FREE VIBRATION OF RODS, BEAMS AND FRAMES USING SPECTRAL ELEMENT METHOD

A THESIS

Submitted by

Anusmita Malik

For the award of the degree

of

MASTER OF TECHNOLOGY

STRUCTURAL ENGINEERING DIVISION
DEPARTMENT OF CIVIL ENGINEERING
NATIONAL INSTITUTE OF TECHNOLOGY, ROURKELA 769008
May 2013

NATIONAL INSTITUTE OF TECHNOLOGY
ROURKELA – 769008, ORISSA
INDIA

CERTIFICATE

This is to certify that the thesis entitled **"FREE VIBRATION OF RODS, BEAMS AND FRAMES USING SPECTRAL ELEMENT METHOD"** submitted by **Anusmita Malik** in partial fulfilment of the requirement for the award of **Master of Technology** degree in **Civil Engineering** with specialization in **Structural Engineering** to the National Institute of Technology, Rourkela is an authentic record of research work carried out by her under my supervision. The contents of this thesis, in full or in part, have not been submitted to any other Institute or University for the award of any degree or diploma.

Project Guide

Rourkela-769008

Dr. Manoranjan Barik

Date:

Associate Professor

Department of Civil Engineering

ACKNOWLEDGEMENT

First and foremost, praise and thanks goes to my God for the blessing that has been bestowed upon me in all my endeavours.

I am deeply indebted to **Dr. Manoranjan Barik,** Associate Professor, my advisor and guide, for the motivation, guidance, tutelage and patience throughout the research work. I appreciate his broad range of expertise and attention to detail, as well as the constant encouragement he has given me over the years. There is no need to mention that a big part of this thesis is the result of joint work with him, without which the completion of the work would have been impossible.

I extend my sincere thanks to the Head of the Civil Engineering Department **Prof. N. Roy**, for his advice and unyielding support over the year.

I would like to take this opportunity to thank my Parents and my sister for their unconditional love, moral support, and encouragement for the timely completion of this project.

I am grateful for friendly atmosphere of the Structural Engineering Division and all kind and helpful professors that I have met during my course.

I express my most sincere admiration to my friends, (in no particular order) **MonalishaPriyadarsini**, **Krishna Kumar**, **VenkyKotipalli** and to all my classmates for their cheering up ability which made this project work smooth.

So many people have contributed to my thesis, my education, and it is with great pleasure to take the opportunity to thank them. I apologize, if I have forgotten anyone.

Anusmita Malik

Roll No:211CE2027

M.Tech. (Structural Engineering)

National Institute of Technology

Rourkela-769008,Odisha, India

PREFACE

In the field of engineering the dynamic characteristics of a structure are of utmost importance for which these need considerable attention for greater accuracy. In today's world of digital computation the researchers' thrust is always towards this accuracy which may be achieved with a computational cost. The finite element method (FEM) has taken its place as a competent numerical tool to analyse the dynamics of a structure. Though it can handle the problems efficiently, it is limited to low frequency wave modes. When the structure vibrates with high frequency, the finite element method needs to be modelled with very large number of elements to capture all necessary frequencies of higher modes. The availability of computational softwares to have analytical solutions with Symbolic Algebra has enabled the researchers to handle the tedious algebraic expressions with less effort. These softwares are MATHEMATICA, MAPLE, Math CAD etc to name a few and MAT LAB to some extent.

The exact solution of the governing differential equations for the vibration problems with higher modes of vibration can be formed by using the shape functions which are frequency dependent. Thus using these frequency dependent shape functions the stiffness matrix which is dependent on the frequency can be formulated which is known as the Dynamic Stiffness Matrix (DSM). Once this Dynamic Stiffness Matrix for an element is formulated the global Dynamic Stiffness Matrix is obtained by following the procedure similar to that of the Finite Element Method (FEM). The great advantage of such a matrix is that even higher frequencies of a structure can be obtained by considering only few elements thus minimizing the computational cost.

In this thesis the higher modes of free vibration frequencies have been obtained for rods, beams and frames. In case of rods and beams up to the twentieth mode have been computed by considering only 2 spectral elements. In case of frame each member is considered as a single element and the frequencies even for higher modes have been achieved accurately. In some examples the number of finite elements required to converge to the exact analytical solution has been shown and it was noted that even for number of finite elements up to 200, the results couldn't converge even for the lower modes.

CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

Although all the principle symbols used in this thesis are defined in the text as they occur, a list of them is presented below for easy reference. On some occasions, a single symbol is used for different meanings depending on the context and thus its uniqueness is lost. The contextual explanations of the symbol as its appropriate place of use is hoped to eliminate the confusion.

English

a	constant vector
A	Cross sectional area
E	Young's modulus
EI	bending rigidity
G	modulus of rigidity(shear modulus)
GJ	torsional rigidity
I	moment of inertia
K	bending-torsion coupling rigidity
k	rate of twist
[K]	stiffness matrix
L	Length
M	bending moment

P	axial load
S	shear force
T	torque
m=ρA	mass per unit length
kAG	shear rigidity
h (y,t)	flexural translation
H (y)	vertical displacement
Θ (y)	bending rotation
Ψ (y)	twist
ω	discrete frequency
θ	angle of rotation
I_α	polar mass moment of inertia per unit length
I_x, I_z	two principal second moment of areas
k_L,k_F	wavenumber
$M_t(x, t)$	transverse shear force
$Q_t(x, t)$	bending moment
$S_R(\omega)$	spectral element matrix
u (x,t)	longitudinal displacement
w (x,t)	transverse displacement

ρ	mass density
φ (y,t)	torsional rotation
{u}	displacement
$\{\dot{u}\}$	velocity
$\{\ddot{u}\}$	acceleration

CHAPTER 1

Introduction

1.1 Theoretical Background

1.1.1 Dynamic Stiffness Method (DSM)

The 1941 work by Kolousek is probably the first to derive the dynamic stiffness matrix for the Bernoulli-Euler beam. Przemieniecki introduced the formulation of the frequency-dependent mass and stiffness matrices for both bar and beam elements. In contrast to the conventional finite element mass and stiffness matrices, which results in the linear eigen problems, the exact dynamic stiffness matrices results in transcendental eigen problems, the coefficients of which are the transcendental functions of frequency. Thus, a drawback of the dynamic stiffness method is that it is not an easy task to compute all natural frequencies accurately by solving the transcendental eigenvalue problems.

In 1971 this difficulty was resolved by Wittrick and Williams by developing by the well-known Wittrick-Williams algorithm for automatic calculation of undamped natural frequencies. The exact dynamic stiffness matrix is used in the DSM. The exact dynamic stiffness matrix is formulated in the frequency domain by using exact dynamic shape functions that are derived from the exact wave solutions to the governing differential equations. To obtain the exact wave solutions in the frequency domain, the time-domain governing differential equations are transformed into the frequency domain by assuming harmonic solutions of a single frequency. Accordingly the exact dynamic stiffness matrix is also frequency dependent and it can be considered as a mixture of the inertia, stiffness and damping properties of a structure element. The dynamic stiffness matrix relates harmonically varying forces to harmonically varying displacements at the nodes of a structural element.

1.1.2 Spectral Analysis Method (SAM)

The solution methods for the governing differential equations formulated in the time-domain can be categorized into two major groups. The first group consists of the time-domain methods, such as the numerical integration methods and the modal analysis method, which is commonly used for the vibration analysis. The second group consists of the frequency-domain methods. The spectral analysis method (SAM) is one of the frequency–domain methods.

In SAM, the solutions to the governing differential equations are represented by the superposition of an infinite number of wave modes of different frequencies. This corresponds to the continuous Fourier transform of the solutions. This approach involves determining an infinite set of spectral components in the frequency domain and performing the inverse Fourier transform to reconstruct the time histories of the solutions. The continuous Fourier transform is feasible only when the function to be transformed is mathematically simple and inverse transform is biggest impediment to most practical cases. Instead of using the continuous Fourier transform, the discrete Fourier transform (DFT) is widely used.

The DFT is an approximation of the continuous Fourier transform. In contrast to the continuous Fourier transform, the solution is represented by a finite number of wave modes of discrete frequencies. We can use the fast Fourier transform (FFT) algorithm to compute the DFT. The use of FFT algorithm make it possible to efficiently take into account as many spectral components as are needed up to the highest frequency of interest.

1.1.3 Spectral Element Method (SEM)

1. The spectral element method (SEM) can be considered as the combination of the key features of the conventional FEM, DSM and SAM.

2. **Key Features of FEM.** Meshing and the assembly of finite elements.
3. **Key Features of DSM.** Exactness of the dynamic stiffness matrix formulated in terms of a minimum number of DOFs.
4. **Key Features of SAM.** Superposition of wave modes via DFT theory and FFT algorithm.

In SEM, exact dynamic stiffness matrices are used as the element stiffness matrices for the finite element in a structure. To formulate an exact dynamic stiffness matrix for the classical DSM, the dynamic responses of a structure are usually assumed to be the harmonic solutions of a single frequency. For the spectral element method (SEM) the dynamic responses are assumed to be the superposition of a finite number of wave modes of a different discrete frequencies based on the DFT theory.

The SEM is stiffness formulated. The spectral elements can be assembled to form a global system matrix equation for the whole problem domain by using exactly the same assembly techniques as used in the conventional FEM. The global system matrix equation is then solved for the global spectral nodal DOFs. We use the inverse-FFT (IFFT) algorithm to compute the time histories of dynamic responses.

1.1.4 Present Scope of Investigation

In the literature many researchers' have formulated the dynamic stiffness matrix (DSM) for the vibration problems of rods, beams and frames by using all of the above methods. They have derived very tedious algebraic expressions by using symbolic computation but the natural frequencies of the above mentioned structural problems are limited to only the first few lower mode frequencies. The higher modes natural frequencies of these skeletal structures are very scanty in literature. Hence in the present thesis the higher modes natural frequencies up to the twentieth mode are presented for the above structures and the results are compared wherever it was possible. The elements have been spectrally formulated to obtain the global dynamic stiffness matrix and hence this method is called spectral element method (SEM).

