ANALYSIS OF GYROSCOPIC EFFECTS IN ROTOR-DISC SYSTEMS

A thesis submitted to National Institute of Technology, Rourkela in partial fulfilment

for the degree of

Master of Technology
in
Mechanical Engineering

by

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June - 2013

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CERTIFICATE

This is to certify that the thesis entitled, "**Analysis of Gyroscopic Effects in Rotor-Disc Systems**", which is submitted by **Mr. Gaurav Maurya** in partial fulfilment of the requirement for the award of degree of M.Tech in Mechanical Engineering to **National Institute of Technology, Rourkela** is a record of candidate's own work carried out by him under my supervision. The matter embodied in this thesis is original and has not been used for the award of any other degree.

Date: 03/06/2013
Rourkela

Dr. H. Roy
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ABSTRACT

This work deals with study of dynamics of a viscoelastic rotor shaft system, where Stability Limit of Spin Speed (SLS) and Unbalance Response amplitude (UBR) are two indices. The Rotor Internal Damping in the system introduces rotary dissipative forces which is tangential to the rotor orbit, well known to cause instability after certain spin speed. There are two major problems in rotor operation, namely high transverse vibration response at resonance and instability due to internal damping. The gyroscopic stiffening effect has some influence on the stability. The gyroscopic effect on the disc depends on the disc dimensions and disc positions on the rotor. The dynamic performance of the rotor shaft system is enhanced with the help of gyroscopic stiffening effect by optimizing the various disc parameters (viz. disc position and disc dimension). This optimization problem can be formulated using Linear Matrix Inequalities (LMI) technique. The LMI defines a convex constraint on a variable which makes an optimization problem involving the minimization or maximization of a performance function belong to the class of convex optimization problems and these can incorporate design parameter constraints efficiently. The unbalance response of the system can be treated with $H\infty$ norm together with parameterization of system matrices. The system matrices in the equation of motion here are obtained after discretizing the continuum by beam finite element. The constitutive relationship for the damped beam element is written by assuming a Kelvin – Voigt model and is used to obtain the equation of motion. A numerical example of a viscoelastic rotor is shown to demonstrate the effectiveness of the proposed technique.

ACKNOWLEDGEMENTS

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As for Girish Kumar Sahu, Mallikarajan Reddy, Kamal Kumar Basumatary, Rakesh Kumar Sonkar, Akhileshwar Singh and Abhinav Deo: You were there for me when really needed and I am yours forever.

Gaurav Maurya

NOMENCLATURE

l	**Length of an Element**
m_d	**Disc Mass**
r_i	**Inner Radius of an Element**
r_o	**Outer Radius of an Element**
s	**Axial Position along an Element**
t	**Time in Seconds**
T	**Kinetic Energy**
P	**Potential Energy**
I_D	**Element Diametral Inertia Per Unit Length**
I_p	**Element Polar Inertia Per Unit Length**
V, W	**Translations in Y and Z Directions**
R_o	**Position Vector of Displaced Centre of Rotation**
E	**Young's Modulus of Elasticity**
M_Y, M_Z	**Bending Moment about Y and Z Axes**
θ, ϕ	**Angles of Rotation about Y and Z Axes**
ζ	**Spin Angle**
Ω	**Spin Speed**
ω	**Whirl Speed**

$\alpha = \Omega/\omega$	**Whirl Ratio**
$\omega_a, \omega_b, \omega_c$	**Angular Rate Components of Cross-Section about Fixed Frame**
μ	**Element Mass Per Unit Length**
η_v	**Viscous Damping Coefficient**
$\{q\},\{p\}$	**Displacement Vectors relative to Fixed and Rotational Frame of Reference**
$\{Q\},\{P\}$	**External Force Vectors relative to Fixed and Rotational Frame of Reference**
$[M_T]$	**Translatory Mass Matrix**
$[M_R]$	**Rotary Mass Matrix**
$[M]=[M_T]+[M_R]$	**Total Mass Matrix of an Element**
$[G],[K]$	**Gyroscopic and Stiffness Matrices**
$[R]$	**Orthogonal Rotation Matrix**
$[\hat{M}],[\hat{G}],[\hat{K}]$	**Transformed Mass, Gyroscopic and Stiffness Matrices**
$[I]$	**Identity Matrix**
$[K_B]$	**Bending Stiffness Matrix**
$[K_C]$	**Skew Symmetric Circulatory Matrix**
$[\Psi]$	**Shape Function Matrix for Displacements**
$[\Phi]$	**Shape Function Matrix for Rotation**

CONTENTS

LIST OF FIGURES

Chapter 1

INTRODUCTION

1.1 Background and importance

By the ISO definition, a rotor is a body suspended through a set of cylindrical hinges or bearings that allow it to rotate freely about an axis fixed in space. The non-rotating supporting structure is defined as a stator. Rotordynamics may be defined as a specialised branch of applied mechanics dealing with the dynamics and stability of rotating machinery. In our daily life, we frequently and unconsciously experience rotordynamics; in the engines of aircraft jets, automobile, the pumps used in household appliances, the drum of the washing machine, computer hard disc drives, spindles of machine tools etc.

As the rotation speed increases, the amplitude of vibration often passes through a maximum that is called a critical speed and corresponding zone is called as resonance. If the amplitude of vibration at a critical speed is excessive catastrophic failure can occur. Moreover, another phenomenon occurs quite often in rotating machinery: instability. Rotors may develop unstable behaviour after certain spin speed. The centrifugal force causes, in some cases, an unbounded growth of amplitude of vibrations in time. The ranges of rotation speeds in which self-excitation occurs are called the instability fields. These so-called self-excited vibrations can result in catastrophic failure.

This is why accurate modelling of the rotordynamic behaviour is crucial in the design of rotating machinery in order to improve product reliability, to increase process efficiency to prolong machinery life and enable safe operation. Developments in computer technology and numerical methods such as Finite Element Method (FEM) have made more accurate and detailed rotordynamic analysis possible. Nowadays, extensive studies are available in the literature about the modelling and testing of the effects of different physical phenomena on rotating machineries.

Rotordynamics analysis date backs to the second half of the nineteenth century because of the necessity of including the rotation speed into dynamic analysis as the rotational speed of many machine elements started to increase. Many scientists tried to provide a theoretical explanation of the rotordynamic behaviour. In 1919, Jeffcott introduced his paper about the dynamic analysis of rotors and presented a simple model that consists of a point mass attached to a massless shaft.

Even though the Jeffcott rotor is much simpler then the real-life rotors, it still provides insight into important phenomena in rotordynamics. But for precise analysis of complex machines such as steam and gas turbines, compressors, pumps, etc., more advanced models are required.

1.2 Analysis Objective

By the turn of 20th century, turbine manufacturers had started to design and operate rotors supercritically. That was only after DeLaval's experiment demonstration of the safely sustained supercritical operation of a steam turbine. His experiment refuted Rankine's hypothesis that rotors, modelled with no Coriolis component, cannot be stable if operated over a speed above the critical one.

No sooner did turbine manufacturers start operating supercritical rotors than they encountered severe vibrations that were at first related to imbalance. The industry was bewildered by the successful supercritical operations of some units but not others of similar constructions. A few researchers hypothesised on the possibility of Rotor Internal Damping (RID) or Material Damping of rotor system being the cause but took it no further until General Electrics (GE) severe problems with their blast furnaces' compressors. In 1924, Kimball came to then apparently illogical reasoning that RID caused such instability as it induced a follower force that is tangential to the rotor's orbit, acts in its direction of precession and increases in magnitude with speed of the rotor. He then argued that this force could overcome the stabilising external damping forces at a certain supercritical onset speed, thereby rendering the rotor unstable.

RID or material damping was the first recognised cause of instability and oil-whip followed shortly after. The material damping (RID) in the rotor shaft introduces rotary dissipative force which is tangential to the rotor orbit, well known to cause instability after certain spin speed. Thus high speed rotor operation suffers from two problems viz. 1) high transverse response due to resonance and 2) instability of the rotor-shaft system over a spin-speed. Both phenomena occur due to material inherent properties and set limitations on operating speed of a rotor. By using light weight and strong rotor, the rotor operating speed can be enhanced. These two parameters have some practical limitations. In other words, the gyroscopic stiffening effect has some influence on the stability. The gyroscopic effect on the disc depends on the disc dimensions and disc position on the rotor. Thus, the proper positioning of the discs and optimised dimensions may ensure high speeds and maximum stability. Optimization of a structural design within the constraints imposed by machine functionality and physical feasibility can minimize the circulatory effects, transmitted noise and vibration, reduce machine wear, and reduce the likelihood of premature failure. A number of researchers have developed numerical methods for optimizing the structural design of a rotor system subject to dynamic performance constraints, with particular focus on critical speed locations.

This work has used finite element method as the basis for assessing vibration behaviour. It reports on a different approach to structural design optimization, where objectives for dynamic performance are formulated as a set of linear matrix inequalities (LMIs) that directly incorporate the design variables to be optimized. Linear matrix inequalities are being increasingly used in the analysis and control of dynamic systems as there are fast and efficient numerical algorithms to solve them. Multiple LMIs relating to performance, stability, or parameter constraints can be combined to form a single LMI, which can be solved using the same generic-algorithms. This flexibility means that LMIs can be used to solve a wide range of optimization problems and create design specifications concerning vibration amplitudes, stability, critical speeds, modal damping levels, and parameter constraints. Moreover, multiple criteria can be combined without destroying the underlying mathematical form of the optimization problem or the algorithm required to solve it. Another advantage of the LMI formulation is that it can deal effectively with uncertain or time varying parameters, particularly those arising from speed dependent dynamics.

