ANALYSIS OF COMPOSITE PLATES USING ELEMENT FREE GALERKIN METHOD

A Thesis

submitted by

S.KRISHNA KUMAR

211CE2029

In partial fulfilment of the

requirements of the degree of

Master of Technology

In

Structural Engineering

Under Guidance of

Prof. K.C.BISWAL

June, 2013

Department Of Civil Engineering

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This is to certify that the thesis entitled, **"ANALYSIS OF COMPOSITE PLATES USING ELEMENT FREE GALERKIN METHOD"** submitted by **KRISHNA KUMAR S.** bearing Roll No. **211CE2029** in partial fulfilment of the requirements for the award of **Master of Technology** Degree in **Civil Engineering** with specialization in **"Structural Engineering"** during 2012-13 session at National Institute of Technology, Rourkela is an authentic work carried out by him under my supervision and guidance.

To the best of my knowledge, the matter embodied in the thesis has not been submitted to any other University/Institute for the award of any Degree or Diploma.

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Place: Rourkela

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S. Krishna Kumar

ABSTRACT

Meshfree methods are a new class of numerical analysis methods that rectify some drawbacks of traditional mesh-based methods like FDM, FEM, BEM & FVM. Composite plates are quite common in aerospace industries and are subject to hostile operating conditions making them prone to cracks and their propagation. Hence meshfree methods have a possible application in the crack propagation of composite laminates. In this work, the Element free Galerkin Method- one of the most popular meshfree methods- is applied to isotropic and composite plates and the behavior of the plates is studied under plane stress and transverse bending. The isotropic plates are analysed using Kirchoff's plate theory and the laminates are analysed using Classical Laminate Theory. To implement the EFG method for analysis of plate, a computer code is developed and executed in MATLAB platform. The current formulation is validated with the exact solutions. The dependence of the performance of the methods on the parameters concerning these methods is analysed and the ways to find optimal parameters are discussed. It is found that EFGM gives excellent results. However, it is dependent heavily on the parameters like support domain size. It is found that the polynomial basis and weight function are the most critical parameters and must be chosen as per the structural theory used. The support domain size, the quadrature order and nodal density also affect the results significantly.

Keywords: Thin plates, Laminates, Meshfree methods, EFG.

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INTRODUCTION

1.1 . AN OVERVIEW

Meshfree methods are among the breed of numerical analysis technique that are being vigorously developed to avoid the drawbacks that traditional methods like Finite Element method possess. FEM, with half a century of passionate research behind it is versatile, time tested and trustworthy. Yet when it comes to specific areas like fracture mechanics and crack propagation, FEM has disadvantages which necessitates the need to have specialist methods dealing with such problems. The structural aspect of 'element' in FEM was found to be restrictive in nature when it comes to the implementation in such problems. Meshfree methods permit an alternative implementation based totally on nodes and devoid of the restriction of the element.

The field of meshfree method is nascent and there have not been any single method that could be versatile enough to rival the FEM. Hence it becomes important for developing methods or applying existing methods for each kind of problem. Until a general meshfree framework is formulated, more and more specialized methods would be conceived and applied to niche problems.

On the other hand, composite laminates are structural elements with a vast range of applications in aerospace and other industries. These elements may undergo rough operating conditions and are prone to cracks and failure. These materials are more complex than isotropic materials, and hence the numerical analysis of such composite elements is also complex. Since fields like Structural health monitoring are gaining application in aerospace industries, it is natural that scientists and engineers are focusing on developing efficient methods for crack propagation, fracture problems etc.

In such a context, a study on the behavior of composite plates using meshfree methods is timely and has immense utility. Further, since the benefits of each of the meshfree methods have not been established so far clearly, it is important to apply different methods to critical problems and gauge their robustness and versatility. Additionally, different theories used to analyze the structures pose different problems in the numerical analysis. It would be interesting to understand the behavior of different meshfree methods vis-à-vis such theories and issues. The prominent trend in scientific communities follows this paradigm. However there is still no final word regarding the suitable method for analysis of composite plates.

Among meshfree methods, only a few like Element Free Galerkin Method and Meshless Local Petrov Galerkin method have gained immense popularity among researchers. Each method has its own merits, that it warrants a comparative study between these methods to decide the superiority of one over the other.

1.2. OBJECTIVE

The objective of this work is to study the response of composite plates under static loading conditions using meshfree methods. This study aims to investigate the optimal parameter settings for obtaining accurate results using meshfree methods.

1.3. SCOPE OF THE STUDY

The present work deals with the application of meshfree methods to the static analysis and plane stress analysis of isotropic plates and composite plates. The scope of the work is-

- To develop Element free Galerkin based algorithms found on the relevant structural theories.

- Apply the meshfree methods for plane stress and plate bending problems in both isotropic plates and laminates. For plate bending, the theory used is thin plate theory and for composite, the analogous Classical Laminate theory is applied.
- The EFGM is dependent on the parameters involved in them, a detailed parametric study is also undertaken to facilitate the understanding of the influence of the parameters on the result.
- Interpretation of result and comparison with literature to assess the validity of the plate theories and performance of the meshfree approaches considered in the work.

1.4. ORGANIZATION OF THESIS

The organization of the thesis is as follows. Chapter 2 gives a detailed review of literature in the field of meshfree methods. Emphasis is on the EFGM and its application to isotropic plates and laminates. The papers followed to arrive at the formulation used in the current work are also reviewed.

Chapter 3 deals with the theoretical formulation behind the analyses. The plane stress equation, thin plate theory, and Classical Laminate theory are also discussed in detail. The requirements of numerical implementation of these theories are also discussed.

Chapter 4 presents a thorough discussion on meshfree methods, their origins, basic concepts behind them and their advantages. Especially EFGM has been discussed thoroughly. The formulations generally used and the formulation used in this work have been discussed elaborately and the issues involved in their implementation are briefly touched upon.

Chapter 5 presents the results of the work and also the inferences that are discerned from these results. The performance of EFGM is analyzed and the parametric behavior of the problem is investigated. The results of these meshfree methods are compared with the exact answers.

Chapter 6 concludes the discussion and presents a holistic view on the results. The broad lines of understanding arrived due to the work are elaborated upon. The future possibilities in the work are also dealt with.

LITERATURE REVIEW

2.1. MESHFREE METHODS

The meshfree methods had their beginnings in the late 1970s when FEM was in the peak of its popularity. With the advent of higher computational power, FEM became ubiquitous. However as FEM was being applied to great variety of fields, its limitations and inhibitive features were also understood. The meshfree methods began as one among the many lines of thought to resolve this issue and to replace or complement FEM in such problems.

The first, method to be developed was the Smooth particle hydrodynamics from the works of *Lucy* (1977), *Gingold and Monaghan* (1977) The method used the global strong form. The trial function is assumed as an integral representation. The ideas like support domain etc. were first introduced in this work. *Liszka and Orkisz* (1980) proposed the Finite point method which was also based on the strong form. The finite differential representation involves Taylor series. Also they used Moving least squares method for approximation.

Nayroles et al (1992) developed the Diffuse Element Method. The method was based on weak form and used Moving Least squares method for approximation of the field variable. They called it by the name Diffuse approximation and it was expected tom be complementary to the FEM. It was expected that the approximation would find use as the smoothing function and also for approximating functions.

Belytschko et al (1994) refined the ideas of Nayroles and developed the Element Free Galerkin Method, the most popular meshfree method till date. They employed the weak form of the governing equation and used MLS for approximating the shape function. They considered certain derivatives that Nayroles had discarded in the interpolation. Also they applied Lagrange Multipliers for the imposition of boundary conditions. However the EFGM was not totally meshfree as simple shaped cells were used for integration.

Slowly the trend veered towards using local weak form for arriving at the system algebraic equations. *Mukherjee and Mukherjee* (1997) introduced boundary node methods, followed by point interpolation method of *Liu et al* (1999). Atluri et al (1999) formulated a meshfree method based on local weak form using Petrov Galerkin approach. The approach attempted to eliminate the need of a mesh for integration cells on a global scale. They called this method as Meshless Local Petrov Galerkin method.

Other methods followed, like the kinds of XFEM (Belytschko, 1999) and Natural Element Method (Sukumar, 1998). There is a push towards computationally efficient, reasonably accurate meshfree methods. However the most popular ones are still the methods of EFGM and MLPG. The next two sections review the literature in these fields.

2.2. ELEMENT FREE GALERKIN METHOD

Belytschko et al (1994) introduced the Element Free Galerkin method and applied it to elasticity and heat conduction problems. The method was an improvement over the methods of Nayroles et al (1992) and showed smoother gradients and reasonable accuracy.

Krysl P. and Belytschko T. (1995) applied the EFGM technique to plate bending. Since the C^1 continuity is easily attainable in the EFGM, the Kirchoff's plate theory was applied instead of the Reissner's theory and alternate approaches as used usually in bending analyses of plates. *Krysl P. and Belytschko T.* (1996) extended the work to thin shells and the issues involved like membrane locking were dealt with. Dolbow et al (1998) wrote a paper on the implementation of EFG which encouraged the use of the method widely.

Kanok-Nukulchai et al (2001) applied EFG to static analysis of plates and also dealt with the elimination of transverse shear locking in shear deformable plates. The formulation was successful in eliminating the shear locking and also providing high order approximation of the displacement and stresses. *Liew K.M et al* (2002) extended EFGM to static analysis laminates and composite beams, especially the laminates with peizo-electric patches. The behavior of the patched laminates and the effects of the actuators in the laminates were studied. The plates theory used was the First Order Shear Deformaion theory.

Peng L.X. et al (2005) applied EFGM to rectangular stiffened plates under uniform loading. The plate theory applied to the static analysis problem was FSDT theory. The advantage in using EFGM to the problem was that the stiffeners could be placed anywhere in the plate unlike FEM where they needed to be along the meshes.

Belinha J. et al (2006) applied EFGM to plates and laminates. They considered the Reissner Mindlin theory for plates and the analogous FSDT for laminates. Different weight functions and basis functions were tried and it was declared that a seventh order weight function was the best. Also the problem of shear locking was encountered by using different shape functions for translations and rotations. *Belinha J. et al* (2007) performed nonlinear analysis on the plates and laminates using EFG. FSDT theory was considered and elastoplastic analysis of the laminates was conducted. A version of Newton-Raphson algorithm was used for solving the nonlinear equations of laminate composite plate.

Valencia et al (2008) studied the effect of various parameters in the EFGM for a 1 dimensional case. Grid irregularity, order of polynomial functions and type of weight functions were studied in the work. *Wu C.P and Yang S.W.* (2011) applied a version of EFGM to 3d analysis of composite and FGM plates. The shape functions were approximated using RMVT based meshless collocation. Wu C.P. and Chiu K.H (2011) performed 3d free vibration analysis of composite plates using a similar approach.

It can be observed that the EFGM implementation of Kirchoff's theory and analogous Classical Laminate theory has been rarely studied. However the EFGM offers an easier implementation of thin plate theory compared to troublesome implementation in FEM. Hence this work focuses on application of Thin Plate theories which would lead to immense computational savings.

