# Riemann problems for hyperbolic systems

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# CERTIFICATE

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*This is to certify that the dissertation entitled**“Riemann problems for hyperbolic systems”** being submitted by **T.Swati** to the Department of mathematics, National Institute of Technology, Rourkela, Odisha, for the award of the degree of Master of Science in mathematics is a record of bonafide research work carried out by them under my supervision and guidance. I am satisfied that the dissertation report has reached the standard fulfilling the requirements of the regulations relating to the nature of the degree.*

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# DECLARATION

I hereby certify that the work which is being presented in thesis entitled ***" RiemannProblems for hyperbolic systems"*** in partial fulfillment of the requirement for the award of the Degree of Master of Science, submitted in the Department of Mathematics, National Institute of Technology, Rourkela is an authentic record of my work carried out under the supervision

of Dr**. Raja SekharTungala.**

The matter embodied in this has not been submitted by me for the award of any other degree.

(**T.Swati**)

This is to certify that the above statement made by the candidate is carried to the best of the Knowledge.

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***T.Swati***

**Abstract**

In this report, we defined hyperbolic system and given some examples. We study the behaviour of hyperbolic system. Later, we revised the exact solution of the Riemann Problem for the non- linear PDE, which in hyperbolic system of the general form of conservation laws which governs one-dimensional isentropic magnetogasdynamics. Lastly, we find the solution using phase plane analysis and interactions of elementary waves between the same families as well as different families.

**INTRODUCTION**

The Riemann problem is defined as the initial value problem for the system with two valued piecewise constant initial data. The Riemann problem is a fundamental tool for studying the interaction between waves. It has played a central role both in the theoretical analysis of systems of hyperbolic conservation laws and in the development and implementation of practical numerical solutions of such systems.

Basically, the Riemann problem gives the micro-wave structural of the flow. One can think of the propagation of the flow as a set of small scale Riemann problem between the wave arising from these Riemann problems.

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# Chapter-1

# Introduction to Hyperbolic Systems

## 1.1Definitions and Examples:

The general form of system of conservation laws in several space variables

$$\frac{\partial u}{\partial t}+\sum_{j=1}^{d}\frac{\partial}{\partial x_j}f_j(u)=0\ . \quad (1.1)$$

Here $\Omega$ be an open subset of $\mathrm{R}^p$, $\mathrm{f_j}:\Omega\rightarrow R^p$; where $u: R^p\times[0,+\infty[\ \rightarrow\ \Omega,$

$u=(u_1,u_2.........,u_p)\ ,X=(x_1,x_2.........,x_d)\in R^d, t>0$.

The set $\Omega$ is called, the set of states and the functions, $f_j=(f_{1j}...........,f_{pj})$ are called flux functions, the system (1.1) is written in conservation form, the conservation of the p real quantities $u_1,u_2,.........,u$ .We have a simplest differential equation model for a fluid flow:

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)=0.$$

This equation is called inviscid Burger's equation, which is also known as one- dimensional conservation law.

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}f(u)+\frac{\partial}{\partial y}g(u)=0\ ,$$

which is a two dimensional equation. From this equation, we get following system of two dimensional equations:

$$\frac{\partial u_1}{\partial t}+\frac{\partial f_1(u_1,u_2)}{\partial x}+\frac{\partial g_1(u_1,u_2)}{\partial y}=0$$

$$\frac{\partial u_2}{\partial t}+\frac{\partial f_2(u_1,u_2)}{\partial x}+\frac{\partial g_2(u_1,u_2)}{\partial y}=0$$

Let D be an arbitrary domain of $\mathrm{R}^p$ and let $n=(n_1,............,n_d)^T$ be the outward unit normal to the boundary $\partial\mathrm{D}$ of D. Then, it follow from (1.1) that,

$$\frac{\partial}{\partial t}\int_D u\ dx+\sum_{j=1}^{d}\int_{\partial D}f_j(u)\ n_j\ ds=0.$$

This is conservation law in integral form. This equation has a physical meaning that the Variation of $\int_D u\ dx$ is equal to the losses through the boundary $\partial\mathrm{D}$.

## 1.2 Hyperbolic System of Conservation Laws:

For all $j = 1,......,d$ , let $A_j(u) = \left( \frac{\partial f_{ij}(u)}{\partial u_k} \right)$ $1 \le i, k \le p$ be an Jacobian matrix of $f_j(u)$;

equation (1.1)is called a hyperbolic system .

If for any $u \in \Omega$ and $w = \left( w_1,........,w_d \right) \in R^d, w \ne 0,$

the matrix $A(u,w) = \sum_{j=1}^{d} w_j A_j(u)$ has p real eigenvalues with Independent eigenvectors

$r_1(u,w), r_2(u,w), ....., r_p(u,w),$ *i.e.*

$A(u,w) r_k(u,w) = \lambda_k(u,w)\, r_k(u,w),\ 1 \le k \le$

$r_k(u,w)$ are right eigenvectors.

$l_k(u,w) A(u,w) = \lambda_k(u,w)\, l_k(u,w), \quad 1 \le k \le p.$

$l_k(u,w)$ are left eigenvectors.

If $A(u,w)$ has p real eigenvalues and p corresponding linear independent eigenvectors, and if $\lambda_k(u,w)$ real distinct eigenvalues, then the system is called strictly hyperbolic.

### Example:

1) $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left( \frac{u^2}{2} \right) = 0$

Let $f(u) = \left( \frac{u^2}{2} \right)$, then $A = \left( \frac{\partial f}{\partial u} \right) = [u]_{1\times 1}$

Here the eigenvalue is 1 and eigenvector is u.

**Example:**

2) $\frac{\partial v}{\partial t} \quad \frac{\partial u}{\partial x} = 0; \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(p(v)) = 0$

$$u = (v,u),\ f = \begin{bmatrix} -u \\ p(v) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial u} \end{bmatrix}$$

$$\lambda^2 + p'(v) = 0$$

$$\lambda^2 = \quad p^{'}(v)$$
$$\lambda = \pm\sqrt{\quad p^{'}(v)} \quad \forall p^{'}(v) < 0.$$

It is hyperbolic system. If the eigenvalues $\lambda_x(u,w)$ are all distinct. The system (1.1) is called strictly hyperbolic.

## 1.3 Cauchy Problem:

Let

$u_t + (f(u))_x = 0,\ x(s) = s, t(s) = 0$

be the partial differential equation with initial data of the curve. We have the surface which contains the curve is called Cauchy problem. $u(x,t): R^d \times [0,+\infty[ \to \Omega$ , for t>0 and $u_0$ is the function of x alone and which have initial value $u_0 : R^d \to \Omega$

$$u_0 = \begin{cases} u_l, x < 0 \\ u_r, x > 0 \end{cases}.$$

Where $u_l$ and $u_r$ are constants, then the Cauchy problem is called Riemann problem.

## 1.4 Riemann Problem:

The conservation laws is given,

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(f(u)) = 0.$$

Let $u_l$ and $u_r$ be two states of $\Omega \subset R^p$; we have for piecewise smooth continuous function $u:(x,t) \to u(x,t)$ solutions of (1.1) that connection $u_l$ and $u_r$ :with initial condition

$$u_0 = \begin{cases} u_l,\ x < 0 \\ u_r,\ x > 0 \end{cases}$$

is called Riemann problem.

**3)Example:**

The equation of gas dynamics in Eulerian coordinate:

In Eulerian coordinates, the Euler equations for a compressible inviscid fluid in the conservation form.

$$\frac{\partial \rho}{\partial t} + \sum_{j=1}^{3} \frac{\partial}{\partial x}(\rho u_j) = 0$$
$$\frac{\partial(\rho u_i)}{\partial t} + \sum_{j=1}^{3} \frac{\partial}{\partial x}(\rho u_j u_i + p\delta_{ij}) = 0, \qquad 1 \le i \le 3,.$$

$$\frac{\partial(\rho e)}{\partial t} + \sum_{j=1}^{3} \frac{\partial}{\partial x}\left((\rho e + p)u_j\right) = 0$$

$\rho =$ density of the fluid , $u = (u_1, u_2, u_3)$ the velocity , $p =$ pressure, $\varepsilon =$ specific internal energy ,

$e = \varepsilon + \frac{|u^2|}{2}$ the specific total energy

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0,$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}\left(\rho u^2 + p\right) = 0,$$

$$\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}\left((\rho e + p)u\right) = 0.$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial}{\partial x}(\rho) + \rho\frac{\partial}{\partial x}(u) = 0$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}\left(\rho u^2 + p\right) = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial \rho u}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}\left((\rho e + p)u\right) = 0$$

$$\frac{\partial}{\partial t}\left(p\frac{\rho}{(\gamma-1)\rho}\right) + \frac{\partial}{\partial x}\left(p\frac{\rho}{(\gamma-1)\rho} + p\right)u = 0$$

$$\frac{\partial}{\partial t}\left(p\frac{1}{(\gamma-1)}\right) + \frac{\partial}{\partial x}\left(p\frac{1}{(\gamma-1)} + p\right)u = 0$$

$$\frac{\partial}{\partial t}p + \gamma p\frac{\partial}{\partial x}u + u\frac{\partial}{\partial x}p = 0$$

$$\begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_t + \begin{bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \gamma & \gamma p u \end{bmatrix} \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_x = 0$$

$$(A - \lambda I) = 0$$

$$\begin{bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \gamma p & u \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} u - \lambda & \rho & 0 \\ 0 & u - \lambda & \frac{1}{\rho} \\ 0 & \gamma & u - \lambda \end{bmatrix} = 0$$

$$(u-\lambda)\begin{vmatrix} u-\lambda & \frac{1}{\rho} \\ \gamma p & u-\lambda \end{vmatrix} = 0$$

$$(u-\lambda)\left[(u-\lambda)^2 - \frac{\gamma p}{\rho}\right] = 0$$

$$(u-\lambda)^2 = \frac{\gamma p}{\rho}$$

$$(u-\lambda) = \pm\sqrt{\frac{\gamma p}{\rho}}$$

eigenvalue s are $\lambda = u + \sqrt{\frac{\gamma p}{\rho}}, u - \sqrt{\frac{\gamma p}{\rho}}, u.$

$for\ \lambda = u,$

$(A - \lambda I)x = 0$

$$\begin{bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \gamma p & u \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} u - \lambda & \rho & 0 \\ 0 & \mathrm{u} - \lambda & \frac{1}{\rho} \\ 0 & \gamma \mathrm{p} & \mathrm{u} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\rho x_2 = 0$

$\frac{1}{\rho} x_3 = 0$

$\gamma p x_2 = 0$

$0 x_1 = 0$

assume $x_1 = 1$

$x_2 = 0$

$x_3 = 0$

The eigenvalue $\lambda = u$ with Corresponding eigenvector is (1,0,0).

2)

$$\lambda = u + \sqrt{\frac{\gamma p}{\rho}}$$

$$\begin{bmatrix} -\sqrt{\frac{\gamma p}{\rho}} & \rho & 0 \\ 0 & -\sqrt{\frac{\gamma p}{\rho}} & \frac{1}{\rho} \\ 0 & \gamma p & -\sqrt{\frac{\gamma p}{\rho}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-\sqrt{\frac{\gamma p}{\rho}}\ x_1 + \rho x_2 = 0 \qquad (1.2)$$

$$-\sqrt{\frac{\gamma p}{\rho}}\ x_2 + \frac{1}{\rho} x_3 = 0 \qquad (1.3)$$

$$\gamma p x_2 + \left( -\sqrt{\frac{\gamma p}{\rho}} x_3 \right) = 0 \qquad (1.4)$$

$$-\sqrt{\frac{\gamma p}{\rho}}\ x_2 + \frac{1}{\rho} x_3 = 0$$

multiple$\left(-\sqrt{\frac{\gamma p}{\rho}}\right)$

$$\rho.\frac{\gamma p}{\rho} x_2 - \sqrt{\frac{\gamma p}{\rho}} x_3 = 0$$

$$\sqrt{\frac{\gamma p}{\rho}} \left[ \rho.\sqrt{\frac{\gamma p}{\rho}} x_2 - x_3 \right] = 0$$

$$-\sqrt{\frac{\gamma p}{\rho}} x_2 + \frac{1}{\rho} x_3 = 0$$

assume $x_3 = 1$

$$-\sqrt{\frac{\gamma p}{\rho}} x_2 = -\frac{1}{\rho} x_3$$

$$x_2 = \frac{1}{\rho}.\sqrt{\frac{\gamma p}{\rho}}$$

putting $x_2$ in equation (1.2)

$$-\sqrt{\frac{\gamma p}{\rho}}x_1+\sqrt{\frac{\rho}{\gamma p}}=0$$

$$-\sqrt{\frac{\gamma p}{\rho}}x_1=-\sqrt{\frac{\rho}{\gamma p}}$$

$$x_1=\sqrt{\frac{\rho}{\gamma p}}.\sqrt{\frac{\rho}{\gamma p}}$$

$$x_1=\frac{\rho}{\gamma p}$$

3)

$$\lambda=u-\sqrt{\frac{\gamma p}{\rho}}$$

$$\begin{bmatrix}\sqrt{\frac{\gamma p}{\rho}} & \rho & 0\\ 0 & \sqrt{\frac{\gamma p}{\rho}} & \frac{1}{\rho}\\ 0 & \gamma p & \sqrt{\frac{\gamma p}{\rho}}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$$

$$\sqrt{\frac{\gamma p}{\rho}}\quad x_1+\rho x_2=0 \qquad (1.5)$$

$$\sqrt{\frac{\gamma p}{\rho}}\quad x_2+\frac{1}{\rho}x_3=0 \qquad (1.6)$$

$$\gamma p x_2+\sqrt{\frac{\gamma p}{\rho}}\quad \mathrm{x}_3=0 \qquad (1.7)$$

from equation,

$$\gamma p x_2+\sqrt{\frac{\gamma p}{\rho}}\quad \mathrm{x}_3=0$$

$$\rho.\frac{\gamma p}{\rho}x_2+\sqrt{\frac{\gamma p}{\rho}}\quad x_3=0$$

$$\sqrt{\frac{\gamma p}{\rho}}[\rho\sqrt{\frac{\gamma p}{\rho}}x_2+\ x_3]=0$$

$$\rho\sqrt{\frac{\gamma p}{\rho}}x_2+\ x_3=0$$

$$\sqrt{\frac{\gamma p}{\rho}}x_2 + \frac{1}{\rho}x_3 = 0$$

$assume\, x_3 = 1$

$$\sqrt{\frac{\gamma p}{\rho}}\; x_2 = -\frac{1}{\rho}x_3$$

$$x_2 = -\frac{1}{\rho}\sqrt{\frac{\rho}{\gamma p}}$$

from equation ,

$$\sqrt{\frac{\gamma p}{\rho}}x_1 + \rho x_2 = 0$$

$$\sqrt{\frac{\gamma p}{\rho}}x_1 + \rho\left[\frac{-1}{\rho}\sqrt{\frac{\rho}{\gamma p}}\right] = 0$$

$$\sqrt{\frac{\gamma p}{\rho}}x_1 - \sqrt{\frac{\rho}{\gamma p}} = 0$$

$$\sqrt{\frac{\gamma p}{\rho}}x_1 = \sqrt{\frac{\rho}{\gamma p}}$$

$$x_1 = \frac{\rho}{\gamma p}$$

The eigenvector is $(x_1 = \frac{\rho}{\gamma p}, x_2 = \frac{-1}{\rho}\sqrt{\frac{\rho}{\gamma p}}, x_3 = 1)$

$$so, (\frac{\rho}{\gamma p}, \frac{-1}{\rho}\sqrt{\frac{\rho}{\gamma p}}, 1)$$

and its eigenvectors are $(1,0,0), (\left(\frac{\rho}{\gamma p}\right), \frac{1}{\rho}\sqrt{\frac{\rho}{\gamma p}}, 1), \left(\frac{\rho}{\gamma p}, -\frac{1}{\rho}\sqrt{\frac{\rho}{\gamma p}}, 1\right)$

and it is a strictly hyperbolic.