1.3 Thesis Outline

Chapter 2 gives a brief history of rotordynamics which is followed by a brief overview of the development of dynamics of rotor shaft systems. It discusses the various rotor models and stability study of systems under various internal and external effects. Then, an overview of various optimisation techniques used in recent past and present. Chapter 3 develops the equation of motion for a viscously damped rotor system. It discusses the finite element modelling of the system which is later used in the optimisation of disc parameters to ensure high stability for a specific configuration of the system. It discusses the Linear Matrix Inequalities (LMIs) technique for optimisation of disc dimensions. Chapter 4 discusses design of a disc for a rotor disc system and a theoretical method to obtain proper disc positions ensuring high working stability for the system. Finally, the conclusions, future scope of the work and references are presented in chapters 5and 6, respectively.

Chapter 2

OVERVIEW OF AVAILABLE LITERATURE

2.1 History

Rotordynamic studies related to technological applications date back to the second half of the nineteenth century, when the increase of the rotational speed of many machine elements made it necessary to include rotation into the analysis of their dynamic behaviour. However, the dynamics of rotating systems, as far as rigid rotors are concerned, was already well understood and the problem of the behaviour of the spinning top had been successfully dealt with by several mathematicians and theoretical mechanicists.

The paper which is considered to be the first paper fully devoted to Rotordynamics is, on the centrifugal force on rotating shafts, published in The Engineer by Rankine [29]. It correctly states that a flexible rotating system has a speed, defined by the author as critical speed, at which very large vibration amplitudes are encountered. However, the author incorrectly predicts that stable running above the critical speed is impossible.

Earlier attempts to build turbines, mainly steam turbines, at the end of nineteenth century led to rotational speeds far higher than those common in other fields of mechanical engineering. At these speeds, some peculiar dynamic problems are usually encountered and must be dealt with to produce a successful design. De Laval had to solve the problem correctly understanding the behaviour of a rotor running at speeds in excess of critical speed, i.e., in supercritical conditions, while designing his famous cream separator and then his steam turbine.

A theoretical explanation of supercritical running was supplied first by Foppl [21], Belluzzo [9], Stodola [6] and Jeffcott [25] in his famous paper of 1919. Although the first turbine rotors were very simple and could be dealt with by using simple models, of the type now widely known as Jeffcott rotor, more complex machines required a more detailed modelling. Actually, although a simplified approach like the above-mentioned Jeffcott rotor can explain qualitatively many important features of real-life rotors, the most important being self-centring in supercritical conditions and the different roles of the damping of the rotor and of the nonrotating parts of the machine, it fails to explain other features, such as the dependence of the natural frequencies on the rotational speed. Above all, the simple Jeffcott rotor does not allow us to obtain a precise quantitative analysis of the dynamic behaviour of complex systems, e.g., those encountered in gas or steam turbines, compressors, pumps, and many other types of machines.

To cope with the increasing complexity of rotating systems, graphical computation schemes were devised. The availability of electromechanical calculators allowed us to develop tabular computational procedures, mainly based on the transfer matrices approach, which eventually substituted graphical computations. In particular, Holzer's method for the torsional vibrations of shafts and the Myklestadt-Prohl method for the computation of the critical speeds of turbine rotors were, and still are, widely used. These methods were immediately automatized when digital computers became available.

The wide diffusion of the finite element method (FEM) deeply influenced also the field of rotordynamics. Strictly speaking, usual general purpose FEM codes cannot be used for rotordynamic analysis owing to the lack of consideration of gyroscopic effects. It is true that a gyroscopic matrix can be forced in the conventional formulation and that several manufacturers use commercial FEM codes to perform rotordynamic analysis, but the rotordynamic field is one of these applications in which purposely written, specialized FEM codes can give their best. Through FEM modelling, it is possible to study the dynamic behaviour of machines containing high-speed rotors in greater detail and consequently to obtain quantitative predictions with an unprecedented degree of accuracy.

2.2 Dynamics of a Viscoelastic Rotor

Extensive studies are available in the literature about the modelling and testing of the effects of different physical phenomena on rotating machineries. Nelson and McVaugh [26] presented a procedure for dynamic modelling of rotor bearing systems which consisted of rigid discs, distributed parameter finite rotor elements, and discrete bearings. They presented their formulation in both a fixed and rotating frame of reference. They developed a finite element model including the effects of rotary inertia, gyroscopic moments, and axial load. The model was based on Euler-Bernoulli beam theory. Later, Zorzi and Nelson [34] extrapolated the same model for rotor with internal damping. Their model consisted of both viscous as well as hysteretic damping. They demonstrated that the material damping in the rotor shaft introduces rotary dissipative forces which are tangential to the rotor orbit, well known to cause instability after certain spin speed. Both forms of internal damping destabilise the rotor system and induce non-synchronous forward precession. This model is one of the best models that explain effects of internal damping and spin speed on the dynamic behaviour of the system in its full entirety.

Correct quantitative prediction is particularly important as the trends of technology toward higher power density, lower weight, and faster machines tend to make worse all problems linked with the dynamic behaviour of rotating machinery.

Higher speeds are often a goal in themselves, like in machine tools or other production machines in which spinning faster means directly increasing productivity. In applications related to power generation or utilization, a faster machine can develop or convert more power manipulating the same torque. As torque is usually the critical factor in dimensioning machine elements (shaft cross section, size of the conductors in electrical machinery, etc.), increasing the speed allows us to make power devices lighter. The use of materials able to withstand higher stresses allows us to reduce the mass and the size of machinery, but stronger materials (e.g., high strength steels or light alloys) are usually not stiffer and then these lighter machines are more compliant and more prone to vibrate and thus, causing them to become unstable.

Researchers like Gunter [23], Dutt and Nakra [19], Genta [3], and Lalanne and Ferraris [4], studied the stability of the system with internal damping. They obtained the results in form of Campbell diagrams and Decay Rate plots. Unbalance response and the threshold spin speed called Stability Limit of Spin Speed were taken as indices of stability. Bulatovic [11-14] performed a great deal of mathematical operations and obtained many theorems, necessary and sufficient conditions for stability of both conservative and non-conservative systems including gyroscopic effect. Gyroscopic effects enhance stability in a damped rotor which has been discussed by M. A. Kandil in his Doctoral Thesis [7].

With the invention of composite materials and their advantages over conventional material, their utilisation became frequent and now most of the machines working over supercritical speeds are made using composite material for their long lives. Accordingly, researchers were not only restricted to a mono-material system but they started analysing multi-material systems. Panda and Dutt [27-28] attempted to predict the frequency dependent material properties of polymeric supports for obtaining optimum performance of rotor-shaft systems i.e. to achieve simultaneously lowest synchronous unbalanced response amplitude as well as highest stability limit of the spin speed. Dutt and Roy [20] studied the stability of polymeric rotor systems, where the equation of motion in the finite element formulation is developed by using the operator based constitutive relationship.

2.3 Optimisation for High Performance

A number of researchers have developed numerical methods for optimizing the structural design of a rotor system subject to dynamic performance constraints, in order to obtain high stability with system running at high speeds. Early work by Bhavikatti [10] and Ranta [30] and co-researchers tackled the problem of minimizing the weight of a rotor subject to constraints on stresses and eigenvalues of the system, respectively. The design variables considered in these studies included the inner radius of hollow rotor sections, the positions of bearings and rigid disc elements, and the bearing stiffnesses. Chen and Wang [16] have tackled similar design optimization problems but used an iterative method to manipulate the eigenvalues of rotor vibration modes. In their study the outer diameter of rotor sections was varied, together with bearing stiffness and damping coefficients. Jafari et al. [24] aimed at finding an optimal disc profiles for minimum weight design using the Karush-Kunh-Tucker (KKT) method as classical optimisation method, Simulated Annealing (SA), and Particle Swarm Optimization (PSO) as two modern optimisation methods. They used the von Mises failure criterion of optimum disc as an inequality constraint to make sure that the rotating disc does not fail. Their result showed that KKT method gives a profile that is slightly less weight while the implementation of PSO and SA is easier and provide flexibility as compared to KKT. A study by Choi and Yang [17] considered using immune genetic algorithms to minimize rotor weight and transmitted bearing forces. Stocki et al. [32] found that the commonly observed nowadays tendency of weight minimization of rotor-shafts of the rotating machinery leads to decrease of shaft bending rigidity making a risk of dangerous stress concentrations and rubbing effects more probable. They aimed at determination of optimal balance between reducing the rotor shaft weight and assuring its admissible bending flexibility. The random nature of residual unbalances of the rotor shaft as well as randomness of journal bearing stiffness have been taken into account in the frame work of robust design optimization. Such a formulation of optimization problem leads to the optimum design that combines an acceptable structural weight with the robustness with respect to uncertainties of residual unbalance – the main source of bending vibrations causing the rubbing effects. They applied robust optimization technique based on Latin Hypercubes in scatter analysis of vibration response. The proposed method has been applied for the optimization of the typical single-span rotor shaft of the 8-stage centrifugal compressor. Further work by Shiau et al. [31] involved a two-stage optimization with a genetic algorithm to find initial values of design variables for further optimization. In their study, various parameter constraints were incorporated in an objective function using a Lagrange multiplier method.

