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Master's Thesis

A LOOK INTO THE OPTIMAL CONTROL USING LIE GROUP SU(3)

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in

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Declaration of Authorship

I do hereby declare that the work which is being presented in the thesis entitled A LOOK INTO OPTIMAL CONTROL USING LIE GROUP SU(3) in partial fulfillment of the requirement for the award of the degree of master of Science, submitted in the Department of Mathematics, National Institute of Technology, Rourkela is an authentic record of my own work carried out under the supervision of Dr. Kishor Chandra Pati .

Further, I declare that the project work with the above mentioned title has not been submitted previously to any other Institution including this college or published at any time before in the present form.

Signed: ______________________________

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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CERTIFICATE

This is to certify that the project report entitled A LOOK INTO OPTIMAL CONTROL USING LIE GROUP SU(3) submitted by Subhasmita Das to the National Institute of Technology Rourkela, Odisha for the partial fulfilment of requirements for the degree of master of science in Mathematics is a bonafide record of review work carried out by her under supervision and guidance of Prof. Kishor Chandra Pati .

The contents of this project, in full or in parts, have not been submitted to any other institute or university for the award of any degree or diploma.

H.O.D Mathematics

Project Guide

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Abstract

Now a day, there is a great deal of interest in the study of control systems on matrix Lie groups in connection with their deep applications in robotics, classical mechanics and engineering. In our study an optimal control problem on the special unitary lie group $SU(3)$ is discussed and some of its geometrical and dynamical properties are point out.

Chapter 1

Introduction to control theory

1.1 Basic Definitions

1.1.0.1 What do you mean by Control ??

A mathematical control is a time-dependent function $u(t)$ that influences a dynamical system $dy/dt = F(u, y)$, with the 'u' such that to minimize some value of the solution for the optimization of some related quantity.

1.1.0.2 What is Control theory ??

The mathematical programming robots and other machines to respond to changing conditions so that they maintain self-control. A Variety of physical systems are controlled by manipulation of their inputs based simultaneous observation of their outputs.

For example, the problem of programming an automatic pilot of an airplane so that the plane doesn't crash. The Control problem is to determine on the basis of available data, the inputs required to achieve a given goal.

1.1.0.3 What Is Mathematical Control Theory?

It is the area of applied mathematics that deals with the fundamental principles underlying the analysis and design of control systems by influencing object behaviours.

1.1.1 System classifications

1. Linear systems control

2. Nonlinear systems control

3. Decentralized/distributed systems control

1.1.2 Main control strategies

- **Adaptive control** : It uses on-line identification of the process parameters, or modification of controller gains to obtain better robustness properties. The Aerospace industry first used this in the 1950s, and have found particular success in that field.

- **Hierarchical control** :A Hierarchical control system is arranged in a hierarchical tree for a set of devices and governing software is a type of Control System. Some links in the tree are implemented by a computer network, with a hierarchical control system is also a form of Networked control system.

- **Intelligent control** :It uses various artificial Intelligence computing approaches like Bayesian probability, machine learning, evolutionary computation and algorithms for genetic to control a dynamic system.

- **Optimal control** : It is the process of determining control and state trajectories on a dynamic system over a repeated duration of time to minimise a performance index.

- **Robust control** :Robust control is a branch of control theory that explicitly deals with uncertainty in its approach to controller design. Different robust control methods are designed to function to restrict the uncertain parameters or disturbances.

- **Stochastic control** :Stochastic control deals with control design with uncertainty in the model. In particular stochastic control problems, initially there exist assumption for random noise and disturbances in the model , one should taken the controller, and the control design for these random deviations.

1.1.3 Control Systems

Definition :A system is an combination of physical components connected or related in a manner as to act as an entire unit. A control system is an arrangement of physical components connected in a way to command, regulate itself or other system. The input is the stimulus with excitation appiled to a control system from an external energy source, generally to produce a particular response from the control system.

1.1.3.1 There are three type of control systems :

- Man-made control system.
- Natural control systems.
- Control systems whose components are both the above.

Example 1.1. *Electric switch is a man-made control system, guiding the flow of electricity, where inputs is on or off condition of the switch and ouput is the state of flow or non-flow of electricity. The control system consisiting of a man driving an automobile has components which are clearly both, biological and man-made.*

1.1.4 Classification of Control systems:

Control sytem are classified into two catagorioes:

1. Open-loop
2. Closed-loop(also known as feedback control system)

Definition 1.1. An open-loop control system is the output independence.

Definition 1.2. Definition : A closed-loop(feedback)control system is one in which action is somehow dependent on the output.

Example 1.2. *A automatic toaster is an open-loop control system because it is controlled by a timer. We just have to set the input time to make the good toast which is determined by the user which is not the part of the system and after which control over the quality of toast(the output) is removed once the time is set.*

Example 1.3. *An automatic mechanism and the airplane it controls is a closed-loop control system. Its purpose is to maintain a specified airplane heading, in a state of atmospheric changes. It performs this task by continously measuring the actual airplane heading and automatic adjusting the air plane control surfaces , hence to bring the actual airplane heading in accordance with the specified heading.*

1.1.5 Feedback:

Feedback is that characteristics of closed-loop control systems which distinguish them from open-loop systems.

Definition 1.3. Feedback is that criteria on a closed-loop system which allows the output to be compared to the system for the appropriate control action may be created, this gives some function of the output and input.

- The concept of the feedback can be understood from the above example of the airplane auto mechanism where the continous checking of the actual heading compared with asked one(input) and according to that the difference is given as the feedback to the system to achieve the desired effect.
- Till now we have seen the basic defination of the control theory but how it is related to mathematics ??

Because, we know that the main theme of getting the concept control theory is to creat and design a control system which can control and ambient a dynamical systems, but, if we recall the characterised the application of differential equation to every dynamical systems c. The process is involved, that is the use of physical laws (Newton's, Kirchhoff's etc) together with various assumptions of linearity etc is known as mathematical modelling.

That is, why it is the interdiscplinary branch of the mathematics and engineering.

1.1.6 The control theory is classified into two catagories:

1. Classical Control theory
2. Modern Control theory

- Classical Control theory

  It is based on Laplace transforms and applies to linear autonomous (time-invariant) systems having single input and output. A function called the transfer function relating to the system is defined.

  - Modern Control theory

  It is not only applicable to linear autonomous systems but also to time-varying systems and is useful when dealing with nonlinear systems. In particularly it is applicable to multiple-input and multiple-output systems in contrast to the classical control theory. Modern Control theory is based on the concept of state.

1.1.7 State-space formulation

The state-space approach is applicable to multivariable systems in contrast to the transfer function techniques(used in classical control theory) which are generally used for single-input, single-output systems.

Consider a system characterised by an n^{th} order differential equation.Assume that the systems are autonomous,which implies that the free system doesnot depend explicitly on time.

Let

$$\dot{y} = \frac{dy}{dt}, \ddot{y} = \frac{d^2y}{dt^2}............, y^{(n)} = \frac{d^n y}{dt^n}$$

The system equation has the form :

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + ... + a_{n-1}\dot{y} + a_n y = u; \qquad ....(\mathbf{1})$$

it is assumed that

$$y(0), \dot{y}(0), ..., y^{(n-1)}(0)$$

are known.

If we define

$$x_1 = y, x_2 = \dot{y}\ , ...x_n = y^{(n-1)}$$

.

then we can write equation(1) as a system of a n simultaneousw differntial equations,each of order 1,namley

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

.

