Lie Group Analysis of Isentropic Gas Dynamics

A Project Report

submitted by

Pooja

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CERTIFICATE

This is to certify that the project report entitled *Lie Group Analysis of Isentropic Gasdynamics* is the bonafide work carried out by *Pooja*, student of M.Sc. Mathematics at National Institute Of Technology, Rourkela, during the year 2012, in partial fulfilment of the requirements for the award of the Degree of Master of Science In Mathematics under the guidance of *Prof. K.C. Pati and Prof. R.S. Tungala*, National Institute of Technology, Rourkela and that the project has not formed the basis for the award previously of any degree, diploma, associateship, fellowship or any other similar title.

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DECLARATION

I hereby declare that the project report entitled ***Lie Group Analysis of Isentropic Gasdynamics*** submitted for the M.Sc. Degree is my original work and the project has not formed the basis for the award of any degree, associate ship, fellowship or any other similar titles.

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ABSTRACT

Firstly, we study the Lie group of transformations including infinitesimal transformations, infinitesimal generators, invariant functions, extended infinitesimal transformations and some theorems. Then, we study the infinitesimal transformations of one-layer shallow water equations. Lastly, we find out the infinitesimals of isentropic gas dynamics by using Lie group analysis.

CONTENTS

INTRODUCTION

Sophus Lie proposed Lie symmetries and by using these symmetries we can obtain the solutions of various partial differential equations. There are several significant studies to apply Lie groups to the differential equations. One of the main characteristics of Lie approach is looking for symmetry groups of differential equations and then reducing to original differential equations with fewer independent variables and investigating the self-symmetry groups partial differential equations can be reduced to an ordinary differential equations. The independent variable of the ordinary differential equations is called a similarity variable. Thus, we can obtain self-similarity solutions of the original equations from the ordinary differential equations. And under the Lie group of transformations the self-similarity solutions are invariant.

The main purpose of this report is to find the self-similarity solution for isentropic gasdynamics by using Lie group analysis and to show that Lie group analysis is a generalization of the dimension analysis.

Gasdynamics is a science in the branch of fluid dynamics, concerned with the motion of gases and its effect on physical systems. Gasdynamics arises from the studies of gas flows in transic and supersonic flights which is on the basis of the principles of the fluid dynamics and thermodynamics.

Several authors have studied the equations of gasdynamics, but we investigate the solutions for isentropic gasdynamics. There is a common approach for analyzing the isentropic gasdynamics and that is to solve hyperbolic shallow water equations with boundary conditions.

In this study, we use the Lie group properties and self-similarity solutions already obtained for one layer shallow water equations to investigate the self-similarity solutions of the isentropic gasdynamics.

CHAPTER 1

Lie Groups of Transformations and Infinitesimal Transformations

1.1 Groups

Definition 1.1.1. A group G is a set of elements with a law of composition ϕ between elements satisfying the following axioms:

1. *Closure property*: For any element a and b of G, $\phi(a, b)$ is an element of G.

2. *Associative property*: For any element a, b and c of G,

$$\phi(a, \phi(b, c)) = \phi(\phi(a, b), c)$$

3. *Identity element*: There exists a unique identity element e of G such that for any element a of G,

$$\phi(a, e) = \phi(e, a) = a$$

4. *Inverse element*: For any element a of G there exists a unique inverse element a^{-1} in G such that

$$\phi(a, a^{-1}) = \phi(a^{-1}, a) = e$$

Definition 1.1.2. A group G is Abelian if $\phi(a, b) = \phi(b, a)$ holds for all elements a and b in G.

Definition 1.1.3. A subgroup of G is a group formed by a subset of elements of G with the same law of composition ϕ.

1.2 Examples of Groups

1. G is the set of all integers with $\phi(a, b) = a + b$. Here $e = 0$ and $a^{-1} = -a$.

2. G is the set of all positive reals with $\phi(a, b) = a.b$. Here $e = 1$ and $a^{-1} = \frac{1}{a}$.

1.3 Groups of Transformations

Definition 1.3.1. Let $\mathbf{x} = (x_1, x_2, x_3, \cdots, x_n)$ lie in region $D \subset \mathbb{R}^n$. The set of transformations

$$\mathbf{x}^* = \mathbf{X}(\mathbf{x}; \epsilon)$$

, defined for each $\mathbf{x}$ in D, depending on parameter lying in set $S \subset \mathbb{R}$, with $\phi(\epsilon, \delta)$ defining a law of composition of parameters ϵ and δ in S, forms a group of transformations on D if:

1. For each parameter ϵ in S the transformations are one-to-one onto D, in particular $\mathbf{x}^*$ lies in D.

2. S with the law of composition ϕ forms a group G.

3. $\mathbf{x}^* = \mathbf{x}$ when $\epsilon = e$, i.e.
$$\mathbf{X}(\mathbf{x}; e) = \mathbf{x}$$

4. If $\mathbf{x}^* = \mathbf{X}(\mathbf{x}; \epsilon)$, $\mathbf{x}^{**} = \mathbf{X}(\mathbf{x}^*; \delta)$,then
$$\mathbf{x}^{**} = \mathbf{X}(\mathbf{x}; \phi(\epsilon, \delta))$$

1.4 One-parameter lie group of transformations

Definition 1.4.1. A group of transformations defines a one-parameter Lie group of transformations if in addition to satisfying axioms (i)-(iv) of definition 1.3.1:
5. ϵ is a continuous parameter, i.e. S is an interval in $\mathbb{R}$. Without loss of generality $\epsilon = 0$ corresponds to the identity element e.
6. $\mathbf{X}$ is infinitely differentiable with respect to $\mathbf{x}$ in D and an analytic function of ϵ in S.
7. $\phi(\epsilon, \delta)$ is an analytic function of ϵ and δ, $\epsilon \in S$, $\delta \in S$.

1.5 Examples of one-parameter lie groups of transformations

A Group of translations in the Plane

$$\mathbf{x}^* = x + \epsilon$$
$$\mathbf{y}^* = y, \quad \epsilon \in \mathbb{R}.$$

Here $\phi(\epsilon, \delta) = \epsilon + \delta$. This group corresponds to motions parallel to the x-axis.

1.6 Infinitesimal transformations

Consider a one-parameter (ϵ) Lie group of transformations

$$\mathbf{x}^* = \mathbf{X}(\mathbf{x}; \epsilon) \tag{1.1}$$

with identity $\epsilon = 0$ and law of composition ϕ. Expanding (1.1) about $\epsilon = 0$, we get

$$\mathbf{x}^* = \mathbf{x} + \epsilon(\frac{\partial \mathbf{X}}{\partial \epsilon}(\mathbf{x}; \epsilon)|_{\epsilon=0}) + O(\epsilon^2).$$

Let

$$\xi(\mathbf{x}) = \frac{\partial \mathbf{X}}{\partial \epsilon}(\mathbf{x}; \epsilon)|_{\epsilon=0} \tag{1.2}$$

The transformation $\mathbf{x} + \epsilon\xi(\mathbf{x})$ is called the infinitesimal transformation of the Lie group of transformations (1.1); the components of $\xi(\mathbf{x})$ are called the infinitesimals of (1.1).

Theorem 1.6.1. *(First Fundamental Theorem of Lie) There exists a parameterisation $\tau(\epsilon)$ such that the Lie group of transformation is equivalent to the solution of the initial value problem for the system of first order differential equations*

$$\frac{d\mathbf{x}^*}{d\tau} = \xi(\mathbf{x}^*),$$

with

$$\mathbf{x}^* = \mathbf{x} \quad \textit{when} \quad \tau = 0.$$

In particular

$$\tau(\epsilon) = \int_0^{\epsilon} \Gamma(\epsilon')d\epsilon'$$

where

$$\Gamma(\epsilon) = \frac{\partial \phi(a, b)}{\partial b}|_{(a,b)=(\epsilon^{-1},\epsilon)}$$

and

$$\Gamma(0) = 1.$$

1.7 Infinitesimal generators

Definition 1.7.1. The infinitesimal generator of the one-parameter Lie group of transformations (1.1) is the operator

$$X = X(\mathbf{x}) = \xi(\mathbf{x}).\nabla = \Sigma\xi_i(\mathbf{x})\frac{\partial}{\partial x_i}, \qquad i = 1, \cdots, n \tag{1.3}$$

where ∇ is the gradient operator,

$$\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n})$$

For any differentiable function $F(\mathbf{x}) = F(x_1, x_2, x_3, \cdots, x_n)$,

$$XF(\mathbf{x}) = \xi(\mathbf{x}) \cdot \nabla F(\mathbf{x}) = \Sigma\xi_i(\mathbf{x})\frac{\partial F(\mathbf{x})}{\partial x_i}. \quad i = 1, \cdots, n.$$

Note that $X\mathbf{x} = \xi(\mathbf{x})$.

