FREE VIBRATION OF 1-D AND 2-D SKELETAL STRUCTURES

A thesis submitted for the partial fulfillments of the requirements for degree of

Bachelor of Technology in "Civil Engineering"

By

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CERTIFICATE

This is to certify that the thesis named, "**Free Vibration of 1-D and 2-D Skeletal Structures,"** submitted by **Ashish Kumar Kanar (B.Tech:109CE0040)** in partial fulfillment of the requirements for the award of **Bachelor of Technology** in **Civil Engineering** during session 2012-2013 at National Institute of Technology, Rourkela is a bonafide record of research work carried out by him under my supervision and guidance.

The candidate has fulfilled all the prescribed requirements.

The thesis which is based on candidates" own work has not been submitted elsewhere for a degree/diploma.

In my opinion, the thesis is of standard required for the award of a Bachelor of Technology degree in Civil Engineering.

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ABSTRACT

A structure is to be designed to achieve a definite set of natural frequencies to avoid resonance, to decrease dynamic stresses, or to provide study materials for certain critical computation of vibration instrumentation. Structures vibrate constantly in certain frequencies under dynamic loadings. So, it is necessary to undertake vibration analysis to avoid resonance with the natural frequency. The natural frequency depends upon stiffness and mass distributions, boundary conditions and the modes in which they are excited. This project "Free Vibrations of 1-D and 2-D Skeletal Structures," aims in computing the free natural frequencies of 1-D and 2-D skeletal structures. FEM (finite Element Analysis) is a versatile numerical technique which is quite effective for vibrational analysis of structures which are under dynamic loads. MATLAB codes are developed using FEM techniques in which inertia and stiffness matrices are constructed. Finally the natural frequencies are computed.

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Chapter - 1

INTRODUCTION

If a body is moving in a oscillating or reciprocating manner, it is called vibration if it involves deformation of body. However if, the reciprocating motion involves only the rigid body movement without involving its deformation, then it is termed as oscillation. The movement of a pendulum is simple harmonic and so is for a ship (if considered as a rigid body) moving over the waves. Unwanted vibration can cause degradation in performance of the structure. The aim of vibrational analysis of a structure is to suppress unwanted vibrations, to generate desirable vibrations, to control or modify it and its isolation to minimize the structural response.

When the excitation force (the driving factor for the initiation of vibration) in a structure doesn't play any role after initiation of motion; then it is called **free vibration**. The corresponding frequencies are called **natural frequencies** of vibration. Among them the lower natural frequenciesare of optimum consideration in the vibrational analysis of a structure.

There are two ways to solve vibration problems.

i. Analytical method
ii. Numerical method

Analytical methods give accurate results but they require symmetry and simplicity of the problem under consideration whereas numerical techniques give approximate results but these are meant for complex problems and time efficient. FEM (finite element method) is one of the most flexible and effective numerical tool for the analysis of vibration of a system.

In this thesis, free vibration of the following structures has been considered.

i. Free torsional vibration of shaft
ii. Free axial vibration of rod
iii. Free vibration of truss
iv. Free vibration of Bernoulli-Euler beam
v. Free vibration of Timoshenko beam
vi. Free vibration of plane frame
vii. Free vibration of stepped beam

Chapter - 2

LITERATURE REVIEW

Young W. Kwon and Hyochoong Bang,[1] in their book, "Finite Element Method using MATLAB," analyzed free vibration of trusses, both Euler and Timoshenko beams, and frame structures to get their natural frequencies using FEM . They compared the FEM results of natural frequencies with the results of analytical methods.

M. Asghar Bhatti,[2] in his book, strived to explain and teach the mechanical aspects of the finite element method and give satisfactory explanations for theoretical questions related to truss, beams and frames.

S.K. Jang and C.W. Bert,[3] in their paper, analyzed the free vibration of stepped beam. In their study, the lowest natural frequency of a stepped beam with two different cross sections was sought for various boundary conditions. Exact solutions had been calculated and are compared with the results obtained by the use of FEM with non-polynomial shape functions and with a commercial code,MSC/pal.

R.D. Blevin's[4] paper was quite helpful in understanding the basics of natural frequency and mode shapes, which are two essential terms in vibration analysis.

T.S. Balasubramanian and G. Subramanian,[5],[6] in their paper compared the frequency values obtained by using 2DPN(degree-per-node) elements and 4DPN elements for uniform, stepped and continuous beams for various boundary conditions to show the superior performance of the 4DPN element. They also studied the beneficial effects of steps on the free vibration response of stepped beams.

Larisse Klein[8] in 1974, is his paper, analyzed the free vibration of elastic beams with non-uniform characteristics by a new method which was seen to combine the advantages of finite element method and Rayleigh-Ritz analysis.

G.M.L. Gladwell[9](Institute of Sound and Vibration Research, University of Southampton, England), in his paper, determined the natural frequencies and principal modes of un-damped free vibration of a plane frame composed of a rectangular grid of uniform beams.

Chapter – 3

FEM FORMULATION

OF

THE PROBLEM

FINITE ELEMENT METHOD :

FEM is a flexible approximate numerical technique for solving partial differential equations which is originated from complex elasticity and structural problems. FEM allows for detailed visualization of stress and strains inside the body of a structure. Any domain/continuum is considered as an assemblage of number of pieces having very small dimensions called **finite elements**. These elements are connected through a number of joints called **nodes**.

Every physical problemisanalyzed/computed by simplifying certain assumptions. Hence the true behavior of structure should be observed within these constraints and certain errors creep into the engineering computations.

The following procedure is performed for FEM.

i. Discretization of the domain
ii. Identification of variables(unknown displacement at each nodes)
iii. Choice of approximating functions
iv. Generation of element stiffness matrix
v. Generation of overall stiffness matrix
vi. Formation of element inertia matrix
vii. Formation of overall inertia matrix
viii. Application of boundary conditions
ix. Solution of simultaneous equations

Vibration analysis of structures comprises of two parts:

a) Static analysis
b) Dynamic analysis

In static analysis formation of global stiffness matrix and global inertia matrix are executed.

In dynamic analysis, computation of natural frequencies is executed.

According to Newton's law,

F=ma

And, kx=ma

Hence, kx=ma

$$\Rightarrow kx-ma=0$$

$$\Rightarrow kx-\omega^2 mx=0 \quad (\text{since } x=a\sin \omega t)$$

$$\Rightarrow (K-\omega^2 m)\,x=0$$

Since x cannot be zero,hence, in matrix form

$$[K]-\omega^2[M]=0$$

Square root of the diagonal elements of the eigen vector will give the values of natural frequencies of the system.

FREE TORSIONAL VIBRATION OF SHAFT :

Let us consider an un-damped free vibration case.

Torsional vibration is nothing but the twisting effect of the ends of a shaft.

A shaft has 2 nodes at its ends.

Let, the displacements at the two ends are $\phi1$ and $\phi2$.

$\phi1$ and $\phi2$ are nothing but the angle of twist.

Let $\phi1=\alpha1$,

$\phi2=\alpha1+ L*\alpha2$

At any point along the shaft, $\phi= \alpha1 +x\alpha2$; which is twist at x distance from node1

We can write,

$$\phi= <1 \; x > \{\alpha1 \;\; \alpha2 \}^T \text{ …………eq(1)}$$

Linear interpolation function can be taken as,

$$\{ \phi1 \;\; \phi2 \}^T=[1 \; 0; 1 \; L]\{\alpha1 \;\; \alpha2 \}^T$$

Or, $\{\alpha1 \;\; \alpha2 \}^T = [1 \; 0; 1 \; L]^{-1}\{ \phi1 \;\; \phi2 \}^T$

Or, $\{\alpha1 \;\; \alpha2 \}^T = (1/L) [L \; 0 ; -1 \; 1]\{\phi1 \;\; \phi2 \}^T$

Putting this value in eq1, we get

$$\Phi= <1 \; x > (1/L) [L \; 0; -1 \; 1]\{\phi1 \;\; \phi2 \}^T$$

$=< (1-x/L), x/L > \{ \phi1, \phi2\}^T$

Or,

$\Phi = < N1,N2 > \{\phi1,\phi2\}^T$

$= < N > \{\underline{\phi}\}$

Where N1 and N2 are shape functions given by N1= (1-x/L) and n2=x/L

$\{\underline{\phi}\}$ = nodal coordinates

Equivalent stress-strain relation of a shaft is,

$T= GJ^* \phi'_x$

Where, T= torsional moment

G= modulus of rigidity

J= moment of inertia in polar form

This eq. is analogous to, $\sigma= C^*\varepsilon$

Where C=GJ and $\varepsilon= \phi'_x$

$\Phi=< (1-x/L),x/L > \{ \phi1, \phi2 \}^T$

Or, $\phi'_x = < -1/L, 1/L > \{ \phi1, \phi2 \}^T$

Now the strain vs displacement relationship is expressed as,

$\varepsilon = \phi'_x = < -1/L, 1/L > \{ \phi1, \phi2 \}^T$

Or, $\varepsilon= [B] \{\underline{\phi}\}$

Or, $\varepsilon=[C] [B] \{\underline{\phi}\}$

Inertia force is $I\phi''_x$ acting on the shaft where I is the moment of inertia/unit length . hence total potential energy can be written as ,

$\prod = ½ \int \varepsilon^T\sigma\, dx + \int I\phi^T \phi''_x dx$

Or, $\prod = ½ \int [B]^T<\phi>^T[C][B] \{\underline{\phi}\} + I\int<N>\{N\}\{\underline{\phi''\}<\phi>}dx$

Or, $\prod = (<\phi>^T/2) (\int[B]^T [C][B]dx) \{\underline{\phi}\} + I\{\underline{\phi}\}\int<N>\{N\}\{\underline{\phi''}\}dx$

The principle of minimum potential energy requires,

$\prod'_{\{\underline{\phi}\}} = 0$

i.e. $(\int B^T CBdx) + I (\int N^T Ndx)\{\underline{\phi''}\} =\{0\}$

this is equivalent to.

