FIBONACCI NUMBER AND ITS DISTRIBUTION MODULO 3^k

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by

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to the

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CERTIFICATE

This is to certify that the work contained in this report entitled **"FIBONACCI NUMBER AND ITS DISTRIBUTION MODULO** 3^k **"** is a bonafide review work carried out by **Mr. Rajesh Bishi**, (**Roll No: 411MA2071.**) student of Master of Science in Mathematics at National Institute of Technology, Rourkela, during the year 2013. In partial fulfilment of the requirements for the award of the Degree of Master of Science in Mathematics under the guidance of **Dr. Gopal Krishna Panda,** Professor, National Institute of Technology, Rourkela and that project has not formed the basis for the award previously of any degree, diploma, associateship, fellowship, or any other similar title.

May, 2013

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DECLARATION

I hereby declare that the project report entitled **Fibonacci Number** submitted for the M.Sc. Degree is a review work carried out by me and the project has not formed the basis for the award of any Degree, Associateship, Fellowship or any other similar titles.

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INTRODUCTION

We know that the Fibonacci numbers are discovered by Leonardo de Fibonacci de pisa. the Fibonacci series was derived from the solution to a problem about rabbits. The problem is: Suppose there are two new born rabbits, one male and the other female. Find the number of rabbits produced in a year if

- Each pair takes one month to become mature:
- Each pair produces a mixed pair every month, from the second month:
- All rabbits are immortal

Suppose, that the original pair of rabbits was born on January 1. They take a month to become mature, so there is still only one pair on February 1. On March 1, they are two months old and produce a new mixed pair, so total is two pair. So continuing like this, there will be 3 pairs in April, 5 pairs in May and so on.

Let $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for n≥2 denote the sequence of the fibonacci numbers. For an integer m≥2, we shall consider Fibonacci numbers in Z_m throughout this chapter. It is known that the sequence $(F_n(mod m))_{n\geq 0}$ is periodic. Let $\pi(m)$ denote the (shortest) period of the sequence. There are some known results on $\pi(m)$ = (2,6,8).

Chapter 1

Preliminaries

In this chapter we recall some definitions and known results on elementary number theory. this chapter serves as base and background for the study of subsequent chapters. We shall keep on refereing back to it as and when required.

1.1 Division Algorithm:

Let a and b be two integers, where $b > 0$. Then there exist unique integers q and r such that $a = bq + r, 0 \leq r < b$.

Definition 1.1.1. *(Divisibility) An integer a is said to be divisible by an integer $d \neq 0$ if there exists some integer c such that $a = dc$.*

Definition 1.1.2. *If a and b are integers, not both zero, then the greatest common divisor of a and b, denoted by* gcd (a,b) *is the positive integer d satisfying*

1. *$d|a$ and $d|b$.*
2. *if $c|a$ and $c|b$ then $c|d$.*

Definition 1.1.3. *(Relatively Prime) Two integer a and b, not both of which are zero, are said to be relatively prime whenever $gcd(a, b) = 1$.*

1.2 Euclidean Algorithm

Euclidean algorithm is a method of finding the greatest common divisor of two given integers. This is a repeated application of division algorithm.

Let a and b two integers whose greatest common divisor is required. since $gcd(a, b) = gcd(|a|, |b|)$, it is enough to assume that a and b are positive integers. Without loss of generality, we assume $a > b > 0$. Now by division algorithm, $a = bq_1 + r_1$, where $0 \leq r_1 < b$. If it happens that $r_1 = 0$, then $b|a$ and $gcd(a, b) = b$. If $r_1 \neq 0$, by division algorithm $b = r_1q_2 + r_2$, where $0 \leq r_2 < r_1$. If $r_2 = 0$, the process stops. If $r_2 \neq 0$ by division algorithm $r_1 = r_2q_3 + r_3$, where $0 \leq r_3 < r_2$. The process continues until some zero reminder appears. This must happens because the reminders $r_1, r_2, r_3, ...$ forms a decreasing sequence of integers and since $r - 1 < b$, the sequence contains at most b non-negative integers. Let us assume that $r_{n+1} = 0$ and r_n is the last non-zero reminder. We have the following relation:

$$a = bq_1 + r_1, 0 < r_1 < b$$

$$b = r_1q_2 + r_2, 0 < r_2 < r_1$$

$$r_1 = r_2q_3 + r_3, 0 < r_3 < r_2$$

$$\vdots$$

$$r_{n-2} = r_{n-1}q_n + r_n, 0 < r_n < r_{n-1}$$

$$r_{n-1} = r_nq_{n+1} + 0$$

Then

$$gcd(a, b) = r_n.$$

1.3 Fundamental Theorem of Arithmetic:

Any positive integer is either 1 or prime, or it can be expressed as a product of primes, the representation being unique except for the order of the prime factors.

1.4 Congruence:

Let m be fixed positive integer. Two integers a and b are said to be congruent modulo m if $a - b$ is divisible by m and symbolically this is denoted by $a\equiv b(modm)$. We also used to say a is congruent to b modulo m.

Some Properties of Congruence:

1. If $a \equiv a(modm)$.

2. If $a \equiv b(modm)$, then $b \equiv a(modm)$.

3. If $a \equiv b(modm), b \equiv c(modm)$, then $a \equiv c(modm)$.

4. If $a \equiv b(modm)$, then for any integer c

$(a + c) \equiv (b + c)(modm); ac \equiv bc(modm)$.

Definition 1.4.1. *(Fibonacci Numbers) Fibonacci Numbers are the numbers in the integer sequence defined by the recurrence relation* $F_n = F_{n-1} + F_{n-2}$ *for all* $n\geq 2$ *with* $F_0 = 0$ *and* $F_1 = 1$.

Definition 1.4.2. *(Lucas Numbers) Lucas Numbers are the numbers in the integer sequence defined by the recurrence relation* $L_n = L_{n-1} + L_{n-2}$ *for all* $n>1$ *and* $L_0 = 2$ *and* $L_1 = 1$.

1.5 Golden Ratio and Golden Rectangle

The Golden Ratio denoted by ϕ, is an irrational mathematical constant, approximately 1.61803398874989. In mathematics two quantities are in the golden ratio of the sum of quantities to the larger quantity is equal to the ratio of the larger quantity to the smaller one. Two quantities a and b are said to be in the golden ratio if

$$\frac{a+b}{a} = \frac{a}{b} = \phi.$$

Then

$$\frac{a+b}{a} = 1 + \frac{a}{b} = 1 + \frac{1}{\phi}$$
$$1 + \frac{1}{\phi} = \phi$$
$$\phi^2 = \phi + 1$$
$$\phi^2 - \phi - 1 = 0$$
$$\phi = \frac{1+\sqrt{5}}{2}$$
$$\phi = 1.61803398874989$$
$$\phi \simeq 1.618.$$

Definition 1.5.1. *(Golden Rectangle) A golden rectangle is one whose side lengths are in golden ratio, that is, approximately 1 :* $\frac{1+\sqrt{5}}{2}$.

Construction of Golden Rectangle A Golden Rectangle can be constructed with only straightedge and compass by this technique

1. Construct a simple square.
2. Draw a line from the midpoint of one side of the square to an opposite corner.
3. Use the line as radius to draw an arc that defines the height of the rectangle.
4. Complete the golden rectangle.

1.6 Divisibility

Theorem 1.6.1. *For any integers a,b,c*

1. *If* $a|b$ *and* $c|d$*, then* $ac|bd$.
2. *If* $a|b$ *and* $b|c$*, then* $a|c$.
3. *If* $a|b$ *and* $a|c$*, then* $a|(bx + cy)$ *for arbitrary integers* x *and* y.

Proof.

1. Since $a|b$ and $c|d$ then there exits $r,s \in Z$ such that $b = ra$ and $d = cs$. Now $bd = ra.sc = rs.ac \Rightarrow ac|bd$.

2. Since $a|b$ and $b|c$ then there exits $r, s \in Z$ such that $b = ra$ and $c = sb$. Now $c = sb = sra \Rightarrow a|c$.