Cole et al. [18] considered optimization of rotor system design using stability and vibration response criteria. Their study included the effects of certain design changes that can be parameterized in a rotor dynamic model through their influence on the system matrices obtained by finite element modelling. They derived a suitable vibration response measure by considering an unknown and axial distribution of unbalanced components having bounded magnitude. They showed that the worst case unbalanced response can be given by an absolute row sum norm of the system frequency response matrix. They minimized this norm through the formulation of linear matrix inequalities (LMIs) that were incorporated with design parameter constraints and the stability criteria. A case study was presented where the method was applied for the optimal selection of bearing support stiffness and damping levels to minimize the worst case vibration of a flexible rotor over a finite speed range. The LMIs are capable of dealing with non-linear systems and they can include design constraints directly. The LMI defines a convex constraint on a variable which makes an optimization problem involving the minimization or maximization of a performance function belong to the class of convex optimization problems and these can incorporate design parameter constraints efficiently. Multiple LMIs relating to performance, stability, or parameter constraints can be combined to form a single LMI, which can be solved using the same usual algorithms. This flexibility means that LMIs can be used to solve a wide range of optimization problems [1].

2.4 Summary

As can be seen from the literature survey a great deal of work has been done in analysing a rotating system both undamped and damped. Researchers established the stability of the system and recognised various factors affecting stability of a particular rotating system. Optimisation techniques have been utilised in order to improve dynamic performance of different configuration of systems. Rotor internal damping which is one of the main factors causing instability can be countered easily if a favourable gyroscopic effect can be maintained in the system. This gyroscopic effect will aid in stability and depends upon various factors such as dimensions of rotor system and positions of mountings such as gears, pulley, wheels, etc. Researchers have used many different techniques like Genetic Algorithm (GA), Particle Swarm Optimisation (PSO), etc., to optimise rotor parameters to obtain favourable working. But these optimisation techniques are cumbersome as they require obtaining both local and global minima and maxima, whereas Linear Matrix Inequalities (LMI) technique can efficiently handle non-linear systems. Many optimisation problems in control, identification and signal processing can be formulated (or reformulated) using Linear Matrix Inequalities. Since an LMI defines a convex constraint on the variable, the optimisation problems involving the minimisation or maximisation of a performance function belong to the class of convex optimisation problems. In Convex Optimisation, the local minima and global minima are same and in case of strictly convex function the minimum value of a function is unique. Thus, if the system can be reduced or formulated to a system of LMIs defining strict convex constraint on the variables, then, the problem remains to find the global optimum. The motivation here was the absentia of such methods which can be applied directly to systems to be controlled. The sole purpose of this piece of work is to represent the factors affecting stability of a system, optimisation of parameters to ensure high stability and a presentation of a less known technique of optimisation for efficient optimisation.

This work includes a theoretical technique to obtain disc positions where the system would be more stable and also, includes optimisation of disc dimensions using LMI technique for ensuring high stability.

Chapter 3

ROTOR SYSTEM MODELLING AND OPTIMISATION

The foundation of analysis is laid here. This chapter includes the Finite Element Modelling of a rotor system followed by optimisation of various disc parameters ensuring high stability of the system.

3.1 Equation of Motion of the System

The typical flexible rotor-bearing system to be analysed consists of a rotor composed of discrete discs, rotor segments with distributed mass and elasticity, and discrete bearings. Such a system is illustrated in Figure (3.1) along with the two reference frames that are utilized to describe the system motion.

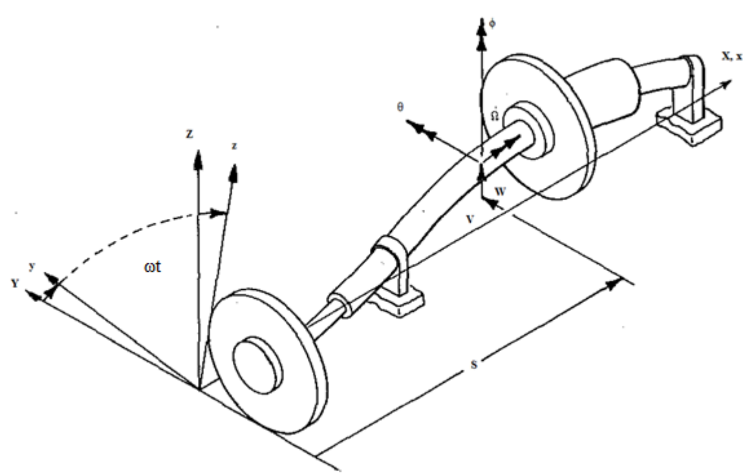


Figure 3.1 Typical rotor disc system configuration.

The XYZ triad is a fixed frame of reference and xyz triad is a rotating frame of reference with X and x being collinear and coincident with the undeformed rotor centre line. Rotating frame is defined with respect to fixed frame with a single rotation ωt about X axis with ω denoting the whirl speed.

A typical cross section of the rotor in a deformed state is defined relative to XYZ by the translations $V(s,t)$ and $W(s,t)$ in the Y and Z directions respectively to locate the elastic centreline and small angle rotations $\theta(s,t)$ and $\phi(s,t)$ about Y and Z respectively to orient the plane of the cross-section. The xyz triad is attached to the cross-section with the "x" axis normal to the cross-section. S is defined by the three successive rotations, illustrated in Figure (3.2),

- ϕ about Z defines a''b''c''
- θ about b'' defines a'b'c'
- ζ about a' defines xyz.

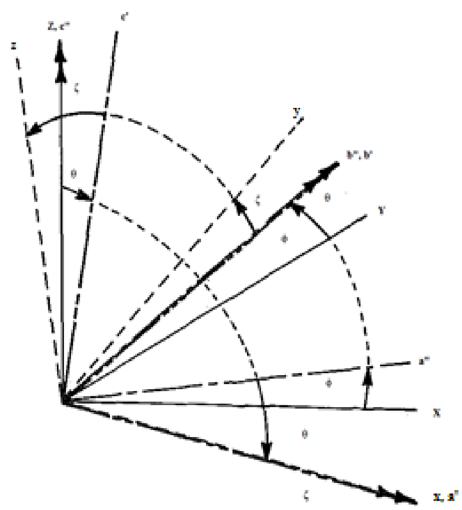


Figure 3.2 Cross section rotation angles

Then the angular velocities relative to XYZ are:

$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{bmatrix} -\sin\theta & 1 & 0 \\ \cos\theta\sin\zeta & 0 & \cos\zeta \\ \cos\theta\cos\zeta & 0 & -\sin\zeta \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ \dot{\zeta} \\ \dot{\theta} \end{Bmatrix} \quad (1)$$

For small deformations the (θ,ϕ) rotations are approximately collinear with the (Y, Z) axes respectively. The spin angle ζ, for a constant speed system and negligible torsional deformation, is Ωt, where Ω denotes the rotor spin speed. The displacements (V,W,θ,ϕ) of a typical cross-section relative to XYZ are transformed to corresponding displacements (v,w,β,γ) relative to xyz by the orthogonal transformation.

$$\{q\}=[R]\{p\} \tag{2}$$

with $$\{q\}=\begin{bmatrix} V \\ W \\ \theta \\ \phi \end{bmatrix},\{p\}=\begin{bmatrix} v \\ w \\ \beta \\ \gamma \end{bmatrix}\&[R]=\begin{bmatrix} \cos\omega t & -\sin\omega t & 0 & 0 \\ \sin\omega t & \cos\omega t & 0 & 0 \\ 0 & 0 & \cos\omega t & -\sin\omega t \\ 0 & 0 & \sin\omega t & \cos\omega t \end{bmatrix}$$

Their time first and second time derivative can be given as

$$\begin{aligned} \{\dot{q}\} &= \omega[S]\{p\}+[R]\{\dot{p}\} \\ \{\ddot{q}\} &= [R]\{\{\dot{p}\}-\omega^2\{p\}\}+2\omega[S]\{\dot{p}\} \end{aligned} \tag{3}$$

with

$$[S]=\frac{1}{\omega}[\dot{R}]=\begin{bmatrix} -\sin\omega t & -\cos\omega t & 0 & 0 \\ \cos\omega t & -\sin\omega t & 0 & 0 \\ 0 & 0 & -\sin\omega t & -\cos\omega t \\ 0 & 0 & \cos\omega t & -\sin\omega t \end{bmatrix}$$

Here the rotor-bearing system is considered to comprise a set of interconnecting components consisting of rigid discs, rotor segments with distributed mass and elasticity, and linear bearings. In this section the rigid disc equation of motion is developed using a Lagrangian formulation. The finite rotor element equation of motion is developed in an analogous manner by specifying spatial shape functions and then treating the rotor element as an integration of an infinite set of differential discs. The bearing equations are not developed and only the linear forms of the equations are utilized in this work.

