Asymptotic Expansion Method for Singular Perturbation Problem

Thesis submitted in partial fulfillment of the requirements for the degree of

Master of Science

by

Gayatri Satapathy

Under the guidance of

Prof. Jugal Mohapatra

Department of Mathematics
National Institute of Technology
Rourkela-769008
India

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Declaration

I hereby certify that the work which is being presented in the thesis entitled "Asymptotic Expansion Method for Singular Perturbation Problem" in partial fulfillment of the requirement for the award of the degree of Master of Science, submitted in the Department of Mathematics, National Institute of Technology, Rourkela is a review work carried out under the supervision of Dr. Jugal Mohapatra. The matter embodied in this thesis has not been submitted by me for the award of any other degree.

(Gayatri Satapathy)
Roll No-410MA2107

This is to certify that the above statement made by the candidate is true to the best of my knowledge.

Place: NIT Rourkela
Date: June 2012

Dr. Jugal Mohapatra
Assistant Professor
Department of Mathematics
NIT Rourkela-769008
India

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Place: NIT Rourkela

Date: June 2012

(Gayatri Satapathy)

Roll No-410MA2107

Abstract

The main purpose of this thesis is to address the application of perturbation expansion techniques for the solution perturbed problems, precisely differential equations. When a large or small parameter 'ε' known as the perturbation parameter occurs in a mathematical model, then the model problem is known as a perturbed problem. Asymptotic expansion technique is a method to get the approximate solution using asymptotic series for model perturbed problems. The asymptotic series may not and often do not converge but in a truncated form of only two or three terms, provide a useful approximation to the original problem. Though the perturbed differential equations can be solved numerically by using various numerical schemes, but the asymptotic techniques provide an awareness of the solution before one compute the numerical solution. Here, perturbation expansion for some model algebraic and differential equations are considered and the results are compared with the exact solution.

Contents

Chapter 1

Motivation

1.1 Introduction

The study of perturbation problem is important as they arises in several branches of engineering and applied mathematics. The word " perturbation" means a small disturbance in a physical system. Mathematically, "perturbation method" is a method for obtaining approximate solution to complex equations (algebraic or differential) for which exact solution is not easy to find. Mainly, such problems which contain at least one small parameter ε known as perturbation parameter. We generally denote ε for the effect of small disturbance in physical system and ε is significantly less than unity.

Mathematicians and engineers study the behavior of the analytical solution of perturbed problems through asymptotic expansion technique which combines a straightforward perturbation expansion using an asymptotic series in the small parameter ε as ε goes to zero. Below, we have discussed the outline of perturbation expansion and the way it works.

Consider a differential equation

$$f(x, y, \frac{dy}{dx}, \varepsilon) = 0 \tag{1.1}$$

with initial or boundary conditions, where x is independent variable, y is dependent variable and $\varepsilon \ll 1$ is the small parameter.

Define an asymptotic series,

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots , \tag{1.2}$$

where y_0, y_1, $y_2, \cdots$ are sufficiently smooth functions. To get the value of y_0, y_1, $y_2, \cdots$ we have to substitute (1.2) in (1.1) after doing the term by term differentiation. After substitution, we may get a sequence of problems and solving for first few terms, we will get y_0, y_1, $y_2, \cdots$. Such solutions obtained form an asymptotic series is called an asymptotic solution.

Numerical analysis and asymptotic analysis are the two principal approaches for solving perturbation problem. The numerical analysis tries to provide quantitative information about a particular problem, whereas the asymptotic analysis tries to gain insight into the qualitative behavior of a family of problems. Asymptotic methods treat comparatively restricted class of problems and require the problem solver to have some understanding of the behavior of the solution. Since the mid-1960s, singular perturbations have flourished, the subject is now commonly a part of graduate students training in applied mathematics and in many fields of engineering. Numerous good textbooks have appeared in this area, some of them are Bush [1], Holmes [2], Kevorkian and Cole [3], Logan [4], Murdock [5], Nayfeh [6], Malley [7].

The main purpose of this work is to describe the application of perturbation expansion techniques to the solution of differential equations. As perturbation problems are arising in many areas, let us discuss a mathematical model for the motion of a projectile in vertically upward direction where, the force caused by air resistance is very small compared to gravity.