3. Since $a|b$ and $a|c$ then there exits $r, s \in Z$ such that $b = ar$ and $c = as$. But then $bx + cy = arx + asy = a(rx + sy)$ whatever the choice of x and y. Since $rx + sy$ is an integer then $a|(bx + cy)$.

Chapter 2

Properties of Fibonacci and Lucas Numbers

2.1 The Simplest Properties of Fibonacci Numbers

Theorem 2.1.1. *The sum of the first n fibonacci numbers is equal to $F_{n+2} - 1$.*

Proof. We have

$$\begin{aligned} F_1 &= F_3 - F_2, \\ F_2 &= F_4 - F_3, \\ &\vdots \\ F_{n-1} &= F_{n+1} - F_n, \\ F_n &= F_{n+2} - F_{n+1}. \end{aligned}$$

Adding up these equations term by term, we get $F_1+F_2+F_3+....+F_n = F_{n+2}-F_2 = F_{n+2}-1$.

Theorem 2.1.2. *The sum of first n fibonacci with odd suffixes is equal to F_{2n}.*

Proof. We know

$$F_1 = F_2,$$

$$F_3 = F_4 - F_2,$$

$$F_5 = F_6 - F_4,$$

$$\vdots$$

$$F_{2n-1} = F_{2n} - F_{2n-2}.$$

Adding up these equations term by term, we obtain

$$F_1 + F_3 + F_5 + ... + F_{2n-1} = F_{2n}.$$

Theorem 2.1.3. $F_1^2 + F_2^2 + ... + F_n^2 = F_n F_{n+1}$.

Proof. We know that

$$F_k F_{k+1} - F_{k-1} F_k = F_k(F_{k+1} - F_{k-1}) = F_k^2$$

$$F_1^2 = F_1 F_2$$

$$F_2^2 = F_2 F_3 - F_1 F_2$$

$$\vdots$$

$$F_n^2 = F_n F_{n+1} - F_{n-1} F_n.$$

Adding up these equations term by term, we get

$$F_1^2 + F_2^2 + ... + F_n^2 = F_n F_{n+1}.$$

Theorem 2.1.4. $F_{n+m} = F_{n-1} F_m + F_n F_{m+1}$.

Proof. We shall prove the theorem by the method of induction on m. for $m = 1$, we get $F_{n+1} = F_{n-1} F_1 + F_n F_{1+1} = F_{n-1} + F_n$ Which is true. Suppose that it is true for $m = k$ and

$m = k + 1$, we shall prove it is also true that $m = k + 2$.

Let

$$F_{n+k} = F_{n-1}F_k + F_nF_{k+1}$$

and

$$F_{n+(k+1)} = F_{n-1}F_{k+1} + F_nF_{k+2}.$$

Adding these two equations, we get

$$F_{n+(k+2)} = F_{n-1}F_{k+2} + F_nF_{k+3}.$$

Hence

$$F_{n+m} = F_{n-1}F_m + F_nF_{m+1}.$$

Theorem 2.1.5. $F_{n+1}^2 = F_nF_{n+2} + (-1)^n$

Proof. We shall prove the theorem by induction on n. We have since, $F_2^2 = F_1F_3 - 1 = 1$, the assertion is true for n = 1. let us assume that the theorem is true for n = $1, 2, ..., k$. Then adding $F_{n+1}F_{n+2}$ to both sides, we get

$$F_{n+1}^2 + F_{n+1}F_{n+2} = F_{n+1}F_{n+2} + F_nF_{n+2} + (-1)^n.$$

Which implies that $F_{n+1}(F_{n+1} + F_{n+2}) = F_{n+2}(F_n + F_{n+1}) + (-1)^n$. This simplifies to $F_{n+1}F_{n+3} = F_{n+2}^2 + (-1)^n$. Finally we have, $F_{n+2}^2 = F_{n+1}F_{n+2} + (-1)^{n+1}$.

2.2 Number-Theoretic Properties of Fibonacci Numbers

Theorem 2.2.1. *For the Fibonacci sequence, gcd*$(F_n, F_{n+1}) = 1$ *for every* $n \geq 1$.