3.1.1 Rigid Disc

The kinetic energy of a typical rigid disc with mass centre coincident with the elastic rotor centreline is given by the expression

$$T_d = \frac{1}{2}\begin{Bmatrix}\dot{V}\\ \dot{W}\end{Bmatrix}^T \begin{bmatrix} m_d & 0 \\ 0 & m_d \end{bmatrix}\begin{Bmatrix}\dot{V}\\ \dot{W}\end{Bmatrix} + \frac{1}{2}\begin{Bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{Bmatrix}^T \begin{bmatrix} I_d & 0 & 0 \\ 0 & I_d & 0 \\ 0 & 0 & I_p \end{bmatrix}\begin{Bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{Bmatrix} \tag{4}$$

With the aid of Eq. (3), Eq. (4) becomes

$$T_d = \frac{1}{2}\begin{Bmatrix}\dot{V}\\ \dot{W}\end{Bmatrix}^T \begin{bmatrix} m_d & 0 \\ 0 & m_d \end{bmatrix}\begin{Bmatrix}\dot{V}\\ \dot{W}\end{Bmatrix} + \frac{1}{2}\begin{Bmatrix}\dot{\theta}\\ \dot{\phi}\end{Bmatrix}^T \begin{bmatrix} I_d & 0 \\ 0 & I_d \end{bmatrix}\begin{Bmatrix}\dot{\theta}\\ \dot{\phi}\end{Bmatrix} - \dot{\zeta}\dot{\phi}\theta I_p \tag{5}$$

The Lagrangian equation of motion of the rigid disc using above equation and the constant spin speed restrictions, $\zeta = \Omega t$, is

$$\left(\left[M_T^d\right]+\left[M_R^d\right]\right)\left\{\ddot{q}^d\right\} - \Omega\left[G^d\right]\left\{\dot{q}^d\right\} = \left\{Q^d\right\} \tag{6}$$

The preceding equation is the equation of motion of the rigid disc referred to the fixed frame of reference with the forcing term including mass unbalance, interconnection forces, and other external disc effects on the disc.

By using Eqs. (2-3) and pre-multiplying by $\left[R\right]^T$, Eq. (6) transforms to

$$\left(\left[M_T^d\right]+\left[M_R^d\right]\right)\left\{\ddot{p}^d\right\} + \omega\left(2\left(\left[\hat{M}_T^d\right]+\left[\hat{M}_R^d\right]\right) - \alpha\left[G^d\right]\right)\left\{\dot{p}^d\right\} - \omega^2\left(\left(\left[M_T^d\right]+\left[M_R^d\right]\right) + \alpha\left[\hat{G}^d\right]\right)\left\{p^d\right\} = \left\{P^d\right\} \tag{7}$$

For the case of thin disc ($I_P = 2I_d$), Eq. (7) becomes

$$\left(\left[M_T^d\right]+\left[M_R^d\right]\right)\left\{\ddot{p}^d\right\} + \omega\left(2\left[\hat{M}_T^d\right] + (1-\alpha)\left[G^d\right]\right)\left\{\dot{p}^d\right\} - \omega^2\left(\left[M_T^d\right] + (1-2\alpha)\left[M_R^d\right]\right)\left\{p^d\right\} = \left\{P^d\right\} \tag{8}$$

The Eq. (8) is the equation of motion of a rigid disc referred to rotating frame with whirl ratio $\alpha = \Omega/\omega$.

3.1.2 Undamped Flexible Shaft

A typical finite rotor element is illustrated in Figure (3.3). The coordinates $\left(q_1, q_2, q_3, \ldots\ldots, q_8\right)$ are the time dependent end point displacements (translations and rotations).

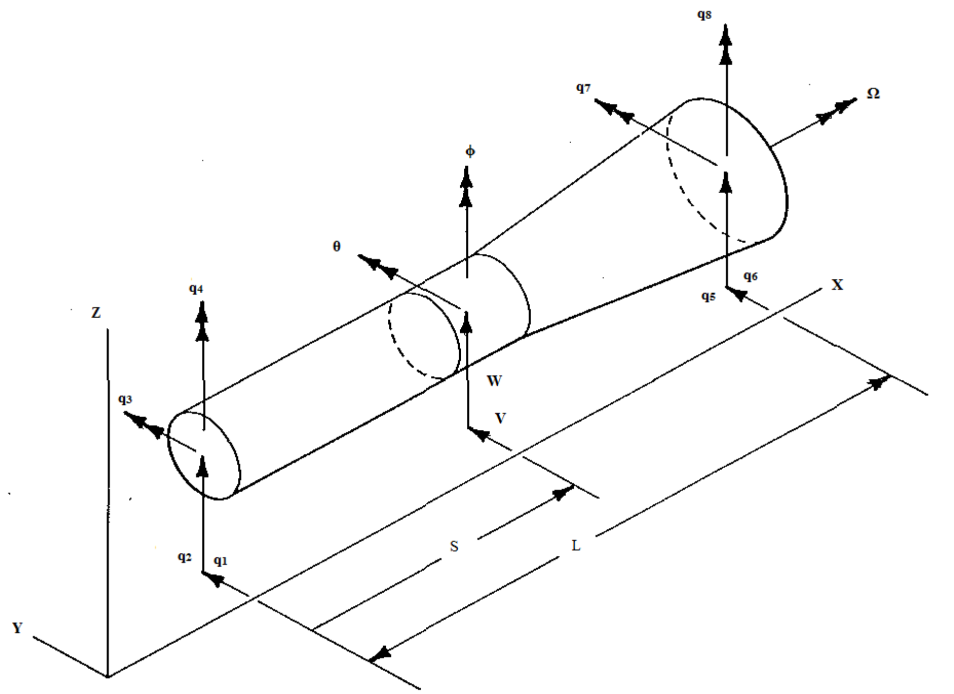


Figure 3.3 Finite rotor elements and coordinates

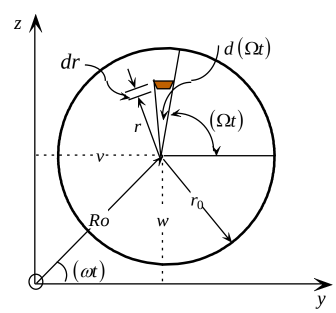


Figure 3.4 Displaced position of the shaft cross-section

Figure 3.4 shows the displaced position of the shaft cross section. An infinitesimal shaded area of differential radial thickness dr at a distance r (where r varies from 0 to r_0) subtending an angle $d(\Omega t)$ where Ω is the spin speed in rad/sec and Ωt varies from 0 to 2π .

It should be noted that here the element time dependent cross section displacements (V,W,θ,ϕ) are also functions of position(s)along the axis of the element. The rotations (θ,ϕ) are related to the translations (V,W) by the equations

$$\theta=\frac{\partial W}{\partial s} \quad \text{and} \quad \phi=\frac{\partial V}{\partial s} \tag{9}$$

The translation of a typical point internal to the element is chosen to obey the relation

$$\begin{Bmatrix} V(s,t) \\ W(s,t) \end{Bmatrix}=[\psi(s)]\{q(t)\} \tag{10}$$

where $\psi(s)$ is the Hermite shape function, which is spatial constraint matrix of displacement functions and is given by

$$[\psi(s)]=\begin{bmatrix} \psi_1 & 0 & 0 & \psi_2 & \psi_3 & 0 & 0 & \psi_4 \\ 0 & \psi_1 & -\psi_2 & 0 & 0 & \psi_3 & -\psi_4 & 0 \end{bmatrix} \tag{11}$$

In this case the individual functions represent the static displacement modes associated with a unit displacement of one of the end point coordinates with all others constrained to zero. These functions are

$$\psi_1=1-3\left(\frac{s}{l}\right)^2+2\left(\frac{s}{l}\right)^3, \psi_2=\left[1-2\left(\frac{s}{l}\right)+\left(\frac{s}{l}\right)^2\right], \psi_3=3\left(\frac{s}{l}\right)^2-2\left(\frac{s}{l}\right)^3$$

And $$\psi_4=l\left[-\left(\frac{s}{l}\right)^2+\left(\frac{s}{l}\right)^3\right]$$

From Eqs. (9-10) the rotations can be expressed in the form $\begin{Bmatrix} \theta \\ \phi \end{Bmatrix}=\left[\Phi(s)\right]\left\{q(t)\right\}$ where $\Phi(s)$ represents a matrix of rotation shape functions and is given as

$$[\Phi(s)]=\begin{bmatrix} 0 & -\psi_1' & -\psi_1' & 0 & 0 & -\psi_3' & \psi_4' & 0 \\ \psi_1' & 0 & 0 & \psi_2' & \psi_3' & 0 & 0 & \psi_4' \end{bmatrix} \tag{12}$$

For a differential disc located at ‘s’ the elastic bending and kinetic energy expressions are respectively,

$$dP = \frac{1}{2}\begin{Bmatrix} V'' \\ W'' \end{Bmatrix}^T \begin{bmatrix} EI & 0 \\ 0 & EI \end{bmatrix} \begin{Bmatrix} V'' \\ W'' \end{Bmatrix} ds \tag{13}$$

$$dT = \frac{1}{2}\begin{Bmatrix} \dot{V} \\ \dot{W} \end{Bmatrix}^T \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} \begin{Bmatrix} \dot{V} \\ \dot{W} \end{Bmatrix} ds + \frac{1}{2}\dot{\zeta}^2 I_p ds + \frac{1}{2}\begin{Bmatrix} \dot{\theta} \\ \dot{\phi} \end{Bmatrix}^T \begin{bmatrix} I_d & 0 \\ 0 & I_d \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ \dot{\phi} \end{Bmatrix} ds - \dot{\zeta}\dot{\phi}\theta I_p ds$$