Theorem 1.7.2. *The one-parameter Lie group of transformations (1.1) is equivalent to*

$$\begin{aligned} \boldsymbol{x}^* &= e^{\epsilon X}\boldsymbol{x} = \boldsymbol{x} + \epsilon X\boldsymbol{x} + \frac{\epsilon^2}{2}X^2\boldsymbol{x} + \dots \\ &= [1 + \epsilon X + \frac{\epsilon^2}{2}X^2 + \dots]\boldsymbol{x} \\ &= \sum_{k=0}^{\infty}\frac{\epsilon^k}{k!}X^k\boldsymbol{x} \end{aligned}$$

where the operator $X = X(\boldsymbol{x})$ is defined by (1.3) and the operator $X^k = XX^{k-1}$, $k = 1, 2, \dots$; in particular $X^kF(\boldsymbol{x})$ is the function obtained by applying the operator X to the function $X^{k-1}F(\boldsymbol{x})$, $k = 1, 2, \dots,$ with $X^0F(\boldsymbol{x}) \equiv F(\boldsymbol{x})$.

1.8 Invariant Functions

Definition 1.8.1. An infinitely differentiable function F($\mathbf{x}$) is an invariant function of the Lie group of transformations (1.1) if and only if for any group transformation (1.1) $F(\mathbf{x}^*) \equiv F(\mathbf{x})$. If $F(\mathbf{x})$ is an invariant function of (1.1), then $F(\mathbf{x})$ is called an invariant of (1.1) and $F(\mathbf{x})$ is said to be invariant under (1.1).

Theorem 1.8.2. *$F(\mathbf{x})$ is invariant under (1.1) if and only if*

$$XF(\mathbf{x}) \equiv 0$$

Theorem 1.8.3. *For a Lie group of transformations (1.1), the identity*

$$F(\mathbf{x}^*) \equiv F(\mathbf{x}) + \epsilon$$

holds if and only if $F(\mathbf{x})$ is such that

$$XF(\mathbf{x}) \equiv 1$$

1.9 Extended Infinitesimal Transformations-One Dependent And One Independent Variable

The one-parameter Lie group of transformations

$$x^* = X(x, y; \epsilon) = x + \epsilon\xi(x, y) + O(\epsilon^2), \tag{1.4}$$

$$y^* = Y(x, y; \epsilon) = y + \epsilon\eta(x, y) + O(\epsilon^2), \tag{1.5}$$

acting on (x, y)-space, has as its infinitesimal

$$\xi(\mathbf{x}) = (\xi(x, y), \eta(x, y)),$$

with corresponding infinitesimal generator

$$\mathbf{X} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}.$$

The k-*th* extension of ((1.4) (1.5)), given by

$$\begin{aligned}
&x^* = X(x, y; \epsilon) = x + \epsilon\xi(x, y) + O(\epsilon^2),\\
&y^* = Y(x, y; \epsilon) = y + \epsilon\eta(x, y) + O(\epsilon^2),\\
&y_1^* = Y_1(x, y, y_1; \epsilon) = y_1 + \epsilon\eta^{(1)}(x, y, y_1) + O(\epsilon^2),\\
&\ \ \vdots\\
&y_k^* = Y_k(x, y, y_1, \ldots, y_k; \epsilon) = y_k + \epsilon\eta^{(k)}(x, y, y_1, \ldots, y_k) + O(\epsilon^2),
\end{aligned}$$

has as its (k-*th* extended) infinitesimal

$$(\xi(x, y), \eta(x, y), \eta^{(1)}(x, y, y_1), \ldots, \eta^{(k)}(x, y, y_1, \ldots, y_k)),$$

with corresponding (k-th extended) infinitesimal generator

$$\mathbf{X}^{(k)} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{(1)}(x,y,y_1)\frac{\partial}{\partial y_1} + \cdots + \eta^{(k)}(x,y,y_1,\cdots,y_k)\frac{\partial}{\partial y_k},$$

$k = 1, 2, \ldots$.

Theorem 1.9.1.

$$\eta^{(k)}(x,y,y_1,\ldots,y_k) = \frac{D\eta^{(k-1)}}{Dx} - y_k\frac{D\xi(x,y)}{Dx}, \qquad k = 1,2,\ldots.$$

where

$$\eta^{(0)} = \eta(x,y).$$

1.10 Lie Group Analysis of A System of Shallow Water Equations

The *one-layer shallow-water equations* can be written as:

$$h_t + hu_x + uh_x = 0 \tag{1.6}$$

$$u_t + uu_x + h_x = 0 \tag{1.7}$$

where h and u are dependent variables and x and t are independent variables.

1.11 Symmetry group analysis of the governing equations

In this section the most general Lie group of transformations which leaves the one-layer shallow-water equation (1.6) and (1.7) invariant are investigated. At first, the Lie group of transformations with independent variables x, t and dependent variables u, h for the problem are considered.

$$\begin{aligned} x^* &= x^*(x,t,u,h;\epsilon) \\ t^* &= t^*(x,t,u,h;\epsilon) \\ u^* &= u^*(x,t,u,h;\epsilon) \\ h^* &= h^*(x,t,u,h;\epsilon) \end{aligned}$$

where ϵ is the group parameter. The infinitesimal generators can be expressed in the following vector form

$$V = \xi^x\frac{\partial}{\partial x} + \xi^t\frac{\partial}{\partial t} + \eta^u\frac{\partial}{\partial u} + \eta^h\frac{\partial}{\partial h}$$

in which ξ^x, ξ^t, η^u, η^h are the infinitesimal functions of the group variables. So, the corresponding one-parameter Lie group of transformations is given by

$$\begin{aligned} x^* &= e^{\epsilon V}(x) = x + \epsilon \xi^x(x,t,u,h) + O(\epsilon^2) \\ t^* &= e^{\epsilon V}(t) = t + \epsilon \xi^t(x,t,u,h) + O(\epsilon^2) \\ u^* &= e^{\epsilon V}(u) = u + \epsilon \xi^u(x,t,u,h) + O(\epsilon^2) \\ h^* &= e^{\epsilon V}(h) = h + \epsilon \eta^h(x,t,u,h) + O(\epsilon^2) \end{aligned}$$

Since the system of one-layer shallow-water equations has at most first-order derivatives, the first prolongations of the generator should be considered as:

$$pr^1 V = V + \tau_x^u \frac{\partial}{\partial u_x} + \tau_t^u \frac{\partial}{\partial u_t} + \tau_x^h \frac{\partial}{\partial h_x} + \tau_t^h \frac{\partial}{\partial h_t} \tag{1.8}$$

where

$$\tau_t^u = \eta_t^u + \eta_u^u u_t + \eta_\rho^u \rho_t - u_x(\xi_t^x + \xi_u^x u_t + \xi_\rho^x \rho_t) - u_t(\xi_t^t + \xi_u^t u_t + \xi_\rho^t \rho_t) \tag{1.9}$$

$$\tau_x^u = \eta_x^u + \eta_u^u u_x + \eta_\rho^u \rho_x - u_x(\xi_x^x + \xi_u^x u_x + \xi_\rho^x \rho_x) - u_t(\xi_x^t + \xi_u^t u_x + \xi_\rho^t \rho_x) \tag{1.10}$$

$$\tau_t^\rho = \eta_t^\rho + \eta_u^\rho u_t + \eta_\rho^\rho \rho_t - \rho_x(\xi_t^x + \xi_u^x u_t + \xi_\rho^t \rho_t) - \rho_t(\xi_t^t + \xi_u^t u_t + \xi_\rho^t \rho_t) \tag{1.11}$$

$$\tau_x^\rho = \eta_x^\rho + \eta_u^\rho u_x + \eta_\rho^\rho \rho_x - \rho_x(\xi_x^x + \xi_u^x u_x + \xi_\rho^x \rho_x) - \rho_t(\xi_x^t + \xi_u^t u_x + \xi_\rho^t \rho_x) \tag{1.12}$$