1.2 Mathematical model

Consider a particle of mass M which is projected vertically upward with an initial speed Y_0. Let Y denote the speed at some general time T. If air resistance is neglected then

the only force acting on the particle is gravity, $-Mg$.(where g is the acceleration due to gravity and the minus sign occurs because the upward direction)
According to Newton's second law the motion of the projectile, *i.e.,*

$$M\frac{dY}{dT} = -Mg. \tag{1.3}$$

Integrating (1.3), we obtain the solution $Y = C - gT$. The constant of integration is determined from the initial condition $Y(0) = Y_0$, so that

$$Y = Y_0 - gT. \tag{1.4}$$

On defining the non-dimensional velocity v, and time t, by $v = Y/Y_0$ and $t = gT/Y_0$, the given equation becomes

$$\frac{dv}{dt} = -1, \quad v(0) = 1, \tag{1.5}$$

with the solution $v(t) = 1 - t$.

Taking account of the air resistance, and is included in the Newton's second law as a force dependent on the velocity in a linear way, we obtain the following linear equation

$$M\frac{dY}{dT} = -Mg - CY, \tag{1.6}$$

where the drag constant C is the dimensions of mass/time. In the non-dimensional variables, it becomes

$$\frac{dv}{dt} = -1 - \left(\frac{CY_0}{Mg}\right) v. \tag{1.7}$$

Let us denote the dimensionless drag constant by ε, then the given equation is,

$$\begin{cases} \dfrac{dv}{dt} = -1 - \varepsilon v, \\ v(0) = 1, \end{cases} \tag{1.8}$$

where $\varepsilon > 0$ is a "small" parameter as the disturbances are very small. The damping constant C in (1.6) is small, since C has the dimensions of mass/time and a small quantity in units of kilograms per second.

1.3 Asymptotic expansion

In the previous section, we have shown the existence of perturbed problem given in (1.8) in nature. Now in this section, we discuss some definitions related perturbation and some basic terminology on asymptotic expansion.

Definition 1.3.1. *The problem which does not contain any small parameter is known as unperturbed problem.*

Example 1.3.2. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 2x^2 - 8x + 4, \quad y(0) = 3, \quad \dfrac{dy}{dx}(0) = 3.$

Definition 1.3.3. *The problem which contains a small parameter is known as perturbed problem.*

Example 1.3.4. $\dfrac{dy}{dx} + y = \varepsilon y^2, \quad y(0) = 1.$

Depending upon the nature of perturbation, a perturbed problem can be divided into two categories. They are

1. Regularly perturbed
2. Singularly perturbed

Definition 1.3.5. *The perturbation problem is said to be regular in nature, when the order (degree) of the perturbed and the unperturbed problem are same, when we set $\varepsilon = 0$. Generally, the parameter presented at lower order terms. The following is an example of regularly perturbed problem.*

Example 1.3.6. $\dfrac{d^2y}{dx^2} + y = \varepsilon y^2, \quad y(0) = 1, \quad \dfrac{dy}{dx}(0) = -1.$

Definition 1.3.7. *The perturbed problem is said to be singularly perturbed, when the order (degree) of the problem is reduced when we set $\varepsilon = 0$. Generally, the parameter presented at higher order terms and the lower order terms starts to dominate. Sometime the above statement is considered as the definition of singularly perturbation problem. The following is an example of singularly perturbed problem.*

Example 1.3.8. $\varepsilon\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 2x + 1, \quad y(0) = 1, \quad y(1) = 4.$

1.3.1 Order symbols

The letters 'O' and 'o' are order symbols. They are used to describe the rate at which the function approaches to limit value.

If a function $f(x)$ approaches to a limit value at the same rate of another function $g(x)$ at $x \to x_0$, then we can write $f(x) = O(g(x))$ as $x \to x_0$. The functions are said to be of same order as $x \to x_0$. We can write it as, $\lim_{x \to x_0} \frac{f(x)}{g(x)} = C$, where C is finite. We can say here **"f is big-oh of g"**. If the expression $f(x) = o(g(x))$ as $x \to x_0$ means $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$. We can say here **"$f$ is little-oh of g"** $x \to x_0$ and $f(x)$ is smaller than $g(x)$ as $x \to x_0$.

