{e^t - 1} + \frac{1}{2} = \frac{1}{2}\left(\frac{e^t + 1}{e^t - 1}\right) = \frac{i}{2}\cot(1/2it).$$

Thus,

$$\begin{aligned}
\left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)\frac{1}{t} &= 2\sum_{n=1}^\infty \frac{1}{t^2 + 4n^2\pi^2}, \\
\zeta(z)\Gamma(z) &= 2\int_0^\infty \left(\sum_{n=1}^\infty \frac{1}{t^2 + 4n^2\pi^2}\right) t^z dt \\
&= 2\sum_{n=1}^\infty \int_0^\infty \frac{1}{t^2 + 4n^2\pi^2} dt \\
&= 2\sum_{n=1}^\infty (2n\pi)^{z-1} \int_0^\infty \frac{t^z}{t^2 + 1} dt \\
&= 2(2\pi)^{z-1}\zeta(1 - z) \int_0^\infty \frac{t^z}{t^2 + 1} dt
\end{aligned}$$

for $-1 < \mathrm{Re\,z} < 0$.

Now for x is a real number with $-1 < x < 0$, the change of variable $S = t^2$ gives

$$\int_0^\infty \frac{t^x}{t^2+1}dt = \frac{1}{2}\int_0^\infty \frac{s^{\frac{1}{2}(x-1)}}{s+1}ds = \frac{1}{2}\pi \csc\left(\frac{1}{2}\pi(1-x)\right) = \frac{1}{2}\pi \sec\left(\frac{1}{2}\pi x\right)$$

So we have, $\dfrac{1}{\Gamma(x)} = \dfrac{\Gamma(1-x)}{\pi}\sin \pi x = \dfrac{\Gamma(1-x)}{\pi}\left(2\sin(\dfrac{1}{2}\pi x)\cos(\dfrac{1}{2}\pi x)\right)$

Riemann's Functional Equation

$\zeta(z) = 2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)Sin(\frac{1}{2}\pi z)$ for $-1 < \mathrm{Re\,z} < 0$.

Theorem 4.24. *The zeta function can be defined to be meromorphic in the plane with only a simple pole at $z = 1$ and $\mathrm{Res}(\zeta; 1) = 1$ for $z \neq 1$ ζ satisfies the functional equation.*

Now since $\Gamma(1-z)\sin(\frac{1}{2}\pi z)$ has a pole at $z = 1, 2, ...$ and ζ is analytic at $z = 2, 3, ...$, we know from the Reimann Functional equation that $\zeta(1-z)\sin(\frac{1}{2}\pi z) = 0$for $z = 2, 3, ...$ furthermore, since the pole of zeros of $\zeta(1-z)\sin(\frac{1}{2}\pi z)$ must be simple. Since $\sin(\frac{1}{2}\pi z) = 0$, whenever z is an even integer; $\zeta(1-z) = 0$ for $z = 3, 5, ...$ that is $\zeta(z) = 0$ for $z = -2, -4, -6, ...$ similar reasoning gives that ζ has no other zeros outside the closed strip $\{z : 0 \leq \mathrm{Re\,z} \leq 1\}$.

Definition 4.25 (Critical strip)**.** The points $z = -2, -4, ...$ are called the trivial zeros of ζ and the strip $\{z : 0 \leq \mathrm{Re\,z} \leq 1\}$ is called the critical strip.

The Reimann Hypothesis

If z is a zero of the ζ function in the critical strip then $\mathrm{Re}\, z = \frac{1}{2}$. The following theorem provides us an important relation between zeta function and number theory.

Theorem 4.26. *If* $\mathrm{Re\,z} > 1$, *then*

$$\zeta(z) = \prod_{n=1}^{\infty}\left(\frac{1}{1-p_n^{-z}}\right)$$

where $\{p_n\}$ is the sequence of prime numbers.

CHAPTER 5

BASIC UNIVALENT FUNCTIONS THEORY AND BIEBERBACH CONJECTURE

In this chapter we introduce the class $\mathcal{S}$ of normalized analytic univalent functions and in the open unit disk and some of its subclasses defined by geometric conditions. We also give some important examples and most of the elementary results concerning the class $\mathcal{S}$ are direct consequences of the area theorem. We also discuss *Bieberbach's theorem, Koebe one-quarter theorem, Growth and distortion theorem* for functions in the class $\mathcal{S}$ and at lastly, we state the *Bieberbach conjecture.*

Definition 5.1 (Univalent function)**.** A single valued function f is said to be univalent in a domain $\mathbb{D} \subset \mathbb{C}$ if it never takes the same value twice, i.e. $f(z_1) \neq f(z_2) \forall z_1, z_2 \in \mathbb{D}$ such that z_1, z_2.