Now, we apply the first prolongation of the infinitesimal generator (1.8) to the system of the partial differential equations (1.6) and (1.7). Firstly we apply (1.8) to the Eq.(1.6),

$$pr^1 V = \left(\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x}\right)|_{h_t = -uh_x - hu_x} = 0$$

then we get,

$$\tau_t^h + \eta^u h_x + u\tau_x^h + \eta^h u_x + h\tau_x^u = 0 \tag{1.13}$$

Similarly by applying (1.8) to the Eq. (1.7),

$$pr^1 V = \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial x}{\partial x}\right)|_{u_t = -uu_x - h_x} = 0$$

then we get

$$\tau_t^u + \eta^u u_x + u\tau_x^u + \tau_x^h = 0 \tag{1.14}$$

It is well-known that one-layer shallow-water equations are the coupled system of partial differential equations. From this, the equations (1.13) and (1.14) must be the coupled system of partial differential equations. These two equations can be arranged by using the above explicit expressions, then each of these equations can get a polynomial in terms of dependent variables. Since we take the problem in a jet space of the system (1.6) and (1.7), one can consider the derivatives of dependent variables w.r.t. the independent variables as independent functions and equate each coefficient of these functions to zero. First we consider equation (1.13)

$$\tau_t^h + \eta^u h_x + u\tau_x^h + \eta^h u_x + h\tau_x^u = 0$$

$\implies (\eta_t^h + \eta_u^h u_t + \eta_h^h h_t - h_x(\xi_t^x + \xi_u^x u_t + \xi_h^t h_t) - h_t(\xi_t^t + \xi_u^t u_t + \xi_h^t h_t))$
$+\eta^u h_x + u(\eta_x^h + \eta_u^h u_x + \eta_h^h h_x - h_x(\xi_x^x + \xi_u^x u_x + \xi_h^x h_x) - h_t(\xi_x^t + \xi_u^t u_x + \xi_h^t h_x)) +$
$\eta^h u_x + h(\eta_x^u + \eta_u^u u_x + \eta_h^u h_x - u_x(\xi_x^x + \xi_u^x u_x + \xi_h^x h_x) - u_t(\xi_x^t + \xi_u^t u_x + \xi_h^t h_x)) = 0$
$\implies \eta_t^h + \eta_u^h u_t + \eta_h^h h_t + h_x \xi_t^x - \xi_u^x u_t h_x - \xi_h^t h_t h_x - h_t \xi_t^t - h_t \xi_u^t u_t - \xi_h^t h_t^2$
$+ \eta^u h_x + u\eta_x^h + u\eta_u^h u_x + u\eta_h^h h_x - uh_x \xi_x^x - uh_x \xi_u^x u_x - u\xi_h^x h_x^2 - uh_t \xi_x^t -$
$uh_t \xi_u^t u_x - uh_t \xi_h^t h_x + \eta^h u_x + h\eta_x^u + h\eta_u^u u_x + h\eta_h^u h_x -$
$u_x h \xi_x^x - h\xi_u^x u_x^2 - u_x h \xi_h^x h_x - hu_t \xi_x^t - hu_t \xi_u^t u_x - h_t \xi_h^t h_x = 0$

Now arranging all the h_t terms and using $h_t = -uh_x - hu_x$, we get

$\implies \eta_t^h + u\eta_x^h + h\eta_x^u + \eta_u^h u_t - \xi_t^x h_x - \xi_u^x h_x u_t + \eta^u h_x + u\eta_u^h u_x + u\eta_h^h h_x - u\xi_x^x h_x -$
$u\xi_u^x h_x u_x - h\xi_u^x u_x^2 - h\xi_h^x h_x u_x - h\xi_x^t u_t - h\xi_u^t u_t u_x - h\xi_h^t u_t h_x - u\eta_h^h h_x + u\xi_h^x h_x^2 +$
$u\xi_t^t h_x + u\xi_u^t h_x u_t + u^2 \xi_x^t h_x^2 + u^2 \xi_u^t u_x h_x + u^2 \xi_h^t h_x^2 - h\eta_h^h u_x + h\xi_h^x h_x u_x + h\xi_t^t u_x +$
$h\xi_u^t u_t u_x + hu\xi_x^t u_x + uh\xi_u^t u_x^2 + uh\xi_h^t h_x u_x - u^2 \xi_h^t h_x^2 - h^2 \xi_h^t u_x^2 - 2uh\xi_h^t h_x u_x = 0$

Comparing the constants and coefficients of the independent functions, we get

$$\eta_t^h + u\eta_x^h + h\eta_x^u = 0$$

$$u_t(\eta_u^h - h\xi_x^t) = 0$$
$$\implies \eta_u^h = h\xi_x^t$$

$h_x(-\xi_t^x + \eta^u + u\eta_h^h - u\xi_x^x + h\eta_h^u - u\eta_h^h + u\xi_t^t + u^2 \xi_x^t) = 0$
$\implies h\eta_h^u + \eta^u = \xi_t^x + u\xi_x^x - u\xi_t^t - u^2 \xi_x^t$
$\implies -\eta_u^h + h\eta_h^u + \eta^u = \xi_t^x + u\xi_x^x - u\xi_t^t - u^2 \xi_x^t - h\xi_x^t$ (since, $\eta_u^h = h\xi_x^t$)
$\implies -\eta_u^h + h\eta_h^u + \eta^u = \xi_t^x + u\xi_x^x - u\xi_t^t - (u^2 + h)\xi_x^t$

$$
\begin{aligned}
& u_x(u\eta^h_u + \eta^h + h\eta^u_u - h\xi^x_x - h\eta^h_h + h\xi^t_t + uh\xi^t_x) = 0 \\
\implies & -h\eta^h_h + h\eta^u_u + \eta_h + u\eta^h_u = h\xi^x_x - h\xi^t_t - uh\xi^t_x \\
\implies & -\eta^h_h + \eta^u_u + \tfrac{\eta_h}{h} + \tfrac{u}{h}\eta^h_u = \xi^x_x - \xi^t_t - u\xi^t_x \\
\implies & -\eta^h_h + \eta^u_u + \tfrac{\eta_h}{h} + \tfrac{u}{h}h\xi^t_x = \xi^x_x - \xi^t_t - u\xi^t_x \qquad (\text{since, } \eta^h_u = h\xi^t_x) \\
\implies & -\eta^h_h + \eta^u_u + \tfrac{\eta_h}{h} = \xi^x_x - \xi^t_t - 2u\xi^t_x
\end{aligned}
$$

$$
\begin{aligned}
& h_x u_t(-\xi^x_u - h\xi^t_h + u\xi^t_u) = 0 \\
\implies & -\xi^x_u - h\xi^t_h + u\xi^t_u = 0
\end{aligned}
$$

Now, Similarly by solving the equation (1.14) we will get the over-determined system of equations are as follows :

$$
\begin{aligned}
\eta^h_t + u\eta^h_x + h\eta^u_x &= 0 \\
-\eta^h_h + \eta^u_u + \frac{\eta^h}{h} &= \xi^x_x - \xi^t_t - 2u\xi^t_x \\
-\eta^h_u + h\eta^u_h + \eta^u &= \xi^x_t - u\xi^t_t + u\xi^x_x - (u^2+h)\xi^t_x \\
-h\xi^t_h - \xi^x_u + u\xi^t_u &= 0 \\
\eta^u_t + u\eta^u_x + \eta^h_x &= 0 \\
-\eta^u_u + \eta^h_h &= \xi^x_x - \xi^t_t - 2u\xi^t_x \\
\xi^x_h - u\xi^t_h + \xi^t_u &= 0 \\
\eta^h_u - h\eta^u_h + \eta^u &= \xi^x_t - u\xi^t_t + u\xi^x_x - (u^2+h)\xi^t_x \\
\xi^x_u - u\xi^t_u + h\xi^t_h &= 0
\end{aligned}
$$

those are called determining equations in terms of infinitesimals and derivatives of the infinitesimal functions w.r.tthe independent and dependent variables. Obtaining the most general Lie groups of the system of PDEs (1.6) and (1.7) is possible by using the solutions of the above determining equations.