## 1.5 Weak solution:

Characteristics curve in one-dimensional case: Let $f : \mathrm{R} \to \mathrm{R}$ be a $C^1$ function. The conservation laws, with initial data:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0, \quad x \in \mathrm{R},\; t > 0, \qquad (1.8)$$

$$u(x,0) = u_0(x), \quad x \in \mathrm{R},$$

Here *u* be a smooth solution, which follows the above equations

$u(x,t) \in C^1$

Let u be smooth solution of Equation (1.1), then the non-conservation form $u_t + f'(u)u_x = 0$.

We take $a(u) = f'(u)$

From above equation, we have non-conservation from

$$\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = 0.$$

The characteristics curve of above condition; it will be define as the solution is integral curve of the differential equation

$$\frac{dx}{dt} = a(u(x,t)). \qquad (1.9)$$

**Theorem (1):** Assume that u is a smooth of (1.1) the characteristic curve are straight lines along, which u is constant.

**Proof:**

Consider a characteristic curve passing through the point $(x_0, 0)$, a solution of the ordinary differential equation is using the Method of characteristics,

$$\frac{dt}{1} = \frac{dx}{f'(u)} = \frac{du}{0}$$

*so,* $$\frac{dx}{dt} = f'(u)$$

with initial value $x(0) = x_0 = C$

Along a curve, u is constant.

$$\frac{d}{dt}u(x(t),t) = \frac{\partial u}{\partial t}(x(t),t) + \frac{\partial u}{\partial x}(x(t),t)\frac{dx}{dt}$$

*i.e,* $$\left(\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x}\right) = 0.$$

By above equation is using by chain rule,

so, $$\frac{d}{dt}u(x(t),t) = 0$$

Hence the characteristic curves are straight lines, whose constant slopes depends on the initial value

$$\frac{dx}{dt} = f'(u)$$

$x(t) = f'(u)t + C$ ( in intergal curve)

$x(t) = f'(u)t + x_0.$

**Example:**

**4)** $u_t + a(u)u_x = 0$ , $u(x,0) = u_0(x)$

**Solution:**

$$u_t + a(u)u_x = 0$$

let $f'(u) = a(u)$

the characteristic curves are

$$x(t) = at + x_0$$

according to initial data,

$$u(x(t),t) = u(x(0),0) = u_0(x_0) = u_0(x - at)$$

$$u(x(t),t) = u_0(x - at)$$

$i.e,$ $u_0$ is smooth function.

**Non-smooth Solution:** $f''(u) > 0$ *and* $f''(u) < 0$ are two cases for convex and concave respectively.

## Existences of non-smooth solution:

We consider convex case i.e, $f''(u) > 0$.

Let $x_1, x_2 \in R$, such that $x_2 > x_1$

if $u_0(x)$ is decreasing function, then $u_0(x_1) > u_0(x_2)$.

Since, $f''(u) > 0$, then $f'(u_0(x_1)) > f'(u_0(x_2))$

$u(x_1, t_1) = u_0(x_1)$, implies that $u_0(x_2) < u_0(x_1)$.

So that characteristics intersect after finite time and form non smooth solution.

**Example:**

**5)** The Burgers' equation (inviscid equation) is $u_t + uu_x = 0,$ with initial condition

$$u(x,0) = \begin{cases} 1, & \text{if } x < 0 \\ 1 - x, & \text{if } 0 \le x \le 1 \\ 0, & \text{if } x > 1 \end{cases}.$$

**Solution:**

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

By solving characteristic curve we get.

$$x(t) = tf'(u) + x_0$$
$$x(t) = t\,u(x,t) + x_0$$
$$x(t) = tu_0(x_0) + x_0$$

In these means, the characteristics curve passes through the point $(x_0,0)$. Then we have

$$x = x(x_0,t) = \begin{cases} x_0 + t, & if\, x_0 \le 0 \\ x_0 + t(1 - x_0), & if\ 0 \le x_0 \le 1 \\ x_0, & if\ x_0 \ge 1 \end{cases}$$

$$\text{we know that,} x_0 = \begin{cases} x - t, & if\, x \le t \\ \dfrac{x\text{-}1}{1\text{-}t}, & if\, t \le x \le 1 \\ x, & if\ x \ge 1 \end{cases}$$

$$u(x,t) = \begin{cases} 1, & if\ x \le t \le 1 \\ \dfrac{x\text{-}1}{1\text{-}t}, & if\, t \le x \le 1 \\ 0, & if\ x \ge 1, t < 1 \end{cases}$$

At t=1, the characteristic intersect

$$u(x,1) = \begin{cases} 1, & if\, x < 1 \\ 0, & if\, x > 1 \end{cases}$$

Now, it is discontinuities may develop after a finite time if f is nonlinear, when $u_0$ is smooth in fig. (1).

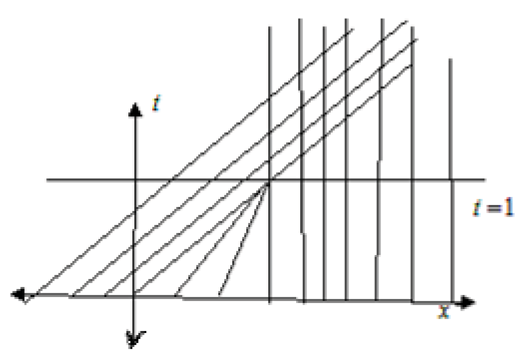


fig. (1)

# Chapter-2

# Riemann Problem for isentropic magnetogasdynamics

## 2.1 Shock and rarefaction waves:

When flow of an isentropic, inviscid and perfectly conducting compressible fluid is subjected to a transverse magnetic field, then conservation form can be written as

$$\frac{\partial}{\partial t}(\rho)+\frac{\partial}{\partial t}(\rho u)=0$$

$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial t}(p+\rho u^2+\frac{B^2}{2\mu})=0,\ t>0,\ x\in R \tag{2.1}$$

Where $\rho \geq 0$ ,$u$, $p \geq 0$, it may represent density, velocity, pressure, $B \geq 0$ transversal magnetic field and $\mu > 0$ denote magnetic permeability, respectively; p and B are functions in which are $p = k_1\rho^{\gamma}$ and $B = k_2\rho$, where $k_1$ and $k_2$ are positive constants and $\gamma$ is the adiabatic constant which lies in the range $1 < \gamma \leq 2$ for most of the gases. The independent variables are t and x.

$$\frac{\partial \rho}{\partial \mathrm{t}}+\frac{\partial}{\partial x}(\rho u)=0$$

$$\Rightarrow \frac{\partial \rho}{\partial \mathrm{t}}+\rho\frac{\partial u}{\partial x}+u\frac{\partial \rho}{\partial x}=0$$

$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}\left(\mathrm{p}+\rho \mathrm{u}^2+\frac{B^2}{2\mu}\right)=0$$

$$\rho\frac{\partial u}{\partial \mathrm{t}}+u\frac{\partial \rho}{\partial t}+\frac{\partial p}{\partial x}+u^2\frac{\partial \rho}{\partial x}+\rho 2u\frac{\partial u}{\partial x}+\frac{1}{\mu}2B\frac{\partial B}{\partial x}=0$$

$$\rho\frac{\partial u}{\partial t}+\frac{\partial p}{\partial x}+\rho u\frac{\partial u}{\partial x}+\frac{2B}{\mu}\frac{\partial B}{\partial x}+u\frac{\partial \rho}{\partial t}+u\rho\frac{\partial u}{\partial x}+u^2\frac{\partial \rho}{\partial x}=0$$

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\frac{\partial p}{\partial x}+\frac{2B}{\mu}\frac{\partial B}{\partial x}+u\left(\frac{\partial \rho}{\partial t}+\rho\frac{\partial u}{\partial x}+u\frac{\partial \rho}{\partial x}\right)=0$$

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\frac{\partial p}{\partial x}+\frac{2B}{\mu}\frac{\partial B}{\partial x}=0$$

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\gamma k_1\rho^{\gamma-1}\frac{\partial \rho}{\partial x}+\frac{2{k_2}^2}{\mu}\rho\frac{\partial \rho}{\partial x}=0$$

where $p = k_1\rho^{\gamma}$, then $\frac{\partial p}{\partial x}=\gamma k_1\rho^{\gamma-1}\frac{\partial \rho}{\partial x}$

$B = k_2\rho,$ it implies that

$$\frac{\partial B}{\partial x} = k_2 \frac{\partial \rho}{\partial x}$$

$$2BB_x/\mu = 2(k_2\rho)(k_2\rho_x)$$

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \gamma k_1\rho^{\gamma-1}\frac{\partial \rho}{\partial x} + \frac{2k_2^{\ 2}}{\mu}\rho\frac{\partial \rho}{\partial x} = 0,$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \gamma k_1\rho^{\gamma-2}\frac{\partial \rho}{\partial x} + \frac{2k_2^{\ 2}}{\mu}\frac{\partial \rho}{\partial x}\right) = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \gamma k_1\rho^{\gamma-2}\frac{\partial \rho}{\partial x} + \frac{2k_2^{\ 2}}{\mu}\frac{\partial \rho}{\partial x} = 0.$$

above equation can be written as

$$\frac{\partial \rho}{\partial \mathrm{t}} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \gamma k_1\rho^{\lambda-2}\frac{\partial \rho}{\partial x} + \frac{2k_2^{\ 2}}{\mu}\frac{\partial \rho}{\partial x} = 0$$

$$\begin{bmatrix}\rho \\ u\end{bmatrix}_t + \begin{bmatrix} u & \rho \\ \gamma k_1\rho^{\gamma-2} + \dfrac{2k_2^{\ 2}}{\mu} & \mathrm{u}\end{bmatrix}\begin{bmatrix}\rho_x \\ u_x\end{bmatrix} = 0$$

for smooth solutions, system (2.1) can be written as

$$U_t + AU_x = 0 \qquad (2.2)$$

where the matrix $A$ is defined as $A = \begin{bmatrix} u & \rho \\ \dfrac{w^2}{\rho} & u\end{bmatrix}$, and $w = \left(c^2 + b^2\right)^{\frac{1}{2}}$ is the magneto-acoustic speed with $c = \left(p'(\rho)\right)^{\frac{1}{2}}$ is the local sound speed and $b = (\dfrac{\left(B^2(\rho)\right)}{\mu\rho})^{\frac{1}{2}}$, which is Alfven speed;

$$U_t + AU_x = 0$$

$$A = \begin{bmatrix} u & \rho \\ \dfrac{w^2}{\rho} & u\end{bmatrix},$$

where, $w = \left(c^2 + b^2\right)^{\frac{1}{2}},$

$$c=\left(p'(\rho)\right)^{\frac{1}{2}}, b=\left(\frac{\left(B^2(\rho)\right)}{\mu\rho}\right)^{\frac{1}{2}}$$

$$(A-\lambda I)=0$$

$$\left[\begin{bmatrix} u & \rho \\ \frac{w^2}{\rho} & u \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]=0$$

$$\begin{bmatrix} u-\lambda & \rho \\ \frac{w^2}{\rho} & u\text{-}\lambda \end{bmatrix}=0$$

$$(u-\lambda)^2-\rho\frac{w^2}{\rho}=0$$

$$(u-\lambda)^2-w^2=0$$

$$(u-\lambda)^2=w^2$$

$$\{((u-\lambda)-w)((u-\lambda)+w)\}=0$$

$$(u-\lambda)-w=0$$

$$-\lambda=w-u$$

$$\lambda_1=u-w$$

$$(u-\lambda)+w=0$$

$$-\lambda_2=-w-u$$

$$\lambda_2=u+w$$

The eigenvalues of $A$ are $\lambda_1=u-w$ and $\lambda_2=u+w$. Thus, the system (2.2) is strictly hyperbolic when w>0. Let $\vec{r}_1=(-\rho,w)^{tr}$ and $\vec{r}_2=(-\rho,w)^{tr}$ are the right eigenvectors corresponding to the eigenvalues $\lambda_1$ and $\lambda_2$ respectively. We have

$$\nabla\lambda_1.\vec{r}_1=\left(\frac{\partial}{\partial\rho}i+\frac{\partial}{\partial u}j\right)(u+w)\begin{pmatrix}-\rho \\ w\end{pmatrix}$$

$$=1+\left(\frac{B^2(\rho)}{2\mu\rho w(\rho)}+\frac{\rho p''(\rho)}{2w(\rho)}\right).$$

when $p''(\rho)\geq 0$, the first characteristic field is genuinely nonlinear. Similarly, it can be shown that the second characteristic field is nonlinear when $p''(\rho)\geq 0$.

The waves associated with $\vec{r_1}$ and $\vec{r_2}$ characteristic field will be either shock or rarefaction waves.

**2.2 Shock:** Let $\rho_l$ and $\rho$ , the left and right hand states of either a shock or a rarefaction wave are $u_l = u(\rho_1), p_l = p(\rho_1), B_l = B(\rho_1)$ and $u = u(\rho), p = p(\rho), B = B(\rho)$ denotes respectively; the system (2.1) are using in Rankine–Hugonist jump conditions, the given by

$$\upsilon[\rho] = [\rho u] \qquad (2.3)$$

$$\upsilon[\rho u] = \left[ p + \rho u^2 + \frac{B^2}{2\mu} \right] \qquad (2.4)$$

where $[.]$ denote the jump across a discontinuity curve $x = x(t)$ and $\upsilon = \frac{dx}{dt}$ is the shock speed.

**Lemma 2.1:**

Let $S_1$ and $S_2$ respectively denote 1- shock and 2-shock associated with $\lambda_1$ and $\lambda_2$ characteristic fields.

Let the states $U_1$ and U satisfy the Rankine-Hugoniot jump conditions (2.4) and (2.3). Then the shock curves satisfy,

$$u = u_l - g(\rho_l, \rho) \qquad (2.5)$$

Where $g(\rho_l, \rho) = \sqrt{\left( p + \frac{B^2}{2\mu} - p_l - \frac{B^2}{2\mu} \right)\left( \frac{\rho - \rho_l}{\rho \rho_l} \right)}$

such that for, $1 < \gamma < 2,$ we have for $\rho > \rho_l, u' < 0$ and $u'' > 0$ on $S_1$, whilst for $\rho < \rho_l$ we have $u' > 0$ and $u'' < 0$ on $S_2$.