Proof. let $gcd(F_n, F_{n+1}) = d > 1$. Then $d|F_n$ and $d|F_{n+1}$. Then $F_{n+1} - F_n = F_{n-1}$ will also be divisible by d. Again, We know that $F_n - F_{n-1} = F_{n-2}$. This implies $d|F_{n-2}$. Working backwards, the same argument shows that $d|F_{n-3}, d|F_{n-4}$,... and finally that $d|F_1 = 1$. This is impossible. Hence $gcd(F_n, F_{n+1}) = 1$ for every $n \geq 1$

Theorem 2.2.2. *For $m \geq 1$, $n \geq 1$, F_{nm} is divisible by F_m*

Proof. We shall prove the theorem by induction on n. For n = 1 the theorem is true. Let us assume that $f_m|F_{nm}$, for n=1,2,3,.....k. Now $F_{m(k+1)} = F_{mk} + F_m = F_{mk-1}F_m = F_{mk}F_{m+1} + F_m$. The right hand site of the equation is divisible by F_m. Hence $d|F_{m(k+1)}$.

Lemma 2.2.1. *if $m = nq + r$, then $gcd(F_m, F_n) = gcd(F_r, F_n)$.*

Proof. Observe that
$gcd(F_m, F_n) = gcd(Fnq + r, F_n) = gcd(F_{nq-1}F_r + F_{qn}F_{r+1}, F_n) = gcd(F_{nq-1}F_r, F_n)$. Now we claim that $gcd(F_{nq-1,F_n}) = 1$. Let $d = gcd(F_{nq-1}, F_n)$. Then $d|F_{nq-1}$ and $d|F_n$. Also that $F_n|F_{nq}$. Therefore $d|F_{nq}$. This d is the positive common divisor of F_{nq} and F_{nq-1}. but $gcd(F_{nq-1}, F_{nq}) = 1$. This is an absurd. Hence $d = 1$.

Theorem 2.2.3. *The greatest common divisor of two Fibonacci number is again a Fibonacci number.*

Proof. Let F_m and F_n be two Fibonacci Number.
Let us assume that $m \geq n$. Then by applying Euclidian Algorithm to m and n, We get the following system of equations

$$m = q_1 n + r_1, 0 \leq r_1 < n$$

$$n = q_2 r_1 + r_2, 0 \leq r_2 < r_1$$

$$r_1 = q_3 r_2 + r_2, 0 \leq r_3 < r_2, ...$$

$$r_{n-2} = q_n r_{n-1} + r_n, 0 \leq r_n < r_{n-1}$$

$$r_{n-1} = q_{n+1} r_n + 0$$

Then form the previous lemma

$$\begin{aligned} gcd(F_m, F_n) &= gcd(F_{r_1}, F_n) \\ &= gcd(F_{r_1}, F_{r_1}) \\ &\vdots \\ &= gcd(F_{r_{n-2}}, F_{r_n}). \end{aligned}$$

Since $r_n|r_{n-1}$, then $F_{r_n}|F_{r_{n-1}}$. Therefore $gcd(F_{r_{n-1}}, F_{r_n}) = F_{r_n}$. But r_n, being the last non-zero reminder Euclidian Algorithm for m and n, is equal to $gcd(m, n)$. Thus $gcd(F_m, F_n) = F_d$, Where $d = gcd(m, n)$.

Theorem 2.2.4. *In the Fibonacci sequence, $F_m|F_n$ if and only if $m|n$.*

Proof. If $F_m|F_n$, then $gcd(F_m, F_n) = F_m$. But we know that $gcd(F_m, F_n) = F_{gcd(m,n)}$. This implies that $gcd(m, n) = m$. Hence $m|n$.