Using Eqs. (10-12) Eq. (13) can be written as

$$dP = \frac{1}{2} EI \{q\}^T [\psi'']^T [\psi''] \{q\} ds$$

$$dT = \frac{1}{2}\mu\{\dot{q}\}^T [\psi]^T [\psi]\{\dot{q}\} ds + \frac{1}{2}\dot{\phi}^2 I_p ds + \frac{1}{2} I_d \{\dot{q}\}^T [\Phi]^T [\Phi]\{\dot{q}\} ds - \dot{\zeta} I_p \{\dot{q}\}^T [\Phi]^T [\Phi]\{\dot{q}\} ds \tag{14}$$

The energy of the complete element can be obtained by integrating Eq. (14) over the length of the element to obtain

$$P + T = \frac{1}{2}\{q\}^T [K_B]\{q\} + \frac{1}{2}\{\dot{q}\}^T \left([M_T] + [M_R]\right)\{\dot{q}\} + \frac{1}{2} I_p \dot{\zeta}^2 + \dot{\zeta}\{\dot{q}\}^T [N]\{\dot{q}\} \tag{15}$$

Where

$$[M_T] = \int_0^l \mu[\psi]^T [\psi] ds \tag{16a}$$

$$[M_R] = \int_0^l I_d [\Phi]^T [\Phi] ds \tag{16b}$$

$$[N] = \int_0^l I_p [\Phi_\phi]^T [\Phi_\theta] ds \tag{16c}$$

$$\left[K_B\right]=\int_0^l EI\left[\psi''\right]^T\left[\psi''\right]ds \tag{16d}$$

$$\left[G\right]=\left[N\right]-\left[N\right]^T \tag{16e}$$

Then, the Lagrangian equation of motion for the finite rotor element and the constant spin speed restriction, $\zeta=\Omega t$, is

$$\left(\left[M_T\right]+\left[M_R\right]\right)\{\ddot{q}\}-\Omega\left[G\right]\{\dot{q}\}+\left[K_B\right]\{q\}=\{Q\} \tag{17}$$

3.1.3 Damped Flexible Shaft

To extend this model to a damped rotor, internal damping is assumed to be of viscous in nature (Zorzi and Nelson [34]) and as such the stress and strain relationship can be given as:

$$\sigma=E\left\{\varepsilon+\eta_v\dot{\varepsilon}\right\}\quad \varepsilon=-r\cos\left[\left(\Omega-\omega\right)t\right]\frac{\partial^2 R_o(x,t)}{\partial x^2} \tag{18a}$$

$$\dot{\varepsilon}=\left(\Omega-\omega\right)r\sin\left[\left(\Omega-\omega\right)t\right]\frac{\partial^2 R_o}{\partial x^2}-r\cos\left[\left(\Omega-\omega\right)t\right]\frac{\partial}{\partial t}\left(\frac{\partial^2 R_o}{\partial x^2}\right) \tag{18b}$$

Where E and η_v are modulus of elasticity and viscous damping coefficient, respectively.

It is noteworthy that some insight into the characteristics of internal damping can be gained from inspection of Eqs. (18a) and (18b). From Eq. (18b), it is apparent that if the system is in a synchronous precessional state, $\Omega=\omega$, and if the orbit is circular, $\left(\partial/\partial t\right)\left(\partial^2 R_o/\partial x^2\right)=0$, then the viscous damping component can offer no alteration of the axial stress σ Eq. (18). Thus, for circular synchronous orbits, the internal viscous damping component cannot produce any out of phase loading to reduce the critical speed orbit. Therefore, either external damping or anisotropic bearings are beneficial here in limiting excursions when traversing critical speeds.

The bending moments at any instant about X and Y-axes are expressed as

$$M_Z = \int_0^{2\pi}\int_0^{r_0} -\left(V + r\cos\left(\Omega t\right)\right)\sigma r dr d(\Omega t)$$
$$M_Y = \int_0^{2\pi}\int_0^{r_0} \left(W + r\sin\left(\Omega t\right)\right)\sigma r dr d(\Omega t) \tag{19}$$

Substituting values from Eq. (18) and performing required integration, the equations for bending moment becomes

$$\begin{Bmatrix} M_Z \\ M_Y \end{Bmatrix} = EI\begin{bmatrix} 1 & \eta_v\Omega \\ \eta_v\Omega & -1 \end{bmatrix}\begin{Bmatrix} V'' \\ W'' \end{Bmatrix} + EI\begin{bmatrix} \eta_v & 0 \\ 0 & -\eta_v \end{bmatrix}\begin{Bmatrix} \dot{V}'' \\ \dot{W}'' \end{Bmatrix} \tag{20}$$

Defining the differential bending energy and dissipation function as:

$$dP = \frac{1}{2}EI\begin{Bmatrix} \phi' \\ \theta' \end{Bmatrix}\begin{bmatrix} 1 & \eta_v\Omega \\ \eta_v\Omega & -1 \end{bmatrix}\begin{Bmatrix} V'' \\ W'' \end{Bmatrix} ds$$
$$dD = \frac{1}{2}EI\begin{Bmatrix} \dot{\phi}' \\ \dot{\theta}' \end{Bmatrix}\begin{bmatrix} \eta_v & 0 \\ 0 & -\eta_v \end{bmatrix}\begin{Bmatrix} \dot{V}'' \\ \dot{W}'' \end{Bmatrix} ds \tag{21}$$

From Eqs. (20-21)

$$dP = \frac{1}{2}EI\{q\}^T[\psi'']^T[\eta][\psi'']\{q\}\,ds$$
$$dD = \frac{\eta_v}{2}EI\{\dot{q}\}^T[\psi'']^T[\psi'']\{\dot{q}\}\,ds \tag{22}$$

Combining this with earlier equations giving the kinetic energy of the system, the Lagrangian equations of motion can be established for damped rotor finite element as

$$\left([M_T]+[M_R]\right)\{\ddot{q}\}+\left(\eta_v[K_B]-\Omega[G]\right)\{\dot{q}\}+\left([K_B]+\eta_v\Omega[K_c]\right)\{q\}=Q(t) \tag{23}$$

All of the matrices of Eq. (23) are symmetric with the exception of the gyroscopic term $[G]$ and the circulation terms $[K_c]$ which are skew symmetric. It is in this circulation matrix $[K_c]$ that the instabilities resulting from internal damping are characterized. It is noteworthy that viscous form of material damping contribute to the circulation effects and also providing a dissipation term, $\eta_v [K_B]\{\dot{q}\}$. Thus the viscous form can provide a stable rotor system providing that this dissipation term dominates. This is achieved when for undamped isotropic supports the spin speed is less than the first forward precessional mode (critical speed).

3.2 Optimisation of Disc Component

The finite element model and the equation of motion so developed in the previous section is utilised here for study of the stability of the system. Optimisation of disc parameters is utilised here to ensure high stability for the system. The disc dimensions are optimised using Linear Matrix Inequalities (LMIs). The optimisation also includes proper placing of discs on rotor shaft system. For this purpose, a theoretical approach of permutation has been adopted as shown later.

3.2.1 Convex Optimisation

Convex minimization, a subfield of optimization, studies the problem of minimizing convex functions over convex sets. The convexity property can make optimization in some sense "easier" than the general case for example, any local minimum must be a global minimum.

Given a real vector space X together with a convex, real-valued function $f: x \to R$ defined on a convex subset $x \in X$, the problem is to fine any point $x^* \in X$ for which $f(x)$ is smallest, i.e., a point x^* such that $f(x^*) \le f(x) \forall x \in X$.

The convexity of f makes the powerful tools of convex analysis applicable. In finite-dimensional normed spaces, the Hahn–Banach theorem and the existence of sub gradients lead to a particularly satisfying theory of necessary and sufficient conditions for optimality, a duality theory generalizing that for linear programming, and effective computational methods.

Convex minimization has applications in a wide range of disciplines, such as automatic control systems, estimation and signal processing, communications and networks, electronic circuit design, data analysis and modelling, statistics (optimal design), and finance. With recent improvements in computing and in optimization theory, convex minimization is nearly as straightforward as linear programming. Many optimization problems can be reformulated as convex minimization problems. For example, the problem of maximizing a concave function can be re-formulated equivalently as a problem of minimizing the function, which is convex.

3.2.2 Linear Matrix Inequalities (LMIs)

In Convex Optimisation, Linear Matrix Inequality is an expression of the form

$$F(x) = F_0 + x_1F_1 + ... + x_nF_n \prec 0 \tag{24}$$

where

- $x = (x_1, x_2, x_3, x_4, ..., x_n)$ is a vector on real numbers called the decision variables.
- $F_0, F_1, ..., F_n$ are real symmetric matrices, i.e., , for $i = 0,1,...,n$.
- The inequality $\prec 0$ means negative definite, i.e., $u^T F(x)u \prec 0$ for all non-zero values of vector u. Because all the eigenvalues of a real symmetric matrix are real, the Eq. (24) is equivalent to saying that all eigenvalues $\lambda(F(x))$ are negative. Equivalently, the maximal eigenvalue $\lambda_{\max}(F(x)) \prec 0$.