- $\frac{1}{3+2x^2} = O(1)$ as $x \to \infty$.
- $\sin x = O(x)$ as $x \to 0$ since $\lim_{x \to 0} \frac{\sin x}{x} = 1$.
- $\frac{1}{1+x^2} = o(1)$ as $x \to \infty$.
- $\sin x^2 = o(x)$ as $x \to 0$ because $\lim_{x \to 0} \frac{\sin x^2}{x} = 0$.

"Big-oh" notation and "Little-oh" notation are generally called "Landau" symbols. The expression $f(x) \sim g(x)$ as $x \to x_0$ means $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$ is called "f is asymptotically equal or approximately equal to g".

1.3.2 Asymptotic expansion

The standard asymptotic sequence is $\{1, \varepsilon, \varepsilon^2, \varepsilon^3, \cdots\}$ as $\varepsilon \to 0$ and $f_n(x)$ represents the members of asymptotic sequence then $f_{n+1}(\varepsilon) = o(f_n(\varepsilon))$ as $x \to a$ *i.e.,* $\lim_{x \to a} \frac{f_{n+1}(\varepsilon)}{(f_n(\varepsilon))} = 0$. The general expression for asymptotic expansion of function $f_n(\varepsilon)$ is the series of terms

$$f(x) = \sum_{n=0}^{N} a_n f_n(\varepsilon) + R_N$$

as $\varepsilon \to 0$, where a_n are constants and $R_N = O(f_{n+1}(\varepsilon))$ as $\varepsilon \to 0$ and $\lim_{n\to\infty} R_N = 0$.

Definition 1.3.9. *The expression* $f(x) = \sum_{n=0}^{N} a_n f_n(\varepsilon) + R_N$, *where* $f(x;\varepsilon)$ *depends on an independent variable* x *and small parameter* ε. *The coefficient of the gauge function* $f_n(\varepsilon)$ *are functions of* x *and the remainder term after* N *terms is a function of both* x *and* ε *is* $R_N = O(f_{n+1}(\varepsilon))$ *is said to be uniform asymptotic expansion, if* $R_N \leq C f_{n+1}(\varepsilon)$, *where* C *is the constant.*

Example 1.3.10. $f(x;\varepsilon) = \dfrac{1}{1 - \varepsilon sinx} = 1 + \varepsilon sinx + \varepsilon^2 (sinx)^2 + \cdots$ *as* $\varepsilon \to 0$.

The remainder term $R_N = 1+\varepsilon \sin x+\varepsilon^2(\sin x)^2+\cdots-\sum_{n=0}^{N} \varepsilon^n (\sin x)^n$, where $\lim_{\varepsilon\to 0}\left(\dfrac{R_N}{\varepsilon^N + 1}\right) = (\sin x)^N + 1$.

Definition 1.3.11. *The expression* $f(x) = \sum_{n=0}^{N} a_n f_n(\varepsilon) + R_N$, *where* $f(x;\varepsilon)$ *depends on an independent variable* x *and small parameter* ε *is said to be non-uniform asymptotic expansion, if there is no constants exists but the relation* $R_N \leq C f_{n+1}(\varepsilon)$ *satisfied is known as non-uniform asymptotic expansion.*

Example 1.3.12. $f(x;\varepsilon) = \dfrac{1}{1 - \varepsilon x} = 1 + \varepsilon x + \varepsilon^2 (x)^2 + \cdots$ *as* $\varepsilon \to 0$.

The remainder term $R_N = 1 + \varepsilon x + \varepsilon^2 (x)^2 + \cdots - \sum_{n=0}^{N} \varepsilon^n (x)^n$, $\lim_{\varepsilon\to 0}\left(\dfrac{R_N}{\varepsilon^N + 1}\right) = (x)^N + 1$. There is no fixed constant C exists such that $R_N \leq C\varepsilon^{N+1}$.

Chapter 2

Perturbation Techniques

In this chapter, we discuss some model perturbed problem both of regular and singular type. The method of asymptotic expansion is applied and approximate solutions are obtained. Then it is compared with the exact solution.

2.1 Regularly perturbed problem

At first, in this section we consider an example of regularly perturbed algebraic equation whose exact roots are known. We find the approximate roots through asymptotic expansion, then we compare the exact roots and the approximate roots.