.

$$\dot{x}_{(n-1)} = x_n$$

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - .... - a_1 x_n + u$$

the last one is form (1). This can be wriiten as a vector as a vector-matrix differential equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ . \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & . & . & . & 0 & 0 \\ 0 & 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & 1 \\ -a_n & -a_{n-1} & . & . & . & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ . \\ . \\ 0 \\ 1 \end{bmatrix} u \qquad ....(2)$$

that is, as

$$\dot{x} = Ax + Bu$$

where x, A and B are defined in equation(2). The output of the system is y, which was defined as $x_1$above and is written in matrix form as

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & . & . & . & . & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} \quad ....(3)$$

that is, as

$$y = Cx$$

Where $C = \begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \end{bmatrix}$

The combination of equations(2)and(3) in the form

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

....................(4)

are known as the state equations of the system considered.The matrix A in equation (2)is said to be in companion form.

The component of x are the state variables $x_1, x_2......x_n$.They can be considered as the coordinate's axes of an n-imensional state-space system which can be represented by a point in the state space.

Example 1.4. *Obtain the forms of state equations of the system defined by*

$$\dddot{y} - 2\ddot{y} + \dot{y} - 2y = u$$

A companion form; Using the state variables defined above, we have

$$x_1 = y, x_2 = \dot{y}, x_3 = \ddot{y}$$

, then

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = 2x_1 - x_2 + 2x_3 + u$$

Hence

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

and

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

1.1.8 Topic in control theory:

1. Stability
2. Controllabilty and Observability

1.1.8.1 Stability

Stability is possibly the most important consideration when designing a control system.The problems involved are not only important but extremly complicated indeed much present day research in control is concerned with stability.

It would intuitively reasonable to define a linear system to be stable if its output is bounded for every bounded input.

There are variety methods which determine the stability of the system but we would not discuss here as it is not necessary for our topic.

1.1.8.2 Controllability and Observability

Kalman introduced in 1960s the concept of Controllability and Observability, the fundamental questions to be answered for a system, in particular in a multivariable system are :

- Can a control function $u(t)$ be found which will transform the initial state x_0 of a system to some desired final state x_f in finite time ??
- Can the state of the system be determined by measuring the system output over a finite time interval ?

The two concepts involved are called controllabilty and observability respectively.So, if the answer is yes to first question ,then the system is Controllable.

Similarly if the answer is yes to second question,then the system is Observable.

Chapter 2

Introduction to Lie algebra and Matrix Exponential

In this section we are going to discuss first about the Matrix exponential and then the Lie algebra. Lie algebra is an indispensable tool in studying with matrix Lie groups. It is well known in one hand that the Lie algebras are simpler than matrix Lie groups, since it is a linear space. Thus, we can understand a lot about Lie algebras doing linear algebra. While on the other way, a matrix Lie group contains much information about that group.

2.0.9 Matrix Lie Groups Definition

The general linear group over the real numbers, denoted by $GL(n, R)$,is the group of all nxn invertible matrices with real entries. The general linear group denoted $GL(n; C)$ over a complex field, is the group of all nxn invertible matrices with complex entries. These are indeed groups under the operation of matrix multiplication: The product of two invertible matrices is invertible, the identity matrix is an identity element for the group, an invertible matrix has an inverse and matrix multiplication is associative.

Definition 2.1. A matrix Lie group G is a subgroup of $GL(n, C)$ in addition to the following properties: Suppose A_m is any sequence of matrices in G, and A_m converges to particular matrix A then either $A \in G$, or A is not invertible.

Counter Examples of a subgroup of $GL(n, C)$ which is not closed is the set of all nxn invertible matrices all of whose entries are real and rational. This is indeed a subgroup of $GL(n, C)$ without the condition of a closed subgroup. Which is one can easily have a sequence of invertible matrices with rational entries convert to an invertible matrix.

2.0.9.1 Properties of matric Lie group

- $\|AB\| \leq \|A\|\|B\|, \qquad \|tA\| = \mid t \mid\| A\|, and \qquad \|A^n\| \leq \|A\|^n$
- $GL(V)$is a dense open subset of $End(V)$.
- The multiplication and inversion on $GL(V)$ are analytic.
- Let G be a matrix Lie group, and X an element of its Lie algebra. Then, e^X is an element of the identity component of G.
- Let G be a matrix Lie group, with Lie algebra g. Let X be an element of g, and A an element of G. Then, $AXA^{-1}$is in g.

2.0.10 The Matrix Exponential

The exponential of a matrix plays a important role in Lie groups.

The exponential enters into the different definition of the Lie algebra of a matrix Lie group with different mechanism for going into information from the Lie algebra to the Lie group.

We know many computations are done easily at level of the Lie algebra, some exponential are the indispensable for matrix Lie groups.

Let an nxn real or complex matrix be X . We wish to define the exponential of X, denoted e^Xor $expX$, by the usual power series

$$e^x = \sum_{m=0}^{n} \frac{x^m}{m!} \qquad ....(1)$$

Here the general convention of using letters such as X and Y for the variable in the matrix exponential.

2.0.10.1 Some elementary or basic properties of the matrix exponential

Let X and Y be arbitrarynxn matrices. Then, we have the following:

1. $e^0 = I$.
2. $adj(e^X) = e^{adj(X)}$.
3. e^X is invertible and $(e^X)^{-1} = e^{-X}$.
4. for all and in C.
5. If $XY = YX$, then $e^{X+Y} = e^X e^Y = e^Y e^X$.
6. If C is invertible, then $e^{CXC^{-1}} = Ce^X C^{-1}$.
7. $||e^x|| \leq e^{||X||}$

Let V be a fnite dimensional vector space equipped with a complete norm $\|.\|$over the field F, where $F = R$ or $F = C$. Let $End(V)$ denote the algebra of linear self-maps on V and let $GL(V)$ denote the general linear group, the group (under composition) of invertible self-maps. If $V = R^n$, then $End(V)$ may be identied with $M_n(R)$, the $n \times n$ matrices, and $GL(V) = GL_n(R)$, the matrices of nonvanishing determinant. We endow $End(V)$ with the usual with the usually the operator and a complete norms, are defined by

$$\|A\| = sup\left\{\|AV\| : \|V\| = 1\right\} = sup\left\{\frac{\|AV\|}{\|V\|} : V \neq 0\right\}$$

; which gives rise to the metric $d(A, B) = \|B - A\|$ on $End(V)$ and, by restriction, on $GL(V)$.

LetΩ be some nonempty subset of $M_n(R)$, called the controls. For any interval J of real numbers we consider the set $\mathcal{U}(J; \Omega)$ of locally bounded measurable functions which in generally defined as control functions, from J into Ω . Each member $U \epsilon \mathcal{U}(J; \Omega)$ determines a corresponding time varying linear differential equation, called the fundamental control equation,

$$\dot{X}(t) = U(t)X(t)$$

. The function U is called a control or steering function and the resulting solutions trajectories. we have the solution of

$$\dot{X}(t) = U(t)X(t), \qquad X(t_0) = X_0$$

is given by $X(t) = \Phi_U(t, t_0)X_0$, where Φ_U is the the fundamental solution for the fundamental equation associated to the coefficient function U. The control set and the dierential equation $\dot{X} = UX$ determine a control system.