PLATE THEORETICAL FORMULATION

3.1 PLANE STRESS

The concept of plane stress involves the consideration of the element as a planar structure with only the in plane stresses being non-zero. The stress across the thickness is assumed to be zero. This criterion is usually valid for thin elements with negligible variation of stresses across the thickness. The plates and laminates considered in this work are also thin ones. Hence before implementing EFGM for plates and laminate bending problems, they are applied to the plane stress problem. In case of plane stress, two degree of freedoms-in plane displacements- are considered. The constitutive matrix for plane stress is

$$D = \left(\frac{Eh^3}{12(1-\nu^2)} \right) \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \qquad \text{(Eq. 3.1)}$$

Where E denotes the Young's modulus for an isotropic material and ν denotes the Poisson's Ratio. This relationship is for the isotropic materials. For the case of orthotropy, the relation is

$$Q_{ij} = \left(\frac{1}{1-\nu_{12}\nu_{21}} \right) \begin{bmatrix} E_1 & \nu_{21}E_1 & 0 \\ \nu_{12}E_2 & E_2 & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \qquad \text{(Eq. 3.2)}$$

This relation holds for one lamina only.

3.2. PLATE THEORIES

The utility of plates and plate like structural components in engineering is due to the two dimensional structural action. However for the same reason the analysis of beams can not be extended to the plates. Different plates theories have been proposed considering varying assumptions, relaxations etc.

Ideally plates must be analyzed as a three dimensional bodies. Since such an approach is computationally reasonable, several assumptions are made and plates are analyzed as two dimensional objects. Plates can be mainly divided into 4 categories [Szilard, 2004]-

1. Stiff plates
2. Membranes
3. Moderately thick
4. Thick plates

The first two come under the category of thin plates. Thin plates are most common and are easy to model as a 2d structure. Different plate theories have been developed for analysis of different types of plates. The first one to be developed was Kirchoff's theory or Thin Plate theory which is being followed in this work. For most of the plates used in engineering applications, the thickness is usually small and in such cases using a three dimensional model is unreasonably time consuming. In addition to this, serious ill conditioning problems occur due to use of higher order theories (Zeinkewicz, 2000). Hence it is often not only convenient but necessary to use thin plate theory for plate structures.

3.2.1. KIRCHOFF'S PLATE THEORY

The theory owes its name to Gustav Kirchoff who formalized the assumptions in 1850 AD. Sophie Germain however had presented the same in 1811. The higher order theories were formulated when Reissner and Mindlin relaxed the criteria in 20^{th} century. Interestingly, though it is simple when it comes to formulation, thin plate theory presents lot of complications when it comes to computer applications. Especially thin plate theory requires that the approximation should satisfy C^1 continuity and this criterion cannot be relaxed. Usually to offset this issue, some alternate methods are implemented. However in case of meshfree methods, most of the issues are avoided and the implementation is a lot easier than in FEM.

The assumptions, formulation and details regarding the thin plate theory are available in standard books like Timoshenko [1959], Szilard [2004] etc. The constitutive matrix for the theory is

$$D = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

(Eq. 3.3)

Usually only one degree of freedom is considered, the transverse displacement. The analysis being a plane analysis, three strains are considered. The strain displacement relation is

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \partial / \partial x \\ \partial / \partial y \\ \partial^2 / \partial x \partial y \end{bmatrix} \{w\}$$ (Eq. 3.4)

The governing equation for the thin plates as per this theory is a 4^{th} order equation which contains second order derivatives. Hence the interpolation or approximation of the displacement should have C^1continuity. Since this is usually difficult, alternative procedures are applied to avoid this. In case of meshfree methods, the presence of weight function enables us to attain arbitrary order of continuity. This avoids most problems associated with the FEM implementation of the thin plate theory.

3.2.2. CLASSICAL LAMINATE THEORY

The theory for thin laminates which is analogous to that of Kirchoff's plate theory is Classical Laminate Theory. The theory does not account for the presence of transverse shear and strain components. Due to the absence of the transverse shear, this theory is applicable only to thin laminates. The strains are assumed to vary linearly over the thickness. Also in the current formulation, three degrees of freedom are considered per node. The expressions for strain are

$$\varepsilon_x = \frac{\partial u}{\partial x}$$
$$\varepsilon_y = \frac{\partial u}{\partial y}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

(Eq. 3.5a)

$$u = u_0 - z\frac{\partial w_0}{\partial x}$$
$$v = v_0 - z\frac{\partial w_0}{\partial y} \qquad \text{(Eq. 3.5b)}$$

Equation 3.5 shows the linear variation of strains which are dependent only on the position along the thickness.

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & -\frac{\partial^2}{\partial x^2} \\ 0 & \frac{\partial}{\partial y} & -\frac{\partial^2}{\partial y^2} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & -\frac{\partial^2}{\partial x \partial y} \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \end{Bmatrix} \qquad \text{(Eq. 3.6)}$$

The equation provided above gives the transformation between displacements and strains. The strain displacement matrix is a 6x3 matrix.

$$\begin{Bmatrix} \varepsilon \\ \kappa \end{Bmatrix} = [B]_{6x3} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \end{Bmatrix}$$

(Eq. 3.7)

An important thing to be noted is that the constitutive matrix is not 3x3 as in thin plates. In laminates there is a possibility of bending extension coupling and this has to be accounted. Also the properties of each lamina, assumed orthotropic, have to be included. The detailed form of this can be understood from R.M.Jones [1999], J.M.Daniel and Ori Ishai [1994] etc.

$$Dmat = \begin{bmatrix} A & B \\ B & D \end{bmatrix}$$

$$A_{ij} = \sum_{k=1}^{n} [(Q_{ij})_k (z_k - z_{k-1})]$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} [(Q_{ij})_k (z_k^2 - z_{k-1}^2)]$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} [(Q_{ij})_k (z_k^3 - z_{k-1}^3)]$$

(Eq. 3.8)

The equation above gives the cumulative behavior of all the laminates together. The A_{ij} refers to the extension coupling, B_{ij} refers to the extension bending coupling, D_{ij} refers to the bending coupling. These values depend on the laminate lay up scheme. For symmetric laminates, B is a null matrix. These values are also dependent on the geometry and material properties. Here Q_{ij} is the transformed laminar elastic constants. These are average values of the stiffness of the multidirectional laminate. Since stresses vary across the dimensions and thickness of the laminate, these equations give the average force and moment resultants.

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \kappa \end{Bmatrix} \qquad \text{(Eq. 3.9)}$$

The strains and curvatures thus refer to the values at the mid-plane of the laminate. Hence the problem reduces to one where we need to only analyze the mid-plane deflection and the behavior at the rest of the points can be derived from the same. The relation between the strains, curvatures and displacements can be found from eq. 3.5-3.7. Hence we find the displacement of the mid-plane alone and the displacements and stresses anywhere else can be found out from that.

To model the transverse shear behavior of laminates there are other theories like First Order Shear Deformation Theory, Third Order Shear Deformation Theory, Higher Order Shear Deformation Theory etc. These are little more complicated but render an accurate picture of complex behaviour of laminates especially the thicker ones. However, in the present study, we focus only on Classical Laminate theory which is suitable for thin laminates.

ELEMENT FREE GALERKIN METHOD

4.1. HISTORICAL OVERVIEW

Most of the engineering problems involve solution of partial differential equations. The complexity of the problems usually makes it impossible to solve them and analytically. Since engineering problems seldom demand complete accuracy, engineers try to obtain approximate solution that can be practically meaningful. Often even if engineers desire absolute accuracy, approximation may be the only possible option and engineers have to settle with that.

Any system or process cannot be analyzed as such and its modeling involves tradeoffs. The physical phenomenon is modeled mathematically by making reasonable assumptions regarding the behavior of the system. This mathematical model is analyzed numerically to get approximate solutions. Hence there are two error components affecting the result- one due to modeling and other due to numerical analysis. Since improving the accuracy of modeling is far more difficult, this generally necessitates the numerical analysis algorithm to be reliable, robust and reasonably accurate.

Historically the works of mathematicians like Newton, Gauss and Euler were the foundations of numerical analysis. The methods were applied to various fields like physics and astronomy. However, great impetus to the field was provided only in the beginning of 20^{th} century. The advent of Finite element methods by 1950s was the pivotal point in the history of FEM. Finite element methods gained popularity due to the robustness and convenience of the method. However in 1970s the limitations of FEM were understood when applying to fracture mechanics problems. This led to the development of meshfree methods as an alternative to FEM in such problems. A detailed historical review was presented in chapter 2.

4.2. MESHFREE METHODS- THE IDEA

The fulcrum of Finite Element Method is the concept of element. The element is used both for interpolating the field variable (displacement in case of structural mechanics problem) and also for performing numerical integration to derive the algebraic system equations. When crack propagates, FEM does not allow crack to cut through the elements, so the cracks have to follow the element boundaries. Due to the propagation of cracks, the mesh has to be redone and it leads to increased computational cost. This necessity of continual remeshing, which is a liability in FEM is due to the fact that the P matrix in eq. 4.1 has to invertible, so it has to be square. This means that the number of monomials- in other words, the number of generalized coordinates- in the assumed polynomial function has to be equal to the number of nodes in the element, neither more nor less. Also the integrity of the element is a prerequisite for the stability of FEM, otherwise the inter-element continuity requirement is violated.

$$\{a_i\} = [P]^{-1}\{u_n\} \qquad \text{(eq. 4.1)}$$

Where $\{a_i\}$ is the vector of generalised coordinates

$[P]^{-1}$ is the inverse of the matrix of monomial values at nodes

$\{u_n\}$ is the vector of nodal displacements

The idea of meshfree methods is to remove the requirement of 'element' and still do the interpolation of displacement and the numerical integration. But any nodal based approach will mean that the number of nodes influencing displacement at a particular node would keep varying across the domain. In that case, P matrix ceases to be square. Hence the FEM shape function interpolation has to be replaced by some nodal based interpolation. In meshfree methods, this is usually done by replacing the interpolation by an approximation procedure used in statistics and data analysis like method of least squares, radial basis functions etc.

Also the numerical integration has to be formulated such that the ‘meshing’ pattern or element shape etc. do not affect the result drastically. This again means the integration domain should be one that is based on the concept of node. In FEM, the integration is conducted within an element. In case of meshfree methods, this is replaced by an integration domain- usually circular or rectangular- for each node. The cells may be formed only to facilitate the placing of Gauss integration points. In ‘truly’ meshfree methods, even the presence of integration cells is avoided. These two ideas- replacing a restrictive shape function and using a node based integration domain are the two fundamental concepts involved in the formulation of any meshfree methods.

The alternative procedures used instead of the FEM approach as described add complications the process of solving PDEs using meshfree methods. Usually trouble occurs in the application of boundary conditions, changes in the weak form- like forming a global weak form, local symmetric weak form etc., computational complexity etc. Also unlike FEM where the inter nodal connectivity is clearly established, in case of meshfree methods, the nodal connectivity and continuity of the field variable interpolation are not explicitly established. Hence due care should be given to these aspects, failing which the concept of using mesh less methods will become void. Another interesting and challenging feature of meshfree nodal domains is that the domains overlap, unlike in FEM where the elements have clearly demarcated boundaries and do not overlap each other. Hence the formulation of a meshfree approach to numerical analysis and approximation can be summarized as follows-

Step1: Formulate a suitable form of the PDE- strong, weak, weakened weak forms, global or local forms, symmetric or asymmetric, Petrov Galerkin or Bubnov Galerkin etc.