$[K]_e\{\underline{\phi}\}_e + [M]_e\{\underline{\phi''}\}_e =\{0\}$

Where,

$[K]_e = (\int B^T CBdx)$

$= \int < -1/L, 1/L>^T GJ< -1/L, 1/L >dx$

$= GJL [1/L^2 -1/L^2; -1/L^2\ 1/L^2]$

$= (GJ/L)[1,-1 ; -1,1]$

And $[M]_e = I\int<(1-x/L), x/L>^T<(1-x/L),x/L>dx$

$= (IL/6) [2, 1; 1, 2]$

If $[M]_e$ is a lumped mass matrix (it is assumed that mass is lumped at nodes), then

$[M]_e= (IL/2)[1, 0; 0,1]$

Though consistent mass is less erroneous, the lumped mass gives better results because both stiffness and mass are overestimated, thus resulting in correcting in correct answer.

When all elements of the element stiffness matrices and mass matrices are assembled and boundary conditions incorporated, the final eq. of vibration is

$[M]\{\Phi''\} + [K]\{\Phi\} = \{0\}$

Assume the solution for $\{\phi\}$ as

$$\{\Phi\} = e^{i\omega t} \{A\}$$

So, $-\omega^2[M]\{A\} + [K]\{A\} = 0$

Or, $(1/ \omega^2)\{A\} = [K]^{-1}[M]\{A\}$

Or, $\lambda\{A\} = [K]^{-1}[M]\{A\}$

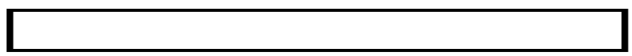
Where $\omega = 1/ \lambda^{½}$

This is a typical eigen value problem

Thus the lower natural frequencies the vibration system can be computed.

FREE AXIAL VIBRATION OF ROD:

The total degrees of freedom for a bar element are the axial displacements at the two ends of the element instead of the angle of twist for torsional vibration .



(fig. 1)

In the above bar, axial displacement at any section is written as,

. $u=< (1-x/L), x/L > \{u1,u2\}^T$

Or, $u=<N > \{\underline{u}\}$

Let’s consider un-damped free vibration case.

The net potential energy may be expressed as,

$$\prod = (<u>^T/2) (\int[B]^T [C][B]dx) \{\underline{\phi}\} + \boldsymbol{\rho A}\{\underline{u}\}\int<N>\{N\}\{\underline{u''}\}dx$$

Since,inertia force is mu’’ = ρAu’’acting on the bar shaft where m is mass/unit length.

Hence, stiffness matrix of an element for axial vibration of rod is the same as torsional vibration of the rod except that GJ is to be replaced by EA.

I in torsional vibration must be replaced by **Aρ** in axial vibration.

Hence, $[K]_e = (EA/L)[1,-1 ; -1,1]$

And, $[M]_e = A\rho\int <(1-x/L),x/L>^T <(1-x/L),x/L> dx$

Thus, when all elements of all the element stiffness and element inertia matrices are assembled and boundary conditions incorporated, the final eq. of free vibration is solved to find the natural frequencies.

$Mu'' + Ku = 0$

Assume the solution for $\{\phi\}$ as

$$\{u\} = e^{i\omega t} \{A\}$$

So, $-\omega^2[M]\{A\} + [K]\{A\} = 0$

Or, $(1/\omega^2)\{A\} = [K]^{-1}[M]\{A\}$

Or, $\lambda\{A\} = [K]^{-1}[M]\{A\}$

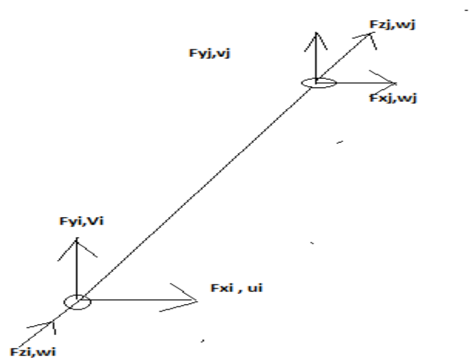
Where $\omega=1/\lambda^{½}$

This is a typical eigen value problem

Thus we can find the lowest natural frequency of the vibration system.

FREE VIBRATION OF PLANAR TRUSS:

Consider a truss element oriented as shown in the global coordinate system



(fig.2)

Since the truss elements are only subjected to axial forces,

$\{Fzi,Fzj\}^T= EA/L[1,-1 ; -1,1]\{w_i,w_j\}^T$

The local displacement $< w_i,w_j >$ can be written as in terms of global coordinates.

$\{w_i,w_j\}^T = [c,s,0,0 ; 0,0,c,s][u_i,v_i,u_j,v_{j\,]}{}^T$ …….eq1

Where c=cosαand s=sinα

$w_i=u_i\cos\alpha+v_i\sin\alpha$

$w_j= u_j\cos\alpha+v_j\sin\alpha$

Eq1 can be written as,

$\{q\}_{1=}[T]\{q\}_g$

Where $\{q\}_l$=local displacement

$\{q\}_g$=global displacement

Since F=Kq

Hence, $\{F\}g= [T]^T\{F\}l$

Or, $\{F\}g = [T]^T\{K\}l\{q\}l$

Or, $\{F\}g = [T]^T\{K\}l[T]\{q\}g$

Or, $\{F\}g= [K]g\{q\}g$

Where,

$[K]g=[T]^T\{K\}l[T]$

$= [c,s,0,0 ; 0,0,c,s]^{T(}EA/L)[1,-1 ; -1,1] [c,s,0,0 ; 0,0,c,s]$

$= (EA/L)[c^2\ \ cs\ \ -c^2\ -cs ; cs\ \ s^2\ \ -cs\ \ -s^2 ; -c^2\ \ -cs\ \ cs ; -cs\ \ -s^2\ \ cs\ \ \ s^2]$

Mass matrix in global system,

$[M]_g = \boldsymbol{\rho}AL/6[2\ 0\ \ 1\ 0 ; 0\ \ 2\ \ 0\ \ 1 ; 1\ \ 0\ \ 2\ \ 0 ; 0\ \ 1\ \ 0\ \ 2]$

Then the dynamic equation,

$[K]g - \omega^2[M]_g= 0$

Solving this equation and by finding the square root of diagonal elements of the eigen vector we can determine the natural frequencies.

FREE VIBRATION OF BERNAULI-EULER BEAM :

Here the beam has 2 degree of freedomper nodes,(1 translation and 1 rotation in both nodes I and j) hence a 4 DOF system.

The displacement function,w= α1 + x α2+x^2 α3+x^3 α4

i.e, $w = < 1\ x\ \ x^2\ x^3 > \{ \alpha_1\ \alpha_2\ \alpha_3\ \ \alpha_4 \}^t$

or, w= <d>{α }

The degree of freedom at the two ends of the element are written as,

$$\begin{Bmatrix} wi \\ \theta i \\ wj \\ \theta j \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0. & 0. \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \{ \alpha \}$$

Since θ= dw/dx (slope)

or, $\{ \underline{w} \} = [A] \{ \alpha \}$

or, $\{ \alpha \} = [A]^{-1} \{ \underline{w} \}$

Hence, $w = <d> [A]^{-1} \{ \underline{w} \}$

or, $w = < N > \{ \underline{w} \}$

Where N is the shape function.