Given a control system arising from Ω and $A, B \epsilon GL(V)$, we say that B is reachable or attainable from A if there exists an interval $[a, b]$ such that a solution of the fundamental control equation X satisfies $X(a) = A$ and $X(b) = B$ for some control function U.

The set of all points reachable from A (including A itself) is called the reachable set to A, and denoted R_A. If we put the focus on B instead of A, then instead of saying that B is reachable from A, we say that A is controllable to B or can be steered to B. The controllability set of B consists of all A that can be steered to B.

Chapter 3

Concept about Unitary and Special unitary group and Hamiltonian system

3.0.11 Hamiltonian systems

A dynamical system of $2n$, is an ordinary differential equations

$$Z^{\cdot} = J\nabla H(z,t),$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

is an n degree-of-freedom (d.o.f.) Hamiltonian system (when it is non-autonomous it has $n + 1/2$ d.o.f.).

Here H is the "Hamiltonian" with smooth scalar function of the extended phase space variables z and time t, the $2n \times 2n$ matrix J is the Poisson matrix and$I$ is the $n \times n$ identity matrix.

These equations eagerlly split into two sets of n equations for canonically conjugate variables $z = (q,p)$,

i.e. $q^{\cdot} = \partial H/\partial p, \qquad \dot{p} = \partial H/\partial q.$

Here the n coordinates$q$ represent the configuration variables of the system (e.g. positions of the component parts) and their canonically conjugate momenta p represent the impetus gained by movement.

3.0.12 Calculation of a Hamiltonian from a Lagrangian

Given a Lagrangian with the generalized coordinates q_i and generalized velocities $\dot{q}_i$ and time:

1. The momenta are calculated simply by differentiating the Lagrangian with respect to the velocites (generalized) : $p_i(q_i, \dot{q}_i, t) = \frac{\partial L}{\partial \dot{q}_i}$.

2. The velocities or generalised velocites $\dot{q}_i$ are expressed in terms of the momenta p_i by inverting the previous expression.

3. The Hamiltonian is calculated using the usual definition of H as the Legendre transformation of L:

$$H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \qquad = \sum_i \dot{q}_i p_i - L$$

Then the velocities or generalised velocites are substituted for using the previous results.

4. Hamilton's equations are used with efficient to obtain the equations of motion of the system.

3.0.13 Deriving Hamilton's equations

Hamilton's equations are derived by the total differential of the Lagrangian depends on (t), q_i and q_i.

$$dL = \sum_i (\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i) + \frac{\partial L}{\partial t} dt$$

Hence the generalized momenta were defined as

$$p_i = \frac{\partial L}{\partial \dot{q_i}}$$

By substituted the above into the total differential of the Lagrangian gives one about

$$\partial L = \sum_i (\frac{\partial L}{\partial q_i} dq_i + p_i \frac{\partial L}{\partial \dot{q_i}} d\dot{q_i}) + \frac{\partial L}{\partial t} dt$$

We can rewrite this as

$$\partial L = \sum_i (\frac{\partial L}{\partial q_i} dq_i + d(p_i \dot{q_i}) - q_i \dot{d} p_i) + \frac{\partial L}{\partial t} dt$$

and rearrange again to get

$$d(\sum_i p_i \dot{q_i} - L) = \sum_i (-\frac{\partial L}{\partial q_i} dq_i + \dot{q_i} dp_i) - \frac{\partial L}{\partial t} dt$$

The term on the left-hand side is just the Hamiltonian that we have defined before, so we find that

$$\partial H = \sum_i (-\frac{\partial L}{\partial q_i} dq_i + q_i \dot{d} p_i) - \frac{\partial L}{\partial t} dt$$

Similarly to the total differential with respect to time of the Lagrangian (with which we started above), independently from the above derivations the total differential of the Hamiltonian is equal to

$$\partial H = \sum_i (\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i) + \frac{\partial H}{\partial t} dt$$

It dramatically follows the path from the previous two independent equations that their right-hand sides are equal with different corresponding. Thus we obtain the equation

$$\sum_i (-\frac{\partial L}{\partial q_i} dq_i + q_i \dot{d} p_i) - \frac{\partial L}{\partial t} dt = \sum_i (\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i) + \frac{\partial H}{\partial t} dt$$

Since this calculation can associate corresponding terms from both sides of this equation to yield:

$$\frac{\partial H}{\partial q_j} = -\frac{\partial L}{\partial q_i}, \quad \frac{\partial H}{\partial p_j} = q_j, \quad \frac{\partial H\dot{}}{\partial t} = -\frac{\partial L}{\partial t}.$$

On-shell, Lagrange's equations tell us that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0.$$

We can rearrange this to get

$$\frac{\partial L}{\partial q_i} = \dot{p}_i$$

Thus Hamilton's equations for many hold on shell:

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j, \qquad \frac{\partial H}{\partial p_j} = \dot{q}_j, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

3.0.14 Unitary Matrix :

A square matrix U is a unitary matrix if $U^{(H)}$=$U^{(-1)}$,

where $U^{(H)}$ denotes the conjugate transpose and $U^{(-1)}$ is the matrix inverse.

For example, $\begin{bmatrix} 2^{\frac{-1}{2}} & 2^{\frac{-1}{2}} & 0 \\ -2^{\frac{-1}{2}}i & 2^{\frac{-1}{2}}i & 0 \\ 0 & 0 & i \end{bmatrix}$

3.0.15 Unitary group:

In mathematics, the unitary group of degree n, denoted $U(n)$, is the group of $n \times n$ unitary matrices, with as usual matrix multiplication group operation.

The unitary group is a subgroup of $GL(n, C)$. A unitary group $U(n)$ is a real Lie group of dimension n^2.

The Lie algebra of $U(n)$ consists of $n \times n$ skew-Hermitian matrices, with the Lie bracket given by the commutator.

3.0.16 Special Unitary Matrix:

A square matrix U is a special unitary matrix if $UU^* = I$,

where I is the identity matrix and U^* is the adjoint matrix with determinant is $detU = 1$.

The first condition means that U is a unitary matrix, and the second condition provides a restriction beyond a general unitary matrix, with determinant $e^{i\theta}$ for θ any R real number.

For example, $\frac{1}{\sqrt{2}}\begin{bmatrix} i & i \\ i & -i \end{bmatrix}$

3.0.17 Special unitary group:

The special unitary group of degree n, denoted $SU(n)$, is the group of $n \times n$ unitary matrices with determinant 1.

The group operation is matrix multiplication.

The special unitary group is a subgroup of the unitary group $U(n)$, consisting of all $n \times n$ unitary matrices, which is itself a subgroup of the general linear group $GL(n, \mathbb{C})$.

The simplest case, $SU(1)$, is the trivial group, having only a single element.

3.0.17.1 Properties :

The special unitary group $SU(n)$ is a real matrix Lie group of dimension $n^2 1$.

Topologically, it is compact with simply connected domain but algebraically, it is a simple Lie group.The center of $SU(n)$ is isomorphic to the cyclic group z_n.

Its outer automorphism group, for $n \geq 3$, is z_2, while the outer automorphism group of $SU(2)$ is the trivial group.

The Lie algebra of $SU(n)$, denoted by $su(n)$ is generated by n^2operators, which satisfy the commutator relationship (for $i, j, k, \ell = 1, 2, ..., n$).

$\left[\hat{O_{ij}}, \hat{O_{kl}}\right] = \delta_{jk}\hat{o}_{il} - \delta_{il}\hat{o}_{kj}$

Additionally, the operator

$$\hat{N} = \sum_{i=1}^{n} \hat{o}_{ii}$$

satisfies

$$\left[\hat{N}, \hat{O}_{ij}\right] = 0$$

This implies the number of independent generators is $n^2 1$.