1.12 The solutions of the determining equations

Here, we will find the solutions ξ^x, ξ^t, η^u, η^h of the above determining Eqs. There is no general method for solving the over-determined system of these determining equations. The power-series method of a solution form is one of these solution techniques for finding the solutions of the determining equations in the Lie group analysis of differential equations. So, at first, we choose the first order of power-series of the infinitesimals which are given by

$$\begin{aligned}
\xi^x &= a_0 + a_{10}x + a_{11}t + a_{12}u + a_{13}\rho \\
\xi^t &= b_0 + b_{10}x + b_{11}t + b_{12}u + b_{13}\rho \\
\eta^\rho &= c_0 + c_{10}x + c_{11}t + c_{12}u + c_{13}\rho \\
\eta^u &= d_0 + d_{10}x + d_{11}t + d_{12}u + d_{13}\rho
\end{aligned}$$

Now, By substituting the above power series forms into the over determining equations, we obtain the equations with powers of the variables x, t, u, ρ and calculate the constant coefficients of the power series forms by equating each coefficient of various powers to zero. Now,

$$\begin{aligned}
&\eta^h_t + u\eta^h_x + h\eta^u_x = 0 \\
&\Longrightarrow \; c_{11} + uc_{10} + hd_{10} = 0 \\
&\Longrightarrow \; c_{11} = 0, \quad c_{10} = 0, \quad d_{10} = 0
\end{aligned}$$

$$\begin{aligned}
&-\eta^h_h + \eta^u_u + \frac{\eta_h}{h} = \xi^x_x - \xi^t_t - 2u\xi^t_x \\
&\Longrightarrow \; -c_{13} + d_{12} + \frac{1}{h}(c_0 + c_{12}u + c_{13}h = a_1 0 - b_{11} - 2ub_{10} \\
&\Longrightarrow \; a_{10} - b_{11} - d_{12} = 0, \;\; c_0 = 0, \;\; c_{12} = 0, \;\; b_{10} = 0
\end{aligned}$$

$$\begin{aligned}
&-\eta^h_u + h\eta^u_h + \eta^u = \xi^x_t + u\xi^x_x - u\xi^t_t - (u^2 + h)\xi^t_x \\
&\Longrightarrow \; hd_{13} + d_0 + d_{11}t + d_{12}u + d_{13}h = a_{11} - ub_{11} + ua_{10} \\
&\Longrightarrow \; 2d_{13}h + d_0 - a_{11} + d_{11}t + u(d_{12} - a_{10} + b_{11} = 0 \\
&\Longrightarrow \; a_{11} - d_0 = 0, \;\; d_{11} = 0, \;\; d_{13} = 0
\end{aligned}$$

$$\begin{aligned}
&-\xi^x_u - h\xi^t_h + u\xi^t_u = 0 \\
&-a_{12} - hb_{13} + ub_{12} = 0 \\
&\Longrightarrow \; a_{12} = 0, \quad b_{13} = 0, \;\; b_{12} = 0
\end{aligned}$$

$$-\eta^u_u + \eta^h_h = \xi^x_x - \xi^t_t - 2u\xi^t_x$$
$$\implies -d_{12} + c_{13} = a_{10} - b_{11}$$
$$\implies a_{10} - b_{11} + d_{12} - c_{13} = 0$$

$$\xi^x_h - u\xi^t_h + \xi^t_u = 0$$
$$\implies a_{13} = 0, \quad b_{11} = 0$$

$$\eta^h_u - h\eta^u_h + \eta^u = \xi^x_t - u\xi^t_t + u\xi^x_x - (u^2 + h)\xi^t_x$$
$$\implies d_0 + d_{12}u = a_{11} - ub_{11} + ua_{10}$$
$$\implies a_{11} - d_0 = 0, \quad a_{10} - b_{11} - d_{12} = 0$$

And from the equations

$$a_{10} - b_{11} - d_{12} - c_{13} = 0$$
$$a_{10} - b_{11} - d_{12} = 0$$

we get

$$c_{13} = a_{10} - b_{11}$$

also

$$d_{12} = c_{13}$$

So, now we get the infinitesimals as follows:

$$\xi^x = a_0 + a_{10}x + a_{11}t$$
$$\xi^t = b_0 + b_{11}t$$
$$\eta^h = c_{13}h$$
$$\eta^u = d_0 + d_{12}u$$

CHAPTER 2

Lie Group Analysis Of Isentropic Gas Dynamics

2.1 Lie Group Analysis Of Isentropic Gas Dynamics

The system of equations which governs the "*isentropic gas dynamics*" can be written as:

$$\rho_t + \rho u_x + u\rho_x = 0 \tag{2.1}$$

$$u_t + uu_x + k_1\gamma\rho^{\gamma-2}\rho_x = 0 \tag{2.2}$$

where u and ρ are the dependent variables and the independent variables are t and x. Equations (2.1) and (2.2) are a quasilinear system of first order PDEs wih two independent and two dependent variables. In this section, we investigate the most general Lie group of transformations which leaves the equations (2.1) and (2.2) invariant. At first, we consider Lie group of transformations with independent variables x,t and dependent variables u,ρ that are:

$$x^* = x^*(x, t, u, \rho; \epsilon)$$
$$t^* = t^*(x, t, u, \rho; \epsilon)$$
$$u^* = u^*(x, t, u, \rho; \epsilon)$$
$$\rho^* = \rho^*(x, t, u, \rho; \epsilon)$$

where ϵ is the group parameter.
The infinitesimal generators can be expressed as a vector form :

$$V = \xi^x \frac{\partial}{\partial x} + \xi^t \frac{\partial}{\partial t} + \eta^u \frac{\partial}{\partial u} + \eta^\rho \frac{\partial}{\partial \rho}$$

in which ξ^x, ξ^t, η^u, η^ρ are infintesimal functions of the group variables. Thus, the corresponding one-parameter Lie group of transformations is given by

$$x^* = e^{\epsilon V}(x) = x + \epsilon\xi^x(x, t, u, \rho) + O(\epsilon^2)$$
$$t^* = e^{\epsilon V}(t) = t + \epsilon\xi^t(x, t, u, \rho) + O(\epsilon^2)$$
$$u^* = e^{\epsilon V}(u) = u + \epsilon\xi^u(x, t, u, \rho) + O(\epsilon^2)$$
$$\rho^* = e^{\epsilon V}(\rho) = \rho + \epsilon\eta^\rho(x, t, u, \rho) + O(\epsilon^2)$$

Since the system of governing equations has atmost first order derivatives, the first prolongation of generator will be :

$$pr^1V = V + \tau_x^u \frac{\partial}{\partial u_x} + \tau_t^u \frac{\partial}{\partial u_t} + \tau_x^\rho \frac{\partial}{\partial \rho_x} + \tau_t^\rho \frac{\partial}{\partial \rho_t} \tag{2.3}$$

where

$$\begin{aligned}
\tau_t^u &= \eta_t^u + \eta_u^u u_t + \eta_\rho^u \rho_t - u_x(\xi_t^x + \xi_u^x u_t + \xi_\rho^x \rho_t) - u_t(\xi_t^t + \xi_u^t u_t + \xi_\rho^t \rho_t)\\
\tau_x^u &= \eta_x^u + \eta_u^u u_x + \eta_\rho^u \rho_x - u_x(\xi_x^x + \xi_u^x u_x + \xi_\rho^x \rho_x) - u_t(\xi_x^t + \xi_u^t u_x + \xi_\rho^t \rho_x)\\
\tau_t^\rho &= \eta_t^\rho + \eta_u^\rho u_t + \eta_\rho^\rho \rho_t - \rho_x(\xi_t^x + \xi_u^x u_t + \xi_\rho^t \rho_t) - \rho_t(\xi_t^t + \xi_u^t u_t + \xi_\rho^t \rho_t)\\
\tau_x^\rho &= \eta_x^\rho + \eta_u^\rho u_x + \eta_\rho^\rho \rho_x - \rho_x(\xi_x^x + \xi_u^x u_x + \xi_\rho^x \rho_x) - \rho_t(\xi_x^t + \xi_u^t u_x + \xi_\rho^t \rho_x)
\end{aligned}$$