$N_1 = (1- 3x^2/L^2 + 2x^3/L^3)$,

$N_2 = (x - 2x^2/L + x^3/L^2)$,

$N_3 = (3x^2/L^2 – 2 x^3/L^3)$,

$N_4 = (-3 x^2/L + x^3/L^2)$

Strain enegy, $U = EI/2 \int (d^2w/dx^2)^2 dx$

$= EI/2 <\underline{w}> \int \{Nxx\} < Nxx > dx \{ \underline{w} \}$

Or, $dU/dx = [K] \{ \underline{w} \}$ (since $U=1/2\ kx^2$ and $dU/dx= kx$)

Hence, $[K] = EI \int \{Nxx\} < Nxx > dx$

Where, $\{Nxx\}^t = < (-6/ L^2 +12x/ L^3), (-4/L+6x/ L^2), (6/ L^2 -12x/L), (-2/L +6x/ L^2)>$

So, element stiffness matrix is given by,

$$[K]e = EI/L^3 \begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2*L^2 & -6*L & 4*L^2 \end{pmatrix}$$

Kinetic energy, $T = w^2 /2 \int \rho A\ w^2 dx$

Or, $T = w^2\rho A/2<\underline{w}> \int \{N\} < N> dx$

$= \omega^2[M]$

Where $[M] = \rho A \int \{N\} < N> dx$

So,

$$[M] = \rho AL/42054 \begin{pmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 13L & 156 & -22L & \\ -13L & -3L^2 & -22L & 4L^2 \end{pmatrix}$$

Thus, when the elements of all element stiffness and inertia matrices of every nodes are assembled followed by incorporation of boundary conditions, the final eq. of free vibration is solved to get the natural frequencies.

$Mu'' + Ku = 0$

Assume the solution for $\{\phi\}$ as

$$\{u\} = e^{i\omega t} \{A\}$$

So, $-\omega^2[M]\{A\} + [K]\{A\} = 0$

Or, $(1/\omega^2)\{A\} = [K]^{-1}[M]\{A\}$

Or, $\lambda\{A\}=[K]^{-1}[M]\{A\}$

Where $\omega=1/\lambda^{1/2}$

This is a typical eigen value problem

Thus we can find the lowest natural frequency of the vibration system .

FREE VIBRATION OF TIMOSHENKO BEAM :

Timoshenko beams are deep beams .with the increasing depth of beam, the effect of transverse shear deformation and rotary inertia become more important.

The deflection function is given by \,

$w = < 1\ x\ \ x^2\ x^3 > \{ \alpha_1\ \alpha_2\ \alpha_3\ \ \alpha_4 \}^t$

the relation between transverse shear strain ϒ, w' and θ is

$w' = \Upsilon + \theta$

where θ denotes the slope deflection curve due to bending deflection alone.

Transverse shear strain may be taken as a constant independent of x,

$\Upsilon = \beta_o$(assumed)

Moment curvature relationship is given by,

$M = -EI\ d\theta/dx$

Shear force V is related to transverse shear strain by,

$V = KAG\ \Upsilon$

Where, K = Timoshenko's shear constant

= 5/6 (rectangular section)

= 9/10 (circular section)

Bending moment and shear force are related as follows .

$V = dM/dx$

we know, $w' = \theta + \Upsilon$

or, $\theta = w' - \Upsilon$

$w = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$

or, $w' = \alpha_2 + 2x\,\alpha_3 + 3x^2\,\alpha_4$

or, $w'' = 2\alpha_3 + 6x\,\alpha_4$

or, $w''' = 6\,\alpha_4$

we have

$d\theta/dx = w''$

or, $d^2\theta/dx^2 = w'''$

we know that

$V = dM/dx = -EI\, d^2\theta/dx^2$

Or, $KAG\,\Upsilon = -EI * 6\,\alpha_4$

Or, $\Upsilon = -6EI\alpha_4 / KAG$

Or, $\beta_0 = -\,\alpha_4\phi L^2/2$ (where $\phi = 12EI/KAGL^2$)

Substituting the values we get nodal displacement as,

$$\begin{Bmatrix} wi \\ \theta i \\ wj \\ \theta j \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -\varphi L^2/2 & 0 \\ 1 & L & L^2 & L^3 \\ 1 & 2L & (3-\varphi/2)L^2 & 0 \end{bmatrix} \{\alpha\}$$

$\{\underline{q}\} = [A]\{\alpha\}$

Or, $\{\alpha\} = [A]^{-1}\{\underline{q}\}$

Hence, $w = < 1\ x\ \ x^2\ \ x^3 > [A]^{-1}\{\underline{q}\}$

Or, $w = < N > \{\underline{q}\}$

Where, N is the shape function for nodal DOF(degree of freedom).

The equation for strain energy,if we add the effect of axial loads,

$$U = EIL/2\int(d\theta/dx)^2 dx + KAG/2\int(dw/dx)^2 dx$$

By simplifying it and comparing it with $U = 1/2Kx^2$

$$[K]_e = EI/(1+\varphi) \begin{pmatrix} 12/L^3 & 6/L^2 & -12/L^3 & 6/L^2 \\ 6/L^2 & (4+\varphi)/L & -6/L^2 & (2-\varphi)/L \\ -12/L^3 & -6/L^2 & 12/L^3 & -6/L^2 \\ 6/L^2 & (2-\varphi)/L & -6/L^2 & (4+\varphi)/L \end{pmatrix}$$

$$+\ p \begin{pmatrix} 6/5L & 1/10 & -6/5L & 1/10 \\ 1/10 & 2L/15 & -1/10 & 6/5L \\ -6/5L & -1/10 & 6/5L & -1/10 \\ 1/10 & -L/30 & -1/10 & 2L/15 \end{pmatrix}$$

Kinetic energy is given by,

$$T = 1/2\int \rho A(dw/dt)^2 dx + 1/2\int PI(d\theta /dt)^2 dx$$

Simplifying and comparing it with $T = 1/2mv^2$,

$$[M]= \rho AL*$$

$$\begin{pmatrix} (13/35 +7\varphi/10+ \varphi^2/3) & & & \text{SYM} \\ (11/210+11\varphi/120+ \varphi^2/24)L & (1/105+ \varphi/60+ \varphi^2/120)L^2 & & \\ (9/10+3\varphi/10+ \varphi^2) & (13/420+3\varphi/40+ \varphi^2/24)L & (13/35+7\varphi/10+ \varphi^2/3)L & \\ -(13/420+3\varphi/40+ \varphi^2/24)L & -(1/140+ \varphi/60+ \varphi^2/120)L^2 & -(11/210+11\varphi/120+ \varphi^2/24)L & (1/105+\varphi/60+\varphi^2/120) \end{pmatrix}$$

$$+\rho I/(1+ \varphi^2)L$$

$$\begin{pmatrix} 6/5 & & & \text{SYM} \\ (.1- \varphi/2)L & (2/15+ \varphi/6+ \varphi^2/3)L^2 & & \\ -6/5 & (-.1+ \varphi/2)L & -6/5 & \\ (1.1- \varphi/2)L & (-1/30- \varphi/6+ \varphi^2/6)L^2 & (-.1+ \varphi/2)L & (2/15+ \varphi/6+ \varphi^2/3)L^2 \end{pmatrix}$$

Thus, when all element stiffness and mass matrices are assembled followed by the incorporation of boundary conditions, the final simultaneous eq. of free vibration is solved.

$$[M]\{u''\} + [K]\{u\} =\{0\}$$

Assume the solution for $\{\phi\}$ as

$$\{u\} = e^{i\omega t} \{A\}$$

So, $-\omega^2[M]\{A\} + [K]\{A\} = 0$

Or, $(1/ \omega^2)\{A\} = [K]^{-1}[M]\{A\}$

Or, $\lambda\{A\}=[K]^{-1}[M]\{A\}$

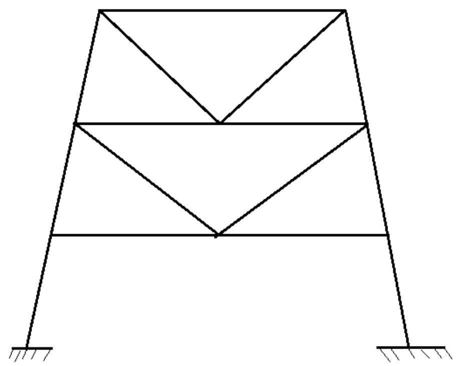
Where $\omega=1/ \lambda^{1/2}$

This is a typical eigen value problem

Thus we can find the lower natural frequencies of the structure.

FREE VIBRATION OF PLANE FRAME

Consider a plane framework as the one shown in figure.



(figure 3)

This plane frame is vibrating in its own plane. When applying finite element method to such structures,the following procedures should be used.

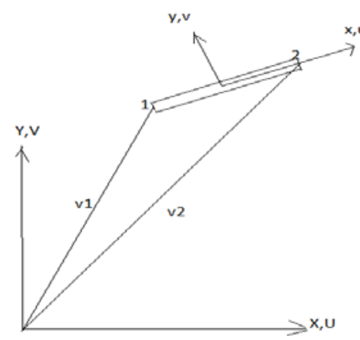
1. Divide each member into appropriate no of elements.
2. Derive the energy expressions for each element in terms of nodal degrees of freedom relative to a local seta of axes.
3. Transform the energy expressions for each element into expressions involving nodal degrees of freedom relative to a common set of global axes.
4. Add the energies of the elements together.