3.0.18 The Lie Algebra of a Matrix Lie Group

Definition 3.1. Let G be a matrix Lie group. The Lie algebra of G, denoted g, is the set of all matrices X such that e^{tX} is in G for all real numbers t.

This implies that X is in g iff parametric subgroup generated by X stays in G. Note that even though G is a subgroup of $GL(n, c)$ (and not essentially for $GL(n, R)$), we do not require that e^{tX} be in G for all complex numbers t, it is purpose only for all real numbers t. Also, it is definitely not enough to have just eX in G. i.e, it has easy to gain an example of an X and a G such that $e^X \in G$ but such that $e^{tX} \notin G$ for some real values of t. Such an X is not in the Lie algebra of G.

3.0.19 Lie algebra of the unitary groups

We know that a matrix is unitary iff $adj(U) = U - 1$.

Thus, e^{tX} is unitary iff

$$adj(e^{tX}) = (e^{tX}) - 1 = e^{-tX}............(3)$$

we know that

$$adj(e^{(tX)}) = e^{adj(tX)},$$

so, equation (3) becomes $adj(e^{(tX)}) = e^{-tX}$ ....(4)

Clearly, a sufficient condition for (4) to hold is that $adj(X) = -X$.

On the other hand, if (4) holds for all t, then by differentiating at $t = 0$,

we see that $adj(X) = -X$ is necessary.

Thus, the Lie algebra of $U(n)$ is the space of all nxn complex matrices X such that $adj(X) = -X$, denoted $u(n)$.

By combining the 2 previous computational approach, we see that the Lie algebra of $SU(n)$ is the space of all nxn complex matrices X such that $adj(X) = -X$ and $trace(X) = 0$, denoted $su(n)$.

Chapter 4

Optimal Control and Calculus of Variations

Definition 4.1. Optimal control : This is the process of getting control and state trajectories for a dynamic system over a span of time to minimise a performance index.

Example 4.1. ***Optimal control problem's examples:***

1. paths of vehicles between two points to minimize fuel or time can be determined.
2. feedback control logic for vehicles or industrial processes to keep them near a theshold operating point in the enviornment of disturbances with acceptable control magnitudes can be determined.

4.0.20 Formulation of optimal control problems

There are different types of optimal control problems based on the performance index, the type of time domain (continuous, discrete),different types of constraints with varities of variables are free to be chosen.

Formulation of an optimal control problem require the following:

- Mathematical modelling of the system to be controlled,
- criteria of the performance index,
- criteria of all boundary values on states conditions, and constraints to be satisfied by states and controls,
- statement checking of what variables are free.

4.0.21 Formal Statement and requirement of Optimal Control System

The Optimal control problem is to getting the optimal control $u^*(t)$ which causes the linear time-invariant system.

$$\dot{x}(t) = Ax(t) + Bu(t)$$

to give the trajectory $x * (t)$ that optimizes or extremizes (minimizes or maximizes) a performance index

$$J = x^{'}(t_f)Fx(t_f) + \int_{t_0}^{t_f} [x^{'}(t)Qx(t) + u^{'}(t)Ru(t)]dt$$

$$\dot{x}(t) = f(x(t), u(t), t)$$

to give the state $x^*(t)$ that optimizes the general performance index

$$J = S(x(t_f), t_f) + \int_{t_0}^{t_f} V(x(t), u(t), t)dt$$

Calculus of variations (CoV) or variational calculus deals with getting the optimum.

4.0.22 Basic Concepts Function and Functional

We discuss some fundamental concepts associated with functionals along side with those of functions.

- **Function**: A variable x is a function of a variable quantity t,(written as $x(t) = f(t)$), if to every value of t over a certain range of t there corresponds a value x; i.e. we have a correspondence to a number t there corresponds a number x.

Note that : here t need not be always time but any independent variable.

Example 4.2. *consider*

$$x(t) = 2t^2 + 1$$

For $t = 1, x = 3, t = 2, x = 9$ and so on.Other functions are $x(t) = 2t$;$x(t_1, t_2) = t_1^2 + t_2^2$.

- **Functional**: A variable J is a functional dependent on a function $f(x)$, written as $J = J(f(x))$, if to each function $f(x)$, there corresponds a value J, i.e., we have a correspondence: to the function $f(x)$ there corresponds a number J.

Example 4.3. *Let $x(t) = 2t^2 + 1$. Then*

$$J(x(t)) = \int_0^1 x(t)dt = \int_0^1 (2t^2 + 1)dt = \frac{2}{3} + 1 = \frac{5}{3}$$

is the area under the curve $x(t)$. If $v(t)$ is the velocity of a vehicle , then

$$J(v(t)) = \int_{t_0}^{t_f} v(t)dt$$

is the path traversed by the vechicle.Thus, here $x(t)$ and $v(t)$are functions of t, and J.

- **Increment of a Function**:

The increment of a function f, denoted by $\triangle f$, is defined as

$$\triangle f \stackrel{\triangle}{=} f(t + \triangle t) - f(t).$$

$\triangle f$ depends on both the independent variable t and the increment of the independent variable $\triangle t$.

Example 4.4. *If*

$$f(t) = (t_1 + t_2)^2$$

Find the increment of the function $f(t)$.

solution : The increment $\triangle f$ becomes

$$\begin{aligned}
\triangle f &\stackrel{\triangle}{=} f(t + \triangle t) - f(t) \\
&= (t_1 + \triangle t_1 + t_2 + \triangle t_2)^2 - (t_1 + t_2)^2 \\
&= (t_1 + \triangle t_1)^2 + (t_2 + \triangle t_2)^2 + 2(t_1 + \triangle t_1)(t_2 + \triangle t_2) - (t_1^2 + t_2^2 + 2t_1 t_2) \\
&= 2(t_1 + t_2)\triangle t_1 + 2(t_1 + t_2)\triangle t_2 + (\triangle t_1)^2 + (\triangle t_2)^2 + 2\triangle t_1 \triangle t_2.
\end{aligned}$$

- **Increment of a Functional**:

The increment of the functional J denoted by $\triangle J$, is defined as

$$\triangle J \triangleq J(x(t) + \delta x(t)) - J(x(t)).$$

Here $\delta x(t)$ is called the variation of the function $x(t)$. we also write increment as $\triangle J(x(t), \delta x(t))$.

Example 4.5. *Find the increment of the functional*

$$J = \int_{t_0}^{t_f} [2x^2(t) + 1]dt.$$

solution : The increment of J is given by

$$\begin{aligned}\triangle J &\triangleq J(x(t) + \delta x(t)) - J(x(t)),\\ &= \textstyle\int_{t_0}^{t_f} [2x(t) + \delta x(t))^2 + 1]dt - \int_{t_0}^{t_f} [2x^2(t) + 1]dt,\\ &= \textstyle\int_{t_0}^{t_f} [4x(t)\delta x(t) + 2(\delta x(t))^2]dt.\end{aligned}$$

4.0.23 Optimization

Optimization is an essential part of design activity in all major disciplines.It is a processes of search that seek to optimize(maximize or minimize)of a mathematical function of several variable subject to certain constraints(equality or inequality constraints). The content of optimization technique is quite general in the sense that it it can be looked in different ways depending on the approach(algebraic or geometric approach). The **nature of variables** may be real,integer,mix of both used in optimization. Again with the optimization classified in two groups:

1. Static optimization problem
2. Dynamic optimization problem

- Static optimization problem

In Static optimization problem,objective function /cost function involves variables that are not changing with time.