3.2.2.1 Definition of LMI

A linear matrix inequality (LMI) is an inequality in the form

$$F(x) \prec 0 \tag{25}$$

Where $F(x)$ is an affine function mapping in a finite dimensional vector space x to either Hermitian or Symmetric vector space.

In most control applications, LMI's arise as functions of matrix variables rather than scalar valued decision variables. This means that we consider inequalities of the form Eq. (25) where $x = R_1^{m_1 \times m_2}$ is the set of real matrices of dimension $m_1 \times m_2$.

3.2.2.2 LMI Formulation

The dynamic performance of a rotor system under linear behaviour can be directly assessed from the transfer function. Rotor unbalance vibration response, stability levels, and critical speed locations are commonly used indicators of dynamic performance, and these generally have equivalent transfer function specifications.

The model in the Eq. (23) can also be represented in state space and transfer function forms (Cole et al. [18]) as follows:

$$\mathrm{E}\dot{x} = \mathrm{A}x + \mathrm{B}f \tag{26}$$

$$\mathrm{T} = (s\mathrm{E} - \mathrm{A})^{-1}\mathrm{B} \tag{27}$$

Where, $\mathrm{E} = \begin{bmatrix} I & 0 \\ G & M \end{bmatrix}, \mathrm{A} = \begin{bmatrix} 0 & I \\ -K & -C \end{bmatrix}, \mathrm{B} = \begin{bmatrix} 0 \\ F \end{bmatrix}, x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix},$ $[F]$ indicates the appropriate nodal location of unbalance force.

The unbalance-induced vibration can be modelled by a vector of external disturbance forces $f = ue^{i\Omega t}$, where the complex components $u = \{u_k\}$, where $k = 1, 2, ..., N$ specify the unbalance force at each rotor section. For the purpose of design optimization, these components are considered to have bounded magnitude.

$$0 \le u_k \le \Omega^2 m$$

Here, m is the upper limit of mass unbalance at rotor sections. Then the vibration magnitude of the n^{th} nodal coordinate is given as

$$Y_n = |\mathrm{C}_n \mathrm{T} u| \tag{28}$$

Where C_n selects the appropriate rows of T.
It can also be written as

$$Y_n = \left|\sum_{k=1}^{N} g_k u_k\right| \le \sum_{k=1}^{N} |g_k||u_k| \tag{29}$$

Where,

$$g = \left[\mathrm{T}_1, \mathrm{T}_2, \ldots\ldots, \mathrm{T}_n\right] = \mathrm{C}_n \mathrm{T}$$

The worst case occurs when the maximum value of u_k is reached which is given as $u_k = \Omega^2 m$.
Therefore, the worst case performance for a system can be given as

$$Y_n = \Omega^2 \sum_{k=1}^{N} |g_k| m_k \tag{30}$$

Thus, the worst-case vibration amplitude at a particular machine location is given by the absolute row-sum of the corresponding frequency response matrix $'g'$ with each input scaled by $\Omega^2 m$. For the purpose of system design, a constraint can be specified in the form $Y \le \gamma f(\Omega)$, giving, for all values of Ω and γ is the scaling factor.

$$\sum_{k=1}^{N} |g_k| m_k \le \gamma f(\Omega) / \Omega^2 \tag{31}$$

Where the bounding function $f(\Omega)$ may be chosen to reflect any design constraints concerning critical speed locations and running speed ranges. The bounding function $f(\Omega)/\Omega^2$ can be treated using a stable transfer function W. With a tight bound $\gamma = \gamma_{\min}$, it follows that there exists $\Omega = \Omega_{wc}$ for which

$$\sum_{k=1}^{N} |g_k| m_k = \gamma_{\min} \tag{32}$$

The objective of the design optimization is to minimize γ. The requirement of stable operation of the rotor system can be further specified in terms of quadratic stability of the system. To treat the row sum norm specification of Eq. (31), consider first the more commonly used L_∞ norm-bound on the system with the input scaled by d_k .

$$\sum_{k=1}^{N} \frac{|\,g_k\,|^2}{d_k^2} \leq \delta \tag{33}$$

If $d_k = 1/m_k$ and the above condition is satisfied with $\delta = \delta_{\min}$, it then followed that

$$\sqrt{\delta_{\min}} \leq \gamma_{\min} \leq \sqrt{\delta_{\min} N}$$

Consequently, the L_∞ norm-bound can provide a loose bound on the row-sum norm.

In an effort to obtain a much tighter bound during an optimisation procedure a direct calculation of the worst-case vibration components $t_k = |g_k|$ can be used to select $d_k = \sqrt{t_k / m_k}$, the input scaling factor. The vibration response criteria of Eq. (29) can then be tackled with an iterative design optimisation procedure by minimising the bound δ in Eq. (33) at each design iteration and then updating t_k . If after a number of design iterations

$$\sum_{k=1}^{N} \frac{|g_k(j\Omega)|^2 m_k}{t_k} \to \sum_{k=1}^{N} \frac{|g_k(j\Omega_{wc})|^2 m_k}{t_k} \tag{34}$$

Then it follows that

$$\sum_{k=1}^{N} \frac{|g_k(j\Omega_{wc})|^2 m_k}{t_k} = \delta_{\min} \tag{35}$$

And thus $\delta_{\min} = \gamma_{\min}$

The time domain equivalent of Eq. (33), is the peak RMS bound

$$\int_0^T y_n^2 dt \leq \delta \int_0^T f^T \mathrm{D}^2 f dt \tag{36}$$

For all $f(t)$, where $\mathrm{D} = diag\{d_1, d_2, d_3, ..., d_k\}$, is a diagonal scaling matrix. Quadratic stability of the system can be proved by the existence of a Lyapunov function of the form.

$$\mathrm{V}(t) = x(t)^T \mathrm{P} x(t) \tag{37}$$

Where P is a positive definite matrix such that $\dot{\mathrm{V}} < 0$ for all possible state variables with $f = [0]$, from Eq. (26), defining $\mathrm{Q} = \left(\mathrm{E}^{-1}\right)^T \mathrm{P}\mathrm{E}^{-1} > 0$

$$\dot{\mathrm{V}} = \left(\mathrm{A}x + \mathrm{B}f\right)^T \mathrm{QE}x + x^T \mathrm{E}^T \mathrm{Q}\left(\mathrm{A}x + \mathrm{B}f\right) \tag{38}$$

Combining Eqs. (36-37) we get

$$\dot{\mathrm{V}} + y_n^2 - \delta \mathrm{f}^T \mathrm{D}^2 \mathrm{f} < 0 \tag{39}$$

With $y_n = C_n x$, the above equation becomes

$$\left(\mathrm{A}x + \mathrm{B}f\right)^T \mathrm{QE}x + x^T \mathrm{E}^T \mathrm{Q}\left(\mathrm{A}x + \mathrm{B}f\right) + x^T \mathrm{C}_n^T \mathrm{C}_n x - \delta \mathrm{f}^T \mathrm{D}^2 \mathrm{f} < 0 \tag{40}$$

Therefore,

$$\begin{bmatrix} x^T \\ \mathrm{f}^T \end{bmatrix} \begin{bmatrix} \mathrm{A}^T \mathrm{QE} + \mathrm{E}^T \mathrm{QA} + \mathrm{C}_n^T \mathrm{C}_n & \mathrm{E}^T \mathrm{QB} \\ \mathrm{B}^T \mathrm{QE} & -\delta \mathrm{D}^2 \end{bmatrix} \begin{bmatrix} x \\ \mathrm{f} \end{bmatrix} < 0 \tag{41}$$

For all $\begin{bmatrix} x^T & \mathrm{f}^T \end{bmatrix}^T \neq 0$ Thus, the design criterion is equivalent to the existence of a symmetric matrix $\mathrm{Q} > 0$ for which the following symmetric matrix is negative definite:

$$\begin{bmatrix} \mathrm{A}^T \mathrm{QE} + \mathrm{E}^T \mathrm{QA} + \mathrm{C}_n^T \mathrm{C}_n & \mathrm{E}^T \mathrm{QB} \\ \mathrm{B}^T \mathrm{QE} & -\delta \mathrm{D}^2 \end{bmatrix} < 0 \tag{42}$$

The state space matrices can be represented as affine parameter depending on the design variable $\mathrm{U}(\theta)$ according to

$$\mathrm{E} = \mathrm{E}_0 + \mathrm{B}_u \mathrm{U} \mathrm{C}_u \tag{43}$$

The design variable matrix $U(\theta)$ can have an arbitrary structure and may be a nonlinear function of the physical design variable. In this case,

$\mathrm{E}_0 = \begin{bmatrix} I & 0 \\ G_0 & M_0 \end{bmatrix}, \mathrm{B}_u = \begin{bmatrix} 0 \\ I \end{bmatrix}, \mathrm{U} = \begin{bmatrix} \Delta G & \Delta M \end{bmatrix} \& \mathrm{C}_u = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ Where $\Delta G \& \Delta M$ are sparse matrices and can be given as:

$$\Delta G = \begin{bmatrix} \ddots & & & & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & -\pi\rho r^4 l/2 & \\ & 0 & 0 & -\pi\rho r^4 l/2 & 0 & \\ & & & & & \ddots \end{bmatrix}, \& \ \Delta M = \begin{bmatrix} \ddots & & & & & \\ & \pi\rho r^2 l & 0 & 0 & 0 & \\ & 0 & \pi\rho r^2 l & 0 & 0 & \\ & 0 & 0 & \pi\rho r^4 l/4 & & \\ & 0 & 0 & & \pi\rho r^4 l/4 & \\ & & & & & \ddots \end{bmatrix}$$

Where, l and r taken as thickness and radius of solid rigid disc and these form the variable to be optimized.