CHAPTER 2

Literature Review

Kolousek [35] is the first to derive the dynamic stiffness matrix for the Bernoulli-Euler beam. Przemieniecki [40] introduced the formulation of the frequency-dependent mass and stiffness matrices for both bar and beam elements.

The Wittrick-Williams algorithm [47] has enhanced the applicability of the dynamic stiffness matrix for automatic calculation of undamped natural frequencies.

Beskos [20] introduced the fundamental concept of the spectral element method for the first time. He derived an exact dynamic stiffness matrix for the beam element and employed FFT for the dynamic analysis of plane frame-works.

Banerjee, Guo and Howson [10] have presented an exact dynamic stiffness matrix of a bending-torsion coupled beam including warping. The work presented in this paper extends the approach by recasting the equations in the form of a dynamic member stiffness matrix. A new procedure is presented, based on the Wittrick-Willliams algorithm [48] for converging with certainty upon any required natural frequency.

Banerjee and Williams [12] presented an elegant and efficient alternative procedure for calculating the number of clamped-clamped natural frequencies of the bending-torsion coupled beam [13] exceeded by any trial frequency, thus enabling the Wittrick-Williams algorithm to be applied with ease when finding the natural frequencies of structure which incorporate such members.

The spectral element method is a high-order finite element technique that combines the geometric flexibility of finite elements with the high accuracy of spectral methods. It exhibits several favourable computational properties, such as the use of tensor products, naturally diagonal mass matrices, and adequacy to implementations in a parallel computer system. Due to these advantages, the spectral element method is a viable alternative to currently popular methods such as finite volumes and finite elements, if accurate solutions of regular problems are sought.

Exact dynamic stiffness matrices have been developed mostly for the 1-D structures including the Timoshenko beams with or without axial force [24, 25, 33, 46], Rayleigh-Timoshenko beams [3, 36] and composite beams [29].

Banerjee and Williams [15] presented the exact dynamic stiffness matrix for a composite beam. It includes the effect of shear deformation and rotatory inertia: i.e., it is for a composite Timoshenko beam. The theory accounts for the coupling between the bending and torsional deformation which usually occurs for such beams due to anisotropic nature of fibrous composites.

Chandrashekhara*et al.* [23] and Abramovich and Livshits [2] considered free bending vibration of composite Timoshenko beams, but without including the coupling between the bending and torsional deformations. In contrast, Teoh and Huang [45] and Teh and Huang [44] included this coupling in their investigation.

Abramovich [1] presented the exact solutions for symmetrically laminated composite beams with ten different boundary conditions, where shear deformation and rotary inertia were included in the analysis.

Banerjee [5] performed the free vibration analysis of axially loaded composite Timoshenko beams by using the dynamic stiffness matrix method with the effects of axial force, shear deformation, rotary inertia and coupling between the bending and the torsional deformations taken into account.

An exact dynamic stiffness matrix is developed by Banerjee [6] and subsequently used for the vibration analysis of a twisted beam whose flexural displacement are coupled in two planes. First the governing differential equations of the motion of the twisted beam undergoing free natural vibration are derived using Hamilton’s principle. Next the general solution of the equations are obtained when the oscillatory motion of the beam is harmonic. The resulting dynamic stiffness matrix is used in connection with the Wittrick-Williams algorithm to compute natural frequencies and mode shapes of a twisted beam with cantilever end conditions.

Fiberg [30] has presented an exact dynamic element stiffness matrix which is a coupled one.

An exact dynamic stiffness matrix for a twisted Timoshenko beam has been developed by Banerjee [8] in order to investigate its free vibration characteristics. First the governing differential equations of motion and the associated natural boundary conditions of a twisted Timoshenko beam undergoing free natural vibration are derived using Hamilton's principle. The inclusion of a given pretwist together with the effects of shear deformation and rotatory inertia, gives rise in free vibration to four coupled second order partial differential equations of motion involving bending displacements and bending rotations in two planes. For harmonic oscillation these four partial differential equations are combined into an eighth order ordinary differential equation, which is identically satisfied by all components of bending displacements and bending rotations.

Banerjee and Williams [16] derived exact analytical expressions for coupled extensional-torsional dynamic stiffness matrix elements of a uniform structural member from the basic governing differential equation of the member in free vibration.

The spectral element method in structural dynamics was presented by Doyle [27] and he published [28] in wave propagation in structures where he introduced an FFT-based spectral analysis methodology. He formulated spectral element for bars, beams and plates using the spectral analysis of wave motion.

Banerjee and Williams [17] derived the analytical expressions for the coupled bending-torsional dynamic stiffness matrix element of a uniform Timoshenko beam in an exact sense by solving the governing differential equations of motion of the element.

Banerjee and Fisher [11] have presented analytical expressions for the coupled bending–torsional dynamic stiffness matrix element of an axially loaded uniform beam.

Banerjee [7] has presented from the governing differential equations of motion in free vibration, the dynamic stiffness matrix of a uniform rotating Bernoulli–Euler beam using the Frobenius method of solution in power series. The derivation includes the presence of an axial force at the outboard end of the beam in addition to the existence of the usual centrifugal force arising from the rotational motion. This makes the general assembly of dynamic stiffness matrices of several elements possible so that a non-uniform (or tapered) rotating beam can be analyzed for its free-vibration characteristics by idealizing it as an assemblage of many uniform rotating beams.

Gopalakrishnan and Doyle [31] formulated spectral element for beam of varying cross section to analyse wave propagation in connected waveguides.

Rosen [43] presented the structural and dynamic behaviour of pretwisted rods and beams.

Capron and Williams [21] have presented exact dynamic stiffness coefficients for an axially loaded Timoshenko member embedded in an elastic medium and shows how to use them in a general theory for finding the natural frequencies of plane or space frames. The member is considered to have a uniform distribution of mass and the stiffness's optionally allow for the separate or combined effects of axial load, rotatory inertia and shear deflection. Axial, torsional and flexural responses are assumed to be uncoupled, as happens for doubly symmetric cross-sections.

Banerjee [9] has outlined a general theory to develop the dynamic stiffness matrix of a structural element. He showed that substantial saving in computer time can be achieved if explicit analytical expressions for the elements of the dynamic stiffness matrix are used instead of numerical methods. Such expressions can be derived with the help of symbolic computation.

Banerjee and Williams [18] in 1985 have used Bernoulli Euler theory and Bessel functions to obtain explicit expression for the exact stiffness for axial, torsional and flexural deformation of an axially loaded beam which is tapered.

The coupled bending-torsional dynamic stiffness matrix terms of an axially loaded uniform Timoshenko beam element are derived by solving the governing differential equations of motion of the element By Banerjee and Williams [19].

Lee [38] has presented the spectral Element Method and its various applications in the structural dynamics.

Chakraborty and Gopalakrishnan [22] applied spectrally formulated finite element for wave propagation analysis in functionally graded beams. Howson and Zare [34] formulated an exact dynamic stiffness matrix for flexural vibration of 3-layered sandwich beams.

Cho and Lee [26] presented An FFT-based spectral analysis method for linear discrete dynamic systems with non-proportional damping.

Doyle and his collegues [42] have applied the SEM mostly to wave propagation in structures and a comprehensive list of the works by this group and other researchers can be found in a book by Doyle [28].

CHAPTER 3

Theoretical Formulation

3.1 SPECTRAL ELEMENT FORMULATION OF ROD

The free longitudinal vibration of a uniform rod is represented by

$$\mathrm{EA}u'' - \rho \mathrm{A}\ddot{u} = 0 \qquad (1)$$

Where u(x,t) is the longitudinal displacement, E is the Young’s modulus, A is the cross-sectional area, and ρ is the mass density. The prime denotes the derivatives with respect to the spatial coordinate x. The internal axial force is given by

$$N_t\,(\mathrm{x},\mathrm{t}) = \mathrm{EA}u'(\mathrm{x},\mathrm{t}) \qquad (2)$$

Where the subscript t is used to denote the quantity in the time domain.

The solution of Equation (1) is assumed in the spectral form as

$$u\,(x,t) = \frac{1}{N}\sum_{n=0}^{N-1} U_n(x;\omega_n)e^{i\omega_n t} \qquad (3)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{N}\sum_{n=0}^{N-1} e^{i\omega_n t}\frac{\partial^2 U_n}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{N}\sum_{n=0}^{N-1} -{\omega_n}^2 e^{i\omega_n t}U_n$$

Substituting Equation (3) into Equation (1) yields eigenvalue problem for a specific discrete frequency as

$$\mathrm{EA}U'' + \omega^2\rho A U = 0 \qquad (4)$$

The general solution to Equation (4) is assumed to be

$$U\,(x) = \mathrm{a}e^{-ik(\omega)x} \qquad (5)$$

$$\frac{\partial^2 U}{\partial x^2} = -k^2 \mathrm{a} e^{-ikx}$$

By substituting Equation (5) into Equation (4), a dispersion relation can be obtained as

$$-\mathrm{EA}k^2 \mathrm{a} e^{-ikx} + \omega^2 \rho A \mathrm{a} e^{-ik(\omega)x} = 0$$

$$\Rightarrow -\mathrm{EA}k^2 + \omega^2 \rho A = 0$$

$$\Rightarrow k^2 - \omega^2 \frac{\rho A}{EA} = 0$$

$$\Rightarrow k^2 - k_L^2 = 0 \qquad (6)$$

Where k_L is the wavenumber for the pure longitudinal wavemode and it is defined by

$$k_L = \omega \sqrt{\frac{\rho A}{EA}} \qquad (7)$$

Equation (6) gives two real roots as

$$k_1 = -k_2 = k_L \qquad (8)$$

For a finite rod element of length L, the general solution of Equation (4) can be then obtained in the form

$$\mathrm{U}(x) = a_1 e^{-ik_L x} + a_2 e^{+ik_L x} = \mathrm{e}(x; \omega)\, \mathbf{a} \qquad (9)$$

where

$$\mathrm{e}(x; \omega) = [e^{-ik_L x}\; e^{+ik_L x}] \qquad (10)$$

$$\mathbf{a} = \{a_1 \;\; a_2\}^T$$

The spectral nodal displacements of the finite rod element can be related to the displacement field as

$$\mathbf{d} = \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} U(0) \\ U(L) \end{Bmatrix} \tag{11}$$