Example 2.1.1. *Consider the algebraic equation* $y^2 - \varepsilon y - 1 = 0$.

Let the roots of Example 2.1.1 is given by

$$\alpha, \beta = \frac{\varepsilon \pm \sqrt{\varepsilon^2 + 4}}{2}.$$

Expanding binomially, we obtain

$$\alpha = 1 + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{8} + \cdots$$

$$\beta = -1 + \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + \cdots$$

Now, we try to find an approximate roots. Take an asymptotic series

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots \tag{2.1}$$

Substituting (2.1) in Example 2.1.1, we get

$$y_0 + 2\varepsilon y_0 y_1 + \varepsilon^2({y_1}^2 + 2y_0 y_1) + \cdots - \varepsilon y_0 - \varepsilon^2 y_1 - \varepsilon^3 y_2 \cdots - 1 = 0. \tag{2.2}$$

Arranging in the order of 'ε' of (2.2), we have

$$\begin{aligned} O(0): &\quad y_0^2 - 1 = 0, \quad \Rightarrow y_0 = \pm 1. \\ O(1): &\quad 2y_0 y_1 - y_0 = 0, \quad \Rightarrow y_1 = \frac{1}{2}. \\ O(2): &\quad y_1^2 + 2y_0 y_2 - y_1 = 0, \quad \Rightarrow y_2 = \pm\frac{1}{8}. \end{aligned}$$

If we will put above value in (2.1), we get

$$\alpha = 1 + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{8} + \cdots$$

$$\beta = -1 + \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + \cdots$$

Here, we observed that the approximate roots obtained by an asymptotic expansion coincide with the exact roots. So for a regularly perturbed problem, we can very well obtain a good approximation to the exact root by an asymptotic series.

Now, in stead of an algebraic equation, let us consider a differential equation in the following example. Here, our aim is to solve the differential equation exactly first and then, approximately by using asymptotic series. Lastly, we compare both the solutions in order to know how much accurate solution can be obtained by an asymptotic method.

Example 2.1.2. *Let us consider a perturbed differential equation of first order*

$$\frac{dy}{dx} + y = \varepsilon y^2, \quad y(0) = 1. \tag{2.3}$$

Solving exactly, we have $\frac{dy}{dx} + y = \varepsilon y^2 \Longrightarrow \frac{dy}{dx} = \varepsilon y^2 - y \Longrightarrow \int \frac{1}{y(\varepsilon y - 1)} dy = \int dx$. Now using partial fraction

$$\frac{1}{y(\varepsilon y - 1)} = \frac{A}{y} + \frac{B}{(\varepsilon y - 1)} \Longrightarrow \frac{1}{y(\varepsilon y - 1)} = \frac{y(A\varepsilon + B) - A}{y(\varepsilon y - 1)}. \tag{2.4}$$

Comparing both the sides, we get $A = -1$ and $B = \varepsilon$. Putting the value of A and B in (2.4) and integrating, we obtain

$$\frac{\varepsilon y - 1}{y} = Ce^x. \tag{2.5}$$

Applying initial condition, we get $\frac{\varepsilon y - 1}{y} = (\varepsilon - 1)e^x$. Simplifying, we have

$$y = e^{-x} + \varepsilon(e^{-x} - e^{-2x}) + \varepsilon^2(e^{-x} - e^{-2x} + e^{-3x}) + \cdots \tag{2.6}$$

which is the exact solution of (2.3).
Now, we wish to find an approximate solution. Let us take an asymptotic series depends on independent variable x and small parameter ε is given by

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots \tag{2.7}$$

Substituting (2.7) in (2.3), we get

$$\frac{dy_0}{dx} + \varepsilon\frac{dy_1}{dx} + \varepsilon^2\frac{dy_2}{dx} + \cdots + y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots = \varepsilon(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots)^2. \tag{2.8}$$

and initial condition can be written as

$$y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots = 1 + 0\varepsilon + 0\varepsilon^2 \cdots \tag{2.9}$$

Arranging in the order of 'ε' of (2.8), we get the following system of differential equations:

$$O(1) : \frac{dy_0}{dx} + y_0 = 0, \quad y_0(0) = 1. \tag{2.10}$$

$$O(\varepsilon) : \frac{dy_1}{dx} + y_1 = y_0^2, \quad y_1(0) = 0. \tag{2.11}$$

$$O(\varepsilon^2) : \frac{dy_2}{dx} + y_2 = 2y_0 y_1, \quad y_2(0) = 0. \tag{2.12}$$