**Proof:** The $\upsilon$ -elimination of

$$\upsilon[\rho] = [\rho u]$$

$$\upsilon[\rho u] = \left[ p + \rho u^2 + \frac{B^2}{2\mu} \right]$$

$$\upsilon[\rho] = [\rho u]$$

$$\upsilon[\rho u] = \left[ p + \rho u^2 + \frac{B^2}{2\mu} \right]$$

$$\frac{[\rho u]^2}{[\rho]} = \left[ p + \rho u^2 + \frac{B^2}{2\mu} \right]$$

$$\left( \frac{(\rho u - \rho_l u_l)^2}{\rho - \rho_l} \right) = \left[ p + \rho u^2 + \frac{B^2}{2\mu} \right]$$

$$(\rho u - \rho_l u_l)^2 = \left[ p + \rho u^2 + \frac{B^2}{2\mu} \right] (\rho - \rho_l)$$

$$\rho^2 u^2 + \rho_l^2 u_l^2 - 2\rho u \rho_l u_l = \left[ p + \rho u^2 + \frac{B^2}{2\mu} - p_l - \rho_l u_l^2 - \frac{B_l^2}{2\mu} \right] (\rho - \rho_l)$$

$$\rho^2 u^2 + \rho_l^2 u_l^2 - 2\rho u \rho_l u_l = \left( p + \frac{B^2}{2\mu} - p_l - \frac{B_l^2}{2\mu} \right) (\rho - \rho_l) + \left( \rho u^2 - \rho_l u_l^2 \right)(\rho - \rho_l)$$

$$\rho^2 u^2 + \rho_l^2 u_l^2 - 2\rho u \rho_l u_l = \left( p + \frac{B^2}{2\mu} - p_l - \frac{B_l^2}{2\mu} \right) (\rho - \rho_l) + \left( \rho^2 u^2 - \rho \rho_l u^2 - p \rho_l u_l^2 + \rho_l^2 u_l^2 \right)$$

$$\rho \rho_l u^2 + \rho \rho_l u_l^2 - 2\rho u \rho_l u_l = \left( p + \frac{B^2}{2\mu} - p_l - \frac{B_l^2}{2\mu} \right) (\rho - \rho_l)$$

$$\rho \rho_l \left( u^2 + u_l^2 - 2u u_l \right) = \left( p + \frac{B^2}{2\mu} - p_l - \frac{B_l^2}{2\mu} \right) (\rho - \rho_l)$$

$$(u - u_l)^2 = \left( p + \frac{B^2}{2\mu} - p_l - \frac{B_l^2}{2\mu} \right) \frac{(\rho - \rho_l)}{\rho \rho_l}$$

$$(u - u_l) = \sqrt{\left( p + \frac{B^2}{2\mu} - p_l - \frac{B_l^2}{2\mu} \right) \frac{(\rho - \rho_l)}{\rho \rho_l}}$$

$$u = u_l - \sqrt{\left( p + \frac{B^2}{2\mu} - p_l - \frac{B_l^2}{2\mu} \right) \frac{(\rho - \rho_l)}{\rho \rho_l}}$$

Let $\psi(\rho) = g^2(\rho, \rho_l)$; on the differentiating (2.5) with respect to $\rho$ we obtain

$$\frac{\partial u}{\partial \rho} = \frac{-1}{2} \frac{\left( p' + \frac{2BB'}{2\mu} \right)\left( \frac{\rho - \rho_l}{\rho \rho_l} \right) + \left( p + \frac{B^2}{2\mu} - p_l - \frac{B^2}{2\mu} \right)}{\sqrt{\left( p + \frac{B^2}{2\mu} - p_l - \frac{B^2}{2\mu} \right)}\left( \frac{\rho - \rho_l}{\rho \rho_l} \right)}$$

$$u' = -\frac{\Psi'(\rho)}{2\sqrt{\Psi(\rho)}}$$

which is negative for $\rho > \rho_1$. We can already show that $\psi$ and $\psi'$ are positive for $\rho > \rho_1$, and $\psi(\rho) = \psi'(\rho_l) = 0$; further, $1 < \gamma < 2,\ \psi'' > 0$, whilst $\psi''' < 0$.

Let $\chi(\rho) = \left((\psi'(\rho))^2 - 2\psi(\rho)\psi''(\rho)\right)$

so that $\chi(\rho_l) = 0$. Since $\chi'(\rho) = -2\psi(\rho)\psi'''(\rho)$, it follows for $1 < \gamma < 2, \chi'(\rho) > 0$. Hence, $\chi(\rho) > \chi(\rho_l)$ for $\rho > \rho_1$. Thus, for $1 < \gamma < 2$, if we differentiate again, we get

$$u'' = \frac{(\psi'(\rho))^2 - 2\psi(\rho)\psi''(\rho)}{4\psi(\rho)^{\frac{3}{2}}} > 0 \text{ on } S_1.$$

Similarly, for $\rho < \rho_1$ and $1 < \gamma < 2$, we have $\frac{du}{d\rho} > 0$ on again differentiating, then $\frac{d^2u}{d\rho} < 0$ on $S_2$.

Now, these shock curves are satisfied the Lax entropy conditions.

**Lemma(2.2):**

If p satisfies $p' > 0$ and $p'' \geq 0$, then the Lax condition hold,

i.e., 1-shock satisfies

$$\upsilon < \lambda_1(U_l),\ \lambda_1(U) < \upsilon < \lambda_2(U)\ (2.6)$$

Whilst the 2-shock satisfies

$$\lambda_1(U_l) < \upsilon < \lambda_2(U_l),\ \ \lambda_2(U) < \upsilon\ \ (2.7)$$

**Proof:**

Let us consider 1-shock curve to prove $\upsilon < \lambda_1(U_l)$. On apply 1-shock, we know that $\rho > \rho_1$, since $p' > 0$ and $p'' \geq 0$, by Lagrange's mean value theorem, there exist exists a $\xi \in (\rho_1, \rho)$ such that $f(a) - f(b) = f'(c)(b - a),\quad c \in (a,b)$

$$f = p,\quad a = \rho_l,\quad b = \rho$$
$$p(\rho) - p(\rho_l) = p'(\xi)(\rho - \rho_l)$$
$$p'(\xi) = \frac{p(\rho) - p(\rho_l)}{(\rho - \rho_l)},\quad \xi \in (\rho_l, \rho),$$

further, since $p'' \geq 0$, we have $p' > 0, p'' \geq 0$ and $p'$ is an increasing function $\xi \in (\rho, \rho_1)$, $\rho_l < \xi$,

$\frac{\rho}{\rho_1} > 1$, $p'(\rho_l) < p'(\xi) = c_l^2$ and thus,

$$c_l^2 < p'(\xi) < p'(\xi)\frac{\rho}{\rho_l}, \qquad \rho > \rho_l, \ \frac{\rho}{\rho_l} > 1$$

and it implies that,

$$p'(\xi)\frac{\rho}{\rho_l} > p'(\xi)$$
$$c_l^2 < \frac{(p - p_l)}{(\rho - \rho_l)} \ \frac{\rho}{\rho_l}. \qquad (2.8)$$

Also, since $\rho + \rho_l > 2\rho_l$ , we have $\frac{\rho + \rho_l}{2} > \rho_l$ it implies that

$$\frac{k_2^2}{2\mu}(\rho + \rho_l) > \frac{k_2^2}{2\mu}\rho_l$$
$$\frac{k_2^2(\rho^2 - \rho_l^2)}{2\mu(\rho - \rho_l)} > \frac{k_2^2}{2\mu}\rho_l$$
$$\frac{\rho}{\rho_l} > 1$$
$$\frac{k_2^2(\rho^2 - \rho_l^2)}{2\mu(\rho - \rho_l)} \cdot \frac{\rho}{\rho_l} > \frac{k_2^2}{2\mu}\rho_l$$

This implies by that

$$\frac{(B^2 - B_l^2)}{2\mu(\rho - \rho_l)}\frac{\rho}{\rho_l} > \frac{(B^2 - B_l^2)}{2\mu(\rho - \rho_l)} > \frac{k_2^2}{2\mu}\rho_l = \frac{k_2^2}{2\mu}\frac{\rho_l^2}{\rho_l}$$
$$\frac{(B^2 - B_l^2)}{2\mu(\rho - \rho_l)}\frac{\rho}{\rho_l} > \frac{(B^2 - B_l^2)}{2\mu(\rho - \rho_l)} > \frac{B_l^2}{2\mu\rho_l}$$
$$\frac{(B^2 - B_l^2)}{2\mu(\rho - \rho_l)}\frac{\rho}{\rho_l} > \frac{(B^2 - B_l^2)}{2\mu(\rho - \rho_l)} > \frac{B_l^2}{2\mu\rho_l}$$

$$b_l^2 > \frac{B_l^2}{2\mu\rho_l}$$

and therefore

$$b_l^2 < \frac{\left(\mathrm{B}^2 - B_l^2\right)}{2\mu(\rho - \rho_l)}\frac{\rho}{\rho_1}. \qquad (2.9)$$

From equation (2.8) and (2.9), we have

$$c_l^2 + b_l^2 < \frac{(p - p_l)}{(\rho - \rho_l)}\ \frac{\rho}{\rho_l} \quad + \frac{\left(B^2 - B_l^2\right)}{2\mu(\rho - \rho_l)}\frac{\rho}{\rho_l}$$

$$w_l^2 < \frac{(p - p_l)}{(\rho - \rho_l)}\ \frac{\rho}{\rho_l} \quad + \frac{\left(B^2 - B_l^2\right)}{2\mu(\rho - \rho_l)}\frac{\rho}{\rho_l}$$

$$w_l^2 < \frac{\rho}{(\rho - \rho_l)\rho_l}\left[(p - p_l) + \frac{\left(B^2 - B_l^2\right)}{2\mu}\right]$$

$$-w_l^2 > -\frac{\rho}{(\rho - \rho_l)\rho_l}\left[(p - p_l) + \frac{\left(B^2 - B_l^2\right)}{2\mu}\right]$$

$$-w_l > -\sqrt{\frac{\rho}{(\rho - \rho_l)\rho_l}\left[(p - p_l) + \frac{\left(B^2 - B_l^2\right)}{2\mu}\right]}$$

$$-w_l > -\sqrt{\frac{\rho(\rho - \rho_l)}{(\rho - \rho_l)^2\rho_l}\left[(p - p_l) + \frac{\left(B^2 - B_l^2\right)}{2\mu}\right]}$$

$$-w_l > -\frac{1}{(\rho - \rho_l)}\sqrt{\left[(p - p_l) + \frac{\left(B^2 - B_l^2\right)}{2\mu}\right]\rho^2\left(\frac{1}{\rho_l} - \frac{1}{\rho}\right)}$$

$$-w_l > -\frac{\rho}{(\rho - \rho_l)}\sqrt{\left[(\mathrm{p} - p_l) + \frac{\left(\mathrm{B}^2 - B_l^2\right)}{2\mu}\right]\left(\frac{1}{\rho_l} - \frac{1}{\rho}\right)}$$

In (2.5), the above inequality holds that $\frac{\rho(u - u_l)}{\rho - \rho_l} < -w_l$, and hence $\upsilon < \lambda_1(U_l) = u_l - w_l$.

In same manner another condition, since $p'' \geq 0$ and $\rho_1 < \rho$ on 1-shock, we have $p'(\eta) = \frac{p - p_l}{\rho - \rho_l} < p'(\rho)$ for some $\eta \in (\rho_l, \rho)$,and hence

$$c^2 > \frac{(p - p_l)}{(\rho - \rho_l)}\ \frac{\rho_l}{\rho}\ . \qquad (2.10)$$

Further, since $\rho > \left(\frac{\rho + \rho_l}{2}\right)$,

$$b^2 > \frac{\left(B^2 - B_l^2\right)}{2\mu(\rho - \rho_l)}\frac{\rho_l}{\rho}, \qquad (2.11)$$

and hence from (2.10) and (2.11), we obtain

$$b^2 + c^2 > \frac{\left(B^2 - B_l^2\right)}{2\mu(\rho - \rho_l)}\frac{\rho_l}{\rho} + \frac{(p - p_l)}{(\rho - \rho_l)}\ \frac{\rho_l}{\rho}, \qquad (2.12)$$

It implies that

$$-w < \sqrt{\left(p - p_l + \frac{\left(B^2 - B_l^2\right)}{2\mu}\right)\frac{\rho_l}{\rho(\rho - \rho_l)}}.$$

From equation (2.3) and (2.5) imply that

$$u\text{-}w < \frac{\rho u\text{-}\rho_l u_l}{\rho - \rho_l} = \upsilon, \text{and hence } \lambda_1(U) < \upsilon.$$

Lastly, we show that $\upsilon < \lambda_2(\mathrm{U})$. In this way, the equation (2.12), which implying that,

$$w > -\sqrt{\frac{\left(B^2 - B_l^2\right)}{2\mu(\rho - \rho_l)}\frac{\rho_l}{\rho} + \frac{(p - p_l)}{(\rho - \rho_l)}\ \frac{\rho_l}{\rho}}.$$

For 1-shock curve, using (2.5) we have $w > \dfrac{(u - u_l)\rho_l}{\rho - \rho_l}$, which implies that $\upsilon < \lambda_2(U)$.

Hence 1-shock satisfied Lax condition; as well as way satisfied by the lax condition for the 2-shock.

Now we will show that the density, pressure, velocity magnetic field vary across a shock. Applying equation (2.1) for 1-shock the left and right states have to satisfies Lax conditions (2.6). Let us define $V = \upsilon\text{-}u$; then since $\upsilon < \lambda_1(U_l)$, it follows that $V_l < -w_l$ implies that $V_l < 0$, and hence $\upsilon < u_l$.

Similarly, by using second condition i.e., $\lambda_1(U) < \upsilon < \lambda_2(U)$, we get

$u\text{-}w < \upsilon < v + w, \text{ so } -w < V < w,$

hence $|V| < w$. From (2.3) we have $\rho V = \rho_l V_l$. Since $\rho$ and $\rho_l$ are positive, both $V$ and $V_l$ must have same sign; since $V_l < 0,$ we have $V < 0$. For 1-shock, the gas speed on the both sides of shock is greater than shock speed, and therefore the particles cross the from left to right.

In case of 2-shock, applying Lax conditions, (2.7) this implying $|V_l| < w_l$ since $\lambda_2(U) < \upsilon$ or equivalently $u + w = \upsilon$, which follows that $w < V$, and hence $V > 0$. In case of 2-shock particles cross from right to left.

Let the states ahead of, and behind the shock be designated the 1- state and 2-state, respectively.

Then, for 1-shock $l = 1, r = 2$, and

hence $V_l^2 > W_l^2 \text{ and } V_2^2 < W_2^{2.}$;

for 2-shock $l = 2 \; r = 1, \; so \; V_l^2 > W_l^2 \; and \; V_2^2 < W_2^{2.}$.

Thus, for both shocks we have

$V_l^2 > W_l^2 \text{ and } V_2^2 < W_2^{2.}$.