Theorem 2.2.5. *The sequence of ratio of successive Fibonacci Numbers $F_{n+1}|F_n$ converges to Golden Ratio i.e., $lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \Phi$.*

Proof. We consider the sequence $r_n = \frac{F_{n+1}}{F_n}$, for n = 1,2,3,....Then by definition of Fibonacci Numbers, we have $r_n = \frac{F_{n+1}}{F_n} = \frac{F_n+F_{n-1}}{F_n} = 1 + \frac{1}{r_{n-1}}$.
When $n \to \infty$, then we can write the above equation in limits:

$$\begin{aligned} x &= 1 + \frac{1}{x} \\ x^2 &= 1 + x = x^2 - x - 1 = 0 \\ x &= \frac{1+\sqrt{5}}{2} = \phi \end{aligned}$$

Hence,

$$lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \Phi$$

2.3 Binet's Formulae for Fibonacci and Lucas Numbers

Lemma 2.3.1. *Let* $\alpha = \frac{1+\sqrt{5}}{2}$ *and* $\beta = \frac{1-\sqrt{5}}{2}$, *so that* α *and* β *are both roots of the equation* $x^2 = x + 1$. *Then* $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$, *for all* $n \geq 1$.

Proof. When $n = 1$, $F_1 = 1$ Which is true. let us assume that it is true for $n = 1, 2, ..., n$. Then $F_{k-1} = \frac{\alpha^{k-1} - \beta^{k-1}}{\sqrt{5}}$ and $F_k = \frac{\alpha^k - \beta^k}{\sqrt{5}}$. Adding these two equations, we get $F_k + F_{k-1} = \frac{\alpha^k}{\sqrt{5}}(1 + \alpha^{-1}) + \frac{\beta^k}{\sqrt{5}}(1 + \beta^{-1})$. Then $F_{k+1} = \frac{\alpha(k+1) + \beta(k+1)}{\sqrt{5}}$.

Lemma 2.3.2. *Let* $\alpha = \frac{1+\sqrt{5}}{2}$ *and* $\beta = \frac{1-\sqrt{5}}{2}$, *so that* α *and* β *are both roots of the equation* $x^2 = x + 1$. *Then* $L_n = \alpha^n + \beta^n$, *for all* $n \geq 1$.

Proof. For n = 1, $L_1 = 1$. Then the theorem is true for n = 1. Let us assume that it is true for n = 1,2,...,k. We have to prove that it is true for n = k+1. Now

$$L_k + L_{k-1} = \alpha^k + \alpha^{k-1} + \beta^k + \beta^{k-1}$$

$$L_{k+1} = \alpha^k(1 + \alpha^{-1}) + \beta^k(1 + \beta^{-1})$$

$$L_{k+1} = \alpha^k(1 + \alpha - 1) + \beta^k(1 + \beta - 1)$$

$$L_{k+1} = \alpha^{k+1} + \beta^{k+1}$$

2.4 Relation Between Fibonacci and Lucas Numbers

Theorem 2.4.1. $L_n = F_{n-1} + F_{n+1}$, *for n>1.*

Proof. We know that

$$L_{k+1} = L_k + L_{k-1}$$

$$L_{k+1} = (F_{k-1} + F_{k+1}) + (F_{k-2} + F_k)$$

$$L_{k+1} = (F_{k-1} + F_{k-2}) + (F_k + F_{k+1})L_{k+1} \qquad = F_k + F_{k+2}$$

Theorem 2.4.2. *If* $gcd(m,n) = 1, 2, 5$ *then* $F_n F_m | F_{nm}$

Proof. Two parts:

1. $$gcd(m,n) = 1, 2 \Rightarrow gcd(F_m, F_n) = 1$$
$$m|n \Rightarrow F_m|F_m$$
$$m|mn \Rightarrow F_m|F_{mn}$$
$$n|mn \Rightarrow F_n|F_{mn}$$
$$gcd(F_m, F_n) = 1 \Rightarrow F_n F_m | F_{nm}$$

2. $$gcd(m,n) = 5, m = 5a, n = 5^k b \text{ mit } gcd(5,a) = 1 = gcd(5,b) = gcd(a,b)$$
$$\Rightarrow gcd(F_m, F_n) = 5$$
$$F_m|F_{mn} \wedge F_n|F_{mn} \wedge 5^{k+1}|F_{mn} \Rightarrow F_n F_m | F_{nm}$$

□

Chapter 3

Distribution of the Fibonacci Numbers Modulo 3^k

3.1 Introduction

Let $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for n≥2 denote the sequence of the Fibonacci numbers. For an integer m≥2, we shall consider Fibonacci numbers in Z_m throughout this chapter. It is known that the sequence $(F_n (\text{mod } m))_{n\geq 0}$ is periodic. Let $\pi(m)$ denote the (shortest) period of the sequence. There are some known results on $\pi(m)$.