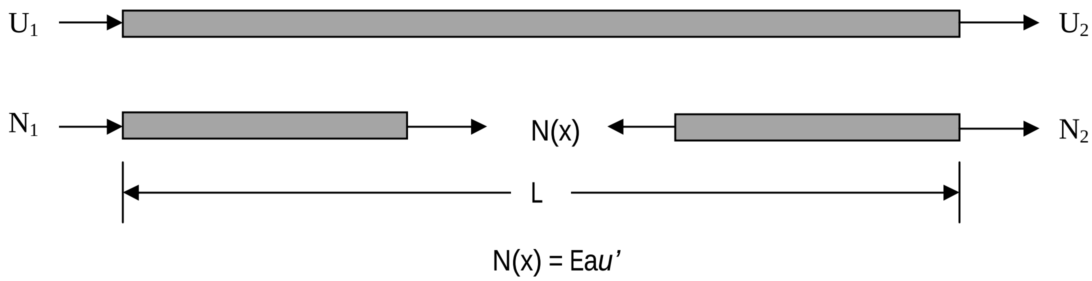

Fig 1.Sign convention for the rod element

Substituting Equation (9) into the right-hand side of Equation (11)gives

$$\mathbf{d} = \begin{bmatrix} e(0;\omega) \\ e(L;\omega) \end{bmatrix} \mathbf{a} = H_{\boldsymbol{R}}(\omega)\ \mathbf{a} \tag{12}$$

where

$$H_{\boldsymbol{R}}(\omega) = \begin{bmatrix} 1 & 1 \\ e^{-ik_L L} & e^{+ik_L L} \end{bmatrix} \tag{13}$$

By eliminating the constant vector **a** from Equation (9) by using the Equation (12), we can represent the displacement field in the finite rod element in terms of the nodal DOFs as

$$\text{U(x)} = N_R(\text{x};\ \omega)\ \mathbf{d} \tag{14}$$

where

N_R(x; ω) = e(x;ω) $H_R^{-1}(\omega)$ = $[N_{R1} N_{R2}]$

$N_{R1}(x;\omega)$=csc $(k_L L)$sin $[k_L(L-x)]$ (15)

$N_{R2}(x;\omega)$ = csc $(k_L L)$ sin$(k_L x)$

From Equation (2), the spectral components of the axial force are related to U(x) by

N(x) = EA$U'(x)$ (16)

The spectral nodal axial forces defined for the finite rod element can be related to the forces defined by the strength of material as

$$f_c(\omega) = \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} = \begin{Bmatrix} -N(0) \\ +N(L) \end{Bmatrix} \quad (17)$$

Substituting Equations (14) and (16) into the right-hand side of Equation (17) gives

$$\boldsymbol{S}_R(\omega)\ \mathbf{d} = \boldsymbol{f}_c(\omega) \quad (18)$$

Where $\boldsymbol{S}_R(\omega)$ is the spectral element matrix for the finite rod element and it is given by

$$\boldsymbol{S}_R(\omega) = \frac{EA}{L}\begin{bmatrix} S_{R11} & S_{R12} \\ S_{R12} & S_{R22} \end{bmatrix} = \boldsymbol{S}_R(\omega)^T \quad (19)$$

where

S_{R11}= S_{R22}= $(k_L L)$ cot $(k_L L)$

S_{R12}= S_{R22}= - $(k_L L)$ csc $(k_L L)$ (20)

3.2 SPECTRAL ELEMENT FORMULATION OF EULER-BERNOULLI BEAM

The free bending vibration of a Bernoulli-beam is represented by

$$\mathrm{EI}w'''' + \rho\mathrm{A}\ddot{w} = 0 \qquad (1)$$

Where w (x,t) is the transverse displacement, E is the Young's modulus, A is the cross-sectional area, I is the area moment of inertia about the neutral axis, and ρ is the mass density. The internal transverse shear force and bending moment are given by

$$M_t(x, t) = \mathrm{EI}w''(x, t), \qquad Q_t(x, t) = -\mathrm{EI}w''(x, t) \qquad (2)$$

Assuming the solution of equation (1) in the spectral form to be

$$W(x, t) = \frac{1}{N}\sum_{n=0}^{N-1} W_n(x; \omega_n) e^{i\omega_n t} \qquad (3)$$

$$\frac{\partial^4 w}{\partial x^4} = \frac{1}{N}\sum_{n=0}^{N-1} e^{i\omega_n t}\frac{\partial^4 W_n}{\partial x^4}$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{1}{N}\sum_{n=0}^{N-1} -{\omega_n}^2 e^{i\omega_n t} W_n$$

Substituting Equation (3) into Equation (1) gives an eigenvalue problem for a specific discrete frequency ($\omega = \omega_n$) as

$$EIw'''' + \rho A\ddot{w} = 0$$

$$EIW'''' - \omega^2 \rho AW = 0 \qquad (4)$$

Assuming the general solution of Equation (4)

$$W(x) = a e^{-ik(\omega)x} \quad (5)$$

$$\frac{\partial^4 W}{\partial x^4} = k^4 a e^{-ikx}$$

Substituting Equation (5) into Equation (4) yields a dispersion relation,

$$EIW'''' - \omega^2 \rho A W = 0$$

$$\Rightarrow EI(k^4 a e^{-ikx}) - \omega^2 \rho A (a e^{-ik(\omega)x}) = 0$$

$$\Rightarrow EI(k^4) - \omega^2 \rho A = 0$$

Then both side divided by EI

$$\Rightarrow k^4 - \omega^2 (\frac{\rho A}{EI}) = 0$$

$$\Rightarrow k^4 - k_f^4 = 0 \quad (6)$$

Wherek_f is a wavenumber for the pure bending (flexural) wave mode defined by

$$k_f = \sqrt{\omega} \left(\frac{\rho A}{EI}\right)^{1/4} \quad (7)$$

Equation (6) gives two pure real roots and two pure imaginary roots as

$$k_1 = k_F = -k_2 \quad , \quad k_3 = ik_F = -k_4 \quad (8)$$

For the finite B-beam element of length L, the general solution of the equation (4) can be obtained in the form

$$W(x;\omega) = a_1 e^{-ik_F x} + a_2 e^{-k_F x} + a_3 e^{+ik_F x} + a_4 e^{+k_F x} = e(x;\omega) \{a\} \quad (9)$$

where,

$$e\,(x;\omega) = [e^{-ik_F x} e^{-k_F x} e^{+ik_F x} e^{+k_F x}] \quad (10)$$

$$\mathbf{a} = \{a_1 \;\; a_2 \;\; a_3 \;\; a_4\}^{\mathbf{T}}$$

The spectral nodal displacements and slope of finite B-beam element can be related to the displacement field by

$$\mathbf{d} = \begin{Bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} W(0) \\ W^1(0) \\ W(L) \\ W^1(L) \end{Bmatrix} \quad (11)$$

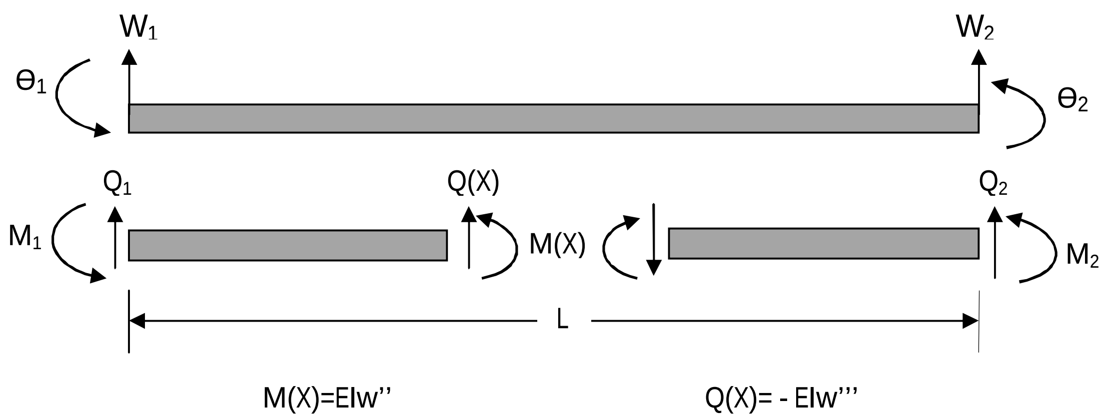

fig. 2 sign convention for the Bernoulli-Euler beam element

Substituting Equation (9) into the right-hand side of Equation (11) gives

$$d = \begin{bmatrix} e(0;\omega) \\ e'(0;\omega) \\ e(L;\omega) \\ e'(L;\omega) \end{bmatrix} a = H_B(\omega)\, a \qquad (12)$$

where

$$H_B(\omega) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -ik_F & -k_F & ik_F & k_F \\ e^{-ik_F L} & e^{-k_F L} & e^{ik_F L} & e^{k_F L} \\ -ik_F e^{-ik_F L} & -k_F e^{-k_F L} & ik_F e^{ik_F L} & k_F e^{k_F L} \end{bmatrix} \qquad (13)$$

The displacement field within the finite B-beam element can be represented in terms of the nodal DOFs vector **d** by eliminating the constant vector **a** from Equation (9) by using Equation (12), thus

$$W(x) = N_B(x;\omega)\mathbf{d} \qquad (14)$$

Where

N_B(X;ω) = e(x;ω) $H_B^{-1}(\omega)$ =[N_{B1} N_{B2} N_{B3} N_{B4}]

$N_{B1}(x) = \eta^{-1}k_F[\cos\bar{x} - \cos(\bar{L}-\bar{x})\cosh\bar{L} - \cos\bar{L}\cosh(\bar{L}-\bar{x}) +$

$\cosh\bar{x} + \sin(\bar{L}-\bar{x})\sinh\bar{L} - \sin\bar{L}\sinh(\bar{L}-\bar{x})]$