Technique to slove static optimization problem:

- Lagrange Multiplier
- Linear/Nonlinear Programming

- **Dynamic optimization problem**

Dynamic optimization problem is concerned with design variables (involved in objective function)are changing with respect to time.thus the time is involoved in the problem statement.

Technique to slove Dynamic optimization problem

- Calculus of Variation
- Dynamic Programming
- Convex Optimization

4.0.24 Optimal control Problem

Optimal Control is the one of getting about control technique in which the control signal optimizes a certain “cost index”.

Example 4.6. *As a simple example, we can considered the problem of a rocket launching a statellite into an orbit above earth. A interesting problem about control would be that of choosing the thrust attitude angle and rate emission of the exhaust gases such that the rocket takes the satellite into its prescribed orbit.An associated optimal control problem is to choose the controls to affect the transfer rate to minimum expenditure of fuel,and time.*

4.0.25 Functionals

- The integrals of the form

$$J = \int_{t_0}^{t_f} F(x, \frac{dx}{dt}, t)dt \qquad (1)$$

Where F is a given function of the function $x(t)$, its derivative $\frac{dx}{dt}$ and the independent variable t.The path $x(t)$ is defined for $t_0 \leq t \leq t_1$.For a given path,say $x =$

$x_1(t)$,equation (1) gives the corresponding value of J, say $J = J_1$.For a second path, say $x = x_2(t)$,equation (1) again gives a value of J, say $J = J_2$.

In general, $J_1 \neq J_2$,and we call integrals of the form (1) functionals.

- As a simple example, Consider the integral

$$J = \int_{t_0}^{t_f} (t^2 + t\frac{dx}{dt})dt$$

where $x = x(t), \qquad 0 \leq t \leq 1$,is some path defined from $t = 0$ to $t = 1$.

It is clear that the value of J depends on the prescribed path $x = x(t)$,$0 \leq t \leq 1$.

For example ,

- $x = t, 0 \leq t \leq 1$ gives $J = \int_0^1 (t^2 + t)dt = \frac{5}{6}$;
- $x = e^t, 0 \leq t \leq 1$ gives $J = \int_0^1 (e^{2t} + te^t)dt = \frac{e^2}{2} + 1$;

4.0.26 Basic Optimal Control Problem

- Earlier we have seen that the state of the system is charaterised by n variables, $x_1, x_2.....x_n$ and we write the state vector as

$$x = (x_1, x_2........x_n)^T..............(2)$$

- These variables satisfy the coupled first order differential equations

$$\frac{dx_i}{dt} = f_i(x_1, x_2, ...x_n; u_1, u_2, ...u_m; t) \quad (1 \leq i \leq n) \ ....(3)$$

on $[0, T]$ and where the m variables $u_1, u_2, ...u_m$ form the control vector u; that is

$$u = (u_1, u_2, ...u_m)^T \ ....(4)$$

- We can write the system in the form

$$\dot{x} = f(x, u, t), \qquad f = (f_1, f_2, ...f_n)^T$$

and if the system is linear

$$\dot{x} = Ax + Bu \quad ....(5)$$

where A is an $n \times n$ matrix, B is an $n \times m$ matrix.

- We assume that $u \epsilon U$, a given control region in m-dimensional space and the $u_j$are assumed continous,except possibly at a finite number of discontinuties.It is assumed that the f_i are contionus with continous partial derivatives. The given initial value of x_i are specified (for example $x = x_0$at time $t = 0$)which means that specifying $u_j,\ 0 \leq t \leq T\ \ (j = 1, 2, 3, ...m)$ determines the $x_i(i = 1, 2, ..., n)$ from equation (3).
- The basic control problem is to choose the control vector $u \epsilon U$ such that the state vector x is transferred from x_0 to a terminal point at time T where some of the state variables are specified; for example suppose x_i are specified at a $t = T$ for $t = 1, 2, 3......, q\ (\leq n)$.The region U is called admissible control region.
- If the transfer can be accomplished, the problem in optimal control is to effect the transfer so that the functional

$$J = \int_0^T f_0(x, u, t)dt \qquad ....(6)$$

is maximised(or minimised or extremum). Here $f_0$is a function of $x_1, x_2, ..., x_n; u_1, u_2, ..., u_m$ and t is continous with partial derivatives.

Example 4.7. ***Examples of finding extremum of*** J

Finding the curve $y(x)$ which gives an extremum value to the functional

$$J = \int_0^1 (y'^2 + 1)dx$$

with $y(0) = 1, y(1) = 2$.

solution :

Before going to the solution one result which is important and to be remembered is the Euler equation which is satisfied by the path $y(x)$ which yield extremum values of the functional

$$J = \int_a^b F(y, y', x)dx$$

and given as follow :

$$F_y - \frac{d}{dx}(F_{y'}) = 0 \quad \text{.......................(7)}$$

Here the integrand $F = y'^2 + 1$, so $F_y = 0$.

Hence the euler equation (7) becomes

$$\frac{d}{dx}(F_{y'}) = 0$$

and on integrating , $F_{y'} = A$, constant

i.e, $2y' = A$.

Integrating again,

$$y = \frac{Ax}{2} + B$$

where B is a constant.To specify the condition $y(0) = 1, y(1) = 2$ requires $B = 1$ and $A = 2$. so the optimal curve is straight line

$$y = x + 1$$

The corresponding extremum value of J is

$$J = \int_0^1 2dx = 2.$$

and this is infact the minimum value of J.

There are different method for determining the extremum of J for different types of constraints implemented which are like Lagrange Multiplier and Hamiltonian Control system(method).

Chapter 5

An Optimal control Problem on the Special Unitary Group SU(3)

Basis Representation for $SU(3)$:

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \qquad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

We get the commutation relation by calculating between all the basis can be given below
:

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
λ_1	0	$2i\lambda_3$	$-2i\lambda_2$	$i\lambda_7$	$-i\lambda_6$	$i\lambda_5$	$-i\lambda_4$	0
λ_2	$-2i\lambda_3$	0	$2i\lambda_1$	$i\lambda_6$	$i\lambda_7$	λ_5	$-i\lambda_5$	0
λ_3	$2i\lambda_2$	$-2i\lambda_1$	0	$i\lambda_5$	$-i\lambda_4$	$-i\lambda_7$	$i\lambda_6$	0
λ_4	$-i\lambda_7$	$-i\lambda_6$	$-i\lambda_5$	0	$(\lambda_3+\lambda_8\sqrt{3})i$	$i\lambda_2$	$i\lambda_1$	$-\sqrt{3}i\lambda_5$
λ_5	$i\lambda_6$	$-i\lambda_7$	$i\lambda_4$	$-(\lambda_3+\lambda_8\sqrt{3})i$	0	$-i\lambda_1$	λ_1	$\frac{1}{\sqrt{3}}(2\lambda_5+i\lambda_4)$
λ_6	$-i\lambda_5$	$i\lambda_4$	$i\lambda_7$	$-i\lambda_2$	$i\lambda_1$	0	$(\sqrt{3}\lambda_8-\lambda_3)i$	$-\sqrt{3}i\lambda_7$
λ_7	$i\lambda_4$	$-i\lambda_5$	$-i\lambda_6$	$-i\lambda_1$	$-i\lambda_2$	$(\lambda_3-\sqrt{3}\lambda_8)i$	0	$\sqrt{3}i\lambda_6$
λ_8	0	0	0	$\sqrt{3}i\lambda_5$	$-\frac{1}{\sqrt{3}}(2\lambda_5+i\lambda_4)$	$\sqrt{3}i\lambda_7$	$-\sqrt{3}i\lambda_6$	0