Step 2: Establish a procedure to define a domain for a node- a domain for establishing the shape functions and same or other domain for the purpose of establishing Gauss points and performing numerical integration.

Step 3: Establish an approximation procedure to formulate the shape functions based on the nodes that fall within the domain of a particular ‘host’ node. In FEM the shape function of a node is calculated from the contributions to the displacement at that host node by the other nodes in the element. Similarly, in case of meshfree methods, the shape function of a particular node is derived from the contribution of the other nodes in its influence domain to the displacement at that particular node.

Step 4: Implement an integration routine consistent with the formulation in Step 1 and also suitable to the domain considered. Quadrature points have to be established and the integrals involving stiffness coefficients and force components at nodes have to be calculated. It is worth noting here that there is no ‘element’ stiffness matrix in case of meshfree methods. Instead each pair of nodes has a matrix having the stiffness coefficients between the two nodes as the values its elements. This matrix is called the nodal stiffness matrix which is added to the global stiffness matrix as in FEM.

Step 5: Application of Boundary conditions. Most meshfree shape functions do not satisfy ‘Kronecker Delta’ property. This means that the boundary conditions can not be imposed directly. Hence some methods like Lagrange Multipliers, Penalty approach are applied to solve this issue.

Step 6: Post Processing. After applying the boundary conditions and solving the equations, the result we get is usually not the displacements themselves- if Kronecker Delta property is not satisfied. So the shape functions are used once more to calculate the displacements from the pseudo displacements or ‘nodal parameters’ [Belytschko, 1994] or ‘fictitious nodal values’ [Atluri,1998]

Thus the given framework is backbone of any meshfree method. The meshfree methods thus provide a wonderful liberation from the constraint of using an element.

4.3. ELEMENT FREE GALERKIN METHOD

Element free Galerkin method (EFGM) mainly deals with the replacement of the FEM shape function by a nodal based approximation. This way the dependence of the field variable on the mesh refinement is removed. The benefit is that the algorithm is easily adaptive. In case of crack growth problem, due to the absence of elements the real crack path can be simulated. So unlike elements where a crack can not cut through an element, in case of EFGM the crack can traverse between two nodes. The nodal influence domains could be easily defined in that case to exclude the nodes on the other side of the crack. Also in case problems like stress concentration, FEM employs finer elements near a hole or any other critical region and then the transformation from finer to coarser mesh has to be properly established. In case of EFGM, higher number of nodes can be arbitrarily placed near the critical zone. There need not be any explicit implementation of a transformation between coarser and finer mesh.

The algorithm of EFG can be divided into the following steps.

Step 1: Displacement approximation using Moving Least Squares (MLS) method.

Step 2: Numerical integration of the stiffness coefficients.

Step 3: Post processing of ‘nodal parameters’ to obtain the displacements

The formulation in EFG is generally a weak form of the system partial differential equations expressed using Bubnov- Galerkin method of weighted residuals. The nodal approximation is using MLS method and the numerical integration is performed by establishing a simple 'mesh' of rectangular cells which are used to define the integration points and their weights. The concept of nodal influence domain is used to get the shape functions as well as for deriving nodal stiffness matrix. A detailed review of EFG and its implementation can be obtained from Nayroles et al (1992), Belytschko et al (1994), Dolbow et al (1998), G.R.Liu (2010) etc.

4.3.1. MOVING LEAST SQUARE METHOD

In 1801 AD, C.F. Gauss, the eminent mathematician of the day developed a method to approximate curves called 'method of least squares'. The name was due to the fact that it essentially tried to minimize the squared sum of errors at each point. He applied it to calculate the orbit of asteroid Ceres. Another famous mathematician, Legendre published results on the application of least squares approach 10 years later. Since then the method has undergone dramatic transformations and is applied in a large range of fields.

In 1981, Lancaster and Salkauskas formulated the Moving Least square approach [Lancaster, 1981]. Nayroles et al (1992) first used it for meshfree approximation and the idea was further formulated into EFGM framework by Belytschko et al (1994).

MLS involves the assumption of the field variable as a summation of series of monomials. The coefficients of the monomials are the unknowns and are calculated such that the squared sum of errors in the domain of a point is minimal. Once the approximation at a point is over, the MLS is 'moved' to another point. Here point may include both the nodes and the Gauss points. The equations leading to MLS shape functions are given below in eq. 4.2.

$$\bar{u} = a_1 + a_2 x + a_3 x^2 + \ldots$$

$$\bar{u} = \sum_{i=1}^{m} p_i(x) a_i(x)$$

$$J = \sum_{i=1}^{n} w(x - x_i) \left[\sum_{j=1}^{m} p_j(x_i) a_j(x) - u_i \right]^2$$

$$N = p^t(x) A^{-1}(x) \; B(x)$$

$$\bar{u} = \sum_{i=1}^{m} N_i(x) u_i \qquad \text{(Eq. 4.2)}$$

The equations show the assumed field $\bar{u}$ and function to be minimized J. ‘J’ involves weights for each points. These weights are functions like cubic weight function, quartic weight, quantic weight, exponential weight etc. The weight functions perform two actions, one as a medium of imparting smoothness or desired continuity to the approximation and other one, more important, is the establishment of the local nature of the approximation. The weight function has a higher value at the host point and reduces to zero at the domain boundary. The choice of weight function greatly affects the result. The polynomial basis- which can be quadratic, cubic, etc., and the weight function together cast a major influence on the performance of the MLS method. Fig 4.1 shows the MLS shape function obtained (thick line in dark blue) for left node and contrasts it with the FEM shape functions for the two nodes. It can be seen that unlike the FEM shape functions which have a value of 1 at host node and zero at others, MLS shape functions do not follow this ‘Kronecker Delta’ property.

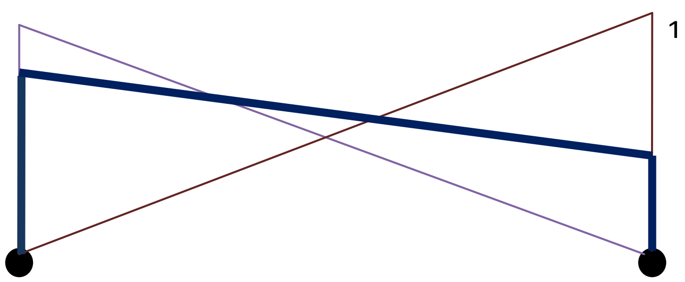

Fig. 4.1. Shape functions of FEM and MLS

This property of MLS shape functions cause difficulty in application of boundary conditions. This can be solved by application of methods like Lagrange Multipliers or Penalty approach.

Another interesting feature, to be noticed in eq. 4.2 is that the inverted matrix is not same as the 'P' in the eq. 4.1. The number of nodes in the domain should be such that this matrix is invertible. To sum it up, MLS can provide great approximations though they come with a price in the form of difficulty in application of boundary conditions. Also they provide only approximations and not interpolations, as shown in Fig 4.2.

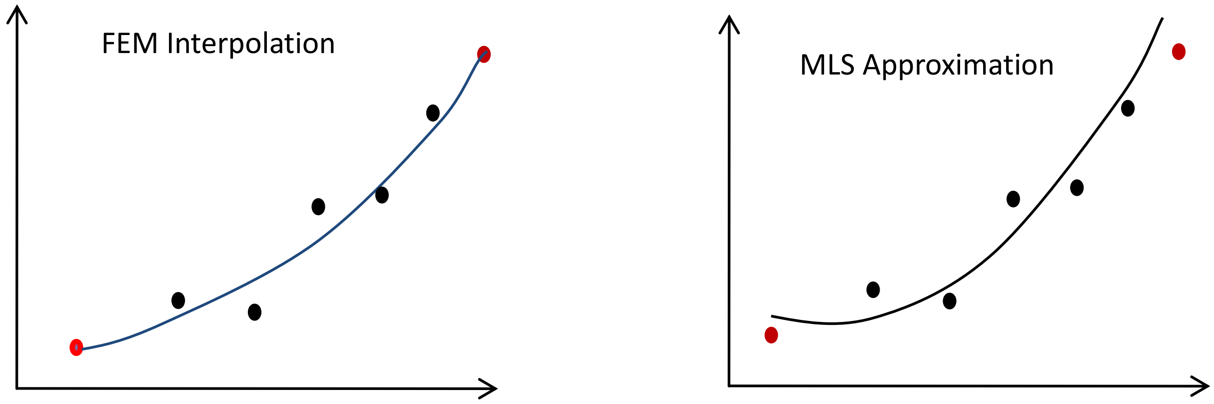

Fig. 4.2 Interpolation and Approximation

4.3.2. WEIGHT FUNCTIONS AND BASIS

A brief note on the weight functions and polynomial bases, especially those used in this work is presented here. The weight function and polynomial basis are most crucial in the MLS approximation and deserve attention, since the continuity of the Meshfree approximation depends on the weight function, its order and properties should be tailored to the problem to which the method, say EFGM, is applied. In fact, the weak formulation's utility lies in the fact that the differentiability condition of the assumed displacement field is relaxed and this burden is transferred to the shoulders of the weight function. This is highly advantageous as the weight function can be easily chosen among the many options available in the literature. In another sense, the weight function also determines how much 'local' the approximation and stiffness coupling should be. Using a compact influence or support domain in the EFGM leads to a sparse and banded stiffness matrix. Use of a weight function determines how much weightage must be given to each node in the approximation and the numerical integration. Hence a ‘steep’ weight function would assign importance to nodes very close to the host node. Likewise, a function like exponential function would not vanish at the domain boundary and hence the compatibility between the overlapping domains collapses. Such criteria must be considered before choosing the function. A list of functions used in current work is given below in eq. 4.3- cubic, quartic, quintic and a seventh order spline used by Belinha et al (2006).

$$w(r) = \begin{cases} \frac{2}{3} - 4r^2 + 4r^3 & for \quad r \le \frac{1}{2} \\ \frac{4}{3} - 4r + 4r^2 - \frac{4}{3}r^3 & for \quad \frac{1}{2} \le r \le 1 \\ 0 \quad for \quad r > 1 \end{cases}$$

$$w(r) = \begin{cases} 1 - 6r^2 + 8r^3 - 3r^4 & for \quad r \le 1 \\ 0 \quad for \quad r > 1 \end{cases}$$

$$w(r) = \begin{cases} 1 - 10r^2 + 20r^3 - 15r^4 + 4r^5 & for \quad r \le 1 \\ 0 \quad for \quad r > 1 \end{cases}$$

$$w(r) = \begin{cases} 1 - \frac{47}{10}r^2 + 12r^4 - 10r^5 + \frac{1}{2}r^6 + \frac{6}{5}r^7 & for \quad r \le 1 \\ 0 \quad for \quad r > 1 \end{cases} \qquad \text{(Eq. 4.3)}$$

It can be seen that each weight function has different characteristics when it comes to parameters like maximum value, steepness, C^n continuity it offers, the nature of derivatives etc. All these influence the result of EFGM. Due consideration must be provided to these and then the weight function chosen. For plates at least C^1 continuity must be available, the weight function be non-negative and the first and second order partial derivatives be non-singular (X.L.Chen, 2003).