The optimization problem takes a new shape as can be seen below:

$$\begin{bmatrix} \mathrm{A}^T \mathrm{Q}(\mathrm{E}_0 + \mathrm{B}_u \mathrm{UC}_u) + (\mathrm{E}_0 + \mathrm{B}_u \mathrm{UC}_u)^T \mathrm{QA} + \mathrm{C}_n^T \mathrm{C}_n & (\mathrm{E}_0 + \mathrm{B}_u \mathrm{UC}_u)\mathrm{C}_u^T \mathrm{QB} \\ \mathrm{B}^T \mathrm{Q}(\mathrm{E}_0 + \mathrm{B}_u \mathrm{UC}_u) & -\delta \mathrm{D}^2 \end{bmatrix} < 0 \quad (44)$$

And it becomes a design problem to find Q and U. With some approximations and use of Schur complement to remove the bi-linearity, the above equation becomes

Minimize γ subject to

$$\begin{bmatrix} \mathrm{A}^T \mathrm{QE} + \mathrm{E}^T \mathrm{QA} + \mathrm{C}_n^T \mathrm{C}_n - \mathrm{A}^T \mathrm{QB}_u \mathrm{B}_u^T \mathrm{QA} & \mathrm{E}^T \mathrm{QB} & \mathrm{A}^T \mathrm{QB}_u + \mathrm{UC}_n \\ \mathrm{B}^T \mathrm{QE} & -\delta \mathrm{D}^2 & 0 \\ \mathrm{B}_u^T \mathrm{QA} + \mathrm{C}_u^T \mathrm{U} & 0 & -I \end{bmatrix} < 0 \quad (45)$$

This inequality is linear in Q and U and therefore, finding a solution for minimal δ is a generalized eigenvalues problem with can be solved using MATLAB programming as there are standard routines are available for solving LMIs [2]. At each iteration, a solution is found to Eq. (45) for minimal δ. The algorithm is halted when either a satisfactory value of the worst-case vibration bound is obtained. Then,

$$\delta_{\min} \to \delta_{opt} = \gamma_{opt}$$

Chapter 4

RESULTS AND DISCUSSIONS

This section involves a design of a solid rotor disc mounted on a rotor shaft as shown in the Figure (4.1). The rotating shaft is supported by bearings at both ends and assumed to be as damped support. The stiffness and the damping effects of the bearing supports are simulated by springs and viscous dampers (k_{yy} = 70 MN/m, k_{zz} = 50 MN/m, d_{yy} = 700 Ns/m and d_{zz} = 500Ns/m) in the two transverse directions. Following Lalanne and Ferraris [4], the material properties of the steel rotor are shown in Table (4.1). The purpose is to design the rotor shaft system in order to ensure low unbalance response amplitude (UBR) and high stability limit of spin speed (SLS). The design variables chosen here are the diameter and thickness of a disc and its position on the system. The initial diameter and thickness and the unbalance on the disc are shown in Table (4.2). The problem involves proper placement of various discs on the rotor shaft system and at the same time to represent the techniques of optimization of various design parameters of the disc for achieving the better gyroscopic stiffening effect. The sole purpose of this study is to represent techniques of optimisation of various parameters of a rotor-shaft-disc system and therefore, to obtain high stability and no feasibility study has been done on the results so obtained.

Material	Density (kg/m^3)	Young's Modulus (GPa)	Length(m)	Diameter(m)	Damping Coefficient (N-s/m)
Mild Steel	7800	200	1.3	0.2	0.0002

Table 4.1: Rotor Material and its Properties.

Diameter(m)	0.20
Thickness(m)	0.05
Mass Unbalance(kg-m)	2e-3

Table 4.2: Disc parameters

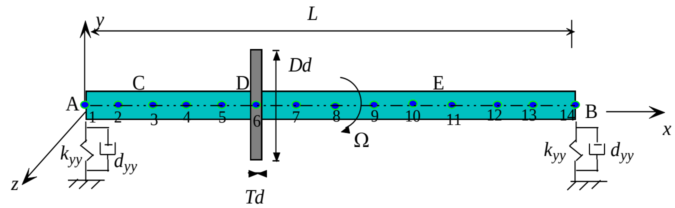


Figure 4.1 Schematic Diagram of the Rotor shaft system

4.1 LMI Optimizations

The dimensions of the disc in the rotor shaft system as shown in Figure (4.1) are optimized here by using LMI technique. The speed-dependent bound on the worst-case vibration response $\gamma f(\Omega)$ is selected for $\gamma = 1$ as shown in Figure (4.2a). The subsequent design optimizations will consider selection of the disc dimensions to minimize the vibration bound γ. The radius to thickness ratio for disc is taken to be 4. The Figure (4.2b) shows the optimization of γ and the final value occurs after 100 iterations. Accordingly, Figure (4.3) shows the worst case response for optimized and unoptimized disc parameters. The optimized or final bound $\gamma f(\Omega)$ is also shown. Figure (4.4), shows the decay rate plot, as can be seen for initial dimensions the slope of the curve is steep whereas for optimized dimensions the slope is less steep showing more stability. The optimized dimension of the disc obtained from LMI technique are shown in Table (4.3) and the corresponding optimised γ and Yn are 9.97e-9 and 3.36e-15m (at 5400 RPM). As can be seen from results, there is a large decrement in the unbalance response amplitude.

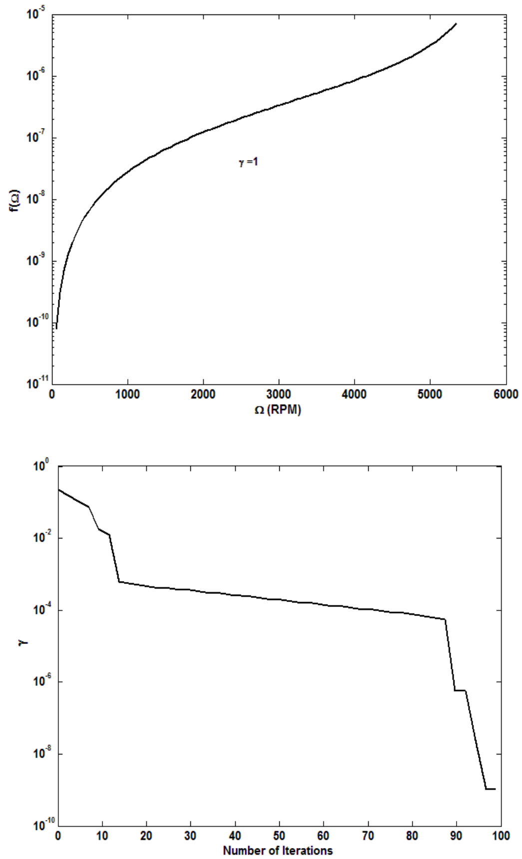


Figure 4.2 Unbalance Respond Bound and Optimization of γ .

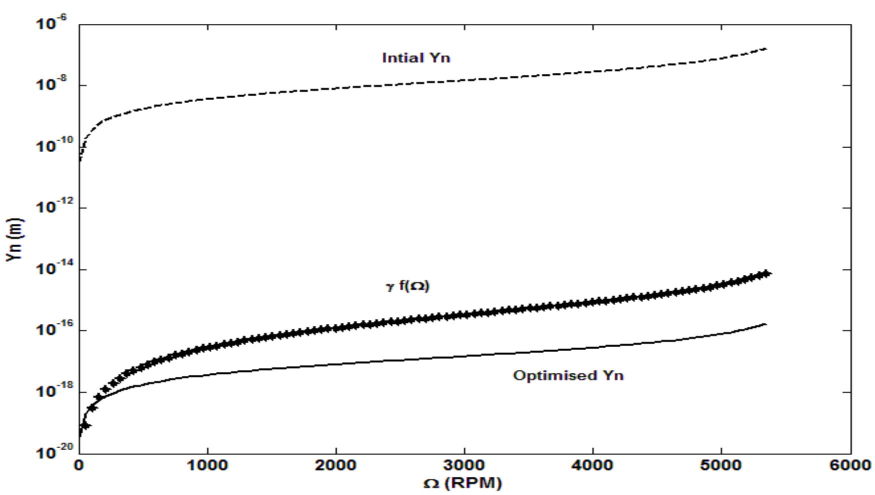


Figure 4.3 Unbalance Response for Optimized and Initial dimensions of disc

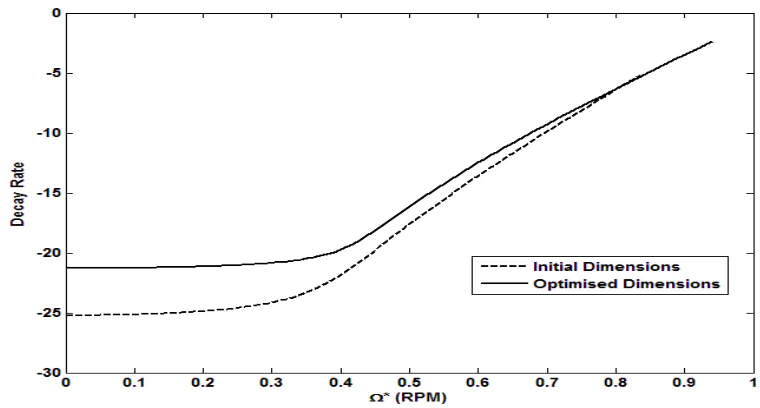


Figure 4.4 Decay rate plot for initial and optimized dimension of disc.