The kinetic energy,

$T_e = 1/2 \int \rho A(u'^2 + v'^2)dx$

The potential energy,

$U_e = \int EA(du/dx)^2dx + 1/2\int EI_z(d^2v/dx^2)^2dx$



(figure 4)

Thedisplacement functions are

$$u = [N_u(\varepsilon)]\{u\}_e$$

$$v = [N_v(\varepsilon)]\{v\}_e$$

the kinetic energy,

$$T_e = 1/2[u_1' \; u_2']^t \rho Aa/3 \; [2 \; 1;1 \; 2][\, u_1' \; u_2'] \; + 1/2$$

$$\begin{pmatrix} v_1' \\ \Theta_{z1}' \\ v_2' \\ \Theta_{z2}' \end{pmatrix}^t \rho Aa/120 \begin{pmatrix} 78 & 22a & 27 & -13a \\ 22a & 8a^2 & 13a & -6a^2 \\ 27 & 13a & 78 & -22a \\ -13a & -6a^2 & -22a & 8a^2 \end{pmatrix} \begin{pmatrix} v_1' \\ \Theta_{z1}' \\ v_2' \\ \Theta_{z2}' \end{pmatrix}$$

This expression may be expressed in a more compacted form as,

$$T_e = 1/2 \; [\, u_1' \; v_1' \; \theta_{z1}' \; u_2' \; v_2' \; \theta_{z2}' \,]^t \rho Aa/105$$

$$\begin{pmatrix} 70 & 0 & 0 & 35 & 0 & 0 \\ 0 & 78 & 22a & 0 & 27 & -13a \\ 0 & 22a & 8a^2 & 0 & 13a & -6a^2 \\ 35 & 0 & 0 & 70 & 0 & 0 \\ 0 & 27 & 13a & 0 & 78 & -22a \\ 0 & -13a & -6a^2 & 0 & -22a & 8a^2 \end{pmatrix} [\, u_1' \; v_1' \; \theta_{z1}' \; u_2' \; v_2' \; \theta_{z2}' \,]$$

$$= ½ \{u'\}^t_e [m]_e \{u'\}_e$$

$$[m]_e = \frac{\rho A a}{105}\begin{pmatrix} 70 & 0 & 0 & 35 & 0 & 0 \\ 0 & 78 & 22a & 0 & 27 & -13a \\ 0 & 22a & 8a^2 & 0 & 13a & -6a^2 \\ 35 & 0 & 0 & 70 & 0 & 0 \\ 0 & 27 & 13a & 0 & 78 & -22a \\ 0 & -13a & -6a^2 & 0 & -22a & 8a^2 \end{pmatrix}$$

Substituting the displacement function into strain energy function we get

$U = 1/2\{u\}^t_e[k]_e\{u\}_e$

Where, $[k]_e$ =

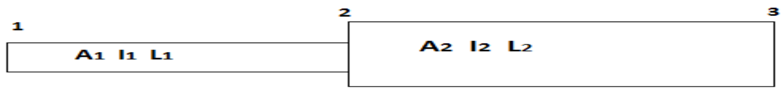
$$\frac{EI_z}{2a^3}\begin{pmatrix} (a/r_z)^2 & 0 & 0 & -(a/r_z)^2 & 0 & 0 \\ 0 & 3 & 3a & 0 & -3 & 3a \\ 0 & 3a & 4a^2 & 0 & -3a & 2a^2 \\ -(a/r_z)^2 & 0 & (a/r_z)^2 & 0 & 0 & \\ 0 & -3 & 00 & 3 & -3a & 0 \\ 0 & 3*a & 2a^2 & 0 & -3a & 4a^2 \end{pmatrix}$$

Thus, when all element stiffness and inertia matrices of each node are assembled and boundary conditions incorporated, the final eq. of free vibration is solved.

$$[k]- \omega^2[M] = 0$$

Square root of the diagonal elements of the eigen vector will give the values of natural frequencies of the system.

FREE VIBRATION OF STEPPED BEAM :



(figure 5)

Where,

A_1= area of 1st beam section

A_2 = area of 1st beam section

I_1= moment of inertia of 1st beam section

I_1= moment of inertia of 2nd beam section

L_1= length of 1st beam section

L_2 = length of 2nd beam section

P = unit weight of beam material

$A_2 = \alpha A_1$ and $I = I_2/I_1$

$I = \alpha^2$ and $L_1 = L_2 = 2a = L/2$

Assume degree of freedom per node = 2

Now, inertia matrix of 1st beam element is given by,

$$[M]_1 = \rho A_1 a/105 \begin{bmatrix} 78 & 22a & 27 & -13a \\ 22a & 8a^2 & 13a & -6a^2 \\ 27 & 13a & 78 & -22a \\ -13a & -6a^2 & -22a & 8a^2 \end{bmatrix}$$

similarly, inertia matrix of 1st beam element is given by,

$$[M]_2 = \rho(I_1)A1a/105 \begin{bmatrix} 78 & 22a & 27 & -13a \\ 22a & 8a^2 & 13a & -6a^2 \\ 27 & 13a & 78 & -22a \\ -13a & -6a^2 & -22a & 8a^2 \end{bmatrix}$$

Inertia mass of the stepped beam,

$$[M] = \rho A_1 a/105 \begin{bmatrix} 78 & 22a & 27 & -13a & 0 & 0 \\ 22a & 8a^2 & 13a & -6a^2 & 0 & 0 \\ 27 & 13a & 78(\sqrt{I}+1) & 22a(\sqrt{I}-1) & 27\sqrt{I} & -13a\sqrt{I} \\ -13a & -6a^2 & 22a(\sqrt{I}-1) & 8a^2(\sqrt{I}+1) & 13a\sqrt{I} & -6a^2\sqrt{I} \\ 0 & 0 & 27\sqrt{I} & 13a\sqrt{I} & 78\sqrt{I} & -22a\sqrt{I} \\ 0 & 0 & -13a\sqrt{I} & -6a^2\sqrt{I} & -22a\sqrt{I} & 8a^2\sqrt{I} \end{bmatrix}$$

Stiffness matrix of 1st beam element,

$$[K]_1 = EI_1/(2a^3) \begin{bmatrix} 3 & 3a & -3 & 3a \\ 3a & 4a^2 & -3a & 2a^2 \\ -3 & -3a & 3 & -3a \\ 3a & 2a^2 & -3a & 4a^2 \end{bmatrix}$$

Stiffness matrix of 2nd beam element,

$$[K]_2 = EI^*I_1/(2a^3) \begin{bmatrix} 3 & 3a & -3 & 3a \\ 3a & 4a^2 & -3a & 2a^2 \\ -3 & -3a & 3 & -3a \\ 3a & 2a^2 & -3a & 4a^2 \end{bmatrix}$$

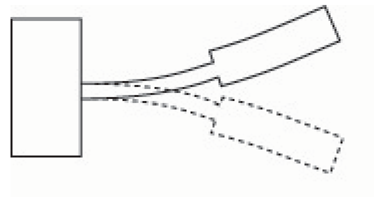
Now the stiffness matrix of the stepped beam is given by;

$$[K] = EI_1/(2a^3) \begin{bmatrix} 3 & 3a & -3 & 3a & 0 & 0 \\ 3a & 4a^2 & -3a & 2a^2 & 0 & 0 \\ -3 & -3a & 3(I+1) & 3a(I-1) & -3I & 3aI \\ 3a & 2a^2 & 3a(I-1) & 4(a^2)(I+1) & -3aI & 2(a^2)I \\ 0 & 0 & -3I & -3aI & 3I & -3aI \\ 0 & 0 & 3aI & 2(a^2)I & -3aI & 4(a^2)I \end{bmatrix}$$

- : BOUNDARY CONDITIONS:-

We have to find out [M] and [K] matrices for the above stepped beam for different boundary conditions.

1) C-F (clamped –free) :-



In this case x_1, θ_1 =0

(figure 6)

Hence we eliminate col(1),col(2),row(1),row(2) from [K] and [M] to find the respective matrices in C-F condition.