Solving (2.10), we obtain $y_0 = C_1 e^{-x}$. Using the initial condition $y_0(0) = 1$, we get $y_0 = e^{-x}$. Solving (2.11), we obtain $y_1 = C_2 e^{-x} - e^{-2x}$ and using the initial condition

$y_1(0) = 0$, we get $y_1 = e^{-x} - e^{-2x}$.
Solving (2.12), we obtain $y_2 = C_3e^{-x} - 2e^{-2x} + e^{-3x}$ and using the initial condition $y_2(0) = 0$, we get $y_2 = e^{-x} - 2e^{-2x} + e^{-3x}$.
Substituting the value of y_0, y_1, y_2 in (2.7), we have

$$y = e^{-x} + \varepsilon(e^{-x} - e^{-2x}) + \varepsilon^2(e^{-x} - 2e^{-2x} + e^{-3x}) + \cdots \tag{2.13}$$

Here, we observed from the above example that the exact solution of (2.3) given by (2.6) and approximate solution given by (2.13) are very well matching. The problem (2.3) is a regularly perturbed differential equation. Also, if we put $\varepsilon = 0$, the (2.3) becomes

$$\frac{dy}{dx} + y = 0. \tag{2.14}$$

Comparing (2.3) and (2.14), one can easily observe that the order of the perturbed differential equation and unperturbed differential equation are same.

2.2 Singularly perturbed problem

In the previous section, we have discussed an algebraic and one differential equation involving ε and both are of regularly perturbed type. Now in this section, we consider an algebraic equation and differential equation involving small parameter, both are of singularly perturbed nature. We will solve these problems through the above procedure to get the behavior singular perturbation.

Example 2.2.1. *Let us consider an algebraic equation*

$$\varepsilon x^2 - x + 1 = 0. \tag{2.15}$$

The roots of (2.15) is given by

$$\alpha, \beta = \frac{1 \pm \sqrt{1 - 4\varepsilon}}{2\varepsilon}.$$

$$\alpha = \frac{1}{\varepsilon} - 1 - \varepsilon - \cdots$$

$$\beta = 1 + \varepsilon + \varepsilon^2 + \cdots$$

which are the two exact roots of (2.15). Let us consider an asymptotic series

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \tag{2.16}$$

Substituting (2.16) in (2.15), we get

$$\varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^2 - (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) + 1 = 0. \tag{2.17}$$

Arranging in the order of ε of (2.17), we get the following system of equations:

$$\begin{aligned} O(0): &\quad -x_0 + 1 = 0, \quad \Rightarrow x_0 = 1. \\ O(1): &\quad x_0^2 - x_1 = 0, \quad \Rightarrow x_1 = 1. \\ O(2): &\quad 2x_0x_1 - x_2 = 0, \quad \Rightarrow x_2 = 2. \end{aligned}$$

Now putting the value of x_0, x_1, x_2 in the (2.16), we get

$$x = 1 + \varepsilon + 2\varepsilon + 2\varepsilon^2 + \cdots, \tag{2.18}$$

which matches to the solved root β up to two terms of the asymptotic expansion. So, we can get only one approximate root through asymptotic expansion. The other root α can not be approximated. So for singularly perturbed problem, we can not get a good approximation to the exact roots by one term asymptotic series.

Now, in stead of an algebraic equation, let us consider differential equations in the following examples. Here, our aim is to solve the differential equation exactly and then, approximately by using asymptotic series. Finally, we compare both the solutions in order to know how much accurate solutions can be obtained by the asymptotic method.

Example 2.2.2. *Consider a differential equation*

$$\varepsilon\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0, \quad y(0) = 0, \quad y(1) = 1. \tag{2.19}$$

Solving exactly and applying the boundary conditions, we obtain

$$y = \frac{e^{m_1x} - e^{m_2x}}{e^{m_1} - e^{m_2}}$$

which is the solution of (2.19), where $m_{1,2} = \dfrac{-1 \pm \sqrt{1+4\varepsilon}}{2\varepsilon}$.