Diameter (m)	Thickness (m)
0.3163	0.0395

Table 4.3: Optimized Results.

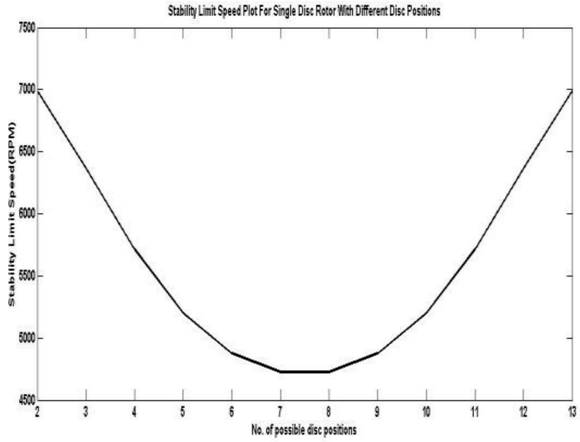
4.2 Disc Positions

The material damping in the rotor shaft introduces rotary dissipative forces which is tangential to the rotor orbit, well known to cause instability after certain spin speed (Zorzi and Nelson [34]). Thus high speed rotor operation suffers from two problems viz. 1) high transverse response due to resonance and 2) instability of the rotor-shaft system over a spin-speed. Both phenomena occur due to material inherent properties and set limitations on operating speed of rotor. By using light weight and strong rotor, the rotor operating speed can be enhanced. These two parameters have some practical limitation. In other words the gyroscopic stiffening effect has some influence on the stability. The gyroscopic effect on the disc depends on the disc dimension and disc position on the rotor. Thus, the optimum positioning of the discs may achieve high speeds and maximum stability. The proper positioning helps ensure high SLS. SLS of the rotor–shaft system has been found out from the maximum real part of all eigenvalues. The system becomes unstable when the real part touches the zero line.

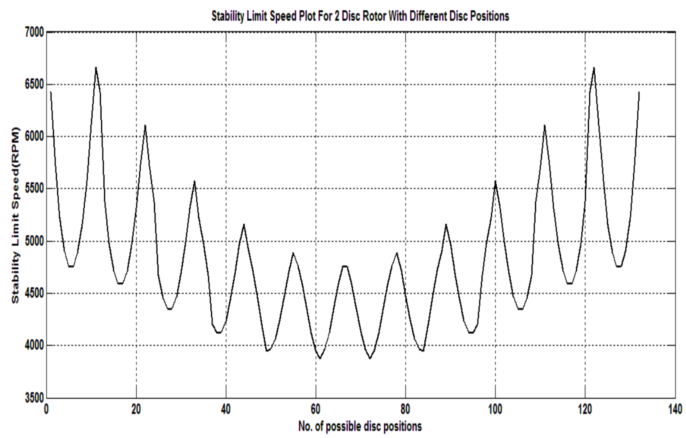
The various disc positions for a single disc rotor are the consecutive nodes. But obtaining the different sets of disc position for a multi disc rotor is not straight forward. It has been done by performing the permutation between the total number of nodes and total number of disc. So the total sets of disc position for a simply supported rotor are given by $N = {}^{n}P_{j}$, where n+2 are the total number nodes and j is the no. of discs.

The SLS plot for different positions so obtained has been shown in Figure (4.5). The method is extrapolated to two discs and three discs cases. In case of multidisc rotors, the discs are taken to be of different dimensions and there can be different ways to put discs on the rotor and therefore, permutation approach has been used to find out a set of positions for discs as they can be of different dimensions and unbalance. For example, if there are 14 nodes and three discs, the discs can be placed on any of these nodes. However, the first and last nodes are taken away by the bearing supports; the number of nodes remaining for the discs is 12. So total sets of disc positions are ${}^{12}P_{3}$=1320 and the discs are located as follows: $DN = [i \quad j \quad k]$; 'DN' is the disc nodes, i, j, k vary from 2 to 13 and when i = j = k DN will be the empty array.

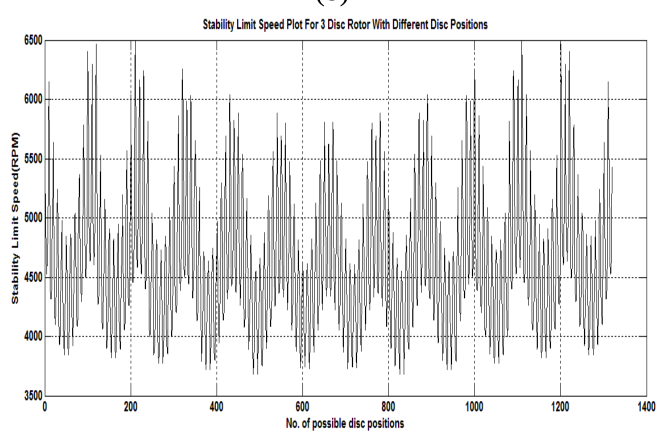
The plots for SLS are found to be symmetric about the vertical axis and it can be seen that the maximum SLS is obtained when the discs are towards the ends of the rotor and if the discs are more towards the centre of the rotor then minimum SLS is obtained. In other words, the rotor will be more stable if the discs are placed towards the ends of the rotor and will be less stable if the discs are placed more towards the centre of the rotor. It is due to the gyroscopic stiffening of the rotor. This stiffening effect will be less when the discs are towards the centre of the rotor.



(a)



(b)



(c)

Figure 4.5 Effect on stability (SLS) of the system with position in case of one disc, two disc and three disc rotors.

Chapter 5

CONCLUSIONS AND FUTURE SCOPE

5.1 Conclusions

The aim of this research is to gain insight into the gyroscopic effects on the rotor disc system. For this purpose mathematical model of the system is used to study its stability. Furthermore, optimisation techniques are utilised in order to ensure high stability of the system under working conditions. The main concentration was speed of the system as nowadays high speed rotors are of prime importance because of the need for speedy works.

A literature survey has been carried out to investigate the developments in modelling and optimisation techniques. It is observed that a huge amount of work has been done in modelling different kinds of rotor with bearings. Most of them involved the Finite Element Modelling. It was also, observed that a number of researchers have used different optimisation techniques with different objective functions in order to make rotor systems stable. However, the existing literature fails to give us a technique that can be combined readily with today's advanced computation techniques and easy to handle both linear and non-linear systems. As a consequence, it was decided to work on some technique that can fulfil the required need.

Here, this work gives the equations of motion of a viscoelastic rotor-shaft system. The linear viscoelastic rotor-material behaviour is represented in the time domain where the damped shaft element is assumed to behave as Voigt model. The finite element model is used to discretize the continuum which is based on Euler-Bernoulli beam theory. Use of LMI technique has been shown here to optimize disc dimension for high dynamic performance of the rotor shaft system. The advantage of the proposed method is the flexibility offered by the LMI formulation, which can be used to create design specifications concerning vibration amplitudes, stability, critical speeds, modal damping levels, and parameter constraints. This work also includes the effects of disc positions in a rotor system. Results are obtained for different sets of disc positions to study the dynamic characteristics, where stability limit of spin speed and unbalance response amplitude are two indices. The rotor will be more stable if the discs are placed towards the ends of the rotor and will be less stable if the discs are placed more towards the centre of the rotor. Thus, proper placement of disc together with optimized dimensions will ensure high stability and less response amplitude.

5.2 Proposed Future Work

Here, a linear system has been taken under study. The LMI technique can handle non-linear systems as well. So, this work can be extrapolated to deal with a non-linear system, which would be more realistic as the real time systems are more non-linear than linear.

Also, the radius to thickness ratio is taken constant here. A polynomial function can be used to specify the variable thickness of the disc and hence shape optimisation of disc is possible. Multi-disc rotor with discs of different dimensions and shapes could possibly be solved using LMIs.

There is a wide scope in the use of LMI technique. But there are currently some drawbacks to the technique due to lack of fast and guaranteed methods to solve bilinear matrix inequalities, which arise through the dependence of the system state-space matrices on the design variables. The importance of developing improved algorithms to solve this problem is widely recognised by researchers in the field of numerical methods. With further development of these numerical tools, methods based on LMIs would prove very useful in active and passive control systems.

Chapter 6

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