The minus Lie-Poisson structure on $SU(3)$

The minus Lie-Poisson structure on $SU(3)$ is given below in the matrix as follow :

$$\begin{bmatrix} 0 & -2ip_3 & 2ip_2 & -ip_7 & ip_6 & -ip_5 & -ip_4 & 0 \\ 2ip_3 & 0 & -2ip_1 & -ip_6 & -ip_7 & -p_5 & ip_5 & 0 \\ -2ip_2 & 2ip_1 & 0 & -ip_5 & ip_4 & ip_7 & -ip_6 & 0 \\ ip_7 & ip_6 & ip_5 & 0 & -(p_3+\sqrt{3}p_8)i & -ip_2 & -ip_1 & -\sqrt{3}ip_5 \\ -ip_6 & ip_7 & -ip_4 & (p_3+\sqrt{3}p_8)i & 0 & ip_1 & -p_1 & -\frac{1}{\sqrt{3}}(2p_5+ip_4) \\ ip_5 & -ip_4 & -ip_7 & ip_2 & -ip_1 & 0 & -(\sqrt{3}p_8-p_3)i & \sqrt{3}ip_7 \\ -ip_4 & ip_5 & ip_6 & ip_1 & ip_2 & -(p_3-\sqrt{3}p_8)i & 0 & -\sqrt{3}ip_6 \\ 0 & 0 & 0 & -\sqrt{3}ip_5 & -\frac{1}{\sqrt{3}}(2p_5+ip_4) & -\sqrt{3}ip_7 & \sqrt{3}ip_6 & 0 \end{bmatrix}$$

Theorem 5.1. *There exist the following type of controllable drift-free left invariant system on $SU(3)$, namely :*

$$\dot{X} = X \cdot (A_1u_1 + A_2u_2 + A_3u_3 + A_4u_4 + A_5u_5 + A_6u_6 + A_7u_7 + A_8u_8) \ldots\ldots\ldots (5.1)$$

5.1 An Optimal Control Problem on SU(3)

Let

$$J(u_{1,}, u_2, u_3) = \frac{1}{2}\int_0^{t_f}(c_1u_1^2 + c_2u_2^2 + c_3u_3^2 + c_4u_4^2 + c_5u_5^2 + c_6u_6^2 + c_7u_7^2 + c_8u_8^2)dt$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 > 0$ be the cost function.

Then the problem which we intend to slove is to find $u_1, u_2, ..., u_8$ that maximize J and steer the system (5.1) from $X = 0$ at $t = 0$ to $X = X_f$ at $t = t_f$

Theorem 5.2. *The optimal controls problem of the above problem for our system* (5.1) *are given by*

$$\begin{aligned}
u_1 &= \frac{p_1}{c_1} \\
u_2 &= \frac{p_2}{c_2} \\
u_3 &= \frac{p_3}{c_3}, \\
u_4 &= \frac{p_4}{c_4}, \\
u_5 &= \frac{p_5}{c_5}, \\
u_6 &= \frac{p_6}{c_6}, \\
u_7 &= \frac{p_7}{c_7}, \\
u_8 &= \frac{p_8}{c_8},
\end{aligned}$$

Where the solutions of the system :

$$\begin{aligned}
\dot{p_1} &= -\frac{2ip_3p_2}{c_2} + 2\frac{ip_2p_3}{c_3} - \frac{ip_7p_4}{c_4} + \frac{ip_6p_5}{c_5} - \frac{ip_5p_6}{c_6} - \frac{ip_4p_7}{c_7} \\
\dot{p_2} &= \frac{2ip_3p_1}{c_1} - 2\frac{ip_1p_3}{c_3} - \frac{ip_6p_4}{c_4} - \frac{ip_7p_5}{c_5} - \frac{p_5p_6}{c_6} + \frac{ip_5p_7}{c_7} \\
\dot{p_3} &= -\frac{2ip_2p_1}{c_1} + 2\frac{ip_1p_2}{c_2} - \frac{ip_5p_4}{c_4} + \frac{ip_4p_5}{c_5} + \frac{ip_7p_6}{c_6} - \frac{ip_6p_7}{c_7} \\
\dot{p_4} &= \frac{ip_7p_1}{c_1} + \frac{ip_6p_2}{c_2} + \frac{ip_5p_3}{c_3} - \frac{ip_2p_6}{c_6} - \frac{ip_1p_7}{c_7} + \frac{\sqrt{3}ip_5p_8}{c_8} - \frac{ip_5(p_3+\sqrt{3}p_8)}{c_5} \\
\dot{p_5} &= -\frac{ip_6p_1}{c_1} + \frac{ip_7p_2}{c_2} - \frac{ip_4p_3}{c_3} + \frac{ip_1p_6}{c_6} - \frac{p_1p_7}{c_7} - \frac{(ip_4+2p_5)p_8}{\sqrt{3}c_8} - \frac{ip_4(-p_3-\sqrt{3}p_8)}{c_4} \\
\dot{p_6} &= \frac{ip_5p_1}{c_1} - \frac{ip_4p_2}{c_2} - \frac{ip_7p_3}{c_3} + \frac{ip_2p_4}{c_4} - \frac{ip_1p_5}{c_5} + \frac{\sqrt{3}ip_7p_8}{c_8} - \frac{ip_5(-p_3+\sqrt{3}p_8)}{c_7} \\
\dot{p_7} &= -\frac{ip_4p_1}{c_1} + \frac{ip_5p_2}{c_2} + \frac{ip_6p_3}{c_3} + \frac{ip_1p_4}{c_4} + \frac{ip_2p_5}{c_5} - \frac{\sqrt{3}ip_6p_8}{c_8} - \frac{ip_6(p_3-\sqrt{3}p_8)}{c_6} \\
\dot{p_8} &= -\frac{\sqrt{3}ip_5p_4}{c_4} - \frac{p_5(ip_4+2p_5)}{\sqrt{3}c_5} - \frac{\sqrt{3}ip_7p_6}{c_6} + \frac{\sqrt{3}ip_6p_7}{c_7}
\end{aligned}$$

With taking suitable extended Hamiltonian H given by

$$H = (p_1u_1 + p_2u_2 + p_3u_3 + p_4u_4 + p_5u_5 + p_6u_6 + p_7u_7 + p_8u_8) -$$

$$(c_1u_1^2 + c_2u_2^2 + c_3u_3^2 + c_4u_4^2 + c_5u_5^2 + c_6u_6^2 + c_7u_7^2 + c_8u_8^2)$$