$N_{B2}(x) = \eta^{-1}[-\cosh(\bar{L}-\bar{x})\sin\bar{L} + \cosh\bar{L}\sin(\bar{L}-\bar{x}) + \sin\bar{x} -$

$\cos(\bar{L}-\bar{x})\sinh\bar{L} + \cos\bar{L}\sinh(\bar{L}-\bar{x}) + \sinh\bar{x}]$ (15)

$N_{B3}(x) = \eta^{-1}k_F[\cos(\bar{L}-\bar{x}) - \cos\bar{x}\cosh\bar{L} -$

$\cos\bar{L}\cosh\bar{x} + \cosh(\bar{L}-\bar{x}) + \sin\bar{x}\sinh\bar{L} - \sin\bar{L}\sinh\bar{x}]$

$N_{B4}(x) = -\eta^{-1}[-\cosh\bar{x}\sin\bar{L} + \cosh\bar{L}\sin\bar{x} + \sin(\bar{L}-\bar{x}) - \cos\bar{x}\sinh\bar{L} + \cos\bar{L}\sinh\bar{x}$

$+\sinh(\bar{L}-\bar{x})]$

$\eta = 2k_F(1 - \cos\bar{L}\cosh\bar{L})$

$\bar{x} = k_F x, \bar{L} = k_F L$

From the Equation (2), the spectral components of the bending moment and transverse shear force can be related to W(x) as

$$Q(x) = -EIW'''(x),\ M(x) = EIW''(x) \quad (16)$$

The spectral nodal transverse shear forces and bending moments defined for the finite B-beam element can be related to the corresponding forces and moments defined by the strength of material by

$$f_c = \begin{Bmatrix} Q_1 \\ M_1 \\ Q_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} -Q(0) \\ -M(0) \\ +Q(L) \\ +M(L) \end{Bmatrix} \qquad (17)$$

Substituting Equation (14) into Equation (6) and the results into the side of Equation

(17)gives

$$S_B(\omega)\,\mathbf{d} = f_c(\omega) \qquad (18)$$

Where $S_B(\omega)$ is the spectral element matrix for the B-beam element given by

$$S_B(\omega) = \frac{EI}{L^3}\begin{bmatrix} s_{B11} & s_{B12} & s_{B13} & s_{B14} \\ s_{B12} & s_{B22} & s_{B23} & s_{B24} \\ s_{B13} & s_{B23} & s_{B33} & s_{B34} \\ s_{B14} & s_{B24} & s_{B34} & s_{B44} \end{bmatrix} = S_B(\omega)^{\mathrm{T}} \qquad (19)$$

where

$s_{B11} = s_{B33} = \Delta_B \bar{L}^3 (\cos\bar{L}\sinh\bar{L} + \sin\bar{L}\cosh\bar{L})$

$s_{B22} = s_{B44} = \Delta_B \bar{L}^3 k_F^{-2} (-\cos\bar{L}\sinh\bar{L} + \sin\bar{L}\cosh\bar{L})$

$s_{B12} = -s_{B34} = \Delta_B \bar{L}^3 k_F^{-1} \sin\bar{L}\sinh\bar{L}$

$s_{B13} = -\Delta_B \bar{L}^3 (\sin\bar{L} + \sinh\bar{L})$

s_{B14} = $-s_{B23}$ = $\Delta_B \overline{L^3} k_F^{-1}(-\cos \overline{L} + \cosh \overline{L})$

s_{B24} = $\Delta_B \overline{L^3} k_F^{-2}(-\sin \overline{L} + \sinh \overline{L})$(20)

$\Delta_B = \frac{1}{1-\cos \overline{L} \cosh \overline{L}}$

$\overline{L}=k_F$L

3.3 SPECTRAL ELEMENT FORMULATION OF TIMOSHENKO BEAM

The free vibration of a uniform Timoshenko beam is written as

$$\kappa GA\,(w'' - \theta') - \rho A\ddot{w} = 0$$

$$EI\theta'' + \kappa GA(w' - \theta) - \rho I\theta'' = 0 \qquad (1)$$

Where w (x, t) and θ (x, t) are the transverse displacement and the slope due to bending, E is the Young's modulus, G is the shear modulus, κ is the shear correction factor, which depends on the shape of the cross-section, A is the cross-sectional area and I is the area moment of inertia about the neutral axis, and ρ is the mass density. The internal transverse shear force and bending moment are given by

$$Q_t(x, t) = \kappa GA\,[W'(x,t) - \theta(x,t)],$$

$$M_t(x, t) = EI\theta'(x, t) \qquad (2)$$

Assuming the solution to equation (1) in the spectral form to be

$$W(x,t) = \frac{1}{N}\sum_{n=0}^{N-1} W_n(x;\omega_n)e^{i\omega_n t}$$

$$\theta(x,t) = \frac{1}{N}\sum_{n=0}^{N-1} \theta_n(x;\omega_n)e^{i\omega_n t} \qquad (3)$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{N}\sum_{n=0}^{N-1} e^{i\omega_n t}\frac{\partial^2 W_n}{\partial x^2}$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{1}{N}\sum_{n=0}^{N-1} -\omega_n{}^2 e^{i\omega_n t}W_n$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{N}\sum_{n=0}^{N-1} e^{i\omega_n t}\frac{\partial^2 \theta_n}{\partial x^2}$$

$$\frac{\partial^2 \theta}{\partial t^2} = \frac{1}{N}\sum_{n=0}^{N-1} -\omega_n{}^2 e^{i\omega_n t}\theta_n$$

Substituting Equation (3) into Equation (1) gives an eigenvalue problem as

$$\kappa GA\,(w'' - \theta') + \rho A\omega^2 W = 0$$

$$EI\theta'' + \kappa GA(W' - \theta) + \rho I\omega^2\theta = 0 \qquad (4)$$

Assuming the general solution of Equation (4) are

$$W(x) = \text{a}e^{-ik(\omega)x}, \qquad \theta(x) = \beta\text{a}e^{-ik(\omega)x} \qquad (5)$$

$$\frac{\partial^2 W}{\partial x^2} = -k^2\text{a}e^{-ikx}, \qquad \frac{\partial^2 \theta}{\partial x^2} = -k^2\beta\text{a}e^{-ikx}$$

Putting the value of Equation (5) into Equation (4) gives

$$\kappa GAk^2 - ik\kappa GA\beta - \rho A\omega^2 = 0$$

$$k^2 EI\beta + \kappa GA\beta - \rho I\omega^2 + ik\kappa GA = 0$$

We obtain the eigenvalue problem as

$$\begin{bmatrix} \kappa GA k^2 - \rho A\omega^2 & -ik\kappa GA \\ ik\kappa GA & EI k^2 + \kappa GA - \rho I \omega^2 \end{bmatrix} \begin{Bmatrix} 1 \\ \beta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{6}$$

Equation (6) gives a dispersion relation as

$$k^4 - \eta k_F^4 k^2 - k_f^4(1 - \eta_1 k_G^4) = 0 \tag{7}$$

where

$$k_F = \sqrt{\omega}(\frac{\rho A}{EI})^{1/4}, \quad k_G = \sqrt{\omega}(\frac{\rho A}{\kappa GA})^{1/4}, \tag{8}$$

and

$$\eta = \eta_1 + \eta_2, \quad \eta_1 = \frac{\rho I}{\rho A}, \quad \eta_2 = \frac{EI}{\kappa GA}, \tag{9}$$

Solving the Equation (6) gives the four roots as

$$k_1 = -k_2 = \frac{1}{\sqrt{2}} k_F \sqrt{\eta k_F^2 + \sqrt{\eta^2 k_F^4 + 4(1 - \eta_1 k_G^4)}} = k_t$$

$$k_3 = -k_4 = \frac{1}{\sqrt{2}} k_F \sqrt{\eta k_F^2 - \sqrt{\eta^2 k_F^4 + 4(1 - \eta_1 k_G^4)}} = k_e \tag{10}$$

$$\beta = \frac{\kappa GAk^2 - \rho A\omega^2}{ik\kappa GA}$$

$$= \frac{1}{ik}(k^2 - k_G^4)$$

From the Equation (6), we can obtain the wave mode ratio as

$$\beta_P(\omega) = \frac{1}{ik_P}(k_P^2 - k_G^4)$$

$$= -\ i r_P(\omega) \qquad (11)$$

where

$$r_P(\omega) = \frac{1}{k_P}(k_P^2 - k_G^4) \qquad (12)$$

By using the four wave numbers, the general solution of Equations (4) can be written as

$$W(x) = a_1 e^{-ik_t x} + a_2 e^{ik_t x} + a_3 e^{-ik_e x} + a_4 e^{+ik_e x} = e_w(x;\omega)\{a\} \qquad (13)$$

$$\Theta(x) = \beta_1 a_1 e^{-ik_t x} + \beta_2 a_2 e^{ik_t x} + \beta_3 a_3 e^{-ik_e x} + \beta_4 a_4 e^{+ik_e x} = e_\theta(x;\omega)\{a\} \qquad (14)$$

where

$$\mathbf{a} = \{a_1 \quad a_2 \quad a_3 \quad a_4\}^{\mathbf{T}} \qquad (15)$$

$$e_w(x;\omega) = [e^{-ik_t x} \quad e^{-ik_t x} \quad e^{+ik_e x} \quad e^{+ik_e x}] \qquad (16)$$

$$e_\theta(x;\omega) = e_w(x;\omega)\,B(\omega)$$

$$B(\omega) = \text{diag}\,[\beta_P(\omega)]$$

The spectral nodal displacements and slope of finite T-beam element of length L can be related to the displacement field by

$$\mathbf{d} = \begin{Bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} W(0) \\ \theta(0) \\ W(L) \\ \theta(L) \end{Bmatrix} \tag{17}$$