To this conditions satisfied the equation (2.4) holds that

$$p_1 + \rho_1 V_1^2 + \frac{B_1^{\;2}}{2\mu} = p_2 + \rho_2 V_2^2 + \frac{B_2^{\;2}}{2\mu};$$

which implies that

$$p_1 + \rho_1 w_1^2 + \frac{B_1^{\;2}}{2\mu} < p_1 + \rho_1 V_1^2 + \frac{B_1^{\;2}}{2\mu} = p_2 + \rho_2 V_2^2 + \frac{B_2^{\;2}}{2\mu} < p_2 + \rho_2 w_2^2 + \frac{B_2^{\;2}}{2\mu};$$

so $p_1 + \rho_1 c_1^2 + (\frac{3k_2^{\;2}}{2\mu})\rho_l^2 < p_2 + \rho_2 c_2^2 + (\frac{3k_2^{\;2}}{2\mu})\rho_2^2$.

the above inequalities follows that $\rho_1 < \rho_2$, and

therefore $p_1 < p_2 \text{ and } B_1 < B_2.$ and from (2.3) we have

$|\rho_1 V_1| = |\rho_2 V_2|$;

Since $\rho_1 > \rho_2$, it follows that $|V_1| > |V_2|$. Since for 1-shock $V_1 < 0$ and $\rho_1 < \rho_2$, and it follows that $V_2 > V_1$, implies that $u_1 > u_2$. Similarly, for 2-shock $V_2 > 0$ and $\rho_1 < \rho_2$, its implying that $V_1 > V_2$ and so $u_1 < u_2$.

## 2.3 Rarefaction waves:

The $U\left(\frac{x}{t}\right)$ which are of the piecewise smooth continuous solutions of (2.2) such that

$$\mathrm{U}(x,t) = \begin{cases} U_l, \frac{x}{t} \leq \lambda_n(U_l) \\ U(\frac{x}{t}), \lambda_n(U_l) \leq \frac{x}{t} \leq \lambda_n(U_r) \\ U_r, \lambda_n(U_r) \end{cases} \quad (2.13)$$

If we take $\eta = \frac{x}{t}$, then the equation (2.2) is a system of ordinary differential equations and it can be written as

$$(A - \eta I)\begin{pmatrix} \dot{\rho}, \dot{u} \end{pmatrix}^{tr} = 0,$$

where $I$ is $2 \times 2$ identity matrix and the differentiating with respect to the variable $\eta$ is denoted by dot. If $\begin{pmatrix} \dot{\rho}, \dot{u} \end{pmatrix}^{tr} = (0,0)$, then $\rho$ and $u$ become constants. If $\begin{pmatrix} \dot{\rho}, \dot{u} \end{pmatrix}^{tr} \neq (0.0)$, then there exist a eigenvector of the matrix $A$ corresponding to the eigenvalue $\eta$. Since it has two real and distinct eigenvalues $\lambda_1 < \lambda_2$, so it has two families of the rarefaction waves $\mathrm{R}_1$ and $\mathrm{R}_2$ which are 1-Rarefaction waves and 2-Rarefaction waves respectively; Let us consider 1-rarefaction waves, since

$$(A - \eta I)\begin{pmatrix} \dot{\rho}, \dot{u} \end{pmatrix}^{tr} = 0$$

and with $\lambda_1 = u - w$ we have

$$\left(\left[\begin{bmatrix} u & \rho \\ \frac{w^2}{\rho} & u \end{bmatrix} - (u-w)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\right) = 0$$

$$\left[\begin{bmatrix} u & \rho \\ \frac{w^2}{\rho} & u \end{bmatrix} - (u-w)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{pmatrix} \dot{\rho} \\ \dot{u} \end{pmatrix} = 0$$

$$\left[\begin{bmatrix} u-(u-w) & \rho \\ \frac{w^2}{\rho} & u-(u-w) \end{bmatrix}\right]\begin{pmatrix} \dot{\rho} \\ \dot{u} \end{pmatrix} = 0$$

$$\begin{bmatrix} w & \rho \\ \frac{w^2}{\rho} & w \end{bmatrix}\begin{pmatrix} \dot{\rho} \\ \dot{u} \end{pmatrix} = 0$$

$$w\dot{\rho} + \rho\dot{u} = 0$$

$$\frac{w^2}{\rho}\dot{\rho} + w\dot{u} = 0$$

$$w\frac{d\rho}{d\eta} + \rho\frac{du}{d\eta} = 0$$

$$\frac{w}{\rho}\frac{d\rho}{d\eta} + \frac{du}{d\eta} = 0$$

$$\Pi_1 = u + \int \frac{w(y)}{y} dy = 0. \qquad (2.14)$$

Where $\Pi_1$ 1-Riemann invariant is (2.14) represents $R_1$ curve. Similarly, $\Pi_2$ 2-Riemann invariant of the 2-Rarefaction wave curve is represent $R_2$ curve

$$\Pi_2 = u - \int \frac{w(y)}{y} dy = 0. \qquad (2.15)$$

## Theorem 2.1

On $R_1$(respectively $R_2$) the Riemann invariant $\Pi_1$(respectively $\Pi_2$) is constant.

**Lemma2.3:**Across1-rarefaction waves (respectively, 2-rarefaction waves), $\rho \le \rho_l$ and $u_l \le u$ (respectively, $\rho \ge \rho_l$ and $u_l \ge u$ if and only if, characteristic speed increases from left hand state to right hand state.

**Proof:** Since, we know that

$$w = \sqrt{c^2 + b^2}$$

$$\frac{dw}{d\rho} = \left( \frac{p''}{2w} + \frac{(B')^2}{2\mu w} \right) > 0$$

$w$ is an increasing function of $\rho$; so it form for 1-rarefaction waves, $w(\rho) \leq w(\rho_l)$ or it can be written as $-w_l \leq -w$. These inequalities are $u_l \leq u$ and $-w_l \leq -w$ show that $\lambda_1(U_l) \leq \lambda_1(U)$. Similarly we can prove $\lambda_2(U_l) \leq \lambda_2(U)$ for 2-rarefaction waves. Then the conversely for 1-rarefaction waves,

since $\lambda_1(U_l) \leq \lambda_1(U)$, we have

$$w\text{-}w_l \leq u - u_l. \qquad (2.16)$$

In further, since 1-rarefaction wave region $\Pi_1$ is constant, then we get

$$u\text{-}u_1 = \int_0^{\rho_l} \frac{w(y)}{y} dy - \int_0^{\rho} \frac{w(y)}{y} dy \ ,$$

the equation (2.16) shows that

$$w\text{-}w_l \leq \int_0^{\rho_l} \frac{w(y)}{y} dy - \int_0^{\rho} \frac{w(y)}{y} dy,$$

which implies that $\rho \leq \rho_1$ and

$$u\text{-}u_1 = \int_0^{\rho_l} \frac{w(y)}{y} dy - \int_0^{\rho} \frac{w(y)}{y} dy \geq 0 .$$

Hence $\rho \leq \rho_l$ and $u_l \leq u$ .Similarly, it can be shown that the 2-rarefaction waves, $\rho \geq \rho_l$ and $u_l \geq u$ .

Introducing a new parameter $\theta$, where $\theta = \frac{\rho_r}{\rho_l}$ obtain from (2.5) the following formulas for shock curves, for 1-shock curve $\theta > 1$ and 2-shock curve $\theta < 1$.(Respectively, rarefaction curves in the term of parameterizations).

From equation (2.5),

$u = u_l - g(\rho_l,\rho)$, where $g(\rho_l,\rho) = \sqrt{\left(p + \frac{B^2}{2\mu} - p_l - \frac{B_l^2}{2\mu}\right)\left(\frac{\rho - \rho_l}{\rho \rho_l}\right)}$

$\frac{\rho_r}{\rho_l} = \theta,\ \frac{p_r}{p_l} = \theta^\gamma$ it implying there by $p = k_1 \rho^\gamma$

$\frac{B_r}{B_l} = \theta$ implies that $B = k_2 \rho$, so that $\frac{B_r}{B_l} = \theta = \frac{\rho_r}{\rho_l}$,

$$\frac{u_r}{u_l} = 1 - \sqrt{\left(p + \frac{B^2}{2\mu} - p_l - \frac{B_l^2}{2\mu}\right)\left(\frac{\rho - \rho_l}{\rho \rho_l}\right)}$$

$$\frac{u_r}{u_l} = 1 - \sqrt{\left(p - p_l + \frac{B^2}{2\mu} - \frac{B_l^2}{2\mu}\right)\left(\frac{\rho - \rho_l}{\rho \rho_l}\right)}$$

$$\frac{u_r}{u_l} = 1 - \sqrt{\left((\frac{p_r}{p_l} - 1) + (\frac{B_r^{\ 2}}{B_l^2} - 1)\right)\left(\frac{\rho - \rho_l}{\rho \rho_l}\right)}$$

$$\frac{u_r}{u_l} = 1 - \sqrt{\left((\frac{p_r}{p_l} - 1) + (\frac{B_r^{\ 2}}{B_l^2} - 1)\right)\left(1 - \frac{1}{\theta}\right)}$$

$$\frac{u_r}{u_l} = 1 - \sqrt{\left[A\left(\theta^\gamma - 1\right) + B\left(\theta^2 - 1\right)\right]\left(1 - \frac{1}{\theta}\right)}. \qquad (2.17)$$

For 1-rarefaction waves $(\theta < 1)$, since 1-Riemann invariant is constant, we have

$\frac{\rho_r}{\rho_l} = \theta,\ \frac{p_r}{p_l} = \theta^\gamma,\ \frac{B_r}{B_l} = \theta,$ so that $\frac{B_r}{B_l} = \theta = \frac{\rho_r}{\rho_l}$

if we set $\theta = t, d\theta = dt$, $\frac{u_r}{u_l} = 1 + \sqrt{\gamma A t^{\gamma-1} + 2Bt\left(1 - \frac{1}{\theta}\right)} dt$

$$\frac{u_r}{u_l} = 1 + \frac{\sqrt{\gamma A t^{\gamma-1} + 2Bt}}{t} dt \qquad (2.18)$$

Similarly, for 2-rarefaction wave $(\theta > 1)$, we have

$\frac{\rho_r}{\rho_l} = \theta, \frac{p_r}{p_l} = \theta^\gamma, \frac{B_r}{B_l} = \theta$

then as well as we set $\frac{B_r}{B_l} = \theta = \frac{\rho_r}{\rho_l}$

$$\frac{u_r}{u_l}=1+\sqrt{\gamma At^{\gamma-1}+2Bt\left(1-\frac{1}{\theta}\right)dt}$$

$$\frac{u_r}{u_l}=1+\frac{\sqrt{\gamma At^{\gamma-1}+2Bt}}{t}dt$$

Thus, for 1-family, either shock or rarefaction wave, we have

$$\frac{u_r}{u_l}=\begin{cases}1+\dfrac{\sqrt{\gamma At^{\gamma-1}+2Bt}}{t}dt, & if\ \theta\le 1,\\ 1-\sqrt{\left[A\left(\theta^{\gamma}-1\right)+B\left(\theta^{2}-1\right)\right]\left(1-\dfrac{1}{\theta}\right)} & ,if\ \theta>1\end{cases} \qquad (2.19)$$

In the similar way,

$$\frac{u_r}{u_l}=\begin{cases}1-\sqrt{\left[A\left(\theta^{\gamma}-1\right)+B\left(\theta^{2}-1\right)\right]\left(1-\dfrac{1}{\theta}\right)}\ , & if\ \theta<1\\ 1+\dfrac{\sqrt{\gamma At^{\gamma-1}+2Bt}}{t}dt, & if\ \theta\ge 1,\end{cases} \qquad (2.20)$$

where $A=\dfrac{k_1\rho_l^{\gamma-1}}{u_l^2}$ and $B=\dfrac{k_2^2\rho_l}{2\mu u_l^2}$ ; is as to expression in above equations (2.19) and (2.20).

## Theorem 3.1:

The $\mathrm{R}_1$ curve is convex and monotonic decreasing while $\mathrm{R}_2$ curve is concave and monotonic increasing.

**Proof:** We know that, 1-rarefaction wave is

$$u=u_l+\int_{\rho}^{\rho_l}\frac{w(y)}{y}dy \quad ,if\ \rho\le\rho_l \qquad (3.1)$$

On differentiating with respect to $\rho,$ we have

$$\frac{du}{d\rho}=-\frac{w}{\rho}<0$$

$$\frac{d^2u}{d\rho^2} = \frac{w}{\rho^2} - \frac{w'}{\rho}. \qquad (3.2)$$

We know that $\mathrm{w} = \sqrt{\mathrm{c}^2 + b^2}$, since $p = k_1\rho^{\gamma}, B = k_2\rho$ in the following equations

$$c^2 = \gamma k_1 \rho^{\gamma-1}, b^2 = \frac{k^2{}_2\rho^2}{\mu\rho}$$

$$w = \sqrt{\gamma k_1 \rho^{\gamma-1} + \frac{k^2{}_2\rho^2}{\mu\rho}}$$

$$w' = \frac{(2bb' + 2cc')}{2\sqrt{c^2 + b^2}}$$

$$w' = \frac{(bb' + cc')}{w}$$

Again, on differentiating with respect to $\rho$ ,we have

$$\frac{\mathrm{d}^2u}{d\rho^2} = \frac{w}{\rho^2} - \frac{w^{'}}{\rho}$$

$$\frac{\mathrm{d}^2u}{d\rho^2} = \frac{w}{\rho^2} - \frac{(bb' + cc')}{w\rho}$$

$$\frac{\mathrm{d}^2u}{d\rho^2} = \frac{w}{\rho^2} - \frac{(bb^{'} + cc^{'})}{w\rho}$$

$$\frac{\mathrm{d}^2u}{d\rho^2} = \frac{w^2 - \rho(bb^{'} + cc^{'})}{w\rho^2}$$

$$\frac{\mathrm{d}^2u}{d\rho^2} = w^2 - \rho(bb^{'} + cc^{'})$$

$$\frac{\mathrm{d}^2u}{d\rho^2} = \gamma \mathrm{k}_1 \rho^{\gamma-1} + \frac{k^2{}_2\rho}{\mu} - \rho(bb^{'} + cc^{'})$$

We know that b and c are differentiating, we have

$$b^2 = \frac{k^2{}_2}{\mu} \quad \text{it implies there } bb^{'} = \frac{k^2{}_2}{2\mu}$$

$$\text{and } c^2 = \gamma k_1 \rho^{\gamma-1} \text{ so that } 2cc^{'} = \gamma(\gamma-1)k_1\rho^{\gamma-2}$$

$$cc^{'} = \frac{\gamma(\gamma-1)k_1\rho^{\gamma-2}}{2}$$

$$\frac{d^2u}{d\rho^2} = \gamma k_1\rho^{\gamma-1} + \frac{k^2{}_2\rho}{\mu} - \frac{k^2{}_2\rho}{2\mu} - \frac{\gamma(\gamma-1)k_1\rho^{\gamma-1}}{2}$$

$$\frac{d^2u}{d\rho^2} = \frac{k^2{}_2\rho}{\mu} + \gamma k_1\rho^{\gamma-1}\left(1-\frac{(\gamma-1)}{2}\right)$$

$$\frac{d^2u}{d\rho^2} = \frac{\frac{k^2{}_2\rho}{\mu} + \frac{(3\gamma-\gamma^2)k_1\rho^{\gamma-1}}{2}}{w\rho^2} > 0$$

$$\frac{d^2u}{d\rho^2} = \frac{\frac{k^2{}_2\rho}{\mu} + \frac{(3\gamma-\gamma^2)k_1\rho^{\gamma-1}}{}}{2\rho^2\sqrt{p' + \frac{BB'}{\mu}}} > 0$$

for $1 \le \gamma \le 2$ hold $\frac{\mathrm{d}^2u}{d\rho^2} = \frac{\frac{k^2{}_2\rho}{\mu} + \frac{(3\gamma-\gamma^2)\mathrm{k}_1\rho^{\gamma-1}}{}}{2\rho^2\sqrt{p' + \frac{BB'}{\mu}}} > 0$

and, therefore, u is convex with respect $\rho$ for1-rarefaction waves. Similarly, we can show for

2-rarefaction waves.