When it comes to basis, the essential requirement is that it should have an order that would provide continuous and smooth derivatives upto the order required. In the current work, quadratic, cubic and linear bases are used (eq. 4.4). The bases are usually chosen to be 'complete' polynomials. Generally, for 2d cases, Pascal's triangle is used to determine the monomial terms to be included in the basis. The requirements for basis are similar to those in FEM. The order of basis affects the minimum number of nodes that must be included in the domain.

Linear- $1+x+y$

Quadratic- $1+x+y+x^2+xy+y^2$

Cubic- $1+x+y+x^2+xy+y^2+x^3+x^2y+xy^2+y^3$ (Eq. 4.4)

The influence domain used in the current EFGM formulation is rectangular. Usually rectangular or circular influence domains are used. The mesh is also a regular one. The benefit of using a regular mesh is that implementation of influence domain size becomes very easy and usually the regular mesh has been reported to be more accurate and easy to handle (Belinha et al, 2006). Further the rectangular domain suits the current problem of rectangular plates and laminates. The rectangular domain's size is defined using a parameter 'd' which is the ratio between the domain size in a direction by the mesh size in that direction.

4.3.3. NUMERICAL INTEGRATION.

The numerical integration in EFGM is done by the discretization of the entire domain into simple rectangular cells. These cells only help to define the quadrature points at which the stiffness coefficients shall be calculated. Each gauss integration point is taken and the support domain of the point is defined. The support domain is defined for a point whereas the influence domain is defined only for a node. The size of these two have to be same as the EFGM uses Bubnov Galerkin method and both the trial and test function have to be same numerically and in extent of influence. The EFGM uses Galerkin weak form and the nodal stiffness matrix obtained is

$$K_{ij} = \int_{\Omega} B_i^T c B_j d\Omega$$

$$K = \begin{bmatrix} K_{11} & K_{22} & K_{33}.... \\ K_{21} & K_{22} & K_{33}... \\ .... & & \\ K_{nt1} & K_{nt2}... & K_{ntnt} \end{bmatrix}$$

$$\begin{bmatrix} K & G \\ G^T & 0 \end{bmatrix} \begin{Bmatrix} U \\ \lambda \end{Bmatrix} = \begin{Bmatrix} F \\ q \end{Bmatrix} \qquad \text{(Eq. 4.5)}$$

The last equation shows an enlarged stiffness matrix which is because of the Lagrange multiplier method used in this work. The penalty method can also be used in the application of the boundary condition but the decision on a suitable penalty parameter is not straight forward and so the process is bit more complicated, though it has other advantages like the stiffness matrix is not enlarged. The Lagrange multiplier approach on the other hand is easy to implement.

Lagrange multiplier essentially works on the principle that the gradient of the main function and the gradient of the constraint function are same at the optimal point that is common to both the function. In other words, the tangents of both the functions coincide. However, the values of the gradients may not be same. Hence we introduce a scaling factor called Lagrange Multiplier. The values of Lagrange multipliers at a node are also found using shape functions, though simpler ones like the Lagrange polynomials are used.

The same integration cell and gauss points are used for integration of forces, traction etc. in the domain and the boundary of the plate. The cells are formed generally (Belinha, 2006) such that they coincide with the nodes, though it can be different as used sometimes in the current work.

The algebraic equations obtained from the eq. 4.5 give the fictitious nodal parameters. So using the shape functions the actual displacements are extracted from the nodal parameters. The stresses, strains etc. can be obtained from the displacements in the same way. The Gauss quadrature used in the work is generally 4x4 gauss quadrature, though in some cases other quadrature patterns like 8x8, 6x6 are used. The quadrature order has considerable influence on the convergence of result.

RESULTS AND DISCUSSION

5.1. OVERVIEW

The EFGM was applied first to standard plane stress problems. EFGM was applied later to isotropic plates and then to laminate plates. The results provided here are also in that order.

5.2. ANALYSIS OF PLANE STRESS CONDITION

The plane stress problem considered is as follows. The plate is a rectangular plate with a side fixed. The other side has traction acting on it. The meshing used in the EFG formulation is rectangular. The plane stress problem is less sensitive to the different parameters involved in the EFGM formulation. Hence the parametric study of plane stress case is not presented here. The optimal 'd' value- d being the number of times the support domain is bigger in a direction than the distance between two nodes- was found after a brief study and the displacement results extracted at that optimal 'd' value are presented here.

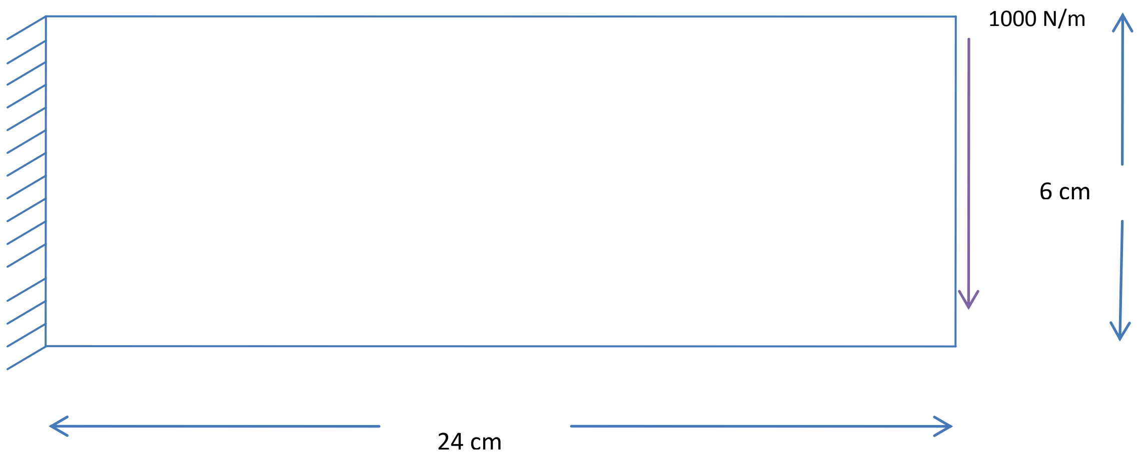

Fig 5.1. The geometry and loading of the plate

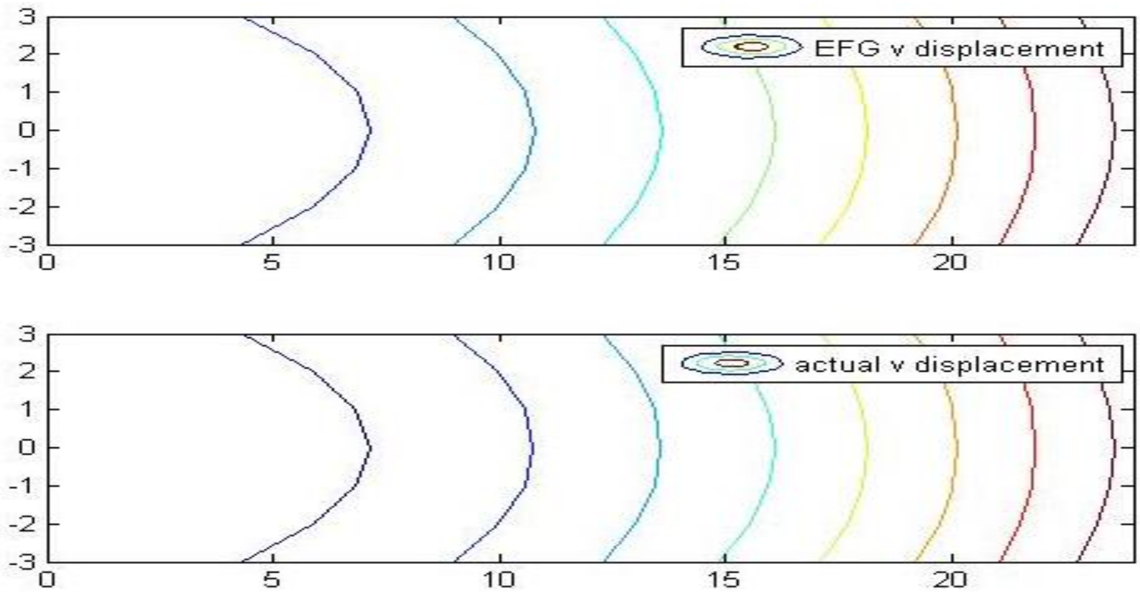

Fig 5.2a. Vertical displacement- EFG result and actual value.

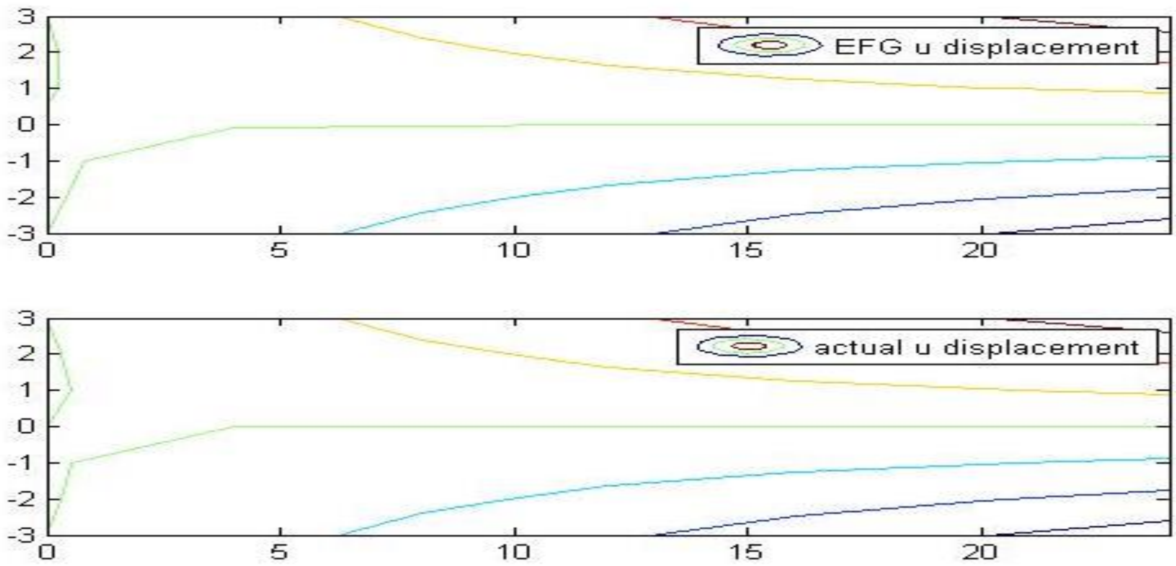

Fig. 5.2b. Horizontal displacement- EFGM results and exact results.

The figures 5.2 shows the displacement in the two in-plane directions. The results are matched with the results in Timoshenko [1970]. Different scales of discretization were attempted and the results are presented in Table 5.1. An error norm has been defined here for the purpose of assessing the efficiency of the algorithm.