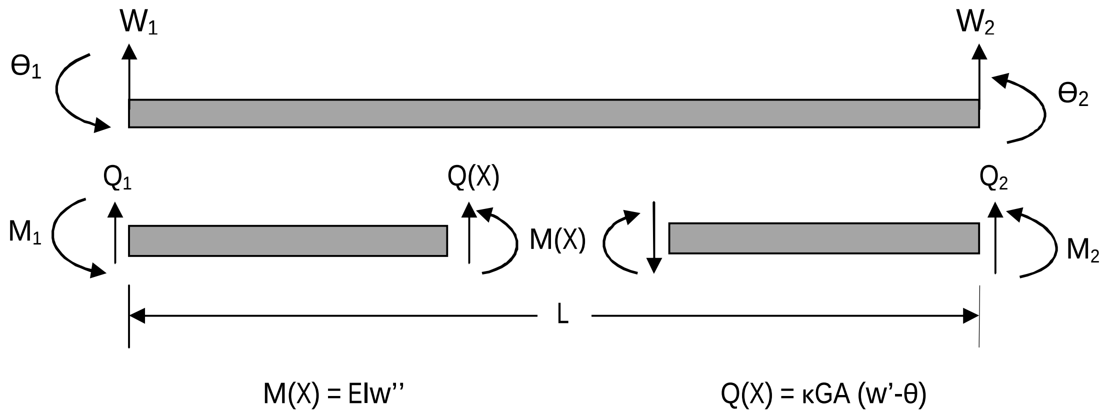

fig. 3 sign convention for the Timoshenko beam element

Substituting Equation (13) and (14) into the right-hand side of Equation (16) gives

$$d = \begin{bmatrix} e_{w'}(0;\omega) \\ e_{\theta'}(0;\omega) \\ e_{w'}(L;\omega) \\ e_{\theta'}(L;\omega) \end{bmatrix} \{a\} = H_T(\omega)\{a\} \tag{18}$$

where

$$H_B(\omega) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -ir_t & ir_t & -ir_e & ir_e \\ e_t & e_t^{-1} & e_e & e_e^{-1} \\ -ir_te_t & ir_te_t^{-1} & -ir_ee_e & ir_ee_e^{-1} \end{bmatrix} \quad (19)$$

Where

$$e_t = e^{-ik_tL}, e_e = e^{-ik_eL},$$

$$r_t = \frac{1}{k_t}(k_t^2 - k_G^4), \quad r_e = \frac{1}{k_e}(k_e^2 - k_G^4)$$

By using Equation (18), the constant vector **a** can be eliminated from Equation (13) and (14) to express the general solutions as

$$W(x) = N_w(x; \omega)\, \mathbf{d}, \qquad \theta(x) = N_\theta(x; \omega)\, \mathbf{d}, \quad (20)$$

Where

$$N_w(x; \omega) = e_w(x; \omega) H_T^{-1}(\omega)$$

$$N_\theta(x; \omega) = e_\theta(x; \omega) H_T^{-1}(\omega) = e_w(x; \omega) B(\omega) H_T^{-1}(\omega) \quad (21)$$

From the Equation (2), the spectral components of the bending moment and transverse shear force can be related to W(x) and θ(x) as

$$Q = \kappa GA(W' - \theta), \quad M = EI\theta' \quad (22)$$

The spectral nodal transverse shear forces and bending moments defined for the finite T-beam element can be related to the corresponding forces and moments defined by the strength of material by

$$f_c(\omega) = \begin{Bmatrix} Q_1 \\ M_1 \\ Q_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} -Q(0) \\ -M(0) \\ +Q(L) \\ +M(L) \end{Bmatrix} \qquad (23)$$

Substituting Equation (20) into Equation (22) and the results to the right-side of Equation

(23) gives

$$S_T(\omega)\,\mathbf{d} = f_c(\omega) \qquad (24)$$

Where $S_T(\omega)$ is the spectral element matrix for the T-beam is given by

$$S_T(\omega) = EI \begin{bmatrix} s_{T11} & s_{T12} & s_{T13} & s_{T14} \\ s_{T12} & s_{T22} & s_{T23} & s_{T24} \\ s_{T13} & s_{T23} & s_{T33} & s_{T34} \\ s_{T14} & s_{T24} & s_{T34} & s_{T44} \end{bmatrix} = S_T(\omega)^{\mathrm{T}} \qquad (25)$$

where

$$s_{T11} = s_{T33} = i\Delta_{\mathrm{T}}\mathrm{A}[(\mathrm{e}_t^2 - 1)(\mathrm{e}_e^2 + 1)\mathrm{r}_t - (\mathrm{e}_t^2 + 1)(\mathrm{e}_e^2 - 1)\mathrm{r}_e](\mathrm{k}_t\mathrm{r}_t - \mathrm{k}_e\mathrm{r}_e)$$

$$s_{T22} = s_{T44} = \Delta_{\mathrm{T}}\mathrm{B}[-(\mathrm{e}_t^2 + 1)(\mathrm{e}_e^2 - 1)\mathrm{r}_t + (\mathrm{e}_t^2 - 1)(\mathrm{e}_e^2 + 1)\mathrm{r}_e]$$

$$s_{T12} = -s_{T34} = \Delta_T A\{(\mathrm{e}_t^2 - 1)(\mathrm{e}_e^2 - 1)(\mathrm{k}_t\mathrm{r}_e + \mathrm{k}_e\mathrm{r}_t) - [(\mathrm{e}_t^2 + 1)(\mathrm{e}_e^2 + 1) - 4\mathrm{e}_t\mathrm{e}_e](\mathrm{k}_t\mathrm{r}_t - \mathrm{k}_e\mathrm{r}_e)\}$$

$s_{T13} = 2i\Delta_T A[(e_t^2 - 1)e_e r_t - (e_e^2 - 1)e_t r_e](-k_t r_t + k_e r_e)$

$s_{T14} = -s_{T23} = -2\Delta_T A(e_t - e_e)(1 - e_t e_e)(-k_t r_t + k_e r_e)$

$s_{T24} = 2\Delta_T B[(e_e^2 - 1)e_t r_t - (e_t^2 - 1)e_e r_e]$

$$\Delta_T = \frac{8k_F^2}{ie_F[2r_t r_e\{(e_t^2+1)(e_e^2+1) - 4e_t e_e\} - (r_t^2 + r_e^2)(e_t^2 - 1)(e_e^2 - 1)]}$$

$$A = (1/8)e_F(1 - \eta_1 k_G^4)^{1/2}$$

$$B = (1/8)e_F[\eta^2 k_F^4 + 4\left(1 - \eta_1 k_G^4\right)]^{1/2}$$

3.4 SPECTRAL ELEMENT FORMULATION OF FRAME

The differential equation of the frame may be written as

$$EI\frac{\partial^4 y}{\partial x^4} + m\frac{\partial^2 y}{\partial t^2} = 0 \qquad (1)$$

where y(t) is the deflection at point x from origin, EI is the flexural rigidity, m is the mass per unit length.

Assuming the solution of the equation (1) in the spectral form to be

$$Y(x, t) = \frac{1}{N}\sum_{n=0}^{N-1} Y_n(x;\omega_n)e^{i\omega_n t} \qquad (2)$$

$$\frac{\partial^4 y}{\partial x^4} = \frac{1}{N}\sum_{n=0}^{N-1} e^{i\omega_n t}\frac{\partial^4 Y_n}{\partial x^4}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{N}\sum_{n=0}^{N-1} -\omega_n{}^2 e^{i\omega_n t} Y_n$$

Putting the above value in equation (1), get the solution as

$$EIY'''' - \omega^2 mY = 0 \quad (3)$$

Assuming the general solution for equation (3)

$$Y(x) = ae^{-ik(\omega)x} \quad (4)$$

$$\frac{\partial^4 y}{\partial x^4} = k^4 ae^{-ikx}$$

The general solution of the equation (3) can be obtained in the form

$$Y(x;\omega) = a_1 e^{-ik_F x} + a_2 e^{-k_F x} + a_3 e^{+ik_F x} + a_4 e^{+k_F x} \quad (5)$$

The equation (5) can be written in the form

$$Y = C_1 \cosh \lambda \frac{x}{l} + C_2 \sinh \lambda \frac{x}{l} + C_3 \cos \lambda \frac{x}{l} + C_4 \sin \lambda \frac{x}{l} \quad (6)$$

where Y is the amplitude of motion,

$$\lambda = \sqrt[4]{\left(\frac{m\omega^2}{EI}\right)L}$$

ω = circular frequency of vibration

and the constants C_1,C_2,C_3,C_4 are dependent on the boundary conditions.

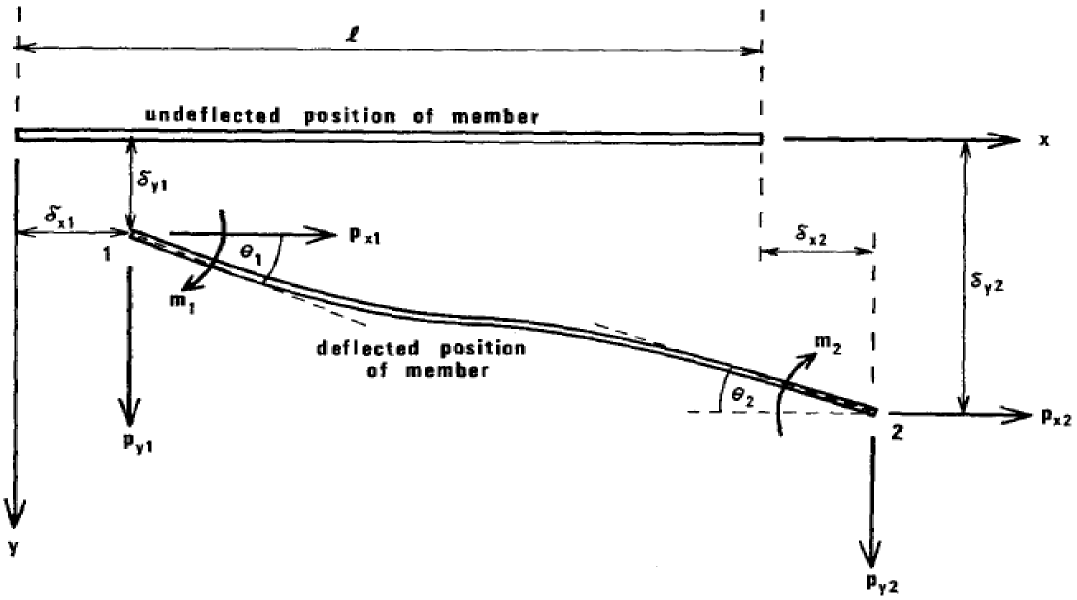

Fig. 4 Forces and Displacements at the ends of a member

Hence the boundary conditions are

Y = 0 at x = 0 and x = L

$\frac{dY}{dx} = 0$ at x = 0

$\frac{dY}{dx} = \theta$ at x = L

By substituting these values in equation (6) we get the values

$$C_1 = -\frac{L\theta}{2\lambda}\frac{\sinh\lambda - \sin\lambda}{1-\cosh\lambda\cos\lambda},$$