Now we prove that the shock curves are starlike with respect to $(\rho_l, u_l)$ for p and B, these has a good geometry in Riemann invariant coordinates whenever $p' > 0$ and $p'' \ge 0$.

## Theorem 3.2:

The 1-shock and 2-shock curve are starlike with respect to $(\rho_1, u_l)$ when $p = k_1\rho^\gamma$ and $B = k_2\rho$ for values of $\gamma$ lying in the range $(1 \le \gamma \le 2)$.

## Proof:

We have to prove that any ray through the point $(\rho_1, u_l)$ be intersected 1-shock curve in at atmost one point for this is sufficient to prove the rays $(\rho_1, u_l)$ through the two different points $(\rho_1, u_1), (\rho_2, u_2)$ on the 1-shock curve, and whose slope are different. The slope of the line joining $(\rho_1, u_l)$ with $(\rho_1, u_1), (\rho_2, u_2)$ is $\frac{u_1 - u_l}{\rho_1 - \rho_l}$ and $\frac{u_2 - u_l}{\rho_2 - \rho_l}$.

For the 1-shock equation (2.5) are implies that

$$\left(\frac{u\text{-}u_l}{\rho\text{-}\rho_l}\right)^2 = f_1(\rho)+f_2(\rho),$$

where $f_1(\rho)=\dfrac{p-p_l}{\rho_l\rho(\rho-\rho_l)}, f_2(\rho)=\dfrac{B^2-B_l^2}{2\mu\rho_l\rho(\rho-\rho_l)}$

we prove that $f_l^{'}(\rho)<0$ and $f_2^{'}(\rho)<0$. When putting $B=k_2\rho$ in $f_2(\rho)$ and differentiating with respect to $\rho$, and we have

$$f_2(\rho)=\frac{B^2-B_l^2}{2\mu\rho_l\rho(\rho-\rho_l)}=\frac{(k_2\rho\ )^2-(k_2\rho_l\ )^2}{2\mu\rho_l\rho(\rho-\rho_l)}$$

$$f_2(\rho)=\frac{k_2^2\left(\rho^2-\rho^2{}_l\right)}{2\mu\rho_l\rho(\rho-\rho_l)}$$

$$f_2(\rho)=\frac{k_2^2(\rho-\rho_l)(\rho+\rho_l)}{2\mu\rho_l\rho(\rho-\rho_l)}$$

$$f_2(\rho)=\frac{k_2^2(\rho+\rho_l)}{2\mu\rho_l\rho}$$

$$f_2(\rho)=\frac{k_2^2}{2\mu}\left(\frac{1}{\rho_l}+\frac{1}{\rho}\right)$$

and its again differentiating with respect to $\rho$, we have

$$f_2^{\,'}(\rho)=\frac{k_2^2}{2\mu}\left(-\frac{1}{\rho^2}\right)$$

$$f_2^{\,'}(\rho)=\frac{-k_2^2}{2\mu\rho^2}<0$$

It also proved that, when $f_1(\rho)$ in $p=k_1\rho^{\gamma}$ and differentiating with respect to $\rho$, and we obtain

$$f_1(\rho)=\frac{p-p_l}{\rho_l\rho(\rho-\rho_l)},$$

$$f_1(\rho)=\frac{k_1\rho^{\gamma}\ -k_1\rho_l^{\gamma}}{\rho_l\rho(\rho-\rho_l)}$$

$$f_1(\rho)=\frac{k_1(\rho^{\gamma}\ -\rho_l^{\gamma})}{\rho_l\rho(\rho-\rho_l)}$$

on differentiating, we have

$$\frac{\partial f_1(\rho)}{\partial \rho}=\frac{\partial \dfrac{k_1(\rho^{\gamma}-\rho_l^{\gamma})}{\rho_l\rho(\rho-\rho_l)}}{\partial \rho}$$

$$\frac{\partial f_1(\rho)}{\partial \rho}=\frac{(\rho_l\rho(\rho-\rho_l))\dfrac{\rho}{\partial\rho}\left(k_1\rho^{\gamma}-k_1\rho_l^{\gamma}\right)-(k_1\rho^{\gamma}-k_1\rho_l^{\gamma})\dfrac{\rho}{\partial\rho}(\rho_l\rho(\rho-\rho_l))}{(\rho_l\rho(\rho-\rho_l))^2}$$

$$\frac{\partial f_1(\rho)}{\partial \rho}=\frac{(\rho_l\rho(\rho-\rho_l)).k_1\beta\gamma\rho^{\gamma-1}-\left(k_1\rho^{\gamma}-k_1\rho_l^{\gamma}\right)\left(2\rho_l\rho-\rho_l^{\ 2}\right)}{(\rho_l\rho(\rho-\rho_l))^2}$$

$$\frac{\partial f_1(\rho)}{\partial \rho}=\frac{(\rho_l\rho(\rho-\rho_l)).k_1\gamma\rho^{\gamma-1}-\left(2\rho_l\rho k_1\rho^{\gamma}-k_1\rho^{\gamma}\rho_l^2-2\rho_l\rho k_1\rho_l^{\gamma}+k_1\rho_l^{\gamma}\rho_l^{\ 2}\right)}{(\rho_l\rho(\rho-\rho_l))^2}$$

$$\frac{\partial f_1(\rho)}{\partial \rho}=\frac{\left(\rho_l\rho^2-\rho_l^2\rho\right)\left(k_1\gamma\rho^{\gamma-1}\right)-\left(2\rho_l\rho k_1\rho^{\gamma}-k_1\rho^{\gamma}\rho_l^2-2\rho_l\rho k_1\rho_l^{\gamma}+k_1\rho_l^{\gamma}\rho_l^{\ 2}\right)}{(\rho_l\rho(\rho-\rho_l))^2}$$

$$\frac{\partial f_1(\rho)}{\partial \rho}=\frac{\left(\rho_l\rho^2k_1\gamma\rho^{\gamma-1}-\rho_l^2\rho\left(k_1\gamma\rho^{\gamma-1}\right)\right)-\left(2\rho_l\rho k_1\rho^{\gamma}-k_1\rho^{\gamma}\rho_l^2-2\rho_l\rho k_1\rho_l^{\gamma}+k_1\rho_l^{\gamma}\rho_l^{\ 2}\right)}{(\rho_l\rho(\rho-\rho_l))^2}$$

$$\frac{\partial f_1(\rho)}{\partial \rho}=\frac{\left(\rho_l\rho^2-\rho_l^2\rho\right)\left(k_1\gamma\rho^{\gamma-1}\right)-\left(2\rho_l\rho k_1\rho^{\gamma}-k_1\rho^{\gamma}\rho_l^2-2\rho_l\rho k_1\rho_l^{\gamma}+k_1\rho_l^{\gamma}\rho_l^{\ 2}\right)}{(\rho_l\rho(\rho-\rho_l))^2}$$

$$\frac{\partial f_1(\rho)}{\partial \rho}=\frac{\rho_l\rho^{\gamma+1}k_1\gamma-k_1\gamma\rho^{\gamma}\rho_l^2-2k_1\rho^{\gamma+1}\rho_l+k_1\rho^{\gamma}\rho_l^2+2\rho\rho_l^{\gamma+1}k_1-k_1\rho_l^{\gamma+2}}{(\rho_l\rho(\rho-\rho_l))^2}$$

$$\frac{\partial f_1(\rho)}{\partial \rho}=\frac{\rho_l\rho^{\gamma+1}k_1\gamma-2k_1\rho^{\gamma+1}\rho_l+k_1\rho^{\gamma}\rho_l^2-k_1\gamma\rho^{\gamma}\rho_l^2+2\rho\rho_l^{\gamma+1}k_1-k_1\rho}{(\rho_l\rho(\rho-\rho_l))^2}$$

$$\frac{\partial f_1(\rho)}{\partial \rho}=\frac{k_1(\gamma-2)\rho_l\rho^{\gamma+1}+k_1(1-\gamma)\rho_l^2\rho^{\gamma}+2k_1\rho_l^{\gamma+1}-k_1\rho_l^{\gamma+2}}{(\rho_l\rho(\rho-\rho_l))^2}$$

Let $g_1(\rho)=k_1(\gamma-2)\rho_l\rho^{\gamma+1}+k_1(1-\gamma)\rho_l^2\rho^{\gamma}+2k_1\rho_l^{\gamma+1}-k_1\rho_l^{\gamma+2}$ ,then $g_1(\rho_l)=0$.

Then $g_1^{'}(\rho)=k_1(\gamma-2)(\gamma+1)\rho_l\rho^{\gamma}+k_1\gamma(1-\gamma)\rho_l^2\rho^{\gamma-1}+2k_1\rho_l^{\gamma+1}-k_1\rho_l^{\gamma+1}$ , $g_1^{'}(\rho_l)=0$.

Since $g_1^{''}(\rho)=\ k_1(\gamma-2)(\gamma+1)\gamma\rho_l\rho^{\gamma-1}+k_1\gamma(1-\gamma)(1-\gamma)\rho_l^2\rho^{\gamma-2}$, if above condition are follows that the values of $\gamma$ in $1\le\gamma\le 2$, we have $g_1^{''}(\rho)<0$. The above equation to be held in 1-shock and then $\rho_1<\rho$

and $g_1^{'}(\rho) < g_1^{'}(\rho_l) = 0$, implying that $g_1(\rho)$ is a decreasing function of $\rho$; and this follows that $g_1(\rho) < g_1(\rho_l)$, it therefore $f_1^{'} < 0$. Thus, $\frac{u - u_1}{\rho - \rho_1}$ is a decreasing function of $\rho$; we hence 1-shock curve is starlike with respect to $(\rho_1 u_l)$, as usually in same way 2-shock curve and is also a starlike with respect to $(\rho_1 u_l)$.

## Lemma 3.1

With $p^{'}(\rho) > 0$ and $p^{''} \geq 0$, the inequalities $0 < \frac{d\Pi_1}{d\Pi_2} < 1$ and $0 < \frac{d\Pi_2}{d\Pi_1} < 1$ hold along 1-shock and 2-shock respectively, with

$$\Pi_1 = u + \int^{\rho} \frac{w(y)}{y} dy, \qquad (3.3)$$

$$\Pi_2 = u - \int^{\rho} \frac{w(y)}{y} dy. \qquad (3.4)$$

**Proof:** From (3.3) and (3.4), we have

$$\frac{d\Pi_1}{d\rho} = \frac{du(\rho)}{d\rho} + \frac{w(\rho)}{\rho} \text{ and}$$
$$\frac{d\Pi_2}{d\rho} = \frac{du(\rho)}{d\rho} - \frac{w(\rho)}{\rho}$$

We know that the above theorem as long a 1-shock curves.

$\frac{du(\rho)}{d\rho} < 0$, it has that $\frac{d\Pi_2}{d\rho} < 0$.

Further, as along 1-shock curves $\left|\frac{d\Pi_1}{d\rho}\right| < \left|\frac{d\Pi_2}{d\rho}\right|$, we have $\left|\frac{d\Pi_1}{d\Pi_2}\right| < 1$. In order to prove that $0 < \left|\frac{d\Pi_1}{d\Pi_2}\right| < 1$, in sufficiently part $\frac{d\Pi_1}{d\rho}$<0. For a condition 1-shock curve, equation (2.5) it imply that

$$\frac{w(\rho)}{\rho} = \frac{\sqrt{\left(p^{'} + \frac{BB^{'}}{\mu}\right)}}{\rho}$$

$$\frac{du(\rho)}{d\rho}+\frac{w(\rho)}{\rho}=\frac{\sqrt{\left(p^{'}+\frac{BB^{'}}{\mu}\right)}}{\rho}\frac{-\left(p^{'}+\frac{2BB^{'}}{2\mu}\right)\left(\frac{\rho-\rho_l}{\rho\rho_l}\right)+\left(p+\frac{B^2}{2\mu}-p_l-\frac{B_l^2}{2\mu}\right)\frac{1}{\rho^2}}{2\sqrt{\left(p+\frac{B^2}{2\mu}-p_l-\frac{B_l^2}{2\mu}\right)\left(\frac{\rho-\rho_l}{\rho\rho_l}\right)}}$$

$$==\frac{\left[\sqrt{\left(p+\frac{B^2}{2\mu}-p_l-\frac{B_l^2}{2\mu}\right)\frac{1}{\rho^2}}-\sqrt{\left(p^{'}+\frac{BB^{'}}{\mu}\right)\left(\frac{1}{\rho_l}-\frac{1}{\rho}\right)}\right]^2}{2\sqrt{\left(p+\frac{B^2}{2\mu}-p_l-\frac{B_l^2}{2\mu}\right)\left(\frac{\rho-\rho_l}{\rho\rho_l}\right)}},$$

hence the above condition holds that

$\frac{du(\rho)}{d\rho}+\frac{w(\rho)}{\rho}<0$, and these implies that $\frac{d\Pi_1}{d\rho}<0$.

Similarly, we show that $0<\frac{d\Pi_1}{d\Pi_2}<1$ along 2-shock curves.

## 4. Riemann Problem:

The system (2.1) be the initial condition as

$$U(x,t_0)=\begin{cases}U_l, if\ x<x_0,\\ U_r, if\ x>x_0\end{cases}, \qquad (4.1)$$

is called as Riemann problem. Where $U_l$ be the state to the left of $x=x_0$ and $U_r$ be the state to the right of $x=x_0$ the constant states are separated by in both waves either a shock waves or rarefaction wave. The Riemann invariant coordinates are

$$\Pi_1=u+\int^{\rho}\frac{w(y)}{y}dy \text{ and } \Pi_2=u-\int^{\rho}\frac{w(y)}{y}dy.$$

### Lemma (4.1):

The mapping $(\rho,u)\to(\Pi_1,\Pi_2)$ is one to one and the Jacobian of this mapping is nonzero when $\rho>0$.

**Proof:** Since, $\Pi_1 = u + \int^{\rho} \frac{w(y)}{y} dy$ and $\Pi_2 = u - \int^{\rho} \frac{w(y)}{y} dy$

On differentiating with respect to $\rho$, we get

$$\frac{\partial \Pi_1}{\partial \rho} = \frac{w}{\rho}, \frac{\partial \Pi_1}{\partial u} = 1,$$
$$\frac{\partial \Pi_2}{\partial \rho} = -\frac{w}{\rho}, \frac{\partial \Pi_2}{\partial u} = 1,$$

Thus, the Jacobian of the mapping $(\rho, u) \to (\Pi_1, \Pi_2)$

$$\begin{vmatrix} \frac{\partial \Pi_1}{\partial \rho} & \frac{\partial \Pi_1}{\partial u} \\ \frac{\partial \Pi_2}{\partial \rho} & \frac{\partial \Pi_2}{\partial u} \end{vmatrix} = \begin{vmatrix} \frac{w}{\rho} & 1 \\ -\frac{w}{\rho} & 1 \end{vmatrix}$$

$$\frac{w}{\rho} + \frac{w}{\rho} = 2\frac{w}{\rho}$$

This is one-one and onto.