Definition 5.2 (Locally univalent function)**.** A function is said to be locally univalent at a point $z_0 \in \mathbb{D}$ if it is univalent in some neighborhood of z_0.

Note:

(1) For analytic functions f, the conditions $f'(z_0) \neq 0$ is equivalent to local equivalence at z_0.

(2) An analytic univalent function is called a conformal mapping because of its angle-preserving property.

We are concerned with the class $\mathcal{S}$ of functions f analytic and univalent in the disk $\mathbb{D} = \{z : |z| < 1\}$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Each $f \in S$ has a Taylor series expansion of the form

$$f(z) = z + \sum_{n\geq 2} a_n z^n, \quad |z| < 1.$$

Examples of functions in $\mathcal{S}$:

(1) The Koebe function $K(z) = z(1-z)^{-2} \in \mathcal{S}$. It maps the disk $\mathbb{D}$ onto the entire plane minus the part of the negative real axis from $\frac{-1}{4}$ to infinity. This can be seen as we express $K(z)$ as follows:

$$K(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}.$$

(2) $f(z) = z$, the identity mapping.

(3) $f(z) = z(1-z)^{-1}$, which maps $\mathbb{D}$ conformally onto the half plane $\operatorname{Re} w > \frac{1}{2}$

(4) $f(z) = \frac{z}{(1-z^2)}$, which maps $\mathbb{D}$ conformally onto the two half lines $\frac{1}{2} \leq x < \infty$ and $-\infty < x \leq -\frac{1}{2}$.

(5) $f(z) = \dfrac{1}{2}\log\left[\dfrac{1+z}{1-z}\right]$, which maps $\mathbb{D}$ onto the horizontal strip $-\frac{\pi}{4} < \operatorname{Im} w < \frac{\pi}{4}$.

(6) $f(z) = z - \frac{1}{2}z^2 = \frac{1}{2}[1-(1-z)^2]$, which maps $\mathbb{D}$ onto the interior of a Cardioid.

Note: Sum two functions in $\mathcal{S}$ need not be univalent. Consider two functions $f = \frac{z}{(1-z)}$ and $g = \frac{z}{1+iz}$. But, $h = f + g$ is not univalent.

Properties of functions in class $\mathcal{S}$

The class $\mathcal{S}$ is preserved under a number of elementary transformations.

(1) Conjugation: If $f \in S$ and $g(z) = \overline{f(\bar{z})} = z + \overline{a_2} z^2 + \overline{a_3} z^3 + \ldots$ then $g \in S$.

(2) Rotation If $f \in S$ and $g(z) = e^{i\theta} f(e^{i\theta} z)$ then $g \in S$.

(3) Dilation: If $f \in S$ and $g(z) = r^{-1} f(rz)$ then $g \in S$.

(4) Disk automorphism: If $f \in S$ and $g(z) = \dfrac{f\left(\dfrac{z+\alpha}{1+\bar{\alpha}z}\right) - f(\alpha)}{(1-|z|^2)f'(\alpha)}$, $|\alpha| < 1$, then $g \in S$.

(5) Range transformation: If $f \in S$ and ψ is function analytic and univalent on the range of f, with $\psi(0) = 0$ and $\psi'(0) = 1$, then $g = \psi \circ f \in S$.

(6) Omitted-value transformation: If $f \in S$ and $f(z) \neq w$, then $g = wf/(w-f) \in S$.

(7) Square-root transformation: If $f \in S$ and $g(z) = \sqrt{(f(z))^2}$, then $g \in S$.

Let the class $\sum$ consists of functions $g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots$, which is analytic and univalent in the domain $\triangle = \{z : |z| > 1\}$ exterior to $\mathbb{D}$, expect for a simple pole at infinity with residue 1. Each function $g \in \sum$ maps $\triangle$ onto the complement of a compact connected set E.

Consider subclass $\sum'$ of $\sum$ for which $0 \in E$; i.e for which $g(z) \neq 0$ in $\triangle$. Any function $g \in \sum$ will belongs to $\sum'$ after suitable adjustment of the constant term b_0, i.e for each $f \in S$, the function

$$g(z) = \left(f\left(\frac{1}{2}\right)\right)^{-1} = z - a_2 + (a_2^2 - a_3)z^{-1} + \cdots$$

which belongs to $\sum'$.

Consider the subclass $\sum_0$ consisting of all $g \in \sum$ with $b_0 = 0$. This can be archived by suitable translation, but it may not be possible to translate a given function $g \in \sum$ simultaneously to both $\sum_0$ and $\sum'$.