Theorem 3.1.1. *If $\pi(p) \neq \pi(p^2)$, then $\pi(p^k) = p^{k-1}\pi(p)$ for each integer $k \geq 1$ and prime p. Also if t is the largest integer with $\pi(p^{`}) = \pi(p)$, then $\pi(p^k) = p^{k-1}\pi(p)$ for $k > t$.*

For any modulus $m \geq 2$ and residue b (mod m) (we always assume $1 \leq b \leq m$). denote by $v(m, b)$ the frequency of b as a residue in one period of the sequence($F_n (\text{mod } m)$). It was proved that $v(5^k.b) = 4$ for each b (mod 5^k) and each $k \geq 1$ by Niederreiter in 1972. Jacobson determined $v(2^k.b)$ for $k \geq 1$ and $v(2^k 5^j.b)$ for $k \geq 5$ and $j \geq 0$ in 1992.

In this chapter we shall partially describe the number $v(3^k.b)$ for $k \geq 1$.

Example 3.1.1. *A period of F_n (mod 27) is listed below:*

F_{8x+y}	1	2	3	4	5	6	7	8	$\leftarrow y$
0	1	1	2	3	5	8	13	21	
1	7	1	8	9	17	26	16	15	
2	4	19	23	15	11	26	10	9	
3	19	1	20	21	14	8	22	3	
4	25	1	26	0	26	26	25	24	
5	22	19	14	6	20	26	19	18	
6	10	1	11	12	23	8	4	12	
7	16	1	17	18	8	26	7	6	
8	13	19	5	24	2	26	1	0	
$x\uparrow$									

so v(27,1) = v(27,26) = 8, v(27,8) = v(27,19) = 5 and v(27,b) = 2 for $b\neq$ 1.8, 19.26.

3.2 Some Known Results

In section 4, we shall consider the frequency of each residue b (mod 3^k) in one period of the sequence(F_n (mod 3^k)). Before considering this problem we list some well-known identities in this section.

$$F_{-n} = (-1)^{n+1}F_n$$

$$F_{n+m} = F_{m-1}F_n + F_mF_{n+1}$$

Let $\alpha(m^k)$ be the first index $\alpha > o$ such that $F_\alpha \equiv 0 \pmod{m^k}$.
Let $\beta(m^k)$ be the largest integer β such that $F_{\alpha(m^k)} \equiv 0 \pmod{m^\beta}$.
i.e $\beta(m^k)$ be the largest exponent β such that $m^\beta / \ F_{\alpha(m^k)}$.
We knoew that $\gcd(F_\alpha, F_{\alpha-1}) = 1$ and $\alpha(m)$ is a factor of $\pi(m)$ for $m \geq 2$

Lemma 3.2.1. *If p is an odd prime and $k \geq \beta(p)$, $\alpha(p^k) = p^{k-\beta(p)}\alpha(p)$*

Proof. we know that $F_\alpha \equiv 0 \pmod{p^k}$

$F_{\alpha(p^k)} = F_{\frac{\pi(p^k)}{2}} = F_{\frac{\pi(p)p^{k-1}}{2}} \equiv 0 \pmod{p^k}$

so $p^k \ / \ F_{\frac{\pi(p)p^{k-1}}{2}}$

i.e $p^k \ / \ F_{\alpha(p)p^{k-1}}$

hence $\alpha(p^k) = \alpha(p)p^{k-1}$, we know that $\beta(p) = 1$

so $\alpha(p^k) = p^{k-\beta(p)}\alpha(p)$ □

Lemma 3.2.2. *For $k \geq 4$. $F_{\pi(3^k)/9-1} \equiv 7.\ 3^{k-2} + 1$ (mod 3^k) and $F_{\pi(3^k)/9} \equiv 4.\ 3^{k-2}$ (mod 3^k).*

Proof. Note that $\pi(3^k) = 8.3^{k-1}$. we prove this lemma by induction on k. When k = 4, we have $F_23 = 28657 \equiv 64 \equiv 7.3^2 + 1$ (mod 3^4) and $F_24 = 46368 \equiv 36 \equiv 4.\ 3^2$ (mod 3^4).