Now, we apply the first prolongation of the infinitesimal generator (2.3) to the system of PDEs (2.1) and (2.2) .
By applying (2.3) to (2.1), we get

$$(\xi^x \frac{\partial}{\partial x} + \xi^t \frac{\partial}{\partial t} + \eta^u \frac{\partial}{\partial u} + \eta^\rho \frac{\partial}{\partial \rho} + \tau_x^u \frac{\partial}{\partial u_x} + \tau_t^u \frac{\partial}{\partial u_t} + \tau_x^\rho \frac{\partial}{\partial \rho_x} + \tau_t^\rho \frac{\partial}{\partial \rho_t})(\rho_t + u\rho_x + \rho u_x) = 0$$

$$\begin{aligned}
&\implies \eta^u \rho_x + \eta^\rho u_x + \rho \tau_x^u + u\tau_x^\rho + \tau_t^\rho = 0\\
&\implies \tau_t^\rho + \eta^u \rho_x + u\tau_x^\rho + \eta^\rho u_x + \rho\tau_x^u = 0
\end{aligned}$$

By applying (2.3) to (2.2), we get

$$(\xi^x \frac{\partial}{\partial x} + \xi^t \frac{\partial}{\partial t} + \eta^u \frac{\partial}{\partial u} + \eta^\rho \frac{\partial}{\partial \rho})(u_t + uu_x + k_1\gamma\rho^{\gamma-2}\rho_x) = 0$$

$$\begin{aligned}
&\implies \tau_t^u + \eta^\rho(\gamma-2)k_1\gamma\rho(\gamma-1)\rho_x + k_1\gamma\rho^{\gamma-2}\tau_x^\rho + \eta^u u_x + u\tau_x^u = 0\\
&\implies \tau_t^u + \eta^u u_x + u\tau_x^u + \eta^\rho k_1\gamma(\gamma-2)\rho^{\gamma-1}\rho_x + k_1\gamma\rho^{\gamma-2}\tau_x^\rho = 0
\end{aligned}$$

We know that the given system of equations are the coupled system of PDEs. So, the above equations must be coupled system of PDEs. These equations can be arranged by using the above explicit expression in, then each of these equations can get a polynomial in terms of dependent variables and in terms of independent variables. Here, one can consider the derivatives of dependent variables w.r.tthe independent variables as independent functions and equate

each coefficients of these functions to zero.
i.e.,
First we consider equation

$$\tau^{\rho}_t + \eta^u \rho_x + u\tau^{\rho}_x + \eta^{\rho} u_x + \rho \tau^u_x = 0$$

$\implies (\eta^{\rho}_t + \eta^{\rho}_u u_t + \eta^{\rho}_{\rho}\rho_t - \rho_x(\xi^x_t + \xi^x_u u_t + \xi^t_{\rho}\rho_t) - \rho_t(\xi^t_t + \xi^t_u u_t + \xi^t_{\rho}\rho_t)) +$
$\eta^u \rho_x + u(\eta^{/}_x rho + \eta^{\rho}_u u_x + \eta^{\rho}_{\rho}\rho_x - \rho_x(\xi^x_x + \xi^x_u u_x + \xi^x_{\rho}\rho_x) - \rho_t(\xi^t_x + \xi^t_u u_x + \xi^t_{\rho}\rho_x))+$
$\eta^{\rho} u_x + \rho(\eta^u_x + \eta^u_u u_x + \eta^u_{\rho}\rho_x - u_x(\xi^x_x + \xi^x_u u_x + \xi_/ rho^x \rho_x) -$
$u_t(\xi^t_x + \xi^t_u u_x + \xi^t_{\rho}\rho_x)) = 0$
$\implies \eta^{\rho}_t + \eta^{\rho}_u u_t + \eta^{\rho}_{\rho}\rho_t + \rho_x \xi^x_t - \xi^x_u u_t \rho_x - \xi^t_{\rho}\rho_t\rho_x - \rho_t\xi^t_t - \rho_t \xi^t_u u_t - \xi^t_{\rho}\rho^2_t + \eta^u\rho_x + u\eta^{\rho}_x +$
$u\eta^{\rho}_u u_x + u\eta^{\rho}_{\rho}\rho_x - u\rho_x\xi^x_x - u\rho_x\xi^x_u u_x - u\xi^x_{\rho}\rho^2_x - u\rho_t\xi^t_x - u\rho_t\xi^t_u u_x - u\rho_t\xi^t_{\rho}\rho_x + \eta^{\rho}u_x +$
$\rho\eta^u_x + \rho\eta^u_u u_x + \rho\eta^u_{\rho}\rho_x - u_x\rho\xi^x_x - \rho\xi^x_u u^2_x - u_x\rho\xi^x_{\rho}\rho_x - \rho u_t\xi^t_x - \rho u_t \xi^t_u u_x - \rho_t\xi^t_{\rho}\rho_x = 0$

Now arranging all the ρ_t terms and using $\rho_t = -u\rho_x - \rho u_x$,we get
$\implies \eta^{\rho}_t + u\eta^{\rho}_x + \rho\eta^u_x + \eta^{\rho}_u u_t - \xi^x_t\rho_x - \xi^x_u\rho_x u_t + \eta^u\rho_x + u\eta^{\rho}_u u_x + u\eta^{\rho}_{\rho}\rho_x -$
$u\xi^x_x\rho_x - u\xi^x_u\rho_x u_x - \rho\xi^x_u u^2_x - \rho\xi^x_{\rho}\rho_x u_x - \rho\xi^t_x u_t - \rho\xi^t_u u_t u_x - \rho\xi^t_{\rho}u_t\rho_x - u\eta^{\rho}_{\rho}\rho_x +$
$u\xi^x_{\rho}\rho^2_x + u\xi^t_t\rho_x + u\xi^t_u\rho_x u_t + u^2\xi^t_x\rho^2_x + u^2\xi^t_u u_x\rho_x + u^2\xi^t_{\rho}\rho^2_x - \rho\eta^{\rho}_{\rho}u_x + \rho\xi^x_{\rho}\rho_x u_x +$
$\rho\xi^t_t u_x + \rho\xi^t_u u_t u_x + \rho u\xi^t_x u_x + u\rho\xi^t_u u^2_x + u\rho\xi^t_{\rho}\rho_x u_x - u^2\xi^t_{\rho}\rho^2_x - \rho^2\xi^t_{\rho}u^2_x - 2u\rho\xi^t_{\rho}\rho_x u_x = 0$

Comparing the constants and coefficients of the independent functions, we get

$$\implies \eta^{\rho}_t + u\eta^{\rho}_x + \rho\eta^u_x = 0$$

$$u_t(\eta^{\rho}_u - \rho\xi^t_x) = 0$$
$$\implies \eta^{\rho}_u = \rho\xi^t_x$$

$\rho_x(-\xi^x_t + \eta^u + u\eta^{\rho}_{\rho} - u\xi^x_x + \rho\eta^u_{\rho} - u\eta^{\rho}_{\rho} + u\xi^t_t + u^2\xi^t_x) = 0$
$\implies \rho\eta^u_{\rho} + \eta^u = \xi^x_t + u\xi^x_x - u\xi^t_t - u^2\xi^t_x$
$\implies -\eta^{\rho}_u + \rho\eta^u_{\rho} + \eta^u = \xi^x_t + u\xi^x_x - u\xi^t_t - u^2\xi^t_x - \rho\xi^t_x \quad (since, \eta^{\rho}_u = \rho\xi^t_x)$
$\implies -\eta^{\rho}_u + \rho\eta^u_{\rho} + \eta^u = \xi^x_t + u\xi^x_x - u\xi^t_t - (u^2 + \rho)\xi^t_x$