We consider Riemann invariants as coordinate system. Let us will take a plane $(\Pi_1, \Pi_2)$ in that plane we draw the curves $S_1, S_2, R_1$ and $R_2$ which divide the plane can into four distinct regions. I, II, III, and IV. Let $U_l$ are left state. Fixing $U_l$ and varying $U_r$. Let us consider $U_r$ belong to any of the four region as fig.4 (a). For $U \in R^2$.

$$S_n(U) = \{(\Pi_1, \Pi_2) : (\Pi_1, \Pi_2) \in S_n\}, n = 1,2.$$
$$R_n(U) = \{(\Pi_1, \Pi_2) : (\Pi_1, \Pi_2) \in R_n\}, n = 1,2 \text{ and } T_n(U) = S_n(U) \cup R_n(U), n = 1,2$$

In an above wave curves, the plane $(\rho, u)$ divides into a four region. To solve the Riemann problem, consider the wave curve $T_2(U_m)$ for $U_m \in T_1(U_l)$. And we have to verify that two curves $T_2(U_m)$ and $T_2(U_{m*})$, where $U_m, U_{m*} \in T_1(U_l)$, so these a two curves are non-intersecting and the set of all such curves to entire half space $\rho > 0$ in the plane $(\Pi_1, \Pi_2)$ in one-one fashion.

If $U_r \in$ I, draw a vertical line $\Pi_2 = \Pi_{2r}$ in fig.4 (a). Which will be intersects $S_1$ uniquely at a point $U_{m_1}$. The solution to Riemann problem is now obvious; we taking on constant state of $U_{m1}$ by a 1-shock and then from $U_{m1}$ to the constant state $U_r$ by a 2-rarefaction wave.

Let $U_r \in$ II region, draw a vertical line $\Pi_2 = \Pi_{2r}$ in fig.4 (a) which is intersect $R_1$ uniquely at a point $U_{m2}$. The solution is going from $U_1$ and $U_{m2}$ by $R_1$ and to from $U_{m2}$ to $U_r$ by $R_2$.

If $U_r \in$ III region, we define the concept of inverse shock curve. The inverse curve denote by $S_2^*$ consists of those states $(\Pi_1, \Pi_2)$ which can be connected to the state $(\Pi_{1r}, \Pi_{2r})$ on the right by $S_2$ shock in fig.4 (a). These represented, from (2.5) by

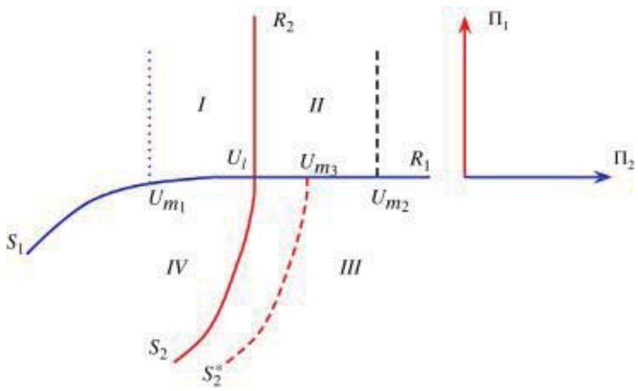


Fig 4(a). Rarefaction curves ($R_1$ and $R_2$) and shock curves ($S_1$ and $S_2$) in the plane $(\Pi_1, \Pi_2)$.

$u = u_r + \sqrt{\left(p + \frac{B^2}{2\mu} - p_r - \frac{B_r^2}{2\mu}\right)\left(\frac{\rho - \rho_r}{\rho \rho_r}\right)}$. The above curve intersect the $R_1$ uniquely a point $U_{m3}$ .Therefore, $U_r$ can be connected with $U_l$ by $R_1$ are followed by $S_2$.

If $U_r \in$ IV region (see fig.4(c)), (as from lemma 3.1) $\frac{d\Pi_2}{d\Pi_1} > 1$ on $S_1$. It also defines in $\frac{d\Pi_2}{d\Pi_1} < 1$ on $S_2^*$. This mean that, the $S_1$ and $S_2^*$ will intersecting uniquely at the point $U_{m_4}$, therefore, the solution consists of 1-shock and 2-shock. Thus we have shown that set $\{T_2(U_m): U_m \in T_1(U_l)\}$, covers the entire half space $\rho > 0,$ in the plane $(\Pi_1 \Pi_2)$ in a one-one way.

When the vacuum state $(\rho = 0)$ it's not satisfied the same condition.

**Lemma (4.2):**

If $\Pi_{1l} \le \Pi_{2r}$, the vacuum occurs.

**Proof:**

From fig.4 (a), $\Pi_{1m} = \Pi_{1l}$ and $\Pi_{2m} = \Pi_{2r}$;

if $\Pi_{1l} \le \Pi_{2r}$, then $\Pi_{1m} - \Pi_{2m} = \Pi_{1l} - \Pi_{2r} \le 0.$

But it will be $\Pi_{1m} - \Pi_{2m} = 2\int_0^{\rho_m} \frac{w(y)}{y} dy$ y

Which implying that that $\rho_m \le 0$. Hence, vacuum occurs.

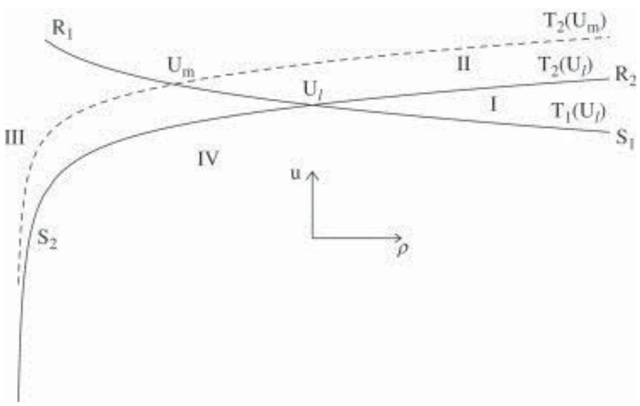


Fig4(b). wave curves in plane $(\rho, u)$.

**Theorem (4.1):**

Assume that $p' > 0, p'' \ge 0$ and that we are given initial states $U_l$ and $U_r$ where $\rho_l > 0$, $\rho_\rho > 0$ for the Riemann problem of system (2.1). Assume that $\Pi_{1l} > \Pi_{2r}$. Then there exists a solution of the Riemann problem for system (2.1). Moreover, the solution is given by 1- wave following by a 2-wave satisfying $\rho > 0$, and the solution is unique in the class of constant states separated by shock waves and rarefaction waves.

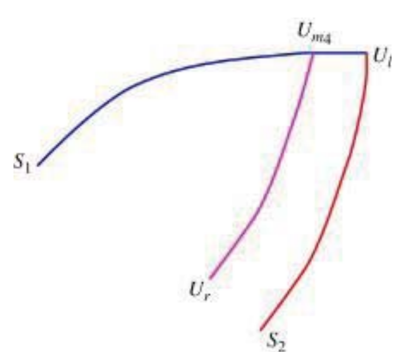

fig.4(c). 2-shock wave and 1-shock wave.

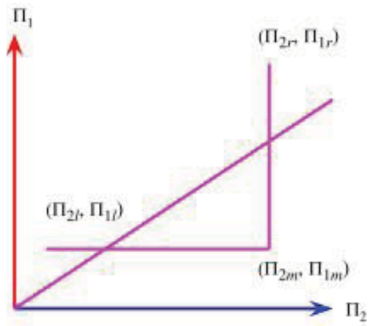


fig4(d). vacuum curve

## 5. Interaction of Elementary waves:

The interaction of elementary waves, obtaining from the Riemann problem (4.1), gives rise to new emerging elementary waves. And then two jump discontinuities at $x_1$ and $x_{2,}$, it as follows:

$$U(x,t_0)=\begin{cases} U_l, if -\infty < x \le x_1 \\ U_*, if\, x_1 < x \le x_2 \\ U_r, if\, x_2 < x < \infty \end{cases} \qquad (5.1)$$

The choice of $U_*$ and $U_r$ in the terms of $U_l$ and an arbitrary $x_1$ and $x_2 \in R$. With the initial data, we have two Riemann problem locally. The first Riemann problem of the elementary wave may interact the second Riemann problem of the elementary wave, and the time of interaction at formed a new Riemann problem at one dimensional Euler equation. It may be found of the interaction of the elementary waves. Here we like $R_2S_1 \to R_2$, it means that a 2-rarefraction waves $(u_l \text{ to } u_*)$ $R_2$, of the first Riemann problem interacts with 1-shock, $S_1$, of the second Riemann problem $u_l$ to $u_r$. Then it interacts to new Riemann problem $u_l$ to $u_r$ via $u_m$ $S_1R_2$. In different families are possible to interaction of elementary waves and as well as the same family are respectively $(S_2S_1, S_2R_1, R_2R_1, R_2S_1)$ and $(S_2S_2, S_1S_1, R_1S_1, S_1R_1, S_2R_2, R_2S_2)$.

## 5.1 Interaction of Elementary waves from different Families:

## (a) Collision of two shocks ($S_2S_1$):

Let $U_l$ is connection to $U_*$ by the 2- shock $S_2$ is a first Riemann problem and $U_*$ is connected to $U_r$ by a 1-shock, $S_1$ of the second Riemann problem. For a given $U_l$, we consider $U_*$ and $U_r$ in such a way that $\rho_* < \rho_l$, from (2.5) we have $u_* = u_l - g(\rho_l, \rho_*)$ in other way that $\rho_* < \rho_r$, then we have written $u_r = u_* - g(\rho_*, \rho_r)$.

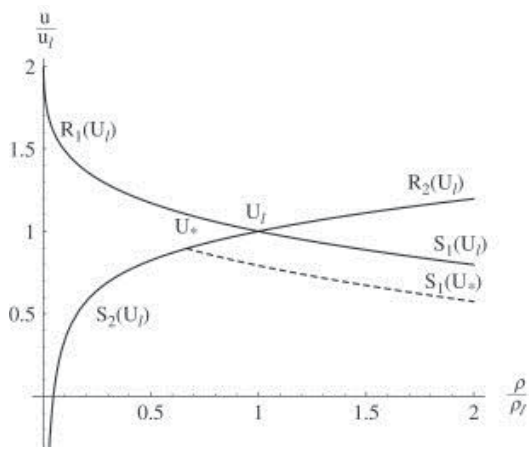


fig. 5.1(a). $S_2S_1$ collision

Since, speed of 1-shock of the second Riemann problem is negative, $S_2$ and speed 2-shock of the first Riemann problem is positive, $S_1$ overtakes $S_2$. Then it shows that for any arbitrary state $U_l$, the state $U_r$ lies in the region IV (in fig.4 (b)). It in sufficient to prove that

$g(\rho_*, \rho) - g(\rho_l, \rho) + g(\rho_l, \rho_*) > 0$ for $\rho_* < \rho_l$ and $\rho_* < \rho$. Let take in contrary that

$g(\rho_*, \rho) - g(\rho_l, \rho) + g(\rho_l, \rho_*) \leq 0.$ If in this, then

$$g^2(\rho_1, \rho_*) + g^2(\rho_*, \rho) + 2g(\rho_1, \rho_*)(\rho_*, \rho) \leq g^2(\rho_1, \rho),$$

Implying thereby that,

$$\left(\mathrm{P}_* - P + \frac{B_*^2 - B^2}{2\mu}\right)\left(\frac{1}{\rho_l} - \frac{1}{\rho_*}\right) + \left(\mathrm{P}_* - P_l + \frac{B_*^2 - B_l^2}{2\mu}\right)\left(\frac{1}{\rho} - \frac{1}{\rho_*}\right) + 2g(\rho_1, \rho_*)g(\rho_*, \rho) \le 0 \tag{5.2}$$

The above equation (5.2), is strictly positive, which is a contradiction. Hence,

$\mathrm{g}(\rho_1, \rho_*) + g(\rho_*, \rho) + g(\rho_l, \rho) > 0$, i.e., the curve $\mathrm{S}_1(U_*)$ are lies below the curves $\mathrm{S}_1(U_l)$, therefore, $U_r$ lies in the region IV. Thus, and it follows interaction results is $\mathrm{S}_2 S_1 \to S_1 S_2$ interaction results, in case of illustrate in fig.5.1 (a).

## (b) Collision of a shock and rarefaction ($S_2R_1$):

Here $U_* \in S_2(U_l)$ and $U_r \in R_1(U_*)$ i.e., for a given $U_l$, Let $U_*$ and $U_r$ such that $\rho_* < \rho$ from equation (2.5), we have $u_* = u_l - g(\rho_l, \rho_*)$ and $\rho_r \le \rho_*$, from equation (2.15) we have

$u_r = u_* + \int_{\rho_r}^{\rho_*} \frac{w(y)}{y} dy$. Since 2-shock of the Riemann problem is positive and 1- rarefaction wave of the second Riemann problem is negative velocity, it follows that $\mathrm{R}_1$ overtakes $\mathrm{S}_1$. Since, for any given $U_l$,

$$\int_{\rho}^{\rho_l} \frac{\mathrm{w(y)}}{y} dy - \int_{\rho}^{\rho_*} \frac{\mathrm{w(y)}}{y} dy + g(\rho_l, \rho_*) > 0$$

for $\rho < \rho_* < \rho_l$, and can be follows that the curve $R_1(U_*)$ lies below the curve $R_1(U_l)$, hence $U_r$ lies in the region III, subsequently $\mathrm{S}_2 R_1 \to R_1 S_2$. The compute results this case in fig.5.1 (b).

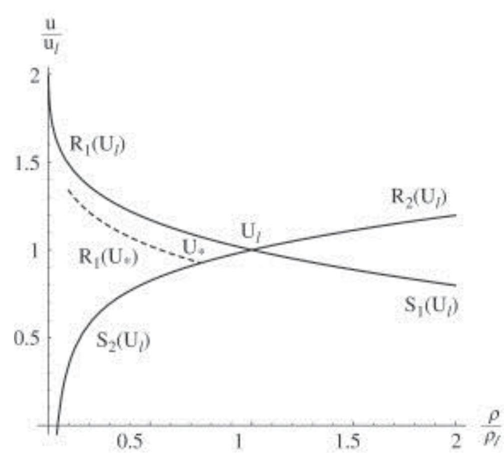


fig. 5.1(b). $S_2R_1$ collision

## (c) Collision of two rarefaction waves ($R_2R_1$):

We consider $U_* \in R_2(U_l)$ and $U_r \in R_1(U_*)$. In any other way, for a given $U_l$, Let $U_*$ and $U_r$ such that $\rho_1 \le \rho_*$, $u_* = u_l + \int_{\rho_l}^{\rho_*} \frac{w(y)}{y} dy$ and $\rho_r \le \rho_*$, then $u_r = u_* + \int_{\rho_r}^{\rho_*} \frac{w(y)}{y} dy$. Since, the trailing end of 2-rarefaction wave has a positive velocity (bounded above) in $(x,t)$- plane and that 1-rarefaction wave has a negative velocity (bounded above), interaction will take place. Since $\rho_1 < \rho_*$ and

$$\int_{\rho}^{\rho_*} \frac{w(y)}{y} dy - \int_{\rho}^{\rho_*} \frac{w(y)}{y} dy + \int_{\rho_l}^{\rho_*} \frac{w(y)}{y} dy > 0,$$

It follows that the curve $R_l(U_*)$ lies above the curve $R_1(U_l)$; hence $U_r$ lies in the region II and the interaction results II and the interaction result is $R_2R_1 \to R_1R_2$. Then computed results, in fig.5.1 (c).