$$C_2 = \frac{L\theta}{2\lambda}\frac{\cosh\lambda - \cos\lambda}{1-\cosh\lambda\cos\lambda},$$

$$C_3 = -C_1$$

$$C_4 = -C_2 \quad (7)$$

Putting the values of C_1,C_2,C_3,C_4 in equation (6) we get the values

$$S = \lambda \left(\frac{\sin\lambda\cosh\lambda - \sinh\lambda\, cos\,\lambda}{1 - \cosh\lambda\cos\lambda}\right), \qquad (8)$$

$$C = \frac{\sinh\lambda - \sin\lambda}{\sin\lambda\cosh\lambda - \sinh\lambda\cos\lambda}, \qquad (9)$$

$$Q = \lambda^2\left(\frac{\sinh\lambda\sin\lambda}{1 - \cosh\lambda\cos\lambda}\right), \qquad (10)$$

$$q = \frac{\cosh\lambda - \cos\lambda}{\sinh\lambda\sin\lambda}, \qquad (11)$$

where

S is the dynamic flexural stiffness,

C is the dynamic flexural carry-over factor,

Q is the dynamic flexural shear stiffness,

q is the dynamic flexure-shear carry-over factor,

K is the ‘’stiffness’’ of the member,

Where $\lambda^4 = \mu\omega^2 l^4/EI$ and μ is the mass per unit length of the member

Other boundary conditions are

$$Y = 0 \quad \text{at} \quad x = 0; \qquad \frac{dY}{dx} = 0 \quad \text{at} \quad x = 0$$

$$Y = \delta \quad \text{at} \quad x = L; \qquad \frac{dY}{dx} = 0 \quad \text{at} \quad x = L$$

By substituting these values in equation (6), we get

$$C_1 = \frac{\delta}{2}\frac{\cos\lambda - \cosh\lambda}{\cos\lambda\cosh\lambda - 1},$$

$$C_2 = \frac{\delta}{2}\frac{\sin\lambda + \sinh\lambda}{\cos\lambda\cos h\lambda - 1},$$

$$C_3 = -\frac{\delta}{2}\frac{\cosh\lambda - \cos\lambda}{\cos\lambda\cos h\lambda - 1},$$

$$C_4 = \frac{\delta}{2}\frac{\cosh\lambda - \cos\lambda}{\cos\lambda\cos h\lambda - 1}, \qquad (12)$$

Substituting the values of equation (12) into equation (3), the values are

$$T = \lambda^3\left(\frac{\sin\lambda\cosh\lambda + \cos\lambda\sinh\lambda}{1 - \cosh\lambda\cos\lambda}\right),$$

$$t = \frac{\sinh\lambda + \sin\lambda}{\sin\lambda\cosh\lambda + \cos\lambda\sinh\lambda},$$

where

T is the dynamic shear stiffness,

t is the dynamic shear carry-over factor, where h = EA/l and $\gamma = \omega l\sqrt{(\mu/EA)}$.

The equation (3) and (10) can be used to obtain a set of member equations which can be expressed in the matrix form

$$\begin{Bmatrix} p_{X1} \\ p_{Y2} \\ m_1 \\ p_{X2} \\ p_{Y2} \\ m_2 \end{Bmatrix} = \begin{bmatrix} h\gamma\cot\gamma & 0 & 0 & -h\gamma\cos ec\gamma & 0 & 0 \\ 0 & Tk/l^2 & Qk/l & 0 & -Ttk/l^2 & Qqk/l \\ 0 & Qk/l & Sk & 0 & -Qkl/l & SCk \\ -h\gamma\cos ec\gamma & 0 & 0 & h\gamma\cot\gamma & 0 & 0 \\ 0 & -Ttk/l^2 & -Qqk/l & 0 & Tk/l^2 & -Qk/l \\ 0 & Qqk/l & SCk & 0 & -Qk/l & Sk \end{bmatrix} \begin{Bmatrix} \delta_{X1} \\ \delta_{Y1} \\ \theta_1 \\ \delta_{X2} \\ \delta_{Y2} \\ \theta_2 \end{Bmatrix}$$

where

p_{x1} = h$\gamma \cot\gamma\delta_{x1} - h\gamma cosec\gamma\delta_{x2}$,

p_{x2}= - hγ cosec $\gamma\delta_{x1} + h\gamma cot\gamma\delta_{x2}$,

p_{Y1} = (k/l) [Q$\theta_1 + Qq\theta_2 + T/l\,\delta_{Y1} - Tt/l\,\delta_{Y2}$],

p_{Y2}= -$Ttk/l^2\,\delta_{Y1} - Qqk/l\theta_1 + Tk/l^2\,\delta_{Y2}$-$Qk/l\theta_2$,

m_1= k [S$\theta_1 + SC\theta_2 + Q/l\delta_{Y1} - Qq/l\,\delta_{Y2}$],

m_2= $Qqk/l\,\delta_{Y1} + SCk\theta_1 - Qk/l\delta_{Y2} - Sk\theta_2$,

$$\textbf{[k]} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

It is the required dynamic stiffness matrix.

where

$$\boldsymbol{k_{11}}=\begin{bmatrix} h\gamma\cot\gamma & 0 & 0 \\ 0 & Tk/l^2 & Qk/l \\ 0 & Qk/l & Sk \end{bmatrix},\boldsymbol{k_{22}}=\begin{bmatrix} h\gamma\cot\gamma & 0 & 0 \\ 0 & Tk/l^2 & -Qk/l \\ 0 & -Qk/l & Sk \end{bmatrix}$$

and

$$\boldsymbol{k_{12}}=\boldsymbol{k_{21}^T}=\begin{bmatrix} -h\gamma\cos ec\gamma & 0 & 0 \\ 0 & -Ttk/l^2 & Qqk/l \\ 0 & \dfrac{-Qqk}{l} & SCk \end{bmatrix}$$

CHAPTER 4

Results and Discussions

TABLE 1: Comparison of Natural frequencies of a fixed-free rod: $\mu = \omega\sqrt{\frac{\rho A}{EA}}.L$

Mode No.	Exact Method (Ref.Yang[50])	Finite Element Method				Spectral Element Method
		20 Elements	40 Elements	100 Elements	200 Elements	2 Elements
1	1.57079	1.57120	1.57090	1.57081	1.57080	1.57080
2	4.71238	4.72330	4.71511	4.71283	4.71250	4.71239
3	- - - -	7.90454	7.86660	7.85600	7.85449	7.85398
4	10.99557	11.13452	11.03023	11.00111	10.99696	10.99557
5	14.13716	14.43302	14.21086	14.14894	14.14011	14.13717
6	17.27875	17.81985	17.41339	17.30026	17.28413	17.27876
7	20.42035	21.31462	20.64276	20.45585	20.42922	20.42035
8	23.56194	24.93598	23.90390	23.61649	23.57557	23.56194
9		28.70050	27.20177	26.78295	26.72338	26.70354
10		32.62062	30.54134	29.95602	29.87283	29.84513
11		36.70140	33.92756	33.13648	33.02412	32.98672
12		40.93568	37.36535	36.32511	36.17746	36.12832
13		45.29700	40.85952	39.52270	39.33302	39.26991
14		49.73019	44.41478	42.73004	42.49101	42.41150
15		54.14013	48.03562	45.94791	45.65162	45.55309
16		58.38110	51.72622	49.17710	48.81505	48.69469
17		62.25172	55.49035	52.41841	51.98149	51.83628
18		65.50344	59.33118	55.67262	55.15113	54.97787
19		67.87056	63.25112	58.94054	58.32417	58.11946
20		69.12216	67.25155	62.22295	61.50082	61.26106

TABLE 2: Comparison of Natural Frequencies of a Fixed-Fixed rod: $\mu = \omega\sqrt{\frac{\rho A}{EA}}.L$

Mode No.	Exact Method (Ref.Yang[50])	Finite Element Method				Spectral Element Method
		20 Elements	40 Elements	100 Elements	200 Elements	2 Elements
1	3.14159	3.14482	3.14240	3.14172	3.14162	3.14159
2	6.28318	6.30905	6.28965	6.28422	6.28344	6.28319
3	9.42477	9.51221	9.44659	9.42827	9.42565	9.42478
4	12.56637	12.77397	12.61911	12.57464	12.56844	12.56637
5	15.70796	16.11416	15.80908	15.72412	15.71200	15.70796
6	18.84955	19.55254	19.02442	18.47747	18.85653	18.84956
7	21.99114	23.10837	22.26905	22.03549	22.00223	21.99115
8	25.13274	26.79944	25.54793	25.19894	25.14928	25.13274
9		30.64057	28.86604	28.36861	28.29789	28.27433
10		34.64102	32.22831	31.54527	31.44823	31.41593
11		38.80046	35.63971	34.72972	34.60052	34.55752
12		43.10311	39.10509	37.92274	37.75495	37.69911
13		47.50949	42.62923	41.12510	40.91170	40.84070
14		51.94588	46.21674	44.33761	44.07098	43.98230
15		56.29303	49.87196	47.56104	47.23297	47.12389
16		60.37737	53.59888	50.79619	50.39788	50.26548
17		63.97166	57.40101	54.04385	53.56589	53.40708
18		66.81354	61.28115	57.30482	56.73721	56.54867
19		68.64717	65.24124	60.57988	59.91203	59.69026
20		-------	69.28203	63.86984	63.09055	62.83185