Suppose the lemma is true for some $k \geq 4$. Since 2k-3 $\geq$ k+1 and $F_{8.3^{k-3}} \equiv 0$ (mod 3). $3(F_{8.3^{k-3}})^2 \equiv$ (mod 3^{k+1}) $(F_{8.3^{k-3}})^3 \equiv$ (mod 3^{k-1}) and $(F_{8.3^{k-3}-1})^3 \equiv (7.3^{k-2}+1)^3 + 1$ (mod 3^{k+1}). By putting n = 8.3^{k-3} in to the above equations and from the induction assumption, we have

$$F_{8.3^{k-2}-1} \equiv (F_{8.3^{k-2}-1})^3 \equiv 7.3^{k-1} + 1 \ (\text{mod } 3^{k+1}).\ F_{8.3^{k-2}} \equiv 3.F_{8.3^{k-3}}\ (F_{8.3^{k-3}-1})^2$$

$$\equiv 3.(4.3^{k-2} + 3^k u)(7.3^{k-2} + 1 + 3^k v)^2 \text{ for some } u, v \in z$$

$$\equiv 4.3^{k-1}[3^{2k-4}(7+9v)^2 + 2.3^{k-2}(7+9v) + 1] \equiv 4.3^{k-1} \ (\text{mod } 3^{k+1}).$$ □

Corollary 3.2.1. *for $k \geq 2$. $F_{\frac{\pi}{3}-1} \equiv 3^{k-1} + 1(mod 3^k)$ and $F_{\frac{\pi}{3}} \equiv 3^{k-1}(mod 3^k)$. Where $\pi = \pi(3^k)$.*

Proof. Suppose k = 2. $F_7 = 13 \equiv 4$ (mod 3^2) and $F_8 = 21 \equiv 3$ (mod 3^2). Suppose k = 3. By the proof of lemma 3.1 we have $f_{23} \equiv 7.362 + 1(mod 3^3)$ and $F_{24} \equiv 4.3^2(mod 3^4)$. This implies $F_{23} \equiv 3^2 + 1(mod 3^3)$ and $F_{24} \equiv 3^2(mod 3^3)$. Suppose $k \geq 4$.We have

$F_{\frac{\pi}{3}-1} = (F_{\frac{\pi}{9}-1})^3 + 3(F_{\frac{\pi}{9}})^2 F_{\frac{\pi}{9}-1} + (F_{\frac{\pi}{9}})^3 \equiv 7.3^{k-1} + 1(mod 3^{k+1}) \equiv 3^{k-1} + 1(mod 3^k)$. Similarly, we have

$F_{\frac{\pi}{3}} \equiv 3F_{\frac{\pi}{9}}(F_{\frac{\pi}{9}}-1)^2(mod 3^{k+1}). \equiv 3.4.3^{k-2}(7.3^{k-2}+1)^2(mod 3^k) \equiv 4.3^{k-1} \equiv 3^{k-1}(mod 3^k).$ □

Proposition 3.2.1. *For* $k \geq 1$. $F_{\frac{\pi}{2}-1} = F_{n(3^k)-1} = -1(mod 3^k)$. *Where* $\pi = \pi(3^k)$.

Proof. we have $F_{\pi-1} = (F_{\frac{\pi}{2}-1})^2 + (F_{\frac{\pi}{2}})^2$. Again we have $(F_{\frac{\pi}{2}-1})^2 \equiv 1(mod 3^k)$. By the definition of π and together with ? , $F_{\frac{\pi}{2}-1} \neq 1(mod 3^k)$. Since the multiplication group of units of Z_{3^k} is cyclic, $F_{\frac{\pi}{2}-1} \equiv -1(mod 3^k)$. □

Proposition 3.2.2. *For* $k \geq 2, F_{n+\frac{\pi}{2}} \equiv -F_n(mod 3^k)$.

Proof. We have $F_{n+\frac{\pi}{2}} = F_{\frac{\pi}{2}-1}F_n + F_{\frac{\pi}{2}}F_{n+1}$. by proposition 3.3 we have $F_{n+\frac{\pi}{2}} \equiv -F_n(mod 3^k)$.