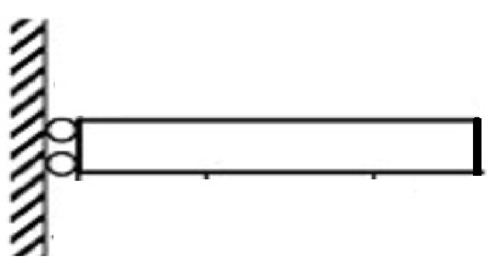
Inertia matrix,

$$[M]_{C\text{-}F} = \rho A_1 a/105 \begin{bmatrix} 78(\sqrt{I}+1) & 22a(\sqrt{I}-1) & 27\sqrt{I} & -13a\sqrt{I} \\ 22a(\sqrt{I}-1) & 8(a\wedge 2)(\sqrt{I}+1) & 13a\sqrt{I} & -6(a\wedge 2)\sqrt{I} \\ 27\sqrt{I} & 13a\sqrt{I} & 78\sqrt{I} & -22a\sqrt{I} \\ -13a\sqrt{I} & -6(a\wedge 2)\sqrt{I} & -22a\sqrt{I} & 8(a\wedge 2)\sqrt{I} \end{bmatrix}$$

Stiffness matrix,

$$[K]_{C\text{-}F} = EI_1/2a^3 \begin{bmatrix} 3(I+1) & 3a(I-1) & -3I & 3aI \\ 13a(I-1) & 4(a\wedge 2)(I+1) & -3aI & 2(a\wedge 2)I \\ -3I & -3aI & 3I & -3aI \\ 3aI & 2(a\wedge 2)I & -3aI & 4(a\wedge 2)I \end{bmatrix}$$

2)F-S (free- slide) :-



In this case, θ_2 =0

Hence eliminate col(6) and row(6) from the general [M],[K] to obtain the corresponding matrices for F-S b condition.

(figure 7)

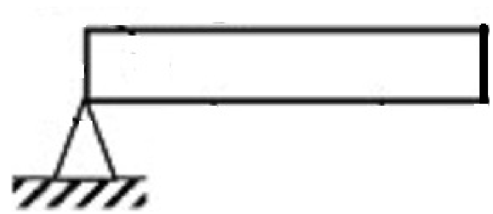
Inertia matrix,

$$[M]_{F\text{-}S}=\rho A_1 a/1055 \begin{bmatrix} 78 & 22a & 27 & -13a & 0 \\ 22a & 8a^2 & 13a & -6a^2 & 0 \\ 27 & 13a & 78(\sqrt{I}+1) & 22a(\sqrt{I}-1) & 27\sqrt{I} \\ -13a & -6a^2 & 22a(\sqrt{I}-1) & 8a^2(\sqrt{I}+1) & 13a\sqrt{I} \\ 0 & 0 & 27\sqrt{I} & 13a\sqrt{I} & 78\sqrt{I} \end{bmatrix}$$

Stiffness matrix,

$$[K]_{F\text{-}S}=EI_1/(2a^3) \begin{bmatrix} 3 & 3a & -3 & 3a & 0 \\ 3a & 4a^2 & -3a & 2a^2 & 0 \\ -3 & -3a & 3(I+1) & 3a(I-1) & -3I \\ 3a & 2a^2 & 3a(I-1) & 4(a^2)(I+1) & -3aI \\ 0 & 0 & -3I & -3aI & 3I \end{bmatrix}$$

3)<u>F-P(free-pinned)</u> :-



In this boundary condition, $x_2=0$

Hence eliminate col(5) and row(5) from the general [M],[K] to obtain the corresponding matrices for F-P boundary condition.

(figure 8)

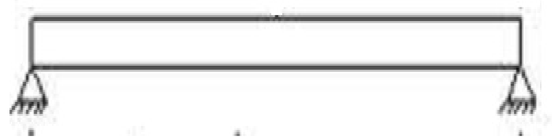
Inertia matrix, $$[M]_{F\text{-}P}= \rho A_1 a/105 \begin{bmatrix} 78 & 22a & 27 & -13a & 0 \\ 22a & 8a^2 & 13a & -6a^2 & 0 \\ 27 & 13a & 78(\sqrt{I}+1) & 22a(\sqrt{I}-1) & -13a\sqrt{I} \\ -13a & -6a^2 & 22a(\sqrt{I}-1) & 8a^2(\sqrt{I}+1) & -6a^2\sqrt{I} \\ 0 & 0 & -13a\sqrt{I} & -6a^2\sqrt{I} & 8a^2\sqrt{I} \end{bmatrix}$$

Stiffness matrix,

. $$[K]_{F\text{-}P}=EI_1/(2a^3) \begin{bmatrix} 3 & 3a & -3 & 3a & 0 \\ 3a & 4a^2 & -3a & 2a^2 & 0 \\ -3 & -3a & 3(I+1) & 3a(I-1) & 3aI \\ 3a & 2a^2 & 3a(I-1) & 4(a^2)(I+1) & 2(a^2)I \\ 0 & 0 & 3aI & 2(a^2)I & 4(a^2)I \end{bmatrix}$$

4) P-P(pinned-pinned) :-

In this case, x_1, x_2=0



Hence eliminate row(1),col(1),col(5) and row(5) from the general [M],[K] to obtain the corresponding matrices for P-P case.

(figure 9)

Inertia matrix,

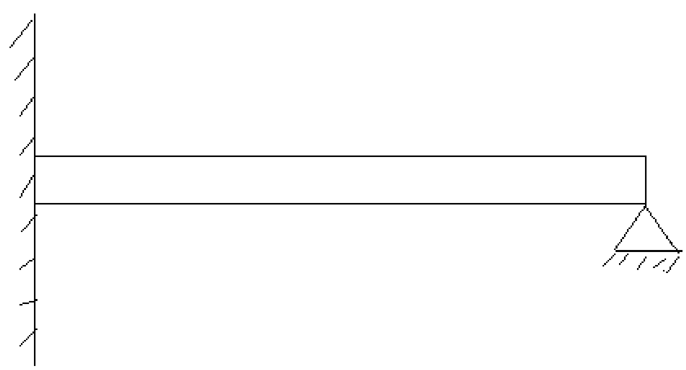
$$[M]_{P\text{-}P}=\rho A_1 a/105\begin{bmatrix} 8a^2 & 13a & -6a^2 & 0 \\ 13a & 78(\sqrt{I}+1) & 22a(\sqrt{I}-1) & -13a\sqrt{I} \\ -6a^2 & 22a(\sqrt{I}-1) & 8a^2(\sqrt{I}+1) & -6a^2\sqrt{I} \\ 0 & -13a\sqrt{I} & -6a^2\sqrt{I} & 8a^2\sqrt{I} \end{bmatrix}$$

Stiffness matrix,

$$[K]_{P\text{-}P}= EI_1/(2a^3)\begin{bmatrix} 4a^2 & -3a & 2a^2 & 0 \\ -3a & 3(I+1) & 3a(I-1) & 3aI \\ 2a^2 & 3a(I-1) & 4(a^2)(I+1) & 2(a^2)I \\ 0 & 3aI & 2(a^2)I & 4(a^2)I \end{bmatrix}$$

5)C-P(clamped-pinned) :-

In this boundary condition, x_1, θ_1, x_2 =0



Hence eliminate row(1),col(1),row(2),col(2),col(5) and row(5)

from the [M] and [K] to get the corresponding matrices for C-P case.

.

(figure 10)

Inertia matrix,

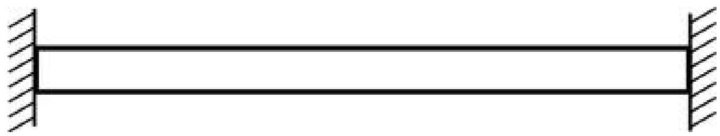
$$[M]_{C\text{-}P}=\rho A_1 a/105\begin{bmatrix} 78(\sqrt{I}+1) & 22a(\sqrt{I}-1) & -13a\sqrt{I} \\ 22a(\sqrt{I}-1) & 8(a^2)(\sqrt{I}+1) & -6(a^2)\sqrt{I} \\ -13a\sqrt{I} & -6(a^2)\sqrt{I} & 8(a^2)\sqrt{I} \end{bmatrix}$$

Stiffness matrix,

$$[K]_{C\text{-}P}= EI_1/(2a^3)\begin{bmatrix} 3(I+1) & 3a(I-1) & 3aI \\ 3a(I-1) & 4a^2(I+1) & 2a^2I \\ 3aI & 2a^2I & 4a^2I \end{bmatrix}$$

6) C-C(clamped-clamped) :-

In this boundary condition, x_1, θ_1, x_2, θ_2 =0



(figure 11)

Hence eliminate row(1),col(1),row(2),col(2),col(5),row(5),col(6) and row(6) from the general [M],[K] to obtain the corresponding matrices for C-C boundary condition.