TABLE 3: Comparison of Natural Frequencies of Euler-Bernoulli Fixed-Free Beam: $\mu = \sqrt{\omega}(\frac{\rho A}{EI})^{1/4}.L$

Mode No.	Exact Method (Ref.Yang[50])	Finite Element Method				Spectral Element Method
		20 Elements	40 Elements	100 Elements	200 Elements	2 Elements
1	1.87510	1.87510	1.87510	1.87510	1.87510	1.87510
2	4.69409	4.69410	4.69409	4.69409	4.69409	4.69409
3	7.85475	7.85482	7.85476	7.85476	7.85476	7.85476
4	10.99554	10.99588	10.99556	10.99554	10.99554	10.99554
5	14.13716	14.13837	14.13724	14.13717	14.13717	14.13717
6	17.27875	17.28200	17.27897	17.27876	17.27876	17.27876
7	20.42035	20.42773	20.42083	20.42036	20.42035	20.42035
8	23.56194	23.57686	23.56292	23.56197	23.56195	23.56194
9		26.73105	26.70535	26.70358	26.70354	26.70354
10		29.89237	29.84828	29.84521	29.84514	29.84513
11		33.06330	32.99189	32.98686	32.98673	32.98672
12		36.24665	36.13643	36.12853	36.12833	36.12832
13		39.44550	39.28216	39.27023	39.26993	39.26991
14		42.66294	42.42940	42.41197	42.41153	42.41150
15		45.90168	45.57855	45.55377	45.55314	45.55309
16		49.16301	48.73001	48.69563	48.69475	48.69469
17		52.44425	51.88426	51.83757	51.83636	51.83628
18		55.73055	55.04185	54.97960	54.97798	54.97787
19		58.96064	58.20334	58.12174	58.11961	58.11946
20		61.82270	61.36938	61.26402	61.26124	61.26106

TABLE 4: Natural frequencies of Euler-Bernoulli Simple-Simple Beam: $\mu = \sqrt{\omega}(\frac{\rho A}{EI})^{1/4}.L$

Mode No.	Exact Method (Ref. Yang [50])	Spectral Element Method
1	3.14159	3.14159
2	6.28318	6.28319
3	9.42477	9.42478
4	12.56637	12.56637
5	15.70796	15.70796
6	18.84955	18.84956
7	21.99114	21.99115
8	25.13274	25.13274
9		28.27433
10		31.41593
11		34.55752
12		37.69911
13		40.84070
14		43.98230
15		47.12389
16		50.26548
17		53.40708
18		56.54867
19		62.83185
20		65.97345

TABLE 5: Natural Frequencies of Euler-Bernoulli Fixed-Simple Beam: $\mu = \sqrt{\omega}(\frac{\rho A}{EI})^{1/4} . L$

Mode No.	Exact Method (Ref. Yang [50])	Spectral Element Method
1	3.92660	3.92660
2	7.06858	7.06858
3	10.21017	10.21018
4	13.35176	13.35177
5	16.49336	16.49336
6	19.63495	19.63495
7	22.77654	22.77655
8	25.91813	25.91814
9		29.05973
10		32.20132
11		35.34292
12		38.48451
13		41.62610
14		44.76770
15		47.90929
16		51.05088
17		54.19247
18		57.33407
19		60.47566
20		63.61725

TABLE 6: Natural Frequencies of Euler-Bernoulli Fixed-Fixed Beam: $\mu = \sqrt{\omega}(\frac{\rho A}{EI})^{1/4}.L$

Mode No.	Exact Method (Ref. Yang [50])	Spectral Element Method
1	4.73004	4.73004
2	7.85320	7.85320
3	10.99560	10.99561
4	14.13716	14.13717
5	17.27875	17.27876
6	20.42035	20.42035
7	23.56194	23.56194
8	26.70353	26.70354
9		29.84513
10		32.98672
11		36.12832
12		39.26991
13		42.41150
14		45.55309
15		48.69469
16		51.83628
17		54.97787
18		58.11946
19		61.26106
20		64.40265

TABLE 7: Comparison of Natural Frequencies of Timoshenko Fixed-Fixed Beam: $\mu = \sqrt{\omega}(\frac{\rho A}{EI})^{1/4}.L$

Mode No.	Classical theory	h/L = 0.002		h/L = 0.005		h/L = 0.01		h/L = 0.02	
		Lee[37]	Present	Lee[37]	Present	Lee[37]	Present	Lee[37]	Present
1	4.73004	4.72998	4.72998	4.72963	4.72963	4.72840	4.72840	4.72350	4.72350
2	7.85320	7.85295	7.85295	7.85163	7.85163	7.84690	7.84691	7.82817	7.82817
3	10.9956	10.9950	10.99498	10.9917	10.99171	10.9800	10.98005	10.9341	10.93412
4	14.1372	14.1359	14.13592	14.1294	14.12937	14.1062	14.10615	14.0154	14.01543
5	17.2788	17.2766	17.27657	17.2651	17.26512	17.2246	17.22459	17.0679	17.06787
6	20.4204	20.4168	20.41685	20.3985	20.39852	20.3338	20.33385	20.0868	20.08680
7	23.5619	23.5567	23.55668	23.5292	23.52918	23.4325	23.43251	23.0682	23.06818
8	26.7035	26.6960	26.69601	26.6567	26.65672	26.5192	26.51919	26.0086	26.00859
9	29.8451	29.8348	29.83477	29.7808	29.78076	29.5926	29.59258	28.9052	28.90522
10	32.9867	32.9729	32.97290	32.9009	32.90092	32.6514	32.65144	31.7558	31.75581
11	36.1283	36.1103	36.11033	36.0168	36.01683	35.6946	35.69458	34.5587	34.55867
12	39.2699	39.2470	39.24701	39.1281	39.12813	38.7209	38.72089	37.3126	37.31261
13	42.4115	42.3829	42.38286	42.2345	42.23446	41.7293	41.72933	40.0169	40.01689
14	45.5531	45.5178	45.51783	45.3355	45.33546	44.7189	44.71894	42.6712	42.67116
15	48.6947	48.6519	48.65186	48.4308	48.43079	47.6888	47.68883	45.2754	45.27541
16			51.78487		51.52011		50.63818		47.82994
17			54.91682		54.60309		53.56627		50.33527
18			58.04763		57.6794		56.47243		52.79212
19			61.17725		60.74874		59.35606		55.20138
20			64.30562		63.81079		62.21665		57.56405

Contd....

Mode No.	Classical theory	h/L = 0.05		h/L = 0.1		h/L = 0.2		h/L=0.25
		Lee[37]	Present	Lee[37]	Present	Lee[37]	Present	Present
1	4.73004	4.68991	4.68991	4.57955	4.57955	4.24201	4.24201	4.05670
2	7.85320	7.70352	7.70352	7.33122	7.33122	6.41794	6.41794	6.00341
3	10.9956	10.6401	10.6401	9.85611	9.85611	8.28532	8.28532	7.66925
4	14.1372	13.4611	13.4611	12.1451	12.1453	9.90372	9.90372	9.09783
5	17.2788	16.1590	16.1590	14.2324	14.2324	11.3487	11.34874	10.37482
6	20.4204	18.7318	18.7318	16.1487	16.1487	12.6402	12.64025	10.89788
7	23.5619	21.1825	21.1825	17.9215	17.9214	13.4567	13.45674	11.57107
8	26.7035	23.5168	23.5168	19.5723	19.5723	13.8101	13.81014	11.98845
9	29.8451	25.7421	25.7420	21.1185	21.1185	14.4806	14.48056	12.66236
10	32.9867	27.8662	27.8661	22.5735	22.5735	14.9383	14.93829	13.22654
11	36.1283	29.8969	29.8969	23.9479	23.9478	15.6996	15.69933	13.68281
12	39.2699	31.8418	31.8418	25.2479	25.2478	16.0040	16.00404	14.38568
13	42.4115	33.7078	33.7077	26.2831	26.2830	16.9621	16.96209	14.71705
14	45.5531	35.5011	35.5011	26.4595	26.4595	16.9999	16.99988	15.33241
15	48.6947	37.2275	37.2275	26.9237	26.9236	17.9357	17.93568	15.86637
16			38.8922		27.5690		18.21437	16.16512
17			40.4998		27.9126		18.82647	16.94264
18			42.0545		28.6752		19.40266	17.02778
19			43.5600		29.0472		19.71075	17.68292
20			45.0196		29.8130		20.36670	18.13375

TABLE 8: Comparison of Natural Frequencies of Timoshenko Pinned-Pinned Beam: $\mu = \sqrt{\omega}(\frac{\rho A}{EI})^{1/4}.L$

Mode No.	Classical theory	h/L = 0.002		h/L = 0.005		h/L = 0.01		h/L = 0.02	
		Lee[37]	Present	Lee[37]	Present	Lee[37]	Present	Lee[37]	Present
1	3.14159	3.14158	3.14158	3.14153	3.14153	3.14133	3.14133	3.14053	3.14053
2	6.28319	6.28310	6.28310	6.28265	6.28265	6.28106	6.28106	6.27471	6.27471
3	9.42478	9.42449	9.42449	9.42298	9.42298	9.41761	9.41761	9.39632	9.39632
4	12.5664	12.5657	12.56569	12.5621	12.56212	12.5494	12.54941	12.4994	12.49941
5	15.7080	15.7066	15.70663	15.6997	15.69966	15.6749	15.67492	15.5784	15.57841
6	18.8496	18.8473	18.84726	18.8352	18.83522	18.7926	18.79263	18.6282	18.62823
7	21.9911	21.9875	21.98750	21.9684	21.96840	21.9011	21.90107	21.6443	21.64431
8	25.1327	25.1273	25.12729	25.0988	25.09882	24.9988	24.99881	24.6227	24.62267
9	28.2743	28.2666	28.26658	28.2261	28.22610	28.0845	28.08450	27.5599	27.55993
10	31.4159	31.4053	31.40529	31.3498	31.34984	31.1568	31.15682	30.4533	30.45331
11	34.5575	34.5434	34.54337	34.4697	34.46969	34.2145	34.21454	33.3006	33.30060
12	37.6991	37.6807	37.68075	37.5853	37.58526	37.2565	37.25647	36.1001	36.10014
13	40.8407	40.8174	40.81736	40.6962	40.69620	40.2815	40.28150	38.8507	38.85074
14	43.9823	43.9531	43.95315	43.8021	43.80214	43.2886	43.28862	41.5517	41.55167
15	47.1239	47.0880	47.08805	46.9027	46.90273	46.2769	46.27686	44.2026	44.20258
16			50.22200		49.99763		49.24534		46.80347
17			53.35494		53.08650		52.19324		49.35460
18			56.48680		56.16900		55.11985		51.85649
19			59.61753		59.24483		58.02451		54.30984
20			62.74706		62.31365		60.90662		56.71552

Contd...