$u_x(u\eta^{\rho}_u + \eta^{\rho} + \rho\eta^u_u - \rho\xi^x_x - \rho\eta^{\rho}_{\rho} + \rho\xi^t_t + u\rho\xi^t_x) = 0$
$\implies -\rho\eta^{\rho}_{\rho} + \rho\eta^u_u + \eta_{\rho} + u\eta^{\rho}_u = \rho\xi^x_x - \rho\xi^t_t - u\rho\xi^t_x$
$\implies -\eta^{\rho}_{\rho} + \eta^u_u + \frac{\eta_{\rho}}{\rho} + \frac{u}{\rho}\eta^{\rho}_u = \xi^x_x - \xi^t_t - u\xi^t_x$

$\implies -\eta^\rho_\rho + \eta^u_u + \frac{\eta_\rho}{\rho} + \frac{u}{\rho}\rho\xi^t_x = \xi^x_x - \xi^t_t - u\xi^t_x (since, \eta^\rho_u = \rho\xi^t_x)$
$\implies -\eta^\rho_\rho + \eta^u_u + \frac{\eta_\rho}{\rho} = \xi^x_x - \xi^t_t - 2u\xi^t_x$
$\rho_x u_t(-\xi^x_u - \rho\xi^t_\rho + u\xi^t_u) = 0$
$\implies -\xi^x_u - \rho\xi^t_\rho + u\xi^t_u = 0$
Now, Let us consider equation

$$\tau^u_t + \eta^u u_x + u\tau^u_x + \eta^\rho k_1\gamma(\gamma-2)\rho^{\gamma-1}\rho_x + k_1\gamma\rho^{\gamma-2}\tau^\rho_x = 0$$

$\implies \eta^\rho k_1\gamma(\gamma-2)\rho^{\gamma-3}\rho_x + \eta^u u_x + \eta^u_t + \eta^u_u u_t + \eta^u_\rho\rho_t - u_x\xi^x_t - \xi^x_u u_x u_t - \xi^x_\rho\rho_t u_x - \xi^t_\rho u_t - \xi^t_u u_t^2 - \xi^t_\rho\rho_t u_t + u(\eta^u_x + \eta^u_u u_x + \eta^u_\rho\rho_x - u_x\xi^x_x - \xi^x_u u_x^2 - u_x\xi^x_\rho\rho_x - u_t\xi^t_x - u_t\xi^t_u u_x - u_t\xi^t_\rho\rho_x) + k_1\gamma\rho^{\gamma-2}(\eta^\rho_x + \eta^\rho_u u_x + \eta^\rho_\rho\rho_x - \rho_x\xi^x_x - \rho_x\xi^x_u u_x - \xi^x_\rho\rho_x^2 - \rho_t\xi^t_x - \rho_t\xi^t_u u_x - \rho_t\xi^t_\rho\rho_x) = 0$
$\implies \eta^u_t + u\eta^u_x + k_1\gamma\rho^{\gamma-2}\eta^\rho_x + \rho_x(\eta^\rho k_1\gamma(\gamma-2)\rho^{\gamma-3} + u\eta^u_\rho + k_1\gamma\rho^{\gamma-2}\eta^\rho_\rho - k_1\gamma\rho^{\gamma-2}\xi^x_x) + u_x(\eta^u - \xi^x_t + u\eta^u_u - u\xi^x_x + k_1\gamma\rho^{\gamma-2}\eta^\rho_u) + u_t(\eta^u_u - \xi^x_u u_x - \xi^t_t - \xi^t_\rho\rho_t - u\xi^t_x - u\xi^t_u u_x - u\xi^t_\rho\rho_x) - \xi^t_u u_t^2 + \rho_x u_x(-u\xi^x_\rho - k_1\gamma\rho^{\gamma-2}\xi^x_u) + \rho_t u_x(-\xi^x_\rho - k_1\gamma\rho^{\gamma-2}\xi^t_u) + \rho_x\rho_t(-k_1\gamma\rho^{\gamma-2}\xi^t_\rho) + \rho_x^2(-k_1\gamma\rho^{\gamma-2}\xi^x_\rho) - u_x^2 u\xi^x_u + \rho_t(\eta^u_\rho - k_1\gamma\rho^{\gamma-2}\xi^t_x) = 0$

Now, By using $u_t = -uu_x - k_1\gamma\rho^{\gamma-2}\rho_x$, we get :
$\implies \eta^u_t + u\eta^u_x + k_1\gamma\rho^{\gamma-2}\eta^\rho_x + \rho_x(\eta^\rho k_1\gamma(\gamma-2)\rho^{\gamma-2} + u\eta^u_\rho + k_1\gamma\rho^{\gamma-2}\eta^\rho_\rho - k_1\gamma\rho^{\gamma-2}\xi^x_x - k_1\gamma\rho^{\gamma-2}\eta^u_u + k_1\gamma\rho^{\gamma-2}\xi^t_t + uk_1\gamma\rho^{\gamma-2}\xi^t_x) + u_x(\eta^u - \xi^x_t + u\eta^u_u - u\xi^x_x + k_1\gamma\rho^{\gamma-2}\eta^\rho_u - u\eta^u_u + u\xi^t_t + u^2\xi^t_x) + \rho_x u_x(-u\xi^x_\rho - k_1\gamma\rho^{\gamma-2}\xi^x_u + u^2\xi^t_\rho + k_1\gamma\rho^{\gamma-2}\xi^x_u + k_1\gamma\rho^{\gamma-2}u\xi^t_u - 2k_1\gamma\rho^{\gamma-2}u\xi^t_u) + \rho_t u_x(-\xi^x_\rho - k_1\gamma\rho^{\gamma-2}\xi^t_u + u\xi^t_\rho) + \rho_x\rho_t(-k_1\gamma\rho^{\gamma-2}\xi^t_\rho + k_1\gamma\rho^{\gamma-2}\xi^t_\rho) + \rho_x^2(-k_1\gamma\rho^{\gamma-2}\xi^x_\rho + k_1\gamma\rho^{\gamma-2}u\xi^t_\rho - (k_1\gamma\rho^{\gamma-2})^2\xi^t_u) + u_x^2(0) + u_t(\eta^u_\rho - k_1\gamma\rho^{\gamma-2}\xi^t_x) = 0$
Comparing the constants and coefficients of the independent functions to zero, we get

$$\eta^u_t + u\eta^u_x + k_1\gamma\rho^{\gamma-2}\eta^\rho_x = 0$$

$$u_t(\eta^u_\rho - k_1\gamma\rho^{\gamma-2}\xi^t_x) = 0$$
$$\implies \eta^u_\rho = k_1\gamma\rho^{\gamma-2}\xi^t_x$$

$\rho_x(\eta^\rho k_1\gamma(\gamma-2)\rho^{\gamma-3} + u\eta^u_\rho + k_1\gamma\rho^{\gamma-2}\eta^\rho_\rho - k_1\gamma\rho^{\gamma-2}\xi^x_x - k_1\gamma\rho^{\gamma-2}\eta^u_u + k_1\gamma\rho^{\gamma-2}\xi^t_t + uk_1\gamma\rho^{\gamma-2}\xi^t_x) = 0$
$\implies \eta^\rho k_1\gamma(\gamma-2)\rho^{\gamma-2} + u\eta^u_\rho + k_1\gamma\rho^{\gamma-2}\eta^\rho_\rho - k_1\gamma\rho^{\gamma-2}\eta^u_u = k_1\gamma\rho^{\gamma-2}\xi^x_x - k_1\gamma\rho^{\gamma-2}\xi^t_t - uk_1\gamma\rho^{\gamma-2}\xi^t_x$ (since $\eta^u_\rho = k_1\gamma\rho^{\gamma-2}\xi^t_x$)
$\implies u\eta^u_\rho + k_1\gamma\rho^{\gamma-2}\eta^\rho_\rho - k_1\gamma\rho^{\gamma-2}\eta^u_u + k_1\gamma(\gamma-2)\rho^{\gamma-3}\eta^\rho = k_1\gamma\rho^{\gamma-2}\xi^x_x - k_1\gamma\rho^{\gamma-2}\xi^t_t -$