Proceeding using the maximum principle, we generally get

$$\frac{\partial H}{\partial u_1} = 0 \qquad \frac{\partial H}{\partial u_2} = 0$$

$$\frac{\partial H}{\partial u_3} = 0 \qquad \frac{\partial H}{\partial u_4} = 0$$

$$\frac{\partial H}{\partial u_5} = 0 \qquad \frac{\partial H}{\partial u_6} = 0$$

$$\frac{\partial H}{\partial u_7} = 0 \qquad \frac{\partial H}{\partial u_8} = 0$$

which leads to

$$\begin{aligned} p_1 &= c_1 u_1 \\ p_2 &= c_2 u_2 \\ p_3 &= c_3 u_3 \\ p_4 &= c_4 u_4 \\ p_5 &= c_5 u_5 \\ p_6 &= c_6 u_6 \\ p_7 &= c_7 u_7 \\ p_8 &= c_8 u_8 \end{aligned}$$

and so reduced Hamiltonian (Optimal Hamiltonian) is given by

$$H = \frac{1}{2}[\frac{p_1^2}{c_1} + \frac{p_2^2}{c_2} + \frac{p_3^2}{c_3} + \frac{p_4^2}{c_4} + \frac{p_5^2}{c_5} + \frac{p_6^2}{c_6} + \frac{p_7^2}{c_7} + \frac{p_8^2}{c_8}]$$

It follows that the reduced Hamiltonian equations have the following expressions

$$\begin{bmatrix} \dot{p_1} \\ \dot{p_2} \\ \dot{p_3} \\ \dot{p_4} \\ \dot{p_5} \\ \dot{p_6} \\ \dot{p_7} \\ \dot{p_8} \end{bmatrix} = \begin{bmatrix} 0 & -2ip_3 & 2ip_2 & -ip_7 & ip_6 & -ip_5 & -ip_4 & 0 \\ 2ip_3 & 0 & -2ip_1 & -ip_6 & -ip_7 & -p_5 & ip_5 & 0 \\ -2ip_2 & 2ip_1 & 0 & -ip_5 & ip_4 & ip_7 & -ip_6 & 0 \\ ip_7 & ip_6 & ip_5 & 0 & -(p_3+\sqrt{3}p_8)i & -ip_2 & -ip_1 & -\sqrt{3}ip_5 \\ -ip_6 & ip_7 & -ip_4 & (p_3+\sqrt{3}p_8)i & 0 & ip_1 & -p_1 & -\frac{1}{\sqrt{3}}(2p_5+ip_4) \\ ip_5 & -ip_4 & -ip_7 & ip_2 & -ip_1 & 0 & -(\sqrt{3}p_8-p_3)i & \sqrt{3}ip_7 \\ -ip_4 & ip_5 & ip_6 & ip_1 & ip_2 & -(p_3-\sqrt{3}p_8)i & 0 & -\sqrt{3}ip_6 \\ 0 & 0 & 0 & -\sqrt{3}ip_5 & -\frac{1}{\sqrt{3}}(2p_5+ip_4) & -\sqrt{3}ip_7 & \sqrt{3}ip_6 & 0 \end{bmatrix} \begin{bmatrix} \frac{p_1}{c_1} \\ \frac{p_2}{c_2} \\ \frac{p_3}{c_3} \\ \frac{p_4}{c_4} \\ \frac{p_5}{c_5} \\ \frac{p_6}{c_6} \\ \frac{p_7}{c_7} \\ \frac{p_8}{c_8} \end{bmatrix} =$$

$$\begin{bmatrix} -\frac{2ip_3p_2}{c_2} + 2\frac{ip_2p_3}{c_3} - \frac{ip_7p_4}{c_4} + \frac{ip_6p_5}{c_5} - \frac{ip_5p_6}{c_6} - \frac{ip_4p_7}{c_7} \\ \frac{2ip_3p_1}{c_1} - 2\frac{ip_1p_3}{c_3} - \frac{ip_6p_4}{c_4} - \frac{ip_7p_5}{c_5} - \frac{p_5p_6}{c_6} + \frac{ip_5p_7}{c_7} \\ -\frac{2ip_2p_1}{c_1} + 2\frac{ip_1p_2}{c_2} - \frac{ip_5p_4}{c_4} + \frac{ip_4p_5}{c_5} + \frac{ip_7p_6}{c_6} - \frac{ip_6p_7}{c_7} \\ \frac{ip_7p_1}{c_1} + \frac{ip_6p_2}{c_2} + \frac{ip_5p_3}{c_3} - \frac{ip_2p_6}{c_6} - \frac{ip_1p_7}{c_7} + \frac{\sqrt{3}ip_5p_8}{c_8} - \frac{ip_5(p_3+\sqrt{3}p_8)}{c_5} \\ -\frac{ip_6p_1}{c_1} + \frac{ip_7p_2}{c_2} - \frac{ip_4p_3}{c_3} + \frac{ip_1p_6}{c_6} - \frac{p_1p_7}{c_7} - \frac{(ip_4+2p_5)p_8}{\sqrt{3}c_8} - \frac{ip_4(-p_3-\sqrt{3}p_8)}{c_4} \\ \frac{ip_5p_1}{c_1} - \frac{ip_4p_2}{c_2} - \frac{ip_7p_3}{c_3} + \frac{ip_2p_4}{c_4} - \frac{ip_1p_5}{c_5} + \frac{\sqrt{3}ip_7p_8}{c_8} - \frac{ip_5(-p_3+\sqrt{3}p_8)}{c_7} \\ -\frac{ip_4p_1}{c_1} + \frac{ip_5p_2}{c_2} + \frac{ip_6p_3}{c_3} + \frac{ip_1p_4}{c_4} + \frac{ip_2p_5}{c_5} - \frac{\sqrt{3}ip_6p_8}{c_8} - \frac{ip_6(p_3-\sqrt{3}p_8)}{c_6} \\ -\frac{\sqrt{3}ip_5p_4}{c_4} - \frac{p_5(ip_4+2p_5)}{\sqrt{3}c_5} - \frac{\sqrt{3}ip_7p_6}{c_6} + \frac{\sqrt{3}ip_6p_7}{c_7} \end{bmatrix}$$

as required.

Theorem 5.3. *The controls $u_1, u_2, u_3, ..., u_8$ are given by sinusoidals, More exactly*

$$\begin{aligned} u_1 &= \frac{l_1}{\sqrt{c_1}} \cos \sqrt{\frac{c_2}{c_1}} \left(\frac{p_2\dot{p}_1 - p_1\dot{p}_2}{p_1^2+p_2^2} \right) t + k_1 \\ u_2 &= \frac{l_2}{\sqrt{c_2}} \sin \sqrt{\frac{c_4}{c_3}} \left(\frac{p_4\dot{p}_3 - p_3\dot{p}_4}{p_3^2+p_4^2} \right) t + k_2 \\ u_3 &= \frac{l_3}{\sqrt{c_3}} \cos \sqrt{\frac{c_6}{c_5}} \left(\frac{p_6\dot{p}_5 - p_5\dot{p}_6}{p_5^2+p_6^2} \right) t + k_3 \\ u_4 &= \frac{l_4}{\sqrt{c_4}} \sin \sqrt{\frac{c_8}{c_7}} \left(\frac{p_8\dot{p}_7 - p_7\dot{p}_8}{p_7^2+p_8^2} \right) t + k_4 \end{aligned}$$

Proof.