Inertia matrix,

$$[M]_{C\text{-}C}=\rho A_1 a/105\begin{bmatrix} 78(\sqrt{I}+1) & 22a(\sqrt{I}-1) \\ 22a(\sqrt{I}-1) & 8a^2(\sqrt{I}+1) \end{bmatrix}$$

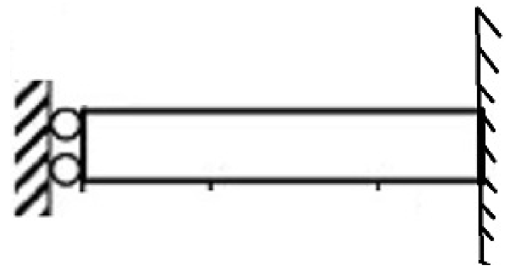
Stiffness matrix,

$$[K]_{C\text{-}C}=EI_1/(2a^3)\begin{bmatrix} 3(I+1) & 3a(I-1) \\ 3a(I-1) & 4a^2(I+1) \end{bmatrix}$$

\

7)C-S(clamped-slide) :-

In this boundary condition, x_1, θ_1, θ_2 =0



(figure 12)

Hence eliminate row(1),col(1),row(2),col(2),col(6) and row(6) from the general [M],[K] to obtain the corresponding matrices for C-S boundary condition.

Inertia matrix,

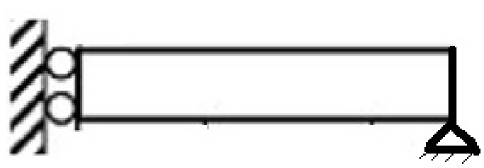
$$[M]_{C\text{-}S}=\rho A_1 a/105\begin{bmatrix} 78(\sqrt{I}+1) & 22a(\sqrt{I}-1) & 27\sqrt{I} \\ 22a(\sqrt{I}-1) & 8(a^2)(\sqrt{I}+1) & 13a\sqrt{I} \\ 27\sqrt{I} & 13a\sqrt{I} & 78\sqrt{I} \end{bmatrix}$$

Stiffness matrix,

$$[K]_{C\text{-}S}= EI_1/(2a^3)\begin{bmatrix} 3(I+1) & 3a(I-1) & -3I \\ 3a(I-1) & 4a^2(I+1) & -3aI \\ -3I & -3aI & 3I \end{bmatrix}$$

8)S-P(slide-pinned) :-

In this boundary condition, θ_2, x_2 =0



(figure 13)

Hence eliminate row(2),col(2),col(5) and row(5) from the general [M],[K] to obtain the corresponding matrices for S-P boundary condition.

Inertia matrix,

$$[M]_{S\text{-}P}=\rho A_1 a/105\begin{bmatrix} 78 & 27 & -13a & 0 \\ 27 & 78(\sqrt{I}+1) & 22a(\sqrt{I}-1) & -13a\sqrt{I} \\ -13a & 22a(\sqrt{I}-1) & 8a\text{^}2(\sqrt{I}+1) & -6a\text{^}2\sqrt{I} \\ 0 & -13a\sqrt{I} & -6a\text{^}2\sqrt{I} & 8a\text{^}2\sqrt{I} \end{bmatrix}$$

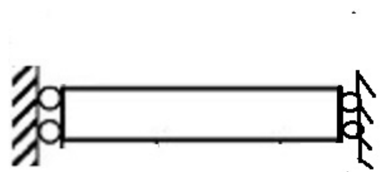
Stiffness matrix,

$$[K]_{S\text{-}P}= EI_1/(2a^3)\begin{bmatrix} 3 & -3 & 3a & 0 \\ -3 & 3(I+1) & 3a(I-1) & 3aI \\ 3a & 3(I-1) & 4(a\text{^}2)(I+1) & 2(a\text{^}2)I \\ 0 & 3aI & 2(a\text{^}2)I & 4(a\text{^}2)I \end{bmatrix}$$

9)<u>S-S(sliding-sliding)</u> :-

In this case, θ1,θ2=0

Hence eliminate row(2),col(2),col(6) and row(6) from the general [M],[K] to obtain the corresponding matrices for S-S condition.



Inertia matrix, ...

(figure 14)

$$[M]_{S\text{-}S}=\rho A_1 a/105\begin{bmatrix} 78 & 27 & -13a & 0 \\ 27 & 78(\sqrt{I}+1) & 22a(\sqrt{I}-1) & 27\sqrt{I} \\ -13a & 22a(\sqrt{I}-1) & 8a\text{^}2(\sqrt{I}+1) & 13a\sqrt{I} \\ 0 & 27\sqrt{I} & 13a\sqrt{I} & 78\sqrt{I} \end{bmatrix}$$

Stiffness matrix,

$$[K]_{S\text{-}S}= EI_1/(2a^3)\begin{bmatrix} 3 & -3 & 3a & 0 \\ -3 & 3(I+1) & 3a(I-1) & -3I \\ 3a & 3(I-1) & 4(a\text{^}2)(I+1) & -3aI \\ 0 & -3I & -3aI & 3I \end{bmatrix}$$

Chapter – 4

RESULTS

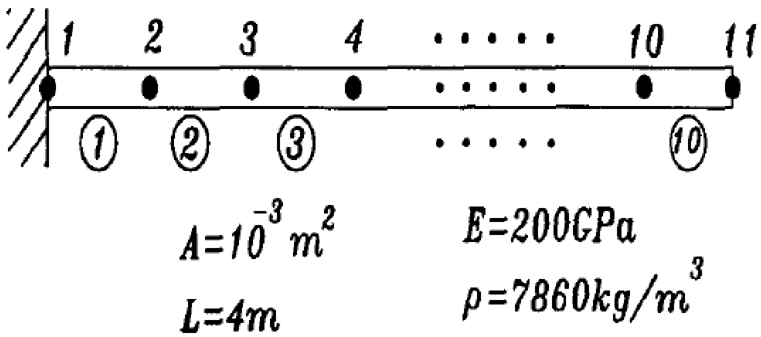
__Example 4.1__:(Natural Frequencies of a Bar)

[Kwon and Bang, example-7.5.1]

Elastic modulus= 200GPa,

Cross sectional area = .001m^2

Density = 7860 kg/m^3



(figure 15)

The natural frequencies are computed using FEM and compared with exact solutions.

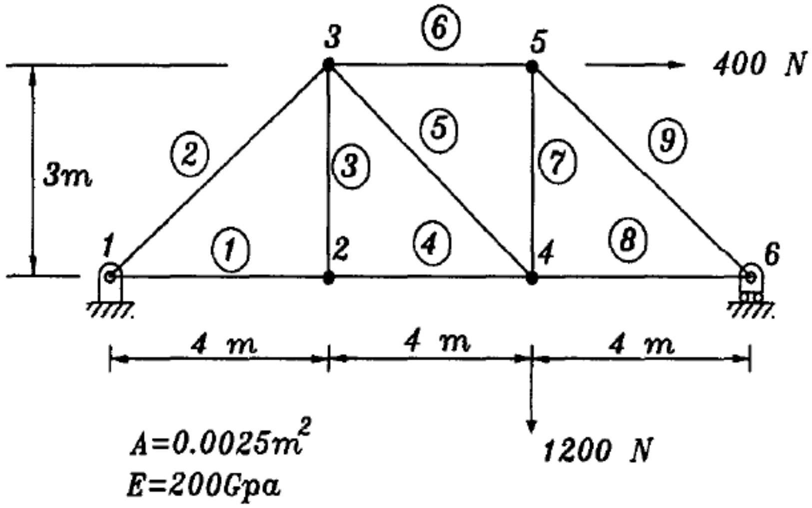
MODES	NATURAL FREQUENCIES		%ERROR
	FEM	Exact Solution	
1	0.	0	0.
2	4060	3962	2.47
3	8737	7924	10.25
4	14198	11895	19.36
5	17474	15847	10.26

(table 1)

__EXAMPLE 4.2__ : (natural frequency of a truss)

[Kwon and Bang,7.5.2]

Each member of the truss shown has the density of 7860 kg/m^3



(figure 16)

The first five natural frequencies of the above truss structure were given below.

1^{st} natural frequency = 240.9 rad/s

2^{nd} natural frequency = 467.9 rad/s

3^{rd} natural frequency = 739.8rad/s

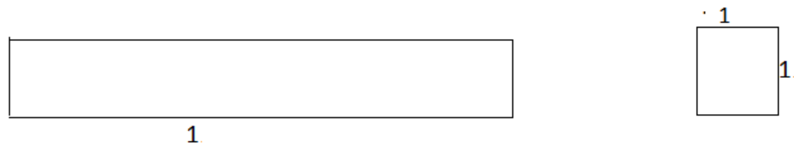
4^{th} natural frequency = 1243rad/s

5^{th} natural frequency = 1633rad/s

EXAMPLE 4.3: (natural frequency of a beam)

[Kwon and Bang, example-8.10.1]

A free beam has unit length. It has a cross section 1 by 1 with unit density. The elastic modulus of the beam is 12. Use 4 elements to model the beam structure, so that non-symmetric mode shapes can be concluded.



(figure 17)

The natural frequencies of the given beam were given below in tabular form.

MODES	FEM solution	Exact solution	% Error
1	0	0	0
2	0	0	0
3	22.400	22.373	.12
4	62.06	61.673	.63
5	121.86	120.90	.79
6	223.29	178.27	25.25

(Table 2)

It was clearly visible that at lower frequencies the two solutions agreed well.