$\tilde{\sum}$ **function** : Let $\tilde{\sum}$ be the subclass of $\sum$ and functions in this class have the omitted set E with two-dimensional Lebegue measure zero. The functions $g \in \tilde{\sum}$ will be called full mapping. Now, let us recall:

Theorem 5.3 (Green's theorem)**.** *Let C be a positively oriented, piecewise smooth, simple, closed curve and let D be the region enclosed by the curve. If P and Q have continuous first order partial derivative on D then,*

$$\int_C P dx + Q dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Theorem 5.4 (Area's theorem)**.** *If $g \in \sum$, then $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$, with equality if and only if $g \in \sum^{\sim}$.*

Proof. Let E be the set omitted by g. For $r > 1$, let c_r be the image under g of the circle $|z| = r$. Since g is univalent, c_r is a simple closed curve which encloses a domain $E_r \supset E$. By Green's theorem, the area of E_r is

$$\begin{aligned} A_r &= \frac{1}{2i}\int_{C_r} \bar{w}dw = \frac{1}{2i}\int_{|z|=r} \overline{g(z)}g^{'}(z)dz \\ &= \frac{1}{2}\int_0^{2\pi}\left(re^{-i\theta} + \sum_{n=0}^{\infty} \bar{b_n}r^{-n}e^{in\theta}\right) \\ &\times \left(1 - \sum_{v=1}^{\infty} vb_v r^{-v-1}e^{-i(v+1)\theta}\right)re^{i\theta}d\theta \\ &= \pi\left(r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right), r > 1. \end{aligned}$$

Letting r decreases to 1, we obtain $m(E) = \pi\left(1 - \sum_{n=1}^{\infty} n|b_n|^2\right)$, where $m(E)$ is the outer measure of E. Since $m(E) \geq 0$, this proves the theorem. □

Corollary 5.5. *If $g \in \sum$, then $|b_1| \leq 1$, with equality if and only if g has the form $g(z) = z + b_0 + b_1/z + \cdots$, $|b_1| = 1$. This is a conformal mapping of $\triangle$ onto the complement of a line segment of length 4.*

Theorem 5.6 (Bieberbach's theorem)**.** *If $f \in S$, then $|a_2| \leq 2$, with equality if and only if f is a rotation of Koebe function.*

Proof. A square-root transformation and an inversion applied to $f \in S$ will produce a function

$$g(z) = (f(1/z^2))^{-1/2} = z - (a_2/2)z^{-1} + \cdots$$

of class $\sum$. Thus $|a_2| \leq 2$, by the corollary to the area theorem. Equality occurs if and only if g has the form $g(z) = z - e^{i\theta}/z$. A simple calculation shows that this is equivalent to $f(\zeta) = \zeta(1 - e^{i\theta\zeta})^{-2} = e^{-i\theta}K(e^{i\theta}\zeta)$, a rotation of the Koebe function. □

Bieberbach has also conjectured in 1916 that absolute value of each Taylor coefficient of the functions in the class $\mathcal{S}$ is bounded by n. Now, we state this conjecture precisely:

Bieberbach conjecture: The Taylor co-efficients of each function $f \in \mathcal{S}$ satisfy $|a_n| \leq n$ for $n = 2, 3, 4, \cdots$. Strict inequality holds for all n unless f is the koebe function or one of its rotation.

We mention here that this conjecture is settled by Louis, de Branges in 1984 (compare [**1**]).

Theorem 5.7 (Koebe One-Quarter theorem)**.** *The range of every function of class* $\mathcal{S}$ *contains the disk* $\{w : |w| < \dfrac{1}{4}\}$.

Proof. If a function $f \in \mathcal{S}$ omits the value $w \in \mathbb{C}$, then

$$g(z) = \frac{wf(z)}{w - f(z)} = z + \left(a_2 + \frac{1}{w}\right)z^2 + \cdots$$

is analytic and univalent in $\mathbb{D}$. This is the omitted-value transformation, which is the composition of f with a linear fractional mapping. Since $g \in \mathcal{S}$, Bieberbach's theorem gives $\left|a_2 + \dfrac{1}{w}\right| \leq 2$. Combined with the inequality $|a_2| \leq 2$, this shows $|1/w| \leq 4$ or $|w| \geq \frac{1}{4}$. Thus every omitted value must lie outside the disk $|w| < \frac{1}{4}$. □

Theorem 5.8. *For each* $f \in \mathcal{S}$,

(5.1)
$$\left|\frac{f''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \leq \frac{4r}{1-r^2}, |z| = r < 1$$

Proof. given $f \in S$, fix $\zeta \in \mathbb{D}$ and perform a disk automorphism to construct

(5.2)
$$F(z) = \frac{f\left(\dfrac{z+\zeta}{1+\bar{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2)(f'(\zeta))} = z + A_2(\zeta)z^2 + \cdots .$$