Error norm = $\sqrt{\{0.5*\int_{\Omega}(e^{num}-e^{ex})^{T}(Dmat)(e^{num}-e^{ex})\}}$

Number of nodes	Error Norm
65	0.0260
121	0.0246
175	0.0146
279	0.0116
637	0.0076

Table 5.1 Convergence of error with increase in the number of nodes

Thus the EFGM successfully analyzed the plate for plane stress condition. The error norms are seen to decrease with the fineness in discretization. The results show that the method worked well for the plane stress analysis. In addition to this problem, a laminate was also analyzed for the plane stress condition. The problem statement is as follows. A laminate composite plate of dimensions 30cm x 30 cm and thickness 1 cm was taken. It was analysed for plane stress condition under a parabolic traction. The formulation was same as before. There were 4 layers of equal thickness in the laminate. The orientation was [0, 90]s. Few results are given here. The displacement contour is given below. Difficulties were observed in the establishment of the displacement continuity across the lamina. Different approaches have been applied in literature to solve this. One such technique is the implementation of truncated shape functions which would get restricted by the boundary of the layer. The boundary of the support domain will not cut across the interface and be limited by it. Also, the nodes on the border, will share a part of the domains on each side. The results show good accuracy. It was also seen that error norm depends only on the support domain.

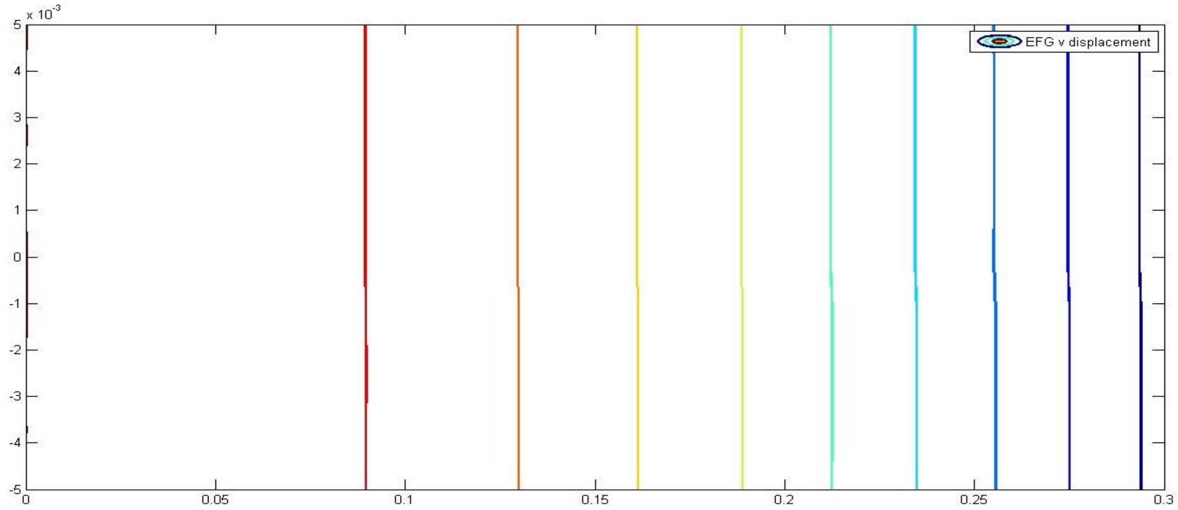

Fig. 5.3. The displacement contour for laminate plate

5.3. ANALYSIS OF THIN PLATES

The isotropic plate considered is the one used in Belinha et al (2006). The details of the plate, geometry, loading etc. are as follows.

Specifications	
E1=E2 (Gpa)	30
Poisson's ratio	0.3
G12 (Gpa)	11.538
q (kN/m^2)	25 (udl)
Geometry	
L=b	20 m
depth	1m
Simply supported all sides	

Table 5.2. Specifications for the isotropic plate

The implementation of EFGM for the isotropic plate involved the parametric study for the method to study the effect of different parameters like polynomial basis, weight function, quadrature order, support domain size, discretization, integration cell size, etc. The dependence of each of the parameter on other was studied. The results for the parametric study are presented here first.

5.3.1. PARAMETRIC STUDY

Four types of weight functions were used in this work as described in Chapter 4. To understand the influence of the weight function on the result, the EFGM routine was run for different weight functions, other things like basis (linear), mesh size (10x10), integration cell mesh (10x10), were left unchanged. For each weight function, the 'd' parameter was kept changing and the variation in result with 'd' was noted. The results are given in the figures given below.

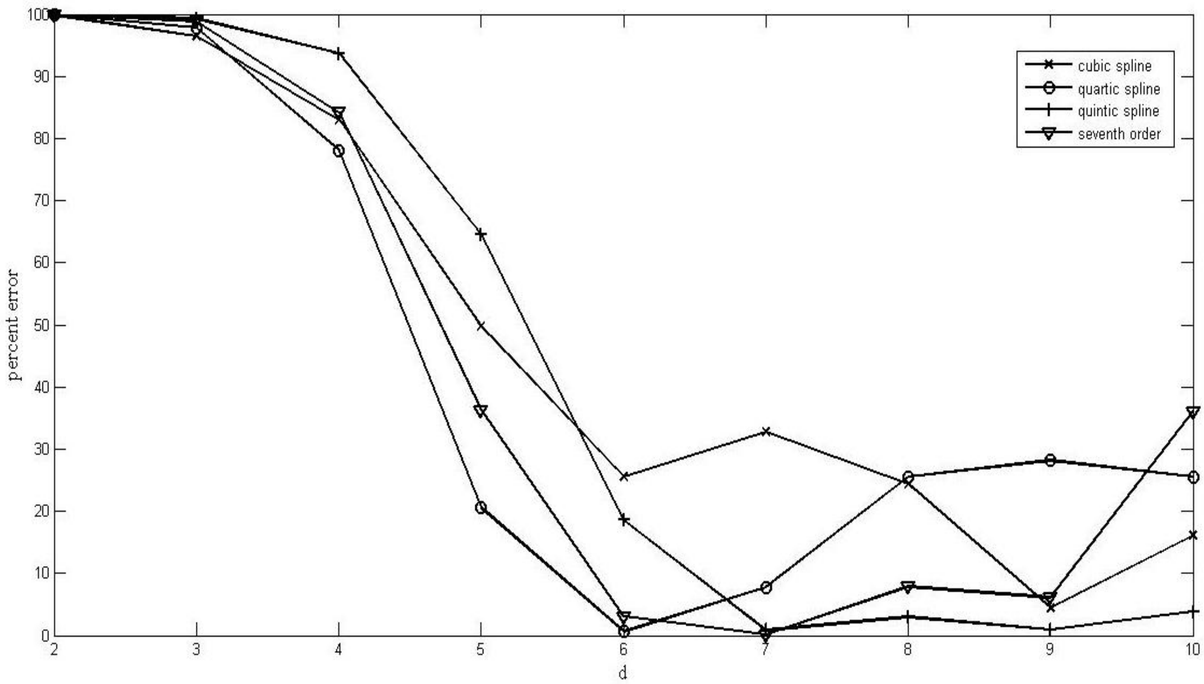

Fig 5.4. Error in percentage vs 'd' parameter for different weight function.

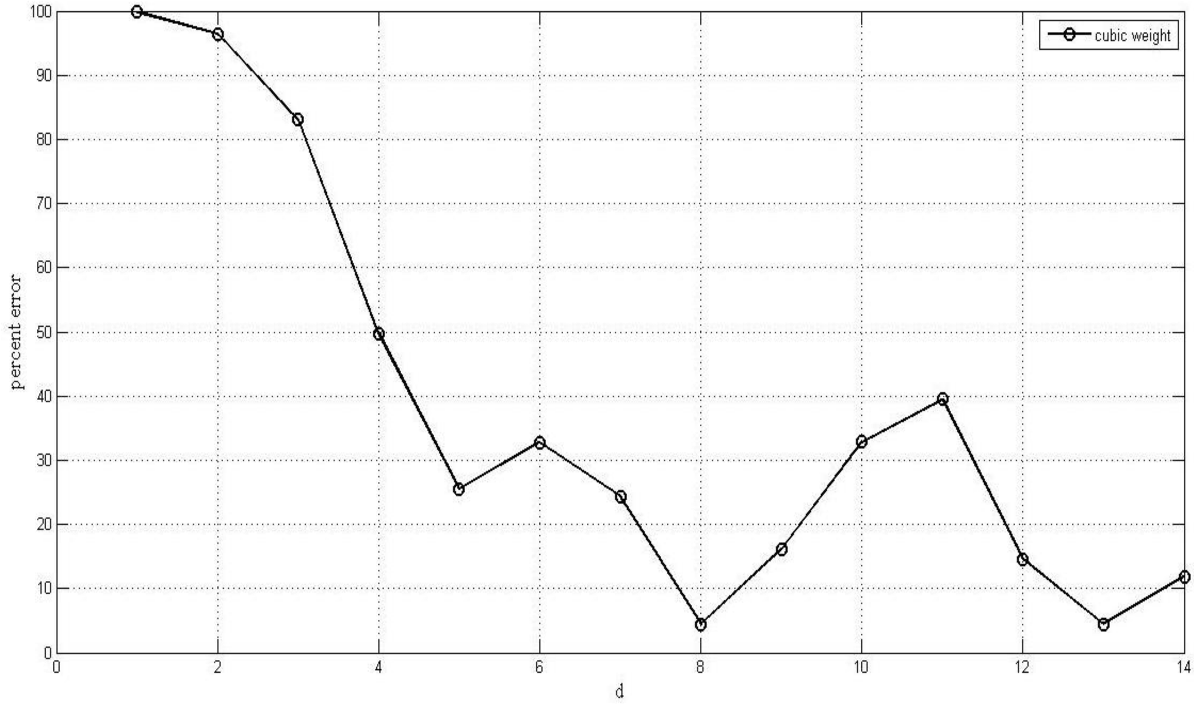

Fig. 5.5. Error in percentage vs ‘d’ for cubic weight case.