Let us take an asymptotic series

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots \tag{2.20}$$

Substituting (2.20) in (2.19), we get

$$\varepsilon(\frac{d^2y_0}{dx^2} + \varepsilon\frac{d^2y_1}{dx^2} + \varepsilon^2\frac{d^2y_2}{dx^2} + \cdots) + (\frac{dy_0}{dx} + \varepsilon\frac{dy_1}{dx} + \varepsilon^2\frac{dy_2}{dx} + \cdots) + (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots) = 0. \tag{2.21}$$

Arranging in the order of ε of (2.21), we get the following system of differential equations:

$$O(0) : \frac{dy_0}{dx} - y_0 = 0, \quad y_0(0) = 0, \quad y_0(1) = 1. \tag{2.22}$$

$$O(1) : \frac{dy_1}{dx} + y_1 = -\frac{d^2y_0}{dx^2}, \quad y_1(0) = 0, \quad y_1(1) = 0. \tag{2.23}$$

Solving (2.22), we obtain

$$y_0(x) = Ce^x. \tag{2.24}$$

From the condition $y_0(0) = 0$ and $y_0(1) = 1$, we get $y_0(x) = 0$ and $y_0 = e^{x-1}$ respectively. Now the function $y_0(x) = 0$ does not satisfy the condition at $x = 1$ and the function $y_0 = e^{(x-1)}$ does not satisfy the condition at $x = 0$ *i.e.*, the solution fails to satisfy one of the boundary condition. Again, we take $\varepsilon = 0$ in differential equation then, we get a first order differential equation

$$\frac{dy}{dx} + y = 0. \tag{2.25}$$

Where the order of perturbed differential equation (2.19) and the unperturbed differential equation (2.25) are different. We conclude here by comparing both the solution that, we cannot get a good approximation to the exact solution by one term asymptotic series.

Example 2.2.3. *Let us consider the boundary value problem*

$$\varepsilon\frac{d^2y}{dx^2} + (1+\varepsilon)\frac{dy}{dx} + y = 0, \quad y(0) = 0, \quad y(1) = 1. \tag{2.26}$$

Solving exactly, we get $y = c_3e^{-x} + c_4e^{\frac{-x}{\varepsilon}}$ using boundary condition. Applying boundary condition, we get

$$y = \frac{e^{-x} - e^{\frac{-x}{\varepsilon}}}{e^{-1} - e^{\frac{-1}{\varepsilon}}}$$

which the solution of (2.26).

Let us take an asymptotic series,

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots \tag{2.27}$$

Substituting (2.27) in (2.26), we get

$$\varepsilon(\frac{d^2y_0}{dx^2} + \varepsilon\frac{d^2y_1}{dx^2} + \varepsilon^2\frac{d^2y_2}{dx^2} + \cdots) + (\frac{dy_0}{dx} + \varepsilon\frac{dy_1}{dx} + \cdots \tag{2.28}$$
$$\varepsilon^2\frac{dy_2}{dx} + \cdots) + (\varepsilon\frac{dy_0}{dx} + \varepsilon^2\frac{dy_1}{dx} + \varepsilon^3\frac{dy_2}{dx} + \cdots) + (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots) = 0.$$

Arranging in the order of ε of (2.28), we get the following system of differential equation:

$$O(0) : \frac{dy_0}{dx} + y_0 = 0, \quad y_0(0) = 0, \quad y_0(1) = 1. \tag{2.29}$$

$$O(1) : \frac{dy_1}{dx} + y_1 = -\frac{d^2y_0}{dx^2} - \frac{dy_0}{dx}, \quad y_1(0) = 0, \quad y_1(1) = 0. \tag{2.30}$$

Solving (2.29), we have

$$y_0(x) = Ce^x. \tag{2.31}$$

Here, with one arbitrary constant both the boundary conditions cannot be satisfied. We conclude here that we cannot get a good approximation to the exact solution for a singularly perturbed problem by one term asymptotic expansion.

From the above examples, it is observed that the asymptotic expansion technique gives approximate solution which matches well to the exact solution for problem of regularly perturbed type. But at the same time, perturbation expansion does not provide a good approximation for singularly perturbed problems. In order to obtain a fitted approximation for singularly perturbed problems, one can use the method of matched asymptotic expansion which is discussed in the next section.