Thus for each b and each n such that $F_n \equiv b(mod 3^k)$ we have $F_{n+\frac{\pi}{2}} \equiv -b(mod 3^k)$. Thus the frequency of $b(mod 3^k)$ and $-b(mod 3^k)$ are equal. That is $v(3^k.b) = v(3^k.3^k - b)$. □

3.3 Frequencies of Fibonacci Numbers Modulo 3^k

In this section, we shall compute some values of $v(3^k, b)$ for $k \geq 1$.

Lemma 3.3.1. *For* $k \geq 1$*, we have*

$$F_{n+\frac{\pi}{3}} \equiv \left\{ \begin{array}{ll} F_n & if\ n \equiv 2,6\ (mod\ 8) \\ F_n + 3^{k-1} & if\ n \equiv 0,5,7\ (mod\ 8) \\ F_n + 2.3^{k-1} & if\ n \equiv 1,3,4\ (mod\ 8) \end{array} \right\} (mod\ 3^k),$$

where $\pi = \pi(3^k)$.

Proof. By previous lemma we have $F_{n+\frac{\pi}{3}} = F_nF_{\frac{\pi}{3}-1} + F_{n+1}F_{\frac{\pi}{3}} \equiv \left(3^{k-1}+1\right)F_n + 3^{k-1}F_{n+1} \equiv$ $Fn + 3^{k-1}F_{n+2}$ (mod 3^k).

since we know that $\pi(3) = 8$ and F_{n+2} (mod $3)_{n\geq\ 0}$ $=\{1, 2, 0, 2, 2, 1, 0, 1, ...\}$. □

Lemma 3.3.2. *For* $k \geq 4$*, we have*

$$F_{n+\frac{\pi}{9}} \equiv \left\{ \begin{array}{ll} F_n & if\ n \equiv 6,18\ (mod\ 8) \\ F_n + 3^{k-1} & if\ n \equiv 10,14\ (mod\ 8) \\ F_n + 2.3^{k-1} & if\ n \equiv 2,22\ (mod\ 8) \end{array} \right\} (mod\ 3^k),$$

where $\pi = \pi(3^k)$.

Proof. By previous lemma we have $F_{n+\frac{\pi}{9}} = F_n F_{\frac{\pi}{9}-1} + F_{n+1} F_{\frac{\pi}{9}} \equiv 3^{k-2}(7F_n + 4F_{n+1}) + Fn \pmod{3^k}$.

Let $U_n = 7F_n + 4F_{n+1}$ since we know that $\pi(9) = 24$ and $U_n \equiv 6, 0, 3, 3, 0, 6 \pmod 9$ when $n \equiv 2, 6, 10, 14, 18, 22 \pmod{24}$ □

Lemma 3.3.3. *For* $k \geq 3$. $v(3^k.b) = 8$ *if* $b \equiv 1 or 26$ $(mod\ 27)$

Proof. We shall prove this theorem by induction on k. Consider $b \equiv 1 \pmod{27}$.

Suppose $k = 3$. We have $v(3^k.b) = 8$.

Suppose $v(3^k.b) = 8$ for $k \geq 3$ Let $b \in \mathbb{Z}_{3^{k+1}}$ with $b \equiv 1 \pmod{27}$.

Let $F_n \equiv b \pmod{3^k}$. $1 \leq i \leq 8$ and $1 \leq n_i \leq \pi(3^k)$.

Since $F_n \equiv b \pmod{27}$. we know that $n_i \equiv 6, 18 \pmod{24}$ there are at least $v(3^k.b) = 8$ where $0 \leq n_i \leq \pi(3^{k+1})$ such that $F_n \equiv b \pmod{3^{k+1}}$. since there are 3ˆk-2 solution in $\mathbb{Z}_{3^{k+1}}$ for the congruence equation $b \equiv 1 \pmod{27}$, $w(3^{k+1}.b) \geq 8.3^{k-2}$.

But we know that $w(3^{k+1}.b) = 8.3^{k-2}$. hence $v(3^k.b) = 8$.

□

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