However discrepancy occurred at larger frequencies.Refinement of finite element mesh is required to lower this discrepancy at higher frequencies.

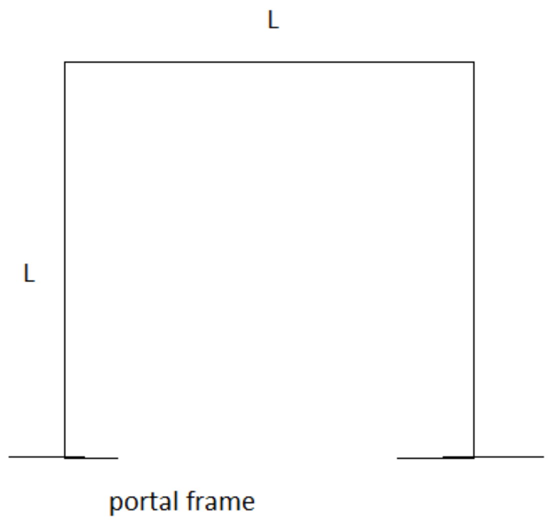
.*EXAMPLE* 4.4: (natural frequencies of a portal frame)

[S. Rajasekaran, example-14.6]

A portal frame is characterized by the following data.

$A = 1.85187 * 10^{-5}$, $I = 2.85785 * 10^{-11}$,

$\rho = 25613.5$ kg/m^3 ,L=.2413m



(figure 18)

The natural frequencies of the portal frame were given below in a tabular form.

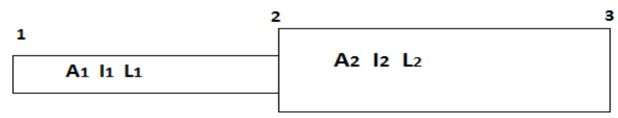
MODE	NATURAL FREQUENCY (rad/s)
1	1.0
2	195.78
3	777.17
4	1274.51
5	1387.34
6	3135.28

(Table3)

EXAMPLE - 4.5 : (natural frequencies of a stepped beam)

[S.K. Jang and C.W. Bert[3]]

A stepped beam is shown in the figure.



(figure 19)

The stepped beam has the following parameters.

A_1= area of 1st beam section . A_2= area of 1st beam section

I_1= moment of inertia of 1st beam section

I_2= moment of inertia of 2nd beam section

L_1= length of 1st beam section

L_2 = length of 2nd beam section .

P = unit weight of beammaterial

$A_2 = \alpha A_1$

$I = I_2/I_1$

$I = \alpha^2$

$L_1 = L_2 = 2a = L/2$

TABLE :-

Natural frequencies,$\omega' = \omega/L^2(EI_1/\rho A_1)^{1/2}$ of fundamental modefor various boundary conditions were given below in a tabular form.

Boundary Conditions	I $(=I_2/I_1)$	Exact Solution	FEM Result	% error
Pinned-pinned (P-P)	1	9.8696	9.9086	.395
	5	10.4129	10.4441	.299
	10	9.8781	9.9002	.223
	20	9.0747	9.0885	.152
	40	8.1369	8.1448	.097
Clamped –clamped (C-C)	1	22.3733	22.7375	1.620
	5	25.9591	26.3573	1.534
	10	27.6807	28.0922	1.486
	20	30.3213	30.7716	1.485
	40	34.3252	34.8702	1.587
Clamped- Free (C-F)	1	3.5160	3.5177	.048
	5	2.4373	2.4376	.012
	10	2.0629	2.0630	.004
	20	1.7418	1.7418	0
	40	1.4685	1.4685	0
Clamped- Pinned (C-P)	1	15.4182	15.5608	.924
	5	16.2811	16.3761	.583
	10	15.5129	15.5783	.421
	20	14.2568	14.2967	.279
	40	12.7501	12.7721	.172
Free - Free (F-F)	1	22.3733	22.4234	.223
	5	24.1650	24.2127	.197
	10	23.5459	23.5809	.137
	20	22.4725	22.5056	.147
	40	21.1907	21.2306	.188
Slide - Slide (S-S)	1	9.8696	9.9101	.410
	5	13.5124	13.5919	.588
	10	15.9066	16.0330	.794
	20	18.2949	18.4909	1.070
	40	20.1954	20.4501	1.261

Boundary Condition	l	Exact solution	FEM solution	%error
Slide – Pinned (S-P)	1	2.4674	2.4680	.024
	5	2.4372	2.4377	.020
	10	2.3292	2.3296	.017
	20	2.1814	2.1844	.013
	40	2.0122	2.0124	.009
Clamped – Slide (C-S)	1	5.5933	5.6007	.132
	5	5.6912	5.6950	.066
	10	5.6321	5.6340	.033
	20	5.3573	5.3590	.031
	40	4.8913	4.8922	.018
Free – Slide (F-S)	1	5.5933	5.5994	.109
	5	9.3624	9.3807	.195
	10	11.0519	11.0924	.336
	20	12.4070	12.4513	.357
	40	13.2947	13.3481	.401
Free- Pinned (F-P)	1	15.4182	15.5142	.622
	5	18.6102	18.6838	.395
	10	18.7641	18.8202	.298
	20	18.4031	18.4566	.290
	40	17.7778	17.8301	.294

(table 4)

TABLE:Natural frequencies of stepped circular beam with various boundary conditions having varying step ratios.

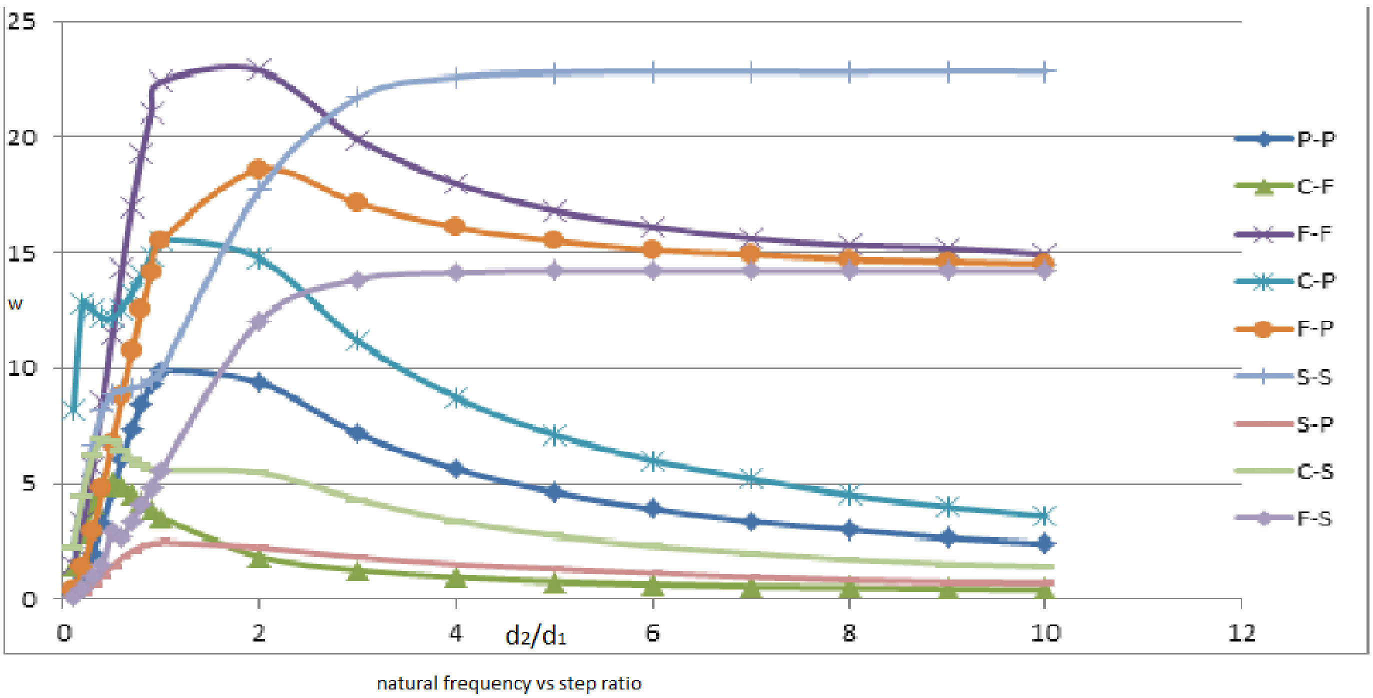
step ratio d_2/d_1 (d)	***Boundary conditions*** Exact solution [FEM solution] (% error)									
	P-P	C-C	C-F	F-F	C-P	F-P	S-S	S-P	C-S	F-S
0.1	.2376 [.2376] (0)	8.9213 [8.987] (.736)	1.4056 [1.4122] (.469)	1.4844 [1.4902] (.39)	6.1522 [8.1617] (32.66)	.3611 [.3611] (0)	2.2435 [2.2797] (1.61)	.0691 [.0691] (0)	2.2350 [2.2712] (1.62)	.1330 [.1333] (.225)
0.2	.9218 [.9219] (.01)	13.2701 [13.6794] (2.08)	2.7851 [2.7983] (.473)	3.3563 [3.3720] (.467)	11.3354 [12.8057] (12.97)	1.3826 [1.3829] (.021)	4.4875 [4.5581] (1.57)	.2733 [.2733] (0)	4.3960 [4.4660] (1.59)	.4797 [.4798] (.02)
0.3	1.9715 [1.9723] (.08)	13.2811 [13.5251] (1.837)	4.0110 [4.0925] (.203)	5.7328 [5.7512] (.32)	12.1092 [12.5134] (3.33)	2.9076 [2.9101] (.08)	6.5555 [6.6519] (1.47)	.5978 [.5978] (0)	6.1135 [6.2007] (1.43)	.9491 [.9500] (.09)
0.4	3.2680 [3.2712] (1.194)	13.6651 [13.8769] (1.549)	4.8249 [4.8449] (.041)	8.4943 [8.5080] (.279)	11.9355 [12.1755] (2.01)	4.7476 [4.7579] (.21)	8.0567 [8.1578] (1.25)	1.0040 [1.0041] (0.009)	6.8477 [6.9190] (1.04)	1.4857 [1.4861] (.02)
0.5	4.6769 [4.6851] (.175)	14.6697 [14.8866] (1.41)	5.0700 [5.0864] (.323)	11.5257 [11.4503] (.654)	11.9969 [12.1787] (1.51)	6.7361 [6.7603] (.36)	8.7794 [8.8638] (.961)	1.4280 [1.4281] (.007)	6.7728 [6.8184] (.67)	2.9076 [2.9073] (.01)
0.6	6.0706 [6.0859] (.252)	16.1756 [16.4184] (1.50)	4.9033 [4.9144] (.226)	14.3073 [14.3393] (.223)	12.4153 [12.5744] (1.28)	8.7463 [8.7889] (.48)	8.9907 [9.0545] (.709)	1.8024 [1.8027] (.016)	6.4247 [6.4527] (.43)	2.6968 [2.6983] (.05)
0.7	7.3431 [7.3664] (.317)	17.9092 [18.1867] (1.549)	4.5635 [4.5705] (.153)	16.9517 [16.9803] (.168)	13.1091 [13.2600] (1.51)	10.6841 [10.7463] (.58)	9.0572 [9.1066] (.545)	2.0884 [2.0888] (.019)	6.0786 [6.0967] (.29)	3.3665 [3.3686] (.06)
0.8	8.4202 [8.4507] (.362)	19.6133 [19.9253] (1.59)	4.1886 [4.1929] (.102)	19.2192 [19.2594] (.209)	13.9286 [14.0770] (1.06)	12.4769 [12.5560] (.63)	9.1843 [9.2258] (.451)	2.2825 [2.2830] (.02)	5.8216 [5.8341] (.21)	4.0764 [4.0795] (.07)