2.3 Matched asymptotic expansion

The method of matched asymptotic expansion was introduced by Ludwig Prandtl's boundary layer theory, in 1905. Mathematically, boundary layer occurs when small parameter is multiplying with highest derivative of differential equation which is of singularly perturbed in nature. Depending upon the presence of boundary layer, the domain of the problem can be divided into two regions, first one is the outer region away from the boundary layer where the solution behaves smoothly and second one is the inner region where the gradient of the solution changes rapidly.

The algorithm for matched asymptotic expansion is given below.

Step-1: We will construct one solution in the outer region through asymptotic expansion away from the boundary layer.

Step-2: On the other hand, we will obtain another solution using stretching variable through asymptotic expansion within the boundary layer.

Step-3: We will match the leading order term of both the solutions by using them using Prandtl's matching condition which is given by

$$\lim_{x\to 0} f^{out}(x) = \lim_{x\to\infty} f^{in}(s).$$

Let us consider a singularly perturbed two-point boundary value problem and apply the method of matched asymptotic expansion.

Example 2.3.1.

$$\varepsilon\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2x + 1, \quad y(0) = 1, \quad y(1) = 4. \tag{2.32}$$

Let us consider an asymptotic series

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots \tag{2.33}$$

Substituting (2.33) in (2.32), we get

$$\varepsilon(\frac{d^2y_0}{dx^2} + \varepsilon\frac{d^2y_1}{dx^2} + \varepsilon^2\frac{d^2y_2}{dx^2} + \cdots) + (\frac{dy_0}{dx} + \varepsilon\frac{dy_1}{dx} + \varepsilon^2\frac{dy_2}{dx} + \cdots) = 2x + 1. \tag{2.34}$$

Arranging in order of ε of (2.34), we get the following system of differential equations:

$$\frac{dy_0}{dx} = 2x + 1, \quad y_0(0) = 1, \quad y_0(1) = 4. \tag{2.35}$$

$$\frac{dy_1}{dx} = -\frac{d^2y_0}{dx^2}, \quad y_1(0) = 0, \quad y_1(1) = 0. \tag{2.36}$$

Here, $y_n(x)$ has to satisfy two boundary conditions. Since the boundary layer appears in the left hand side of the domain, the boundary condition $y(0) = 1$ cannot be satisfied. Again the problems (2.35) and (2.36) are of first order, solving (2.35), we get $y_0(x) = x^2 + x + c$, from the condition $y_0(1) = 4$, we obtain

$$y_0(x) = x^2 + x + 2. \tag{2.37}$$

Similarly, solving (2.36), we get

$$y_1(x) = -2(x - 1). \tag{2.38}$$

Therefore, the outer expansion up to second term is

$$y^{out}(x;\varepsilon) = (x^2 + x + 2) + \varepsilon 2(1 - x). \tag{2.39}$$

Here, "outer" label is used to indicate that the solution is invalid in $x = 0$. Now the exact solution of (2.32) $y = y_c + y_p$ and $y_c = C_1 + C_2 e^{\frac{-x}{\varepsilon}}$, $y_p = x^2 + x(1 - 2\varepsilon)$. So,

$$y(x) = C_1 + C_2 e^{\frac{-x}{\varepsilon}} + x^2 + x(1 - 2\varepsilon). \tag{2.40}$$

Applying boundary condition, $y(0) = 1$, $y(1) = 4$, we get

$$C_1 + C_2 = 1. \tag{2.41}$$

$$C_1 + C_2 e^{\frac{-1}{\varepsilon}} + 2 - 2\varepsilon = 4. \tag{2.42}$$

Solving (2.41)and (2.42), we get $C_1 = 2(1 + \varepsilon)$, $C_2 = -(1 + 2\varepsilon)$.

Now putting the value of C_1 and C_2 in (2.40), we get

$$y_(x) = 2(1 + \varepsilon) - (1 + 2\varepsilon)e^{\frac{-x}{\varepsilon}} + x^2 + x(1 - 2\varepsilon)$$

Simplifying,

$$y_(x) = (x^2 + x + 2) - e^{\frac{-x}{\varepsilon}} + \varepsilon\big(2(1 - x) - 2e^{\frac{-x}{\varepsilon}}\big). \tag{2.43}$$

Comparing (2.39) and (2.43), we observe the term $e^{\frac{-x}{\varepsilon}}$ is absent in outer solution.