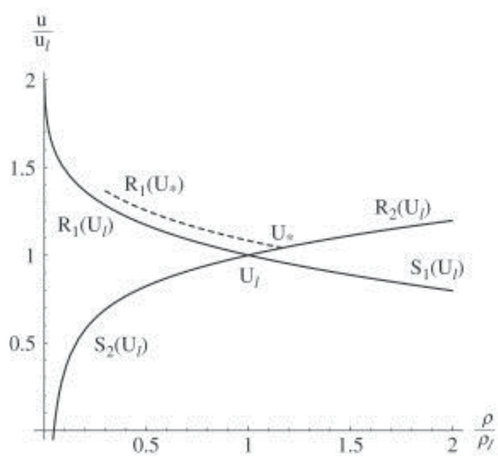


fig. 5.1(c) $R_2R_1$ collision

### (d) Collision of a rarefaction wave and a shock ($R_2S_1$):

Here $U_* \in R_2(U_l)$ and $U_r \in S_1(U_*)$, i.e., for given $U_l$, we choose $U_*$ and $U_r$ such that $\rho_1 \le \rho_*$,

$u_* = u_l + \int_{\rho_l}^{\rho_*} \frac{w(y)}{y} dy$ and $\rho_* < \rho_r$ and $u_l = u_* - g(\rho_*, \rho_r)$. Since, the second Riemann problem of 1-shock speed is less than 2-rarefaction wave of first Riemann problem of the speed of trailing end in $(x,t)$-plane, and therefore $S_1$ penetrates $R_2$. For any given $U_l$. It show that $U_r \in$ I, then it, to show that

$$\int_{\rho_l}^{\rho_*} \frac{w(y)}{y} dy + g(\rho_l, \rho) - g(\rho_*, \rho) > 0. \qquad (5.3)$$

Since $g(\rho_l, \rho)$ is a decreasing function with respect to the first variables $\rho_1$, then we will have

$g(\rho_l, \rho) > g(\rho_*, \rho)$ for $\rho_1 < \rho_*$. Hence, the equalities (5.3), there imply that curves $S_1(U_*)$ lies above the curve $S_1\ (U_l)$ and $U_r$ lies in the region I. Thus the interaction result is $R_2S_1 \rightarrow S_1R_2$; and its computed results in fig. 5.1(d).

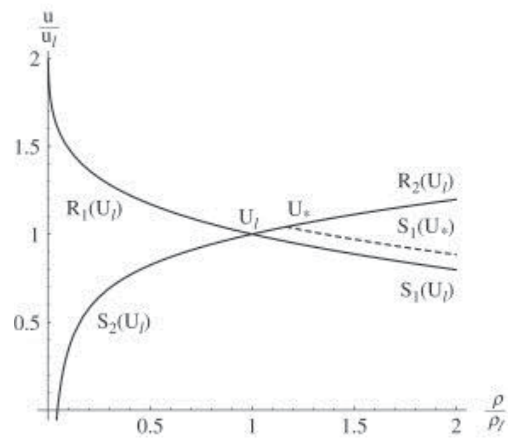


fig. 5.1(d) $R_2S_1$ collision.

## 5.2. Interaction of Elementary waves from same family:

### (a) 2- shock wave overtakes another 2-shock wave ($S_2S_2$):

We consider the situation in which $U_l$ is connection to $U_*$ by a shock of the first Riemann problem and $U_*$ is connected to $U_r$ by a 2- shock of the second Riemann problem. In other situation a given left state $U_l$, the intermediate state $U_*$, and the right state $U_r$ are chose such that $\rho_* < \rho_l$ and $u_* = u_l - g(\rho_l . \rho_*)$ with Lax conditions satisfy

$$\lambda_1(U_l) < \upsilon_2(U_l, U_*) < \lambda_2(U_l), \quad \lambda_2(U_*) < \upsilon_2(U_l, U_*), \qquad (5.4)$$

and $\rho_r < \rho_*$, and $u_r = u_* - g(\rho_*, \rho_r)$ with Lax stability conditions

$$\lambda_1(U_*) < \upsilon_2(U_*, U_r) < \lambda_2(U_*), \quad \lambda_2(U_*) < \upsilon_2(U_*, U_r) \qquad (5.5)$$

where $\upsilon_2(U_l, U_*)$ is the speed of shock connection $U_l$ to $U_*$, and similarly, $\upsilon_2(U_*, U_r)$ is the speed of shock connecting $U_*$ to $U_r$. From (5.4) and (5.5), we obtained $\upsilon_2(U_*, U_r) < \upsilon_2(U_l, U_*)$, i.e., the second Riemann problem of 2-shock overtakes the first Riemann problem of 2-shock at a finite time, then its give rise to new Riemann problem with data $U_l$ and $U_r$. To prove this problem. We must have to determine the region in which $U_r$ lies respect to $U_l$. Let be claim that $U_r$ vary lies in region III so this have solution of the new Riemann problem consists of $\mathrm{R}_1$ and $\mathrm{S}_2$. In any more way, to show that to our claim: we have to prove that $\mathrm{S}_2(U_*)$ lies in entirely in the region III; to prove this required to show that for $\rho < \rho_* < \rho_l$.

$g(\rho_l, \rho)$- $g(\rho_*, \rho) - g(\rho_l, \rho_*) > 0.$ We consider, on the contradiction that

$g(\rho_l, \rho)$- $g(\rho_*, \rho) - g(\rho_l, \rho_*) \leq 0.$ for $\rho < \rho_* < \rho_l$. Then the follow that, if we take, then

$$\mathrm{g}^2(\rho_1, \rho) + g^2(\rho_*, \rho_l) + 2g(\rho_1, \rho_*)(\rho_*, \rho) \leq g^2(\rho_*, \rho), \tag{5.6}$$

Implying there by that,

$$\left(p - p_l + \frac{B^2 - B_l^2}{2\mu}\right)\left(\frac{1}{\rho_l} - \frac{1}{\rho_*}\right) + \left(\mathrm{p}_* - p_l + \frac{B_*^2 - B_l^2}{2\mu}\right)\left(\frac{1}{\rho_l} - \frac{1}{\rho}\right) \leq 2g(\rho_1, \rho_*)g(\rho_l, \rho) \leq 0$$

Proving that

$$\left[\left(p - p_l + \frac{B^2 - B_l^2}{2\mu}\right)\left(\frac{1}{\rho_l} - \frac{1}{\rho_*}\right) + \left(p_* - p_l + \frac{B_*^2 - B_l^2}{2\mu}\right)\left(\frac{1}{\rho_l} - \frac{1}{\rho}\right)\right] \leq 0, \tag{5.7}$$

which is contradiction on as left hand of inequalities (5.7) is positive. Hence, $\mathrm{S}_2\mathrm{S}_2 \to \mathrm{R}_1 S_2$; and computed results in above situation fig.5.2(a).

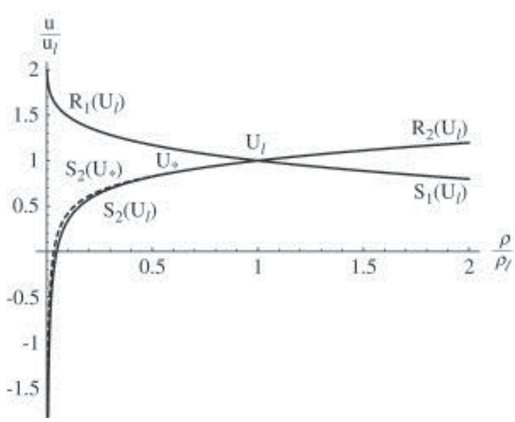


fig. 5.2(a) $S_2$ overtakes $S_2$.

**(b) 1-shock wave overtakes another 1- shock wave ($S_1S_1$):**

Let $U_r$ lies in a region I, so that $S_1S_1 \to S_1R_2$, is similarly to the previous case and its above situation illustrate computer results.

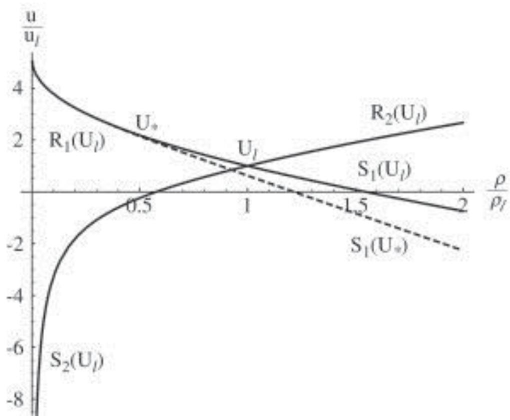


fig. 5.2(b) $S_1$ overtakes $R_1$.

## (C) 1-shock wave overtakes 1-Rarefaction wave ($R_1S_1$):

In case, the Riemann problem of the $U_l$ is connected to $U_*$ by 1- rarefaction wave and the second Riemann of the $U_*$ is connected to $U_r$ by 1-shock. i.e., a given $U_l$, Let $U_*$ and $U_r$ in such a way that $\rho_* \le \rho_l,\ u_* = u_l + \int_{\rho_*}^{\rho_l} \frac{w(y)}{y} dy$ and $\rho_* < \rho_r, u_r = u_* - g(\rho_*, \rho_r)$. So we show that $S_1(U_*)$ lies below of the $R_1(U_l)$ for $\rho_* < \rho \le \rho_l$, in other way, for $\rho_* < \rho \le \rho_l$,

$$g(\rho_*, \rho) + \int_{\rho}^{\rho_l} \frac{w(y)}{y} dy - \int_{\rho_*}^{\rho_l} \frac{w(y)}{y} dy > 0 \qquad (5.8)$$

Let us define, $F_1(\rho) = g(\rho_*, \rho) + \int_{\rho}^{\rho_l} \frac{w(y)}{y} dy - \int_{\rho_*}^{\rho_l} \frac{w(y)}{y} dy$ so that $F_1(\rho_*) = 0$. On differentiating $F_1(\rho)$ with respect to $\rho$, we obtain $F_1'(\rho) > 0$, implying that $F_1(\rho_*) < F_1(\rho)$. i.e., $F_1(\rho) > 0$, and hence $S_1(U_*)$ lies below the curve $R_1(U_l)$ for $\rho_* < \rho \le \rho_l$. In another to show that it is sufficiently to $S_1(U_l)$ lies above the curve $S_1(U_*)$ for $\rho_1 \le \rho$; the sufficiently part, the claim has

$$g(\rho_*,\rho)-g(\rho_l,\rho)-\int_{\rho_*}^{\rho_l}\frac{w(y)}{y}dy>0,\forall\rho_1\leq\rho\,; \qquad (5.9)$$

Let us define $F_2(\rho_l)=g(\rho_*,\rho)-g(\rho_l,\rho)-\int_{\rho_*}^{\rho_l}\frac{w(y)}{y}dy$

So that $F_2(\rho_l)=F_1(\rho_l)>0$.

Let us consider that $g(\rho_*,\rho)-g(\rho_*,\rho_l)\leq g(\rho_l,\rho)\text{ for }\rho_*<\rho_l<\rho,$

implying that,

$$g^2(\rho_*,\rho)-g^2(\rho_*,\rho_l)-2g(\rho_*,\rho)g(\rho_*,\rho_l)\leq g^2(\rho_l,\rho),$$

implying thereby that

$$\left(P_1-P_*+\frac{B_l^{\,2}-B_*^{\,2}}{2\mu}\right)\left(\frac{1}{\rho_*}-\frac{1}{\rho_l}\right)+\left(P\ -P_*+\frac{B_l^2-B_*^2}{2\mu}\right)\left(\frac{1}{\rho_*}-\frac{1}{\rho_l}\right)\leq 2g(\rho_*,\rho)g(\rho_*,\rho_l);$$

or equivalently,

$$\left[\left(P_1-P_*+\frac{B^2-B_l^{\,2}}{2\mu}\right)\left(\frac{1}{\rho_*}-\frac{1}{\rho_l}\right)-\left(P\ -P_*+\frac{B^2-B_*^2}{2\mu}\right)\left(\frac{1}{\rho_*}-\frac{1}{\rho_l}\right)\right]^2\leq 0. \qquad (5.10)$$

But the left hand side of inequality (5.10) is positive, which leaves us with a contradiction.

Hence, $g(\rho_*,\rho)\text{-}g(\rho_1,\rho)>g(\rho_*,\rho_l)\text{ for }\rho_*<\rho_l<\rho,$ implying that,

$$g(\rho_*,\rho)-g(\rho_l,\rho)-\int_{\rho_*}^{\rho_l}\frac{w(y)}{y}dy>g(\rho_*,\rho_l)-\int_{\rho_*}^{\rho_l}\frac{w(y)}{y}dy=F_2(\rho_l)>0$$

We define a new function,

$$F_3(\rho)=g(\rho_*,\rho)-g(\rho_l,\rho)-\int_{\rho_*}^{\rho_l}\frac{w(y)}{y}dy\text{ for }\rho_*\leq\rho\leq\rho_l.$$

At some point $(\tilde{\rho}_1,\tilde{u}_1)$ intersected in $S_2(U_l)$ and $S_1(U_*)$, for $\rho_*<\tilde{\rho}_1<\rho_l$. Since, $F_3(\rho)>0$ and $F_3(\rho_*)<0,$ it is intermediate value property, there exists a $\tilde{\rho}_1,$ between $\rho_*$ and $\rho_1,$ such that $F_3(\tilde{\rho}_1)=0,$ by virtue of monotonicity. Thus, $S_2(U_l)$ and $S_1(U_*)$ is uniquely determined of the intersection, and the computer results in fig. 5.2(b). We distinguished three cases to depending on the value of $\rho_r$,

(a) If $\rho_r < \tilde{\rho}_1$, indeed 1-shock is weak as compared to 1- rarefaction wave, when $U_r \in$ III and the interaction results is $R_1 S_1 \to R_1 S_2$.

(b) If $\rho_r = \tilde{\rho}_1$, indeed two waves of first family interact, they annihilate each other, and give rise to wave of second family, when $U_r$ lies on $S_2(U_l)$ and the interaction result is $R_1 S_1 \to S_2$.

(c) If $\rho_r > \tilde{\rho}_1$, and the interaction result is $R_1 S_1 \to S_1 S_2$, on $U_r \in IV$; indeed , the 1-rarefaction of the first Riemann problem is weak as compare to the 1-shock of second Riemann problem, which is stronger, overtakes and the trailing end of 1-rarefaction wave a reflected shock

$S_2(U_m, U_r)$, and a connection new connection constants state $U_m$ on the left to $U_r$ on the right, is produced. The transmitted wave, after interaction, is the 1-shock that joins state $U_l$ on the left and $U_m$ on the right.