$uk_1\gamma\rho^{\gamma-2}\xi^t_x$

$\implies u(k_1\gamma\rho^{\gamma-2}\xi^t_x) + k_1\gamma\rho^{\gamma-2}\eta^\rho_\rho - k_1\gamma(\gamma-2)\rho^{\gamma-3}\eta^u_u + k_1\gamma(\gamma-2)\rho^{\gamma-3}\eta^\rho = k_1\gamma\rho^{\gamma-2}\xi^x_x - k_1\gamma\rho^{\gamma-2}\xi^t_t - uk_1\gamma\rho^{\gamma-2}\xi^t_x$

$\implies \eta^\rho_\rho - \eta^u_u + (\gamma-2)\frac{\eta^\rho}{\rho} = \xi^x_x - \xi^t_t - 2u\xi^t_x$

$u_x(\eta^u - \xi^x_t - u\xi^x_x + k_1\gamma\rho^{\gamma-2}\eta^\rho_u + u\xi^t_t + u^2\xi^t_x) = 0$

$\implies \eta^u - \xi^x_t - u\xi^x_x + k_1\gamma\rho^{\gamma-2}\eta^\rho_u + u\xi^t_t + u^2\xi^t_x = 0$

$\implies \eta^u + k_1\gamma\rho^{\gamma-2}\eta^\rho_u = \xi^x_t + u\xi^x_x - u\xi^t_t - u^2\xi^t_x$

$\implies \eta^u + k_1\gamma\rho^{\gamma-2}\eta^\rho_u - \eta^u_\rho = \xi^x_t + u\xi^x_x - u\xi^t_t - (u^2 + k_1\gamma\rho^{\gamma-2})\xi^t_x$

$\rho_x u_x(-u\xi^x_\rho + u^2\xi^t_\rho - uk_1\gamma\rho^{\gamma-2}\xi^t_u) = 0$

$\implies -\xi^x_\rho + u\xi^t_\rho - k_1\gamma\rho^{\gamma-2}\xi^t_u = 0$

$\implies \xi^x_\rho - u\xi^t_\rho + k_1\gamma\rho^{\gamma-2}\xi^t_u = 0$

So, the over-determined system of equations are as follows :

$$\begin{aligned}
\eta^\rho_t + u\eta^\rho_x + \rho\eta^u_x &= 0 \\
\eta^\rho_u - \rho\xi^t_x &= 0 \\
-\xi^x_u - \rho\xi^t_\rho + u\xi^t_u &= 0 \\
-\eta^\rho_\rho + \eta^u_u + \frac{\eta_\rho}{\rho} &= \xi^x_x - \xi^t_t - 2u\xi^t_x \\
-\eta^\rho_u + \rho\eta^u_\rho + \eta^u &= \xi^x_t + u\xi^x_x - u\xi^t_t - (u^2+\rho)\xi^t_x \\
\eta^u_\rho - k_1\gamma\rho^{\gamma-2}\xi^t_x &= 0 \\
\eta^\rho_\rho - \eta^u_u + (\gamma-2)\frac{\eta^\rho}{\rho} &= \xi^x_x - \xi^t_t - 2u\xi^t_x \\
\eta^u + k_1\gamma\rho^{\gamma-2}\eta^\rho_u - \eta^u_\rho &= \xi^x_t + u\xi^x_x - u\xi^t_t - (u^2 + k_1\gamma\rho^{\gamma-2})\xi^t_x \\
\xi^x_\rho - u\xi^t_\rho + k_1\gamma\rho^{\gamma-2}\xi^t_u &= 0 \\
\eta^u_t + u\eta^u_x + k_1\gamma\rho^{\gamma-2}\eta^\rho_x &= 0
\end{aligned}$$

that are called determining equations in terms of infinitesimals.

2.2 Solutions of Determining Equations

Here, we find the solutions ξ^x, ξ^t, η^u, η^ρ of the above determining equations. There are several solution techniques to deal with the determining equations in the Lie group analysis of differential equations. The power se-

ries of a solution form is one of these solution techniques.
So, here we first choose the first order of power series of the infinitesimals which are given by:

$$\xi^x = a_0 + a_{10}x + a_{11}t + a_{12}u + a_{13}\rho$$
$$\xi^t = b_0 + b_{10}x + b_{11}t + b_{12}u + b_{13}\rho$$
$$\eta^\rho = c_0 + c_{10}x + c_{11}t + c_{12}u + c_{13}\rho$$
$$\eta^u = d_0 + d_{10}x + d_{11}t + d_{12}u + d_{13}\rho$$

Now, by substituting the above power series forms into the determining equations, we obtain the equations with powers of the variables x, t, u, ρ and calculate the constant coefficients of the power series forms by equating each coefficient of various powers to zero. Now,

$$\eta^\rho_t + u\eta^\rho_x + \rho\eta^u_x = 0$$
$$\implies c_{11} + uc_{10} + \rho d_{10} = 0$$
$$\implies c_{11} = 0, \quad c_{10} = 0, \quad d_{10} = 0$$

$$\eta^\rho_u - \rho\xi^t_x = 0$$
$$\implies c_{12} - \rho b_{10} = 0$$
$$\implies c_{12} = 0, \quad b_{10} = 0$$

$$-\xi^x_u - \rho\xi^t_\rho + u\xi^t_u = 0$$
$$\implies -a_{12} - \rho b_{13} + ub_{12} = 0$$
$$\implies a_{12} = 0, \quad b_{13} = 0, \quad b_{12} = 0$$

$$-\eta^\rho_\rho + \eta^u_u + \frac{\eta_\rho}{\rho} = \xi^x_x - \xi^t_t - 2u\xi^t_x$$
$$\implies -c_{13} + d_{12} + \frac{1}{\rho}(c_0 + c13\rho) = a_{10} - b_{11}$$
$$\implies c_0 = 0, \quad d_{12} - a_{10} + b_{11} = 0$$

$$-\eta^\rho_u + \rho\eta^u_\rho + \eta^u = \xi^x_t + u\xi^x_x - u\xi^t_t - (u^2 + \rho)\xi^t_x$$
$$\implies \rho d_{13} + d_0 + d_{11}t + d_{12}u + d13\rho = a_{11} + ua_{10} - ub_{11}$$
$$\implies 2d_{13}\rho + d_0 - a_{11} + d_{11}t + u(d_{12} - a_{10} + b_{11} = 0$$
$$\implies d_{13} = 0, \quad d_{11} = 0, \quad d_0 = a_{11}, \quad d_{12} - a_{10} + b_{11} = 0$$

$$\eta^\rho_\rho - \eta^u_u + (\gamma-2)\frac{\eta^\rho}{\rho} = \xi^x_x - \xi^t_t - 2u\xi^t_x$$
$$\implies c_{13} - d_{12} + (\gamma-2)c_{13} = a_{10} - b_{11}$$
$$\implies c_{13} = 0, \quad d_{12} + a_{10} - b_{11} = 0$$

$$\eta^u + k_1\gamma\rho^{\gamma-2}\eta^\rho_u - \eta^u_\rho = \xi^x_t + u\xi^x_x - u\xi^t_t - (u^2 + k_1\gamma\rho^{\gamma-2})\xi^t_x$$
$$\implies d_0 + d_{12}u = a_{11} + a_{10}u - b_{11}u$$
$$\implies d_0 = a_{11}, \quad d_{12} - a_{10} + b_{11} = 0$$

$$\xi^x_\rho - u\xi^t_\rho + k_1\gamma\rho^{\gamma-2}\xi^t_u = 0$$
$$\implies a_{13} = 0.$$

And from the Eqs.:

$$d_{12} + a_{10} - b_{11} = 0$$
$$d_{12} - a_{10} + b_{11} = 0$$

we get:

$$2d_{12} = 0$$
$$\implies d_{12} = 0$$

So, now we get the infinitesimals as follows:

$$\xi^x = a_0 + a_{10}x + a_{11}t$$
$$\xi^t = b_0 + a_{10}t$$
$$\eta^\rho = 0$$
$$\eta^u = a_{11}$$

2.3 Similarity Analysis

In this section, we want to reduce PDEs (2.1) and (2.2) to a system of ODEs by constructing similarity variables. Let us denote ξ^t, ξ^x, η^u, η^ρ by