The leading order term in the exact solution,

$$y_(x) = (x^2 + x + 2) - e^{\frac{-x}{\varepsilon}} \tag{2.44}$$

$$\frac{dy}{dx} = 2x + 1 + \frac{1}{\varepsilon}e^{\frac{-x}{\varepsilon}} \tag{2.45}$$

$$\frac{d^2y}{dx^2} = 2 - \frac{1}{\varepsilon^2}e^{\frac{-x}{\varepsilon}} \tag{2.46}$$

since the boundary layer is at left, choosing $x = s\varepsilon$ in singularly perturbed differential equation, we get

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = \varepsilon + 2\varepsilon^2 s. \tag{2.47}$$

Assuming the boundary layer expansion called inner expansion

$$y(s;\varepsilon) = y_0(s) + \varepsilon y_1(s) + \cdots \tag{2.48}$$

The inner expansion satisfy the boundary condition at $x = s = 0$. Now substituting (2.48) in (2.47) and arranging in the order of ε, we get the following system of differential equations:

$$\frac{d^2y_0}{ds^2} + \frac{dy_0}{ds} = 0, \quad y_0(0) = 1. \tag{2.49}$$

$$\frac{d^2y_1}{ds^2} + \frac{dy_1}{ds} = 0, \quad y_1(0) = 0. \tag{2.50}$$

Solving (2.49) and (2.50), we have

$$y_0 = C_3 + (1 - C_3)e^{-s} \tag{2.51}$$

$$y_1 = C_4 - C_4e^{-s} + s \tag{2.52}$$

If we equate the leading order term of outer and inner expansion at $x = 5\varepsilon$, we get

$$x^2 + x + 2 = C_3 + (1 - C_3)e^{-5} \tag{2.53}$$

$C_3 = \dfrac{2 + 5\varepsilon + 25\varepsilon^2 - e^{-5}}{1 - e^{-5}}$. If we choose $x = 6\varepsilon$, we get $C_3 = \dfrac{2 + 6\varepsilon + 36\varepsilon^2 - e^{-6}}{1 - e^{-6}}$. Applying matching condition, $\lim_{x\to 0}(x^2 + x + 2) = \lim_{s\to\infty} C_3 + (1 - C_3)e^{-s}$. We have $C_3 = 2$. Thus leading order terms in the expansion, outer region $y_0 = (x^2 + x + 2)$ for $x = O(1)$. Inner region $y_0 = 2 - e^{\frac{-x}{\varepsilon}}$ for $x = O(\varepsilon)$. To prove these are valid leading order terms we consider, if $x = O(1)$ then $y_0 = (x^2 + x + 2)$ if $x = O(\varepsilon)$ then $y_0 = 2 - e^{\frac{-x}{\varepsilon}}$. Here conclude that the matching condition has correctly predicted the leading order terms.

Chapter 3

Conclusion and Future work

3.1 Conclusion

In this thesis, we have discussed the application of asymptotic expansion method for some model problems involving small parameter ε. In Chapter 1, we have defined some basic terminologies, definition of regular and singular perturbation, Landau symbols, uniform and non-uniform asymptotic expansions. In Chapter 2, at first we have applied the method of asymptotic expansion for some model algebraic equations and differential equations containing ε. Then, we have applied the same method on some model singularly perturbed problems. We have observed that the asymptotic expansion method gives very good result for regularly perturbed problems where as one term asymptotic expansion does not work well for singularly perturbed problems. To get a good approximation for singularly perturbed problems, the method of matched asymptotic expansion is discussed in the last section of Chapter 2 and we conclude that through matched asymptotic expansion, one can obtain a good approximation for singularly perturbed problems.

3.2 Future work

The work of the thesis can be extended in various directions. Some of the future works to be carried out are listed below.

1. The problems considered here are linear one dimensional problems. One can also extend the idea of asymptotic expansion discussed here to problems in higher dimension. It can also be extended for non-linear differential equation which is difficult to solve exactly.

2. The solution obtained through asymptotic expansion gives us an awareness of the exact solution before solving the problem analytically or numerically. So, one can get some idea about the solution of perturbation problems through asymptotic expansion before solving it.

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