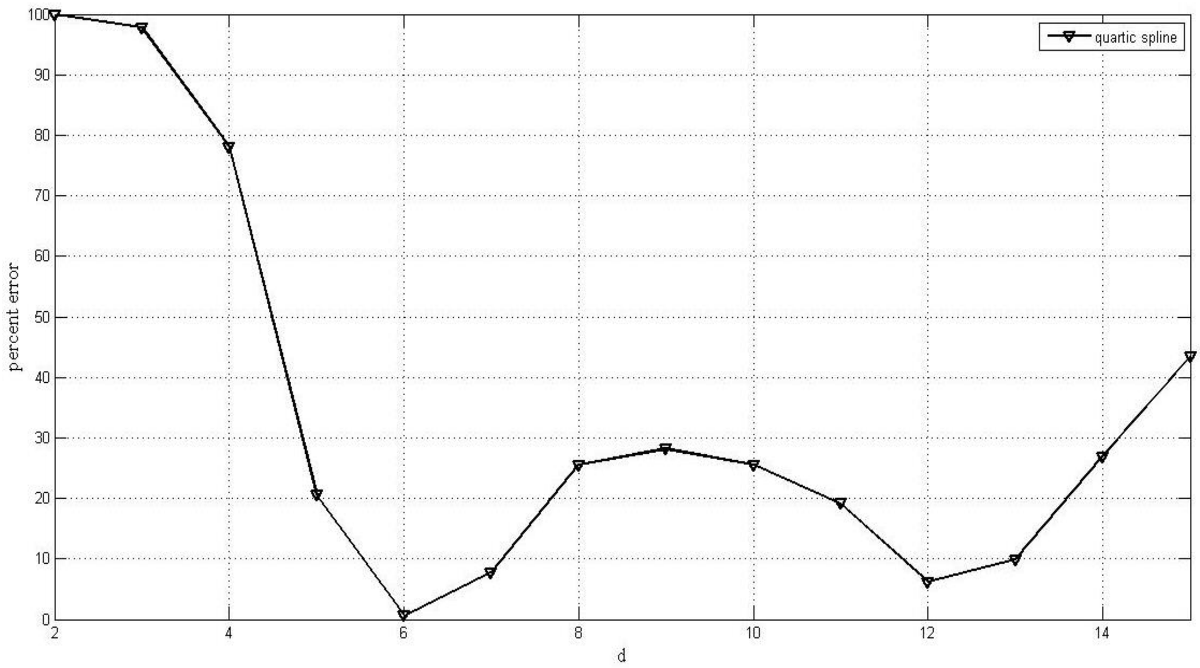

Fig. 5.6. Error in percentage vs ‘d’ parameter for quartic spline

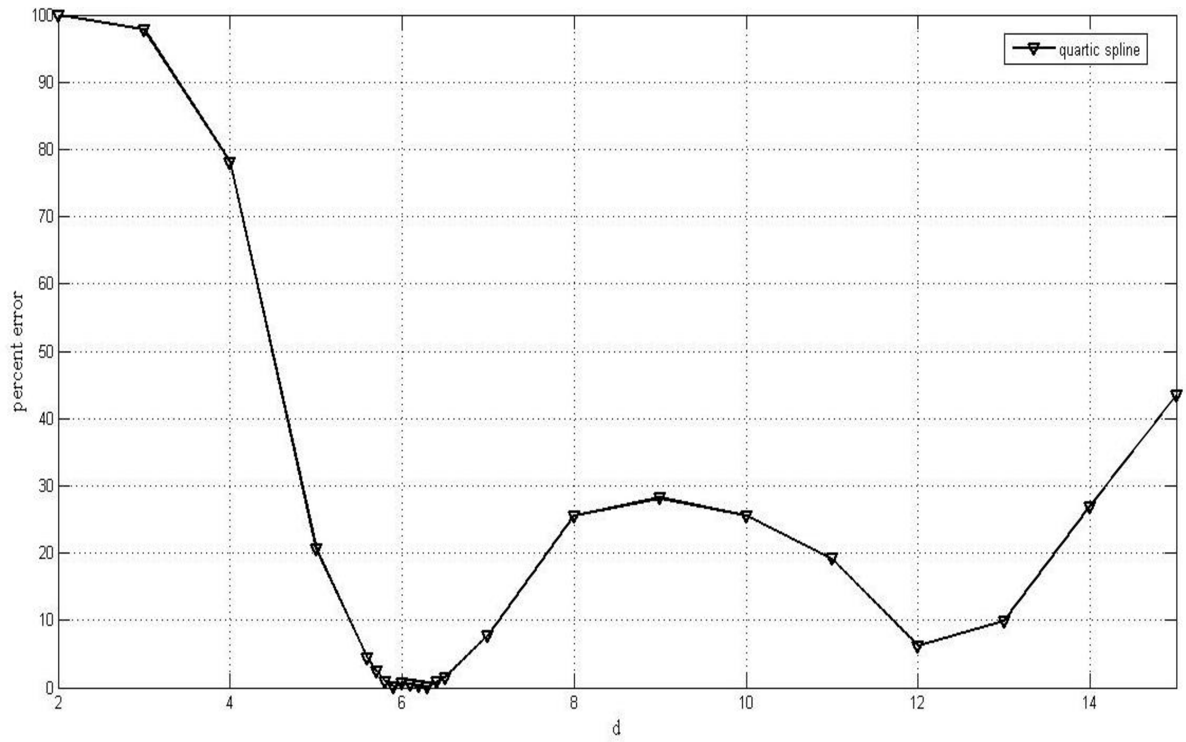

Fig. 5.7. Replot of Fig 5.5 with more points in the prospective optimal region

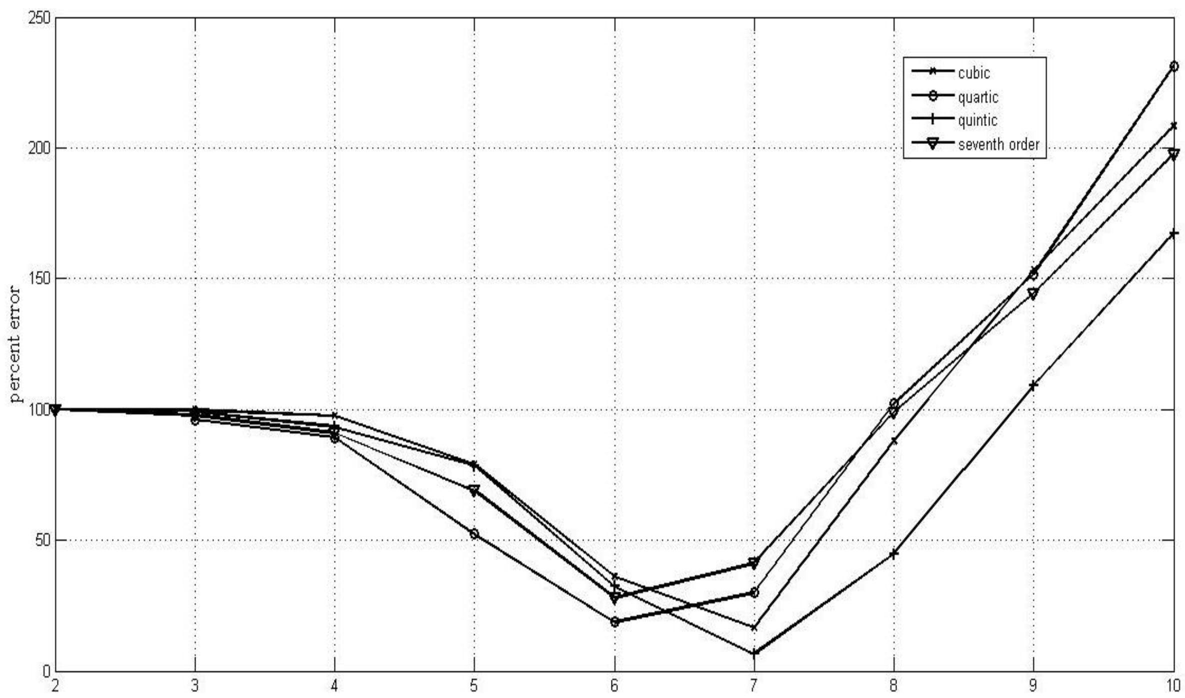

Fig. 5.8. Percent error vs 'd' for quadratic basis

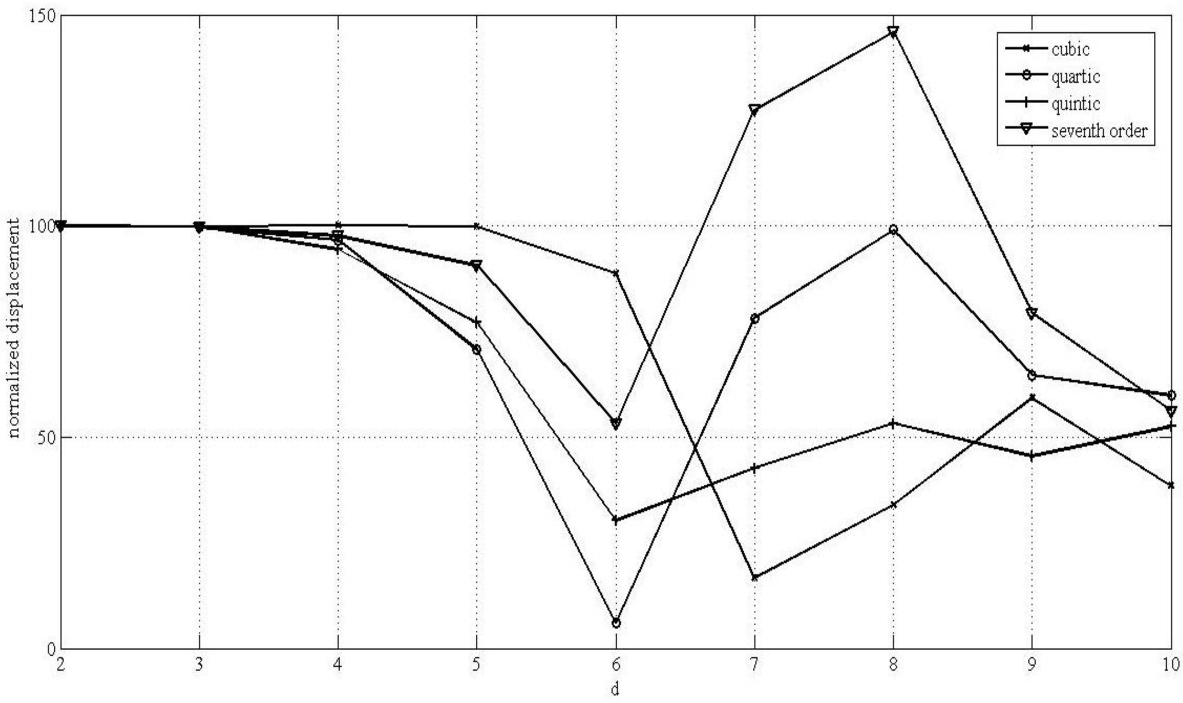

Fig 5.9. Percent error vs ‘d’ for cubic basis.

The graphs presented above clearly show that linear basis and quintic spline function are the best options for the case of isotropic plate. It can be seen that the higher bases interestingly deliver poor results. This may be due to the fact that higher the order, more the optimal ‘d’ would be, since more points are required to be within the domain. It can also be seen that even in the worst case, quintic weight performs better than other functions. Most importantly, quintic weight gives the most stable variation of result with ‘d’, that is the result almost converges after reaching the minimum ‘d’ required to obtain near exact result. This can’t be said of the other functions which fluctuate widely even after obtaining the near exact results once. Especially, the seventh order spline function performs poorly. This may be due to the fact that the first and second derivatives for seventh order spline are singular, which is strictly prohibited for thin plates and laminates (Chen X.L., 2003). This is the reason why the quintic spline weight function performs well but seventh order spline, though costs more time, fails to provide a consistent result. Hence the choice of the weight function is crucial for the efficiency of EFGM algorithm. The weight function should suit the continuity requirement of the plate formulation.

Fig. 5.5 shows the complex relationship between the ‘d’ and accuracy of result. The error value does not vary monotonously with the ‘d’. Rather there is a lot of fluctuation of the result with variation in the ‘d’. This is counter intuitive as one expects the d to increase monotonously with ‘d’ as it does for lower ‘d’ values. Also worth noting is the fact that the result varies even when ‘d’ is unrealistically large. In Fig. 5.5, the d values above 10 are meaningless since the number of elements in a direction is 10 and any increase of ‘d’ above it is not going to add any nodes to the domain of a point. Still, we can see that the result varies a lot. The reason for this strange behavior is that, though the number of nodes in the domain is same after a ‘d’ of 10, the weight function value at a given node varies. This is because the weight function is defined for the ratio of distance of a node from a point and the support size (i.e. ‘d’). This change in the value of weights assigned to the nodes influence the numerical integration result and also the MLS shape functions.