Then $F \in \mathcal{S}$ and a calculation gives

$$A_2(\zeta) = \frac{1}{2}\left((1-|\zeta|^2)\frac{f''(\zeta)}{f'(\zeta)} - 2\bar{\zeta}\right)$$

But by Bieberbach's theorem, $|A_2(\zeta)| \leq 2$. Simplifying this inequality and replacing ζ by z, we obtain the inequality (5.1). A suitable rotation of the Koebe function shows that the estimate is sharp for each $z \in \mathbb{D}$. □

Theorem 5.9 (Distortion theorem)**.** *For each $f \in \mathcal{S}$,*

$$\frac{1-r}{(1+r)^3} \leq |f^{'}(z)| \leq \frac{1+r}{(1-r)^3}, |z| = r < 1. \tag{5.3}$$

For each $z \in \mathbb{D}$, equality occurs if and only if f is a suitable rotation of the Koebe function.

Proof. Since an inequality $|\alpha| \leq c$ implies $-c \leq \mathrm{Re}\{\alpha\} \leq$ c, it follows from 5.1 that

$$\frac{2r^2-4r}{1-r^2} \leq \mathrm{Re}\left(\frac{zf^{''}(z)}{f^{'}(z)}\right) \leq \frac{2r^2}{1-r^2}.$$

Because $f^{''}(z) \neq 0$ and $f^{'}(0) = 1$, we can choose a single-valued branch of $\log f^{'}(z)$ which vanishes at the origin. Now observe that

$$\mathrm{Re}\left(\frac{zf^{''}(z)}{f^{'}(z)}\right) = r\frac{\partial}{\partial r}\mathrm{Re}\,(\log f^{'}(z)), z = e^{i\theta}.$$

Hence,

$$\frac{2r-4}{1-r^2} \leq \frac{\partial}{\partial r}\log|f^{'}(re^{i\theta})| \leq \frac{2r+4}{1-r^2}. \tag{5.4}$$

Taking θ fixed, integer with respect to r from 0 to R. A calculation then produces the inequality

$$\log\frac{1-R}{(1+R)^3} \leq \log|f^{'}(Re^{i\theta})| \leq \log\frac{1+R}{(1-R)^3},$$

and the distortion theorem follows by exponentiation. A suitable rotation of the Koebe function, whose derivative is $K^{'}(z) = \dfrac{1+z}{(1-z)^3}$, shows that both estimates of $|f^{'}(z)|$ are best possible. Furthermore, whenever equality occurs for $z = e^{i\theta}$ in either the upper or the lower estimate of (5.3), then equality must hold in the corresponding part of (5.4) for all r, $r \leq R$. In particular, $\mathrm{Re}\left[\left(\frac{f^{''}(0)}{f^{'}(0)}\right)e^{i\theta}\right] = \pm 4$, Which Implies that $|a_2| = 2$. Hence by Bieberbach's theorem, f must be a rotation of the Koebe function. ☐

Theorem 5.10 (Growth theorem)**.** *For each $f \in S$,*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2} \tag{5.5}$$

For each $z \in \mathbb{D}$, $z \neq 0$, equality occurs if and only if f is a suitable rotation of Koebe function.

Proof. Let $f \in S$ and fix $z = re^{i\theta}$ with $0 < r < 1$. Observe that $f(z) = \int_0^r f^{'}(\rho e^{\theta i})e^{i\theta}d\rho$, since $f(0) = 0$. thus by the distortion theorem,

$$|f(z)| \leq \int_0^r |f^{'}(\rho e^{\theta i})|d\rho \leq \int_0^r \frac{1+\rho}{(1-\rho)^3}d\rho = \frac{r}{(1-r)^2}.$$

The lower estimate is more subtle. It holds trivially if $|f(z)| \geq \frac{1}{4}$, since $r(1+r)^{-2} < \frac{1}{4}$ for $0 < r < 1$. If $|f(z)| < \frac{1}{4}$, the Koebe one-quarter theorem implies the radial segment from 0 to $f(z)$ lies entirely in the range of f. Let C be preimage of this segment. Then C is a simple arc from 0 to z, and

$$f(z) = \int_C f^{'}(\zeta)d\zeta.$$

But $f^{'}(\zeta)g\zeta$ has constant signum along C, by construction, so the distortion theorem gives

$$|f(z)| = \int_C |f^{'}(\zeta)||d\zeta| \geq \int_0^r \frac{1-\rho}{(1+\rho)^3}d\rho = \frac{r}{(1+r)^2}.$$

Equality in either part of (5.5) implies equality in the corresponding part of (5.3), which implies that f is a rotation of the Koebe function. $\square$

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