Let

$$\frac{p_1^2}{c_1} + \frac{p_2^2}{c_2} = l_1^2$$

$$\frac{p_3^2}{c_3} + \frac{p_4^2}{c_4} = l_2^2$$

$$\frac{p_5^2}{c_5} + \frac{p_6^2}{c_6} = l_3^2$$

$$\frac{p_7^2}{c_7} + \frac{p_8^2}{c_8} = l_4^2$$

The reduced Hamiltonian is obviously a constant of motion. So we may write

$$\frac{p_1^2}{c_1} + \frac{p_2^2}{c_2} + \frac{p_3^2}{c_3} + \frac{p_4^2}{c_4} + \frac{p_5^2}{c_5} + \frac{p_6^2}{c_6} + \frac{p_7^2}{c_7} + \frac{p_8^2}{c_8} = l^2$$

i.e,

$$l_1^2 + l_2^2 + l_3^2 + l_4^2 + l_5^2 + l_6^2 + l_7^2 + l_8^2 = l^2$$

If we take now :

$$p_1 = l_1\sqrt{c_1}\cos\theta_1$$

$$p_2 = l_1\sqrt{c_2}\cos\theta_2$$

$$p_3 = l_3\sqrt{c_3}\cos\theta_3$$

$$p_4 = l_4\sqrt{c_4}\cos\theta_4$$

$$p_5 = l_5\sqrt{c_5}\cos\theta_5$$

$$p_6 = l_6\sqrt{c_6}\cos\theta_6$$

$$p_7 = l_7\sqrt{c_7}\cos\theta_7$$

$$p_8 = l_8\sqrt{c_8}\cos\theta_8$$

So, we get

$$u_1 = \frac{p_1}{c_1} = \frac{l_1\cos\theta_1}{\sqrt{c_1}}$$

$$u_2 = \frac{l_1}{\sqrt{c_2}} \sin\theta_1$$

$$u_3 = \frac{l_2}{\sqrt{c_3}} \cos\theta_2$$

$$u_4 = \frac{l_2}{\sqrt{c_4}} \sin\theta_2$$

$$u_5 = \frac{l_3}{\sqrt{c_5}} \cos\theta_3$$

$$u_6 = \frac{l_3}{\sqrt{c_6}} \sin\theta_3$$

$$u_7 = \frac{l_4}{\sqrt{c_7}} \cos\theta_4$$

$$u_8 = \frac{l_4}{\sqrt{c_8}} \sin\theta_4$$

Now ,

$$\frac{p_1}{p_2} = \sqrt{\frac{c_1}{c_2}} \cot\theta_1 \qquad \Longrightarrow \theta_1 = \sqrt{\frac{c_2}{c_1}} \arctan(\frac{p_1}{p_2})$$

$$\dot{\theta}_1 = \sqrt{\frac{c_2}{c_1}} \left(\frac{p_2\dot{p}_1 - p_1\dot{p}_2}{p_1^2 + p_2^2} \right)$$

similarlly

$$\dot{\theta}_2 = \sqrt{\frac{c_2}{c_1}} \left(\frac{p_4\dot{p}_3 - p_3\dot{p}_4}{p_3^2 + p_4^2} \right)$$

$$\dot{\theta}_3 = \sqrt{\frac{c_6}{c_5}} \left(\frac{p_6\dot{p}_5 - p_5\dot{p}_6}{p_5^2 + p_6^2} \right)$$

$$\dot{\theta}_4 = \sqrt{\frac{c_8}{c_7}} \left(\frac{p_6\dot{p}_7 - p_7\dot{p}_8}{p_7^2 + p_8^2} \right)$$

It follow that

$$\theta_1 = \sqrt{\frac{c_2}{c_1}} \left(\frac{p_2\dot{p}_1 - p_1\dot{p}_2}{p_1^2 + p_2^2} \right) t + k_1$$

$$\theta_2 = \sqrt{\frac{c_4}{c_3}} \left(\frac{p_4\dot{p}_3 - p_3\dot{p}_4}{p_3^2 + p_4^2} \right) t + k_2$$

$$\theta_3 = \sqrt{\frac{c_6}{c_5}} \left(\frac{p_6\dot{p}_5 - p_5\dot{p}_6}{p_5^2 + p_6^2} \right) t + k_3$$

$$\theta_4 = \sqrt{\frac{c_8}{c_7}} \left(\frac{p_8\dot{p}_7 - p_7\dot{p}_8}{p_7^2 + p_8^2} \right) t + k_4$$

then

$$u_1 = \frac{l_1}{\sqrt{c_1}} \cos \sqrt{\frac{c_2}{c_1}} \left(\frac{p_2\dot{p}_1 - p_1\dot{p}_2}{p_1^2 + p_2^2} \right) t + k_1$$

$$u_2 = \frac{l_2}{\sqrt{c_2}} \sin \sqrt{\frac{c_4}{c_3}} \left(\frac{p_4\dot{p}_3 - p_3\dot{p}_4}{p_3^2 + p_4^2} \right) t + k_2$$

$$u_3 = \frac{l_3}{\sqrt{c_3}} \cos \sqrt{\frac{c_6}{c_5}} \left(\frac{p_6\dot{p}_5 - p_5\dot{p}_6}{p_5^2 + p_6^2} \right) t + k_3$$

$$u_4 = \frac{l_3}{\sqrt{c_4}} \sin \sqrt{\frac{c_8}{c_7}} \left(\frac{p_8\dot{p}_7 - p_7\dot{p}_8}{p_7^2 + p_8^2} \right) t + k_4$$

Conclusion

In this short project, we have studied mathematical control theory, in more particularly optimal control on the compact special unitary group $SU(3)$. We have restricted our-selves only in the mathematical aspects. However this group is related with many physical problem , which are interesting to physicist .We hope our study will be lead to a small step towards such type of investigation.

Bibliography

1. DAlessandro, D. 2002. *The Optimal Control Problem on SO(4) and Its Applications to Quantum Control.* IEEE Transactions on Automatic Control 47, no. 1: 8792.

2. Storm, J. *Time Optimal Control of a Right Invariant System on a Compact Lie Group*: 134.

3. Saccon, A., J. Hauser, and A.P. Aguiar. *Optimal Control on Lie Groups : Implementation Details of the Projection Operator Approachno. 3.*

4. Sachkov, Y.L. *Control Theory on Lie Groups Control Theory on Lie Groupsno.* March 2006.

5. Munchen, T.U. *A PRACTICAL GUIDE TO THE SOLUTION OF REAL-LIFE OPTIMAL CONTROL.*

6. Kara, A. 2007. *On Observability of Affine Control Systems* 2, no. 17: 825829.

7. Struemper, H. 1998. *Nilpotent Approximation and Nilpotentization for Underactuated Systems on Matrix Lie Groups.* Proceedings of the 37th IEEE Conference on Decision and Control (Cat. No.98CH36171) 4, no. 2: 41884193.

8. OPTIMAL CONTROL SYSTEMS by Desineni Subbaram Naidu in CRC PRESS.

9. Introduction To Control Theory Including Optimal Control by David Burghes and Alexander Graham (Ellis Horwood Limited books).

10. Camelia Pop An Optimal Control Problem on the Heisenberg Lie group H(3) ,Politehnica University of Timisora.

11. D is TEFANO 3 ,STUBBERUD,WILLIAMS 'Theory and Problems of feedback and Controls system ' Schaum's Outline Series (McGraw Hill books)

12. G C.Walsh and Richard Montgomery and S.Shankar Sastry 'Optimal path planning on Matrix Lie groups', Proceedings of the 33^{rd} Conference on Decision and Control Lake Buena Vista,Fl-December 1994 .

13. Harriet Pollatsek 'Lie group-A problem oriented introduction via Matrix groups' , @ 2009 Mathematical Assocation of America.

14. Harold Cohen 'Mathematics for Scientist and Engineers' , Prentice Hall.