Fig. 5.6 and 5.7 show that if the ‘d’ is varied even in the range of second decimal point, change in result can be obtained. Thus it is obvious the support size is very crucial to the functioning of the EFGM. Also it is obvious that the polynomial basis, the choice of weight function etc. influence heavily the variation of result with ‘d’. It is also obvious that since the curves do not converge except a few cases, finding an optimal ‘d’ for a problem where we do not know the exact result may be extremely difficult.

5.3.2. RESULTS FOR ISOTROPIC PLATE

The result for the isotropic plate is validated with the exact analytical results provided by Timoshenko [1959].

Method used	Central Transverse Displacement(mm)
Exact (Timoshenko)	5.91126
EFG- linear basis, cubic weight	6.17410
EFG- linear basis, quartic weight	5.94995
EFG- linear basis, quintic weight	5.96184
EFG- linear basis, seventh order weight	5.90258

Table 5.3 Results for isotropic plate for various weight functions.

The results show that the current EFGM implementation has excellent compliance with the exact result. The result for linear basis has been presented as it was the best among the different bases used.

Mesh	Integration cells	Central Transverse Displacement(mm)
10 x 10	10 x 10	5.96184
20 x 20	10 x 10	5.96184

Table 5.4. Results for change in node density

Gauss Points	Central Transverse Displacement(mm)
4 x 4	6.17410
6 x 6	6.17199
8 x 8	6.17092
9 x 9	6.16646

Table 5.5. Results for change in quadrature rule

The tables 5.4 and 5.5 show that the use of a very fine mesh or a higher order quadrature rule alone does not guarantee better results. In fact, the improvement in the result is negligible in both cases. This indicates that the choice of a suitable basis and weight function for the problem in hand is far more important than discretization. The same behavior was observed for other bases and weight functions, the results of which have not been presented here. It was also observed that changing the mesh discretization caused slight or sometimes severe changes in the optimal support size. Also it was observed that placing the integration cells over the mesh is far more convenient than both being of different density, in which case two different support sizes have to be defined, further complicating the process. Hence, regular mesh and coinciding nodal mesh and integration mesh are found to be convenient.

5.4. ANALYSIS OF LAMINATES

The formulation for laminates is slightly different from that of isotropic plates. Most importantly, the presence of curvature terms in the Classical laminate theory formulation makes it necessary to have a higher order of continuity. Also it is observed that this behavior along with other factors that distinguish this problem from the isotropic plate problem causes changes to the effect the parameters have on the result. Hence a detailed parametric study is conducted for the laminate static analysis problem. The results of the same are presented in this section. The problem statement is as follows-

Specifications	
E1 (Gpa)	250
E2 (Gpa)	10
ν 12	0.25
ν 21	0.01
G12 (Gpa)	11.538
q (kN/m^2)	100

Geometry	
L=B	20 m
D	0.2 m
Simply supported all sides	

Table 5.6. Specifications for laminate plate

The plate specifications are as given in Belinha et al (2006). The parametric study and the results for laminates are discussed herby. The laminate is analyzed for different lay up schemes. The lay up schemes are as given in Belinha et al (2006). The results are validated with the exact results as per Reddy [33] presented in Belinha et al (2006).

The list of laminate sequences used are-

0^0

$0^0/90^0/0^0$

$0^0/90^0/90^0/0^0$

$0^0/90^0/0^0/90^0/0^0$

$0^0/90^0/90^0/0^0/90^0/90^0/0^0$

5.4.1. PARAMETRIC STUDY FOR EFGM

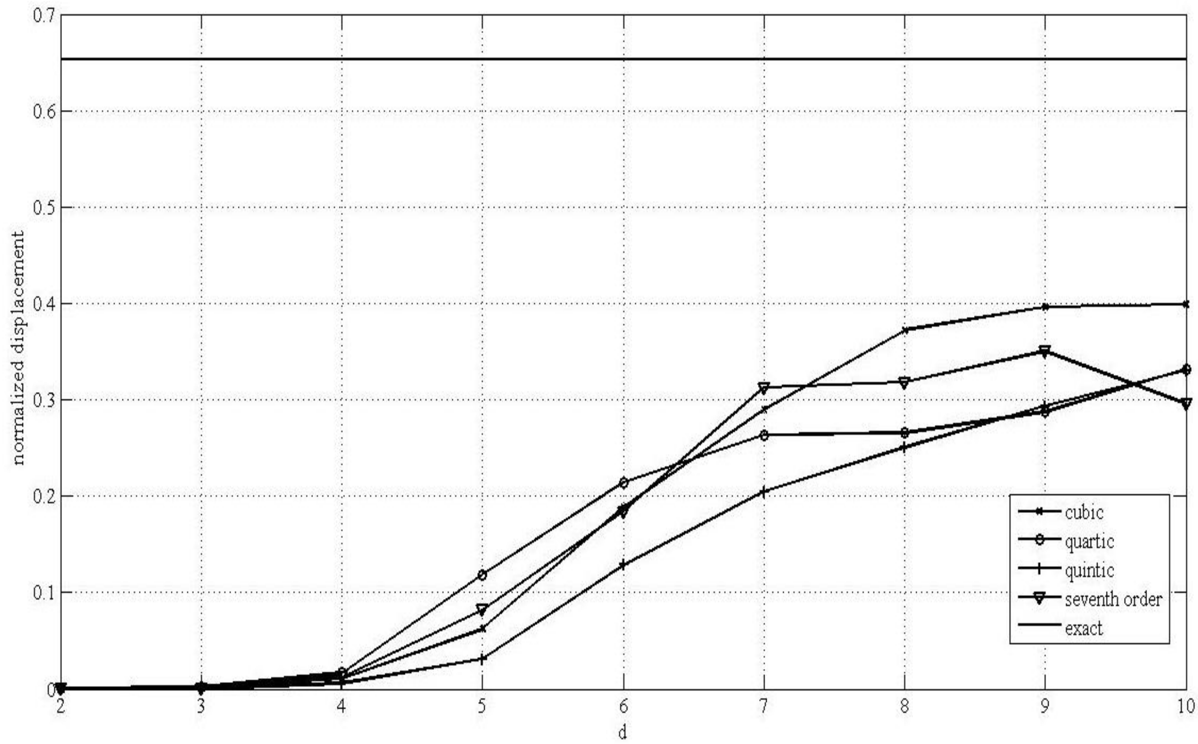

Fig.5.10. Effect of ‘d’ for linear basis

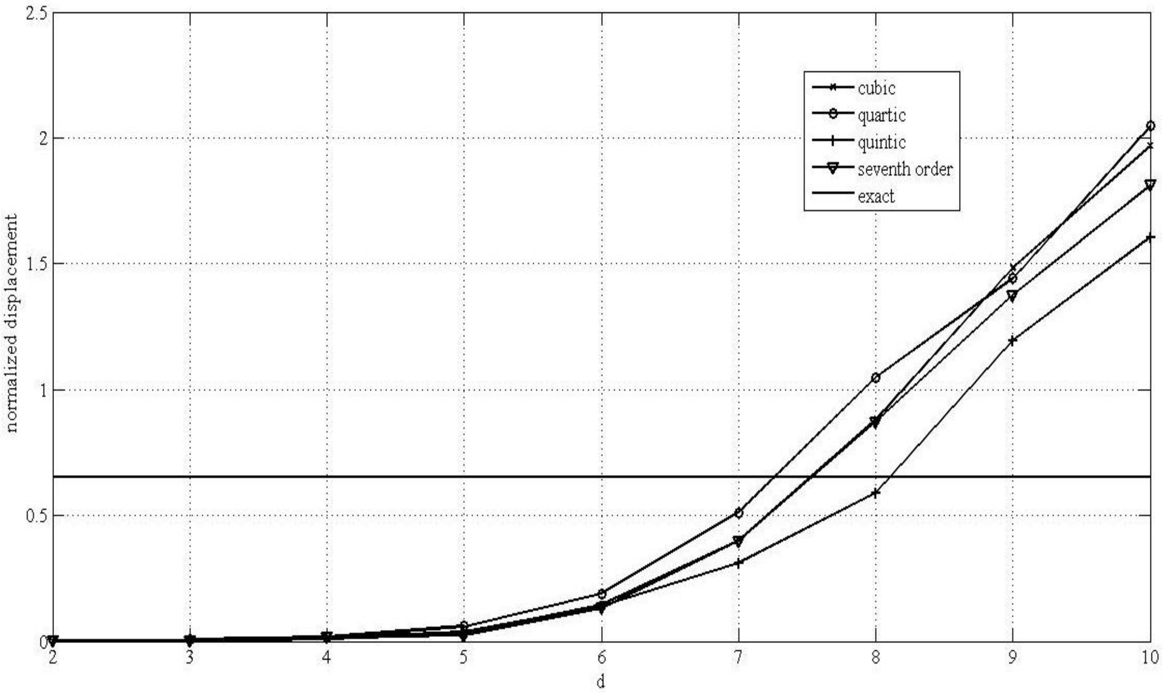

Fig. 5.11. Effect of ‘d’ for quadratic basis

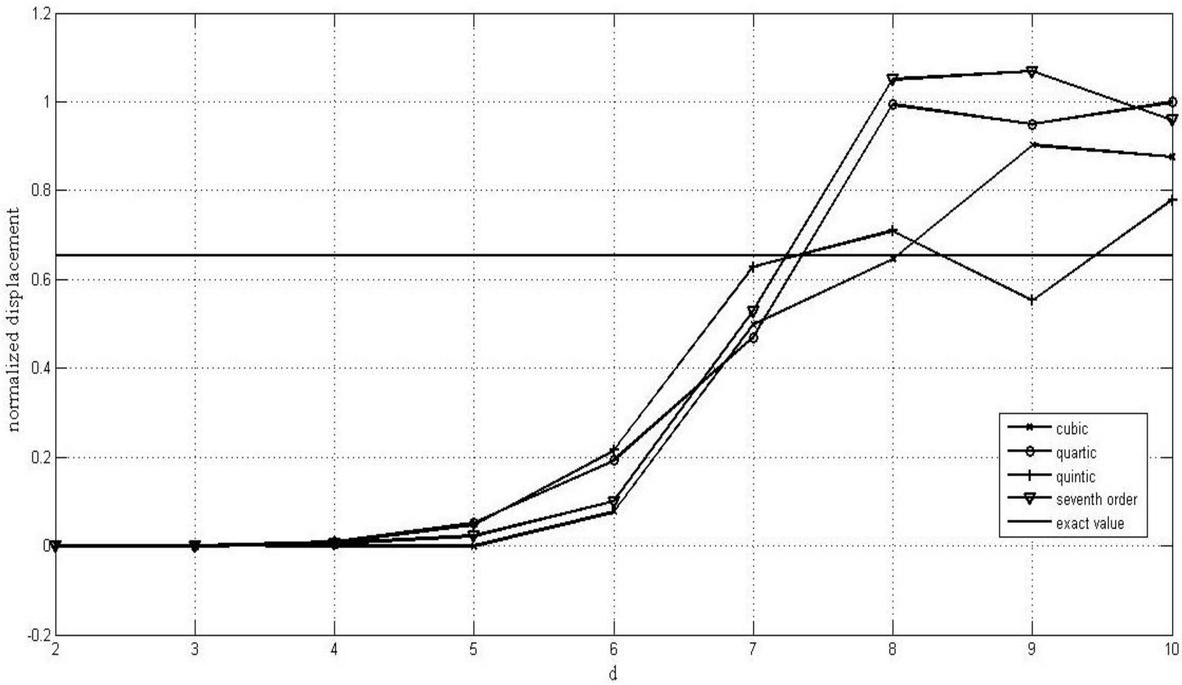

Fig. 5.12. Effect of ‘d’ for cubic basis.