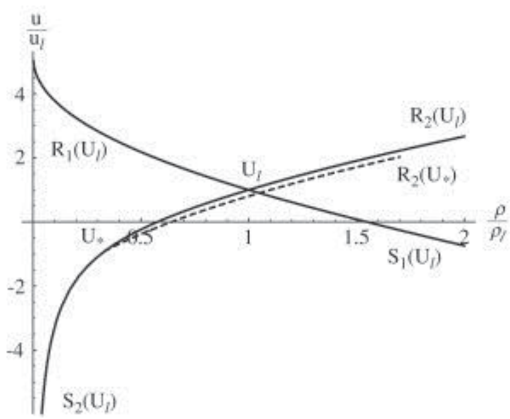


fig.5.2(c) $R_2$ overtakes $S_2$

## (d) 1- Rarefaction wave overtakes 1-shock wave ($S_1R_1$):

Here for a given $U_l$, we consider $U_*$ and $U_r$, such that $U_* \in S_1(U_l)$ and $U_r \in R_1(U_*)$, i.e., $\rho_* > \rho_l$ from equation (2.5) we have $u_* = u_l - g(\rho_l, \rho_*)$ and $\rho_* \geq \rho_l$, from equation (2.5) we have $u_r = u_* + \int_{\rho_*}^{\rho_l} \frac{w(y)}{y} dy$. In the plane *(x,t)* the speed of trailing end of, $\lambda_1(U_*)$ is less than 1-shock speed $\upsilon_1(U_l, U_*)$ and therefore the 1-rarefaction wave from right overtakes 1-shock from left a finite time.

We show that the curve $R_1(U_*)$ lies below the curve $S_1(U_l)$ for $\rho_1 \leq \rho < \rho_*$; for this we have to that

$g(\rho_1,\rho_*)-g(\rho_l,\rho)-\int_{\rho}^{\rho_*}\frac{w(y)}{y}dy>0$ for $\rho_1\le\rho<\rho_*$. Let us a new function $G_1(\rho)$, to show that

$G_1(\rho)=g(\rho_1,\rho_*)-g(\rho_l,\rho)-\int_{\rho}^{\rho_*}\frac{w(y)}{y}dy>0$ for $\rho_1\le\rho<\rho_*$, then in this way it define the for $w(\rho)$ and $g(\rho_l,\rho)$, to hold that on $G_1(\rho)$ differentiating, we have that

$$G_1'(\rho)=\frac{\left[\left(\mathrm{p}-\mathrm{p}_1+\frac{B^2-B_l^2}{2\mu}\right)\frac{1}{\rho^2}-\left(p'+\frac{BB'}{\mu}\right)\left(\frac{1}{\rho_l}-\frac{1}{\rho}\right)\right]^2}{2g(\rho_l,\rho)}<0$$

Implying thereby that $G_1(\rho)>G_1(\rho_*)$, since $G_1(\rho_*)=0$, we have $G_1(\rho)>0$, then we prove that, $R_1(U_*)$ lies below the curve $R_1(U_l)$ for $\rho\le\rho_1<\rho_*$, then

$g(\rho_1,\rho_*)+\int_{\rho}^{\rho_l}\frac{w(y)}{y}dy-\int_{\rho}^{\rho_*}\frac{w(y)}{y}dy>0$ for $\rho\le\rho_1<\rho_*$. Since the left hand side of this inequalities, for $\rho\le\rho_1<\rho_*$, to be $G_1(\rho_l)$, which is $G_1(\rho_l)$is positive, so the conclusion above. So, we show that to $R_1(U_*)$ and $S_2(U_l)$ intersect into uniquely at some points $(\tilde{\rho}_2,\tilde{u}_2)$; to show that, for this $g(\rho_1,\rho_*)-g(\rho_1,\rho)-\int_{\rho}^{\rho_*}\frac{w(y)}{y}dy=0$ a uniquely root has $\tilde{\rho}_2$ such that $\tilde{\rho}_2<\rho_l$. To show a new function, we define $G_2(\rho)$, there implying that $G_2(\rho)=0$, $g(\rho_1,\rho_*)-g(\rho_1,\rho)-\int_{\rho}^{\rho_*}\frac{w(y)}{y}dy$; and we know, $G_2(\rho)\le 0$ it takes negative as close to zero, then $G_2(\rho_l)>0$. The curves are interest uniquely at the $R_1(U_*)$ and $S_2(U_l)$ it follows that the intermediate value property, and in view of monotonically; here three cases the value of $\rho_r$ on depending, we distinguish like,

(i) when $\rho_r>\tilde{\rho}_2$, and $U_r\in IV$; the interaction result is $S_1R_1\to S_1S_2$; indeed in sufficient case the both curves are interaction and then the 1-rarefaction wave is weak compared to the 1-shock is stronger, which is produced a new elementary wave.

(ii) when $\rho_r=\tilde{\rho}_2$, and $U_r\in S_2(U_l)$, the interaction result is $S_1R_1\to S_2$ i.e., interaction of the first family of elementary waves. Gives rise to a second family of a new elementary wave.

(iii) when $\rho_r<\tilde{\rho}_2$, and $U_r\in III$ the interaction result is $S_1R_1\to R_1S_2$.

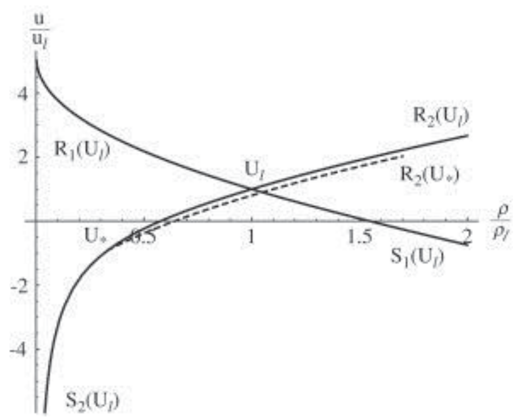


Fig.5.1(d) $R_2$ overtakes $S_2$.

## (e) 2-Rarefaction wave overtakes 2- shock wave ($S_2R_2$):

When $U_* \in S_2(U_l)$ and $U_r \in R_2(U_*)$ of the $S_2R_2$ interaction takes place, in any another word, in a given $U_1$, we have consider $U_*$ and $U_r$ are in a such way that $\rho_* < \rho_l$, from (2.5), we have $u_* = u_l - g(\rho_l, \rho_*)$ and $\rho_* \le \rho_l$, from (2.15) we have $u_r = u_* + \int_{\rho_*}^{\rho_r} \frac{w(y)}{y} dy$. We show that for $\rho_* < \rho \le \rho_l$, $S_2(U_l)$ lies above the curve $R_2(U_*)$, i.e.,

$$g(\rho_1, \rho_*) - g(\rho_1, \rho) - \int_{\rho}^{\rho_*} \frac{w(y)}{y} dy > 0 \quad \forall \rho \in (\rho_*, \rho_l) \qquad (5.11)$$

We define, to show that $M_1(\rho)$, $M_1(\rho) = g(\rho_1, \rho_*) - g(\rho_1, \rho) - \int_{\rho_*}^{\rho} \frac{w(y)}{y} dy$. Since there implying by that $S_2(U_1)$ lies above $R_2(U_*)$,On differentiation $M_1(\rho)$, since $M_1'(\rho) > 0$, we have $M_1(\rho) > M_1(\rho_*)$, since $M_1(\rho_*) = 0$, it follows that $M_1(\rho) > 0$, we prove that the $R_2(U_1)$ lies above the curve $R_2(U_*)$ for $\rho_* < \rho \le \rho_l$; to show that for this it is enough $M_1(\rho_l) = g(\rho_1, \rho_*) - \int_{\rho_*}^{\rho} \frac{w(y)}{y} dy - \int_{\rho_l}^{\rho} \frac{w(y)}{y} dy > 0$ for $\rho_* < \rho_l \le \rho$ and the curve $R_2(U_1)$ lies above the curve $R_2(U_1)$ for $\rho_* < \rho_1 \le \rho$, $M_1(\rho_l) > 0$; the left hand side of this inequalities is $M_1(\rho_l)$ which to positive, we show that $R_2(U_*)$ of the intersect uniquely $S_1(U_1)$ at point $(\tilde{\rho}_3, \tilde{u}_3)$ for $\rho_* < \rho_1 < \tilde{\rho}_3$. We define $M_2(\rho) = g(\rho_l, \rho)\text{-}g(\rho_1, \rho_*) - \int_{\rho_*}^{\rho} \frac{w(y)}{y} dy$, $for\ \rho_* < \rho_1 \le \rho$ so that $M_2(\rho) < 0$, and we consider a constant $K > 0$, such that $M_2(\rho) > 0$ for all $\rho > K$. Then there exists a $\tilde{\rho}_3$ such that $M_2(\tilde{\rho}_3) = 0$. Thus, $R_2(U_*)$ and $S_1(U_l)$ are intersect uniquely at $(\tilde{\rho}_3, \tilde{u}_3)$ as $R_2(U_*)$ and $S_1(U_l)$ in a terms of monotone, and the computed results shown in fig.5.2(c). Here three cases are following,

(i) If $\rho_r < \tilde{\rho}_3,\ U_r \in IV$ the interaction result is $S_2R_2 \to S_1S_2$, indeed, the strength of $R_2$ is small compared to the elementary wave $S_2$, and $S_2$ annihilates $R_2$ in a finite time. The strength of the reflected $S_1$ is small compared to the incident waves $S_2$ and $R_2$.

(ii) When $\rho_r = \tilde{\rho}_3$ and $U_r \in S_1(U_l)$ the interaction result is $S_2R_2 \to S_1$ indeed $S_1$ is weaker than $R_2$ compared to the incident waves $R_2$ and $S_2$.

(iii) If $\rho_r > \tilde{\rho}_3$, the interaction results is $S_2R_2 \to S_1R_2$; indeed, $R_2$ is stronger than $S_2$.

## (f) 2-Shock waves overtakes 2-Rarefaction($R_2S_2$):

For a given $U_1$, we have $U_*$ and $U_r$, here $U_* \in R_2(U_l)$ and $U_r \in S_1(U_l)$ such that $\rho_1 \le \rho_*$,

$u_* = u_l + \int_{\rho_l}^{\rho_*} \frac{w(y)}{y} dy$ and $\rho_r \le \rho_*, u_r = u_* - g(\rho_*, \rho_r)$. We prove that $R_2(U_l)$ lies above the curve

$S_2(U_*)$ for $\rho_1 \le \rho < \rho_*$.

$$g(\rho_*, \rho) + \int_{\rho_{l*}}^{\rho} \frac{w(y)}{y} dy - \int_{\rho_{l*}}^{\rho_*} \frac{w(y)}{y} dy > 0, \quad \forall \rho_1 \le \rho < \rho_* \qquad (5.12)$$

To show that, we have a new function

$$N_1(\rho) = g(\rho_*, \rho) + \int_{\rho_{l*}}^{\rho} \frac{w(y)}{y} dy - \int_{\rho_{l*}}^{\rho_*} \frac{w(y)}{y} dy \ \text{ for } \rho_1 \le \rho \le \rho_*; \text{ so that } N_1(\rho) = 0.$$

This , in view of the expression for $w(\rho)$ and $g(\rho_*, \rho)$, yields

$$N_1'(\rho) = -\frac{\left[\left(p' + \frac{BB'}{\mu}\right)\left(\frac{1}{\rho} - \frac{1}{\rho_*}\right) - \frac{1}{\rho^2}\left(p_* - p + \frac{B_*^2 - B^2}{2\mu}\right)\right]}{2g(\rho_*, \rho)} < 0$$

There implying by that,. Hence this result $N_1(\rho) > N_1(\rho_*) = 0$ we show that $S_2(U_l)$ lies above the curve $S_2(U_*)$ for $\rho \le \rho_1 < \rho_*$; to show, it is sufficient for this

$g(\rho_*,\rho)-g(\rho_l,\rho)-\int_{\rho_l}^{\rho_*}\frac{w(y)}{y}dy>0$ for $\rho\le\rho_1<\rho_*$. If $g(\rho_*,\rho)-g(\rho_l,\rho)>g(\rho_*,\rho_l)$ then

$g(\rho_*,\rho)-g(\rho_l,\rho)-\int_{\rho_l}^{\rho_*}\frac{w(y)}{y}dy>g(\rho_*,\rho_l)-\int_{\rho_l}^{\rho_*}\frac{w(y)}{y}dy=\mathrm{N}_1(\rho_l)>0$ . We consider, which is contradiction that $g(\rho_*,\rho)-g(\rho_l,\rho)\le g(\rho_*,\rho_l)$. Thus, it we have that $(\rho_*,\rho)-g(\rho_*,\rho_l)\le g(\rho_l,\rho)$.

There implies by that, $g^2(\rho_*,\rho)-g^2(\rho_*,\rho_l)\text{-}2g(\rho_*,\rho)g(\rho_*,\rho_l)\le g^2(\rho_l,\rho)$; this expression, in terms of $g(\rho_*,\rho), g(\rho_*,\rho_l)$ and $g(\rho_l,\rho)$ yields

$$\left(p_l-p_*+\frac{B_l^2-B_*^2}{2\mu}\right)\left(\frac{1}{\rho_*}-\frac{1}{\rho}\right)+\left(p-p_*+\frac{B^2-B_*^2}{2\mu}\right)\left(\frac{1}{\rho_*}-\frac{1}{\rho}\right)\le 2g(\rho_*,\rho)g(\rho_*,\rho_l);$$

or equivalently

$$\left[\left(p_l-p_*+\frac{B_l^2-B_*^2}{2\mu}\right)\left(\frac{1}{\rho_*}-\frac{1}{\rho}\right)+\left(p-p_*+\frac{B^2-B_*^2}{2\mu}\right)\left(\frac{1}{\rho_*}-\frac{1}{\rho}\right)\right]^2\le 0. \qquad (5.13)$$

Which is contraction, the above equation (5.13) is positive for $\rho\le\rho_1<\rho_*$, hence, $g(\rho_*,\rho)-g(\rho_l,\rho)>g(\rho_*,\rho_l)$ for $\rho\le\rho_1<\rho_*$, we proved that, a point $(\tilde{\rho}_4,\tilde{u}_4)$ at $\mathrm{S}_2(U_*)$ and $\mathrm{S}_1(U_l)$ intersect uniquely for $\rho_l<\tilde{\rho}_4<\rho_*$. Here again we distinguish three cases depending on the value of $\rho_\mathrm{r}$.

(i) If $\rho_\mathrm{r}>\tilde{\rho}_4, \mathrm{U}_\mathrm{r}\in I$, the interaction results is $\mathrm{R}_2S_2\to S_1R_2$, indeed, the elementary wave $\mathrm{R}_2$ is stronger compared to $\mathrm{S}_2$, the strength of reflected $\mathrm{S}_1$ is small compared to the incident waves $\mathrm{S}_2$ and $\mathrm{R}_2$.

(ii) If $\rho_\mathrm{r}=\tilde{\rho}_4$, and $\mathrm{U}_\mathrm{r}\in S_1(U_l)$ the interaction result is $\mathrm{R}_2S_2\to S_1$.

(iii) If $\rho_\mathrm{r}<\tilde{\rho}_4$, and $\mathrm{U}_\mathrm{r}\in IV$ the interaction result is $\mathrm{R}_2S_2\to S_1S_2$; indeed, $\mathrm{S}_2$ is stronger than compared to the elementary wave $\mathrm{R}_2$ is weaker.

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