0.9	9.2635 [9.2994] (.387)	21.1232 [21.4641] (1.613)	3.8338 [3.8365] (.07)	21.0307 [21.0770] (.22)	14.7321 [14.8786] (.99)	14.0680 [14.1587] (.64)	9.4500 [9.4885] (.407)	2.4018 [2.4025] (.03)	5.6652 [5.6746] (.16)	4.8215 [4.8259] (.09)
1.0	9.8696 [9.9086] (.395)	22.3733 [22.7359] (1.62)	3.5160 [3.5177] (.048)	22.3733 [22.4210] (.213)	15.4182 [15.5608] (.92)	15.4182 [15.5142] (.62)	9.8696 [9.9101] (.41)	2.4674 [2.4680] (.02)	5.5933 [5.6007] (.13)	5.5933 [5.5994] (.109)
2	9.3538 [9.3701] (.174)	29.3393 [29.7732] (1.478)	1.8397 [1.8397] (0)	22.8515 [22.8852] (.147)	14.6995 [14.7468] (.32)	18.5595 [18.6100] (.27)	17.5587 [17.7275] (.961)	2.2342 [2.2346] (.018)	5.4699 [5.4719] (.04)	12.0199 [12.0611] (.34)
3	7.1485 [7.1526] (.057)	40.0354 [40.7176] (1.70)	1.2332 [1.2332] (0)	19.8611 [19.9003] (.019)	11.1549 [11.1660] (.09)	17.0666 [17.1201] (.31)	21.4171 [21.7198] (1.41)	1.8200 [1.8201] (.005)	4.3144 [4.3149] (.01)	13.7883 [13.8471] (.42)
4	5.6310 [5.6322] (.021)	53.0768 [54.2360] (2.18)	.9264 [.9261] (.034)	17.9407 [18.0013] (.337)	8.7256 [8.7287] (.03)	16.0259 [16.0800] (.34)	22.2601 [22.6001] (1.527)	1.4958 [1.4959] (.006)	3.3700 [3.3701] (.002)	14.0871 [14.1504] (.45)
5	4.6089 [4.6097] (.017)	66.3507 [68.3972] (3.08)	.7417 [.7420] (.04)	16.7816 [16.8102] (.17)	7.1104 [7.1100] (.005)	15.4132 [15.5120] (.64)	22.4377 [22.7906] (1.572)	1.2559 [1.3000] (3.51)	2.7325 [2.7326] (.003)	14.1349 [14.2012] (.47)
6	3.8887 [3.9013] (.324)	78.4064 [82.7020] (5.47)	.6183 [.6183] (0)	16.0549 [16.1232] (.425)	5.9831 [6.0016] (.309)	15.0391 [15.1003] (.41)	22.4702 [22.8276] (1.59)	1.0765 [1.1000] (2.18)	2.2901 [2.2901] (0)	14.1348 [14.2031] (.48)
7	3.3582 [3.3496] (.256)	85.8217 [97.0236] (13.05)	.5301 [.5345] (.83)	15.5772 [15.6213] (.283)	5.1577 [5.2000] (.82)	14.7981 [14.9078] (.74)	22.4676 [22.8276] (1.60)	.9392 [.9301] (.97)	1.9685 [2.0001] (1.6)	14.1256 [14.2009] (.53)
8	2.9528 [3.0012] (1.639)	88.1235 [98.1254] (15.64)	.4639 [.5003] (7.7)	15.2493 [15.3098] (.396)	4.5296 [4.5001] (.65)	14.6352 [14.7103] (.51)	22.4569 [22.8157] (1.583)	.8316 [.8012] (3.65)	1.7252 [1.7101] (.87)	14.1159 [14.2001] (.59)
9	2.6335 [2.6351] (.060)	88.8330 [99.2354] (16.01)	.4124 [.4456] (8.05)	15.0156 [15.1908] (.987)	4.0363 [4.0011] (.87)	14.5203 [14.6011] (.55)	22.4456 [22.8434] (1.77)	.7453 [.7230] (2.9)	1.5351 [1.5129] (1.45)	14.1076 [14.2023] (.67)
10	2.3758 [2.4001] (1.02)	89.1225 [103.3244] (19.87)	.3718 [.4120] (10.81)	14.8439 [14.9103] (.447)	3.6391 [3.6000] (1.07)	14.4366 [14.5130] (.53)	22.4354 [22.8396] (1.80)	.6748 [.7010] (3.9)	1.3825 [1.4012] (.97)	14.1008 [14.2034] (.73)

(Table 5.)

GRAPH:

Graph of natural frequencies vs step ratio for various boundary conditions, was plotted below.



(Figure 20)

Chapter – 5

CONCLUSION

AND

DISCUSSION

CONCLUSION :

FEM results showed quite fair accuracy when compared with the analytical methods.

%error in FEM can be minimized by increasing the finite elements and refinement of finite mesh which is not so difficult to compute by the help of computer.

FEM was quite efficient and better than the analytical in case of any irregularity or complexity of the structures.

FEM calculations in addition with MATLAB codes form a powerful medium for vibration analysis of irregular structures with many advantages over all other methods out there.

DISCUSSION:

FEM is effective in analyzing the physical properties, which are complex for any closed bound solution.

FEM is quite effective and time saving tool for solving the vibration analysis of any kind of structures from linear to complex non-linear structures.
It is simple to use unlike the tedious analytical methods. It is meant for modern world problems and it can be easily programmed with computers and, various structural softwares heavily depend upon FEM.

When a continuum is discretized, an infinite degrees of freedom system is converted into a model having finite number of degrees of freedom. The accuracy depends to a great extent on the mesh grading of the continuum.

In regions of high strain gradient, higher gradation of finite element mesh is needed whereas in the regions of lower strain, the mesh chosen may be coarser. As the element size decreases, the discretization error reduces.

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