The results in these graphs are for the one layer scheme. It can be observed that the linear basis which worked best for the thin plates performs badly for laminates. As discussed earlier, this is due to the fact that there are higher order derivatives of shape function involved and the shape function has to be C^2 continuous. The quadratic basis also behaves the same way and fails to give a good result, though it performs better than linear basis and is theoretically sufficient. The cubic basis performs best among the three and gives excellent results. Especially the combination of cubic basis and quintic function works very well and is stable versus the support size change. As in the case of the plates, the increase in 'd' does not always cause monotonous change in results. There seems to be no clear relation between the 'd' and the displacement arrived at by EFGM.

Similar results have been obtained for other laminate stacking sequence too. The results of these are presented below

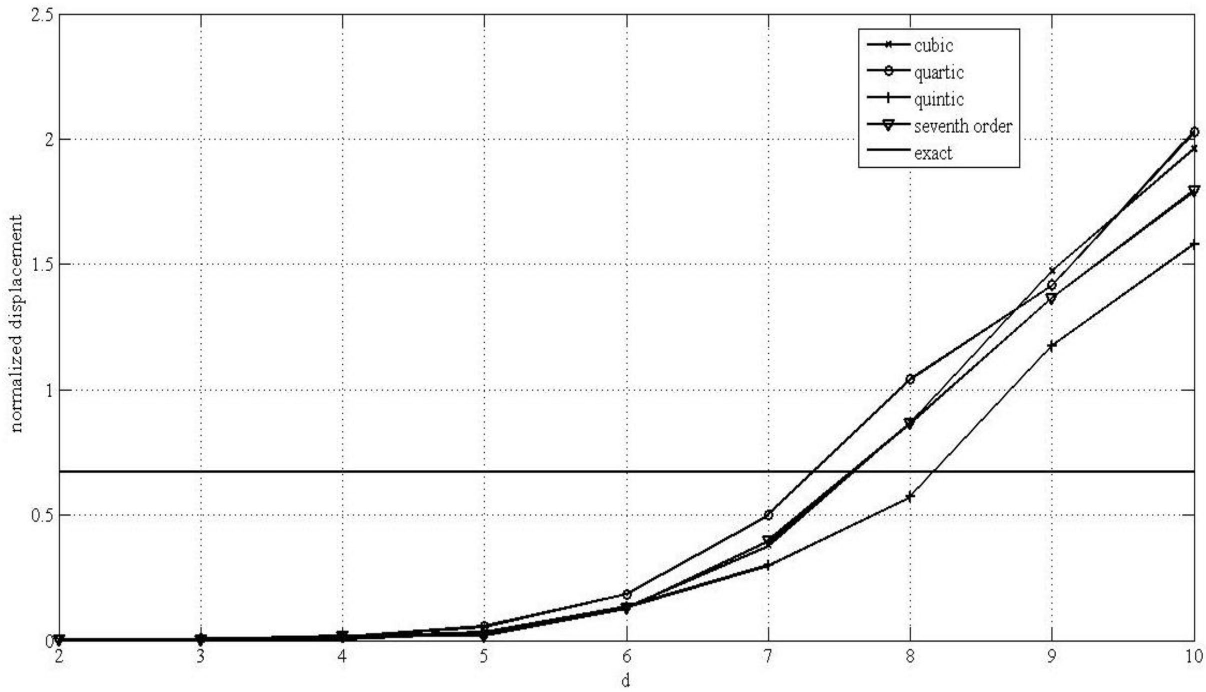

Fig. 5.13. Effect of 'd' for quadratic basis for 3 layer sequence

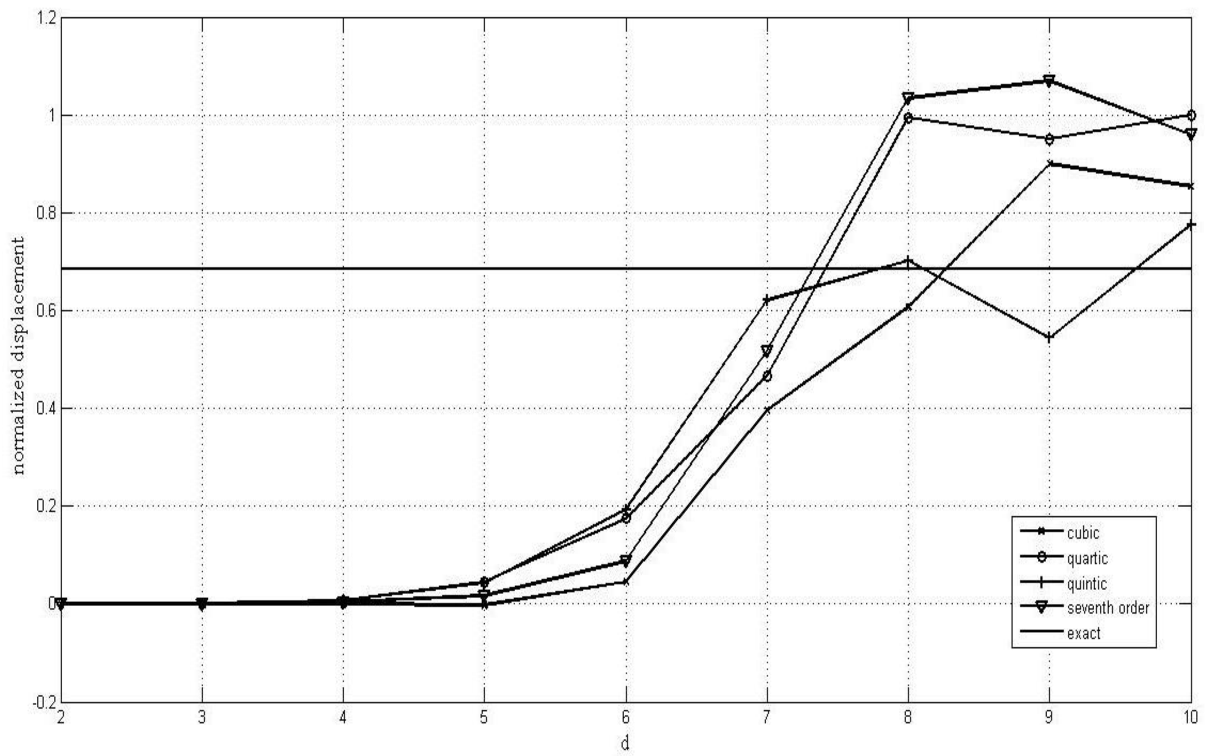

Fig. 5.14. Effect of ‘d’ for cubic basis and 3 layer sequence

The figures 5.13 and 5.14 are similar to those obtained for the one layer scheme. This denotes that the plate stacking sequence does not have an impact on the optimal support size parameter. This is significant because once an optimal ‘d’ value is found for a laminate of particular specifications, the result can be applied to any laminate sequence of similar specifications.

The graphs also show that the seventh order function as applied by Belinha et al (2006) do not give consistent and accurate results as suggested in their work. It may be reasoned that such a weight function may be useful in some areas like elimination of shear locking etc., but it shows poorer performance the quintic function.

5.4.2. RESULTS FOR STATIC ANALYSIS BY EFGM

The central transverse deflections for various cases of stacking sequences are presented here. The EFGM results match well with the analytical results.

	Exact	Present EFG
0^0	0.6528	0.6465
$0^0/90^0/0^0$	0.6697	0.6074
$0^0/90^0/90^0/0^0$	0.6833	0.6923
$0^0/90^0/0^0/90^0/0^0$	0.6874	0.6878
$0^0/90^0/90^0/0^0/90^0/90^0/0^0$	0.6896	0.6835

Table 5.7. Results for optimal ‘d’=8 for different lay up schemes for h/t=1/100

It is obvious that the current formulation gives excellent results for all the cases. It must be noted that the optimal ‘d’ was taken as 8. However if we take upto two decimal points and then find the optimal ‘d’, the result would be still be close to the exact results. For example for the 1 layer scheme and the ‘d’ of 8.3, the result obtained is 0.6521 which is very close to the exact result. Hence with little more effort in choosing optimal ‘d’, still closer results could be achieved. Since choosing such a subtle parameter may not be convenient always, table 5.7 provides results for a ‘d’ of 8. Since the current formulation was based on computationally efficient Kirchoff’s plate theory, it seems to be successful in matching theoretical results while costing lesser computer time.

CONCLUSION

The present work has considered the response of plates and laminates under plate bending and plane stress conditions. The analysis has been conducted using EFGM. Though the analysis of plates and laminates has been long done successfully using FEM, this work has its significance in the fact that this forms the basis for future extension to crack propagation and fracture mechanics problems of laminate where the innate uniqueness of meshfree methods would be fully harnessed. Also it must be understood that the actually difficult implementation of Kirchoff's plate theory has been handled without much trouble using EFG method. The relative ease with which this was achieved as compared to the FEM shows the superiority of meshfree approaches and the latent potential of these methods. The work also underscores the importance of understanding the effect of the parameters in EFGM. The tailoring of the bases, weight functions, domain size, quadrature rule, nodal density etc. are very crucial to the proper performance of the algorithms. The present work also highlights that the laminate plate analysis is more complex than the isotropic plate analysis. Crucially, the order of derivatives required in CLT is higher than the Kirchoff's plate theory. However, using an appropriate weight function easily solves this issue. Arbitrary completeness and continuity can be achieved by using a suitable weight function.

The conclusions derived from current work are summarized as follows-

1. The EFGM provides reasonably accurate results for all the problems considered- plane stress, plate bending, and laminate bending.
2. The parameters of EFGM play a crucial role in determining the efficiency of the algorithm. The present work analyzed the major trends of variation caused by these parameters and possible lines along which they may be optimized.

3. The basis function and the weight function, which influence the trial and test function, are the most important parameters which affect the result. They have to be chosen in accordance to the theory employed for analysis.
4. The laminates needed a larger domain than the plate problem which in turn needs larger domain than the plane stress problem. This is due to the order of approximation required in each case. Higher the order, more the minimum number of neighbouring nodes required to obtain a decent approximation is.
5. The error in approximation does not simply decrease with the increase in 'd'. The relation between these two is somewhere arbitrary.

FUTURE SCOPE OF RESEARCH

1. The optimal 'd' parameter must be studied in depth to find possible ways to find the optimal parameter value even when one does not know the exact result for comparison.
2. The application of higher order laminate theories can extend the formulation presented to thicker plates. The formulation can also be extended to stiffened plates.
3. Dynamic analysis and 3D analysis can be conducted on laminates as an extension of current work.
4. Problems like stress concentration, delamination study, etc can be conducted on the composites to lead to further studies in crack propagation.

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