ϕ_1, ϕ_2, ψ_1 and ψ_2 respectively. So,

$$\phi_1 = b_0 + a_{10}t$$
$$\phi_2 = a_0 + a_{10}x + a_{11}t$$
$$\psi_1 = a_{11}$$
$$\psi_2 = 0$$

The characteristic equations are :

$$\frac{dt}{\phi_1} = \frac{dx}{\phi_2} = \frac{du}{\psi_1} = \frac{d\rho}{\psi_2} \tag{2.4}$$

Solving these characteristic equations, we get similarity variable θ which is given as a constant in the solution. Here, we distinguish 4 cases:
Case 1: $a_0 = 0, a_{10} = 0\,, a_{11} = 0$
Case 2: $a_0 = 0, a_{10} \neq 0, a_{11} \neq 0$
Case 3: $a_0 \neq 0, a_{10} = 0, a_{11} \neq 0$
Case 4: $a_0 \neq 0, a_{10} \neq 0, a_{11} \neq 0$
We solve the above distinguished cases one by one to get the similarity variable in each case.
CASE 1: $a_0 = 0, \quad a_{10} = 0, \quad a_{11} = 0$
Using the characteristic equation (2.4)

$$\frac{dt}{b_0 + a_{10}t} = \frac{dx}{a_0 + a_{10}x + a_{11}t} = \frac{du}{a_{11}} = \frac{d\rho}{0}$$

$$\implies \frac{dt}{b_0} = \frac{dx}{0}$$
$$\implies dx = 0$$
$$\implies \theta = x$$

and

$$\frac{dt}{b_0} = \frac{du}{0}$$
$$\implies U(\theta) = u(x,t)$$

also

$$\frac{dt}{b_0} = \frac{d\rho}{0}$$
$$\implies R(\theta) = \rho(x,t)$$

So, we get

$$\theta = x$$
$$U(\theta) = u(x,t)$$
$$R(\theta) = \rho(x,t)$$

where $U(\theta)$ and $R(\theta)$ are the integration constants and are the new dependent variables.
Substituting these new dependent variables into equation (2.1) and (2.2), we get a system of ODEs with independent variable θ.

1. $\rho_t + \rho u_x + u\rho_x = 0$

$$\rho_t = \frac{\partial R}{\partial \theta}.\frac{\partial \theta}{\partial t} = R'.0 = 0$$
$$\rho_x = \frac{\partial R}{\partial \theta}.\frac{\partial \theta}{\partial x} = R'$$
$$u_x = \frac{\partial U}{\partial \theta}.\frac{\partial \theta}{\partial x} = U'$$

$$\implies UR' + RU' = 0$$

$u_t + uu_x + k_1\gamma\rho^{\gamma-2}\rho_x = 0$

$$u_t = \frac{\partial U}{\partial \theta}.\frac{\partial \theta}{\partial t} = 0$$
$$\implies UU' + k_1\gamma R^{\gamma-2}R' = 0$$

Solving the above system of ODEs, we get

$$\begin{aligned}
UR' + RU' &= 0 \\
\implies \frac{R'}{R} + \frac{U'}{U} &= 0 \\
\implies \ln R + \ln U &= \ln k_1 \\
\implies R &= \frac{k_1}{U} \\
\implies \rho &= \frac{k_1}{U} \quad (since, R(\theta) = \rho(x,t))
\end{aligned}$$

and

$$\begin{aligned}
UU' + k_1\gamma R^{\gamma-2}R' &= 0 \\
\implies \frac{u^2}{2} + \frac{k_1\gamma}{\gamma-1}R^{\gamma-1} &= k_2 \\
\implies \frac{u^2}{2} + \frac{k_1\gamma}{\gamma-1}\rho^{\gamma-1} &= k_2 \quad (since, R(\theta) = \rho(x,t))
\end{aligned}$$

Case 2: $a_0 = 0, \quad a_{10} \neq 0, \quad a_{11} \neq 0$
Using the characteristic equation (2.4)

$$\frac{dt}{b_0 + a_{10}t} = \frac{dx}{a_0 + a_{10}x + a_{11}t} = \frac{du}{a_{11}} = \frac{d\rho}{0}$$

$$\begin{aligned}
\implies \frac{dt}{b_0} = \frac{dx}{a_{10}t + a_{11}} &= \frac{du}{a_{10}} = \frac{d\rho}{0} \\
\implies (a_{10} + a_{11})dt &= b_0 dx \\
\implies b_0 x - a_{10}\frac{t^2}{2} - a_{11}t &= \theta
\end{aligned}$$

and

$$\begin{aligned}
\frac{dt}{b_0} &= \frac{du}{a_{10}} \\
\implies \frac{a_{10}}{b_0}t + U(\theta) &= u(x,t) \\
\implies U(\theta) &= u(x,t) - \frac{a_{10}}{b_0}t
\end{aligned}$$

also

$$\frac{dt}{b_0} = \frac{d\rho}{0}$$
$$\implies R(\theta) = \rho(x,t)$$

where $U(\theta)$ and $R(\theta)$ are the integration constants and are the new dependent variables.
Substituting these new dependent variables into equation (2.1) and,(2.2) we get a system of ODEs with independent variable θ.

1. $\rho_t + \rho u_x + u\rho_x = 0$

$$\rho_t = \frac{\partial R}{\partial \theta}.\frac{\partial \theta}{\partial t} = R'.(-a_{10}t - a_{11})$$

$$\rho_x = \frac{\partial R}{\partial \theta}.\frac{\partial \theta}{\partial x} = R'(b_0)$$
$$u_x = \frac{\partial U}{\partial \theta}.\frac{\partial \theta}{\partial x} = U'(b_0)$$

$$u_t = \frac{\partial U}{\partial \theta}.\frac{\partial \theta}{\partial t} = U'(b_0 t - a_{11})$$
$$\implies R'(-a_{10}t - a_{11}) + (U + \frac{a_{10}}{b_0}t)R'(b_0) + b_0 R U' = 0$$

$$\implies R'(b_0 U - a_{11}) + b_0 R U' = 0$$
$$u_t + u u_x + k_1 \gamma \rho^{\gamma-2} \rho_x = 0$$

$$\implies U'(-a_1 0t - a_{11}) + \frac{a_{10}}{b_0} + (b_0 U + a_{10}t)U' + b_0 k_1 \gamma R^{\gamma-2} R' = 0$$
$$\implies U'(b_0 U - a_{11}) + k_1 \gamma a_4 R^{\gamma-2} R' + \frac{a_{10}}{b_0} = 0$$

Now solving the above ODEs,we get for $U = \frac{a_{11}}{b_0}$,
the above equation will be:

$$k_1 \gamma a_4 R^{\gamma-2} R' + \frac{a_1 0}{b_0} = 0$$

$$\implies \frac{k_1\gamma a_4 R^{\gamma-1}R'}{\gamma-1} + \frac{a_10}{b_0}(b_0x - \frac{a_10}{2}t^2 - a_11t) = c_2$$

$$\implies k_1\gamma a_4 R^{\gamma-1}R' + \frac{a_10}{b_0}(b_0x - \frac{a_10}{2}t^2 - a_11t)(\gamma-1) = (\gamma-1)c_2$$

$$\implies k_1\gamma a_4 R^{\gamma-1}R' = \frac{a_10}{b_0}(\gamma-1)(\frac{a_10}{2}t^2 - a_11t - b_0x) + c_3$$

$$\implies R^{\gamma-1} = [\frac{\frac{a_10}{b_0}(\gamma-1)(\frac{a_10}{2}t^2 + a_11t - b_0x)}{k_1\gamma b_0}] + c_3$$

$$\implies R = [\frac{a_10(\gamma-1)(\frac{a_10}{2}t^2 + a_11t - b_0x)}{k_1\gamma {b_0}^2}] + c_3$$

CONCLUSION

We studied the systematic procedure for solving partial differential equations by using Lie group analysis. We applied this procedure to isentropic gasdynamics. By using infinitesimal transformations, we reduced the partial differential equations to ordinary differential equations; in some cases, we obtained the exact solution and in the other one can solve numerically.

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