Mode No.	Classical theory	h/L = 0.05		h/L = 0.1		h/L = 0.2		h/L=0.25
		Lee[37]	Present	Lee[37]	Present	Lee[37]	Present	Present
1	3.14159	3.13498	3.13498	3.11568	3.11568	3.04533	3.04533	2.99853
2	6.28319	6.23136	6.23136	6.09066	6.09066	5.67155	.67155	5.44527
3	9.42478	9.25537	9.25537	8.84052	8.84052	7.83952	7.83952	7.38372
4	12.5664	12.1813	12.18132	11.3431	11.34310	9.65709	9.65709	8.97763
5	15.7080	14.9926	14.99264	13.6132	13.61317	11.2220	11.22204	10.33928
6	18.8496	17.6810	17.68103	15.6790	15.67904	12.6022	12.60221	10.42586
7	21.9911	20.2447	20.24467	17.5705	17.57050	13.0323	13.03233	10.92329
8	25.1327	28.6862	28.68621	19.3142	19.31418	13.4443	13.44427	11.53748
9	28.2743	25.0111	25.01111	20.9325	20.93255	13.8433	13.84329	12.03020
10	31.4159	27.2263	27.22633	22.4441	22.44408	14.4378	14.43776	12.61546
11	34.5575	29.3394	29.33944	23.8639	23.86391	14.9766	14.97658	13.30784
12	37.6991	31.3581	31.35808	25.2044	25.20442	15.6676	15.66764	13.60154
13	40.8407	33.2896	33.28962	26.0647	26.06465	16.0241	16.02413	14.51506
14	43.9823	35.1410	35.14099	26.2814	26.28141	16.9584	16.95842	14.59352
15	47.1239	36.9186	36.91862	26.4758	26.47583	17.0019	17.00192	15.36975
16			38.62837		26.88855		17.92180	15.83950
17			40.27558		27.68657		18.24190	16.17562
18			41.86510		27.78724		18.79278	16.94024
19			43.40130		28.84371		19.49290	17.03343
20			44.88816		28.87553		19.62186	17.66942

TABLE 9: Natural Frequencies of Timoshenko Fixed-Free Beam:

$$\mu = \sqrt{\omega}\left(\frac{\rho A}{EI}\right)^{1/4} . L$$

Mode No.	$h/L =$ 0.002	$h/L =$ 0.005	$h/L =$ 0.01	$h/L =$ 0.02	$h/L =$ 0.05	$h/L =$ 0.1	$h/L =$ 0.2	$h/L =$ 0.25
1	1.87510	1.87509	1.87503	1.87481	1.87324	1.86771	1.84656	1.83160
2	4.69404	4.69376	4.69279	4.68888	4.66204	4.57241	4.28529	4.11957
3	7.85455	7.85346	7.84956	7.83407	7.73048	7.41542	6.61128	6.23346
4	10.99500	10.99217	10.9820	10.94238	10.68618	9.98735	8.51863	7.90934
5	14.13606	14.13026	14.10965	14.02901	13.53185	12.32243	10.15839	9.33062
6	17.27678	17.26644	17.22979	17.08781	16.25592	14.44589	11.57215	10.39475
7	20.41714	20.40035	20.34107	20.11409	18.85511	16.38833	12.78239	11.00900
8	23.55708	23.53162	23.44206	23.10370	21.33116	18.17662	13.34954	11.39753
9	26.69652	26.65984	26.53138	26.05309	23.68906	19.83284	13.95152	12.16053
10	29.83540	29.78465	29.60770	28.95929	25.93557	21.37405	14.33794	12.54755
11	32.97367	32.90566	32.66975	31.81996	28.07826	22.81248	15.10606	13.25192
12	36.11125	36.02251	35.71635	34.63325	30.12479	24.15363	15.47281	13.72904
13	39.24810	39.13482	38.74635	37.39788	32.08259	25.38752	16.28664	14.28009
14	42.38413	42.24224	41.75872	40.11299	33.95861	26.21866	16.56093	14.81276
15	45.51929	45.34441	44.75247	42.77813	35.75923	26.55586	17.41799	15.33210
16	48.65353	48.44099	47.72670	45.39321	37.49025	26.88207	17.63785	15.74028
17	51.78676	51.53163	50.68057	47.95845	39.15690	27.57973	18.40087	16.43068
18	54.91894	54.61601	53.61334	50.47430	40.76383	27.91035	18.78009	16.58401
19	58.05001	57.69379	56.52432	52.94143	42.31512	28.66725	19.31484	17.31563
20	61.17989	60.76466	59.41292	55.36067	43.81436	29.05975	19.85378	17.58006

TABLE 10: Natural Frequencies of Timoshenko Fixed-Pinned Beam:

$$\mu = \sqrt{\omega}(\frac{\rho A}{EI})^{1/4}.L$$

Mode No.	$h/L =$ 0.002	$h/L =$ 0.005	$h/L =$ 0.01	$h/L =$ 0.02	$h/L =$ 0.05	$h/L =$ 0.1	$h/L =$ 0.2	$h/L =$ 0.25
1	3.92657	3.92641	3.92581	3.92345	3.90714	3.85176	3.66561	3.55283
2	7.06843	7.06761	7.06469	7.05308	6.97477	6.73057	6.07268	5.74997
3	10.20974	10.20744	10.19926	10.16692	9.95632	9.36591	8.07437	7.53281
4	13.35083	13.34589	13.32836	13.25959	12.83069	11.75841	9.78617	9.04182
5	16.49163	16.48257	16.45046	16.32572	15.58540	13.93309	11.28676	10.34740
6	19.63209	19.61708	19.56406	19.36045	18.21559	15.92091	12.62393	10.56807
7	22.77213	22.74904	22.66773	22.35947	20.72198	17.75055	13.14153	11.38453
8	25.91170	25.87805	25.76006	25.31909	23.10895	19.44613	13.78451	11.59139
9	29.05073	29.00374	28.83970	28.23621	25.38303	21.02728	13.95633	12.54004
10	32.18915	32.12572	31.90540	31.10832	27.55169	22.50984	14.90650	12.73656
11	35.32691	35.24363	34.95591	33.93348	29.62270	23.90648	15.10788	13.58536
12	38.46394	38.35710	37.99012	36.71025	31.60364	25.22660	15.99984	13.96370
13	41.60018	41.46576	41.00694	39.43767	33.50168	26.11891	16.33126	14.51824
14	44.73557	44.56926	44.00537	42.11523	35.32343	26.45166	16.99988	15.19475
15	47.87004	47.66725	46.98450	44.74274	37.07498	26.57075	17.59154	15.39885
16	51.00353	50.75939	49.94348	47.32034	38.76180	27.29895	17.93806	16.16266
17	54.13597	53.84534	52.88153	49.84845	40.38890	27.69326	18.74656	16.45160
18	57.26732	56.92478	55.79795	52.32768	41.96074	28.31082	18.91640	16.94267
19	60.39750	59.99738	58.69213	54.75884	43.48138	28.84570	19.61756	17.57853
20	63.52645	60.06284	61.56351	57.14285	44.95446	29.46095	20.09759	17.70131

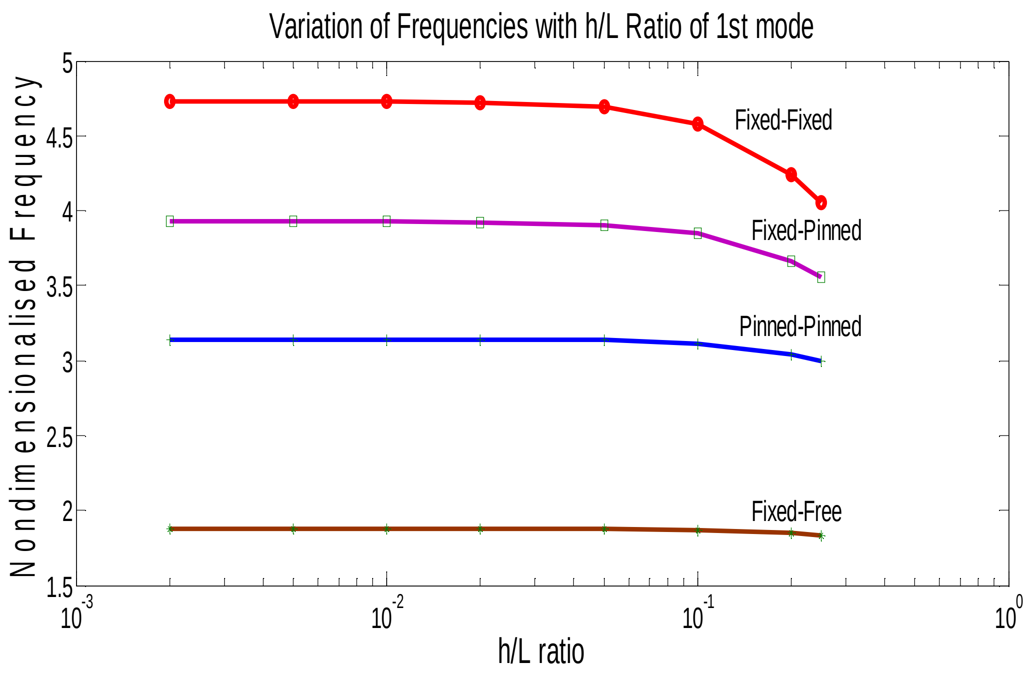

Fig4. Natural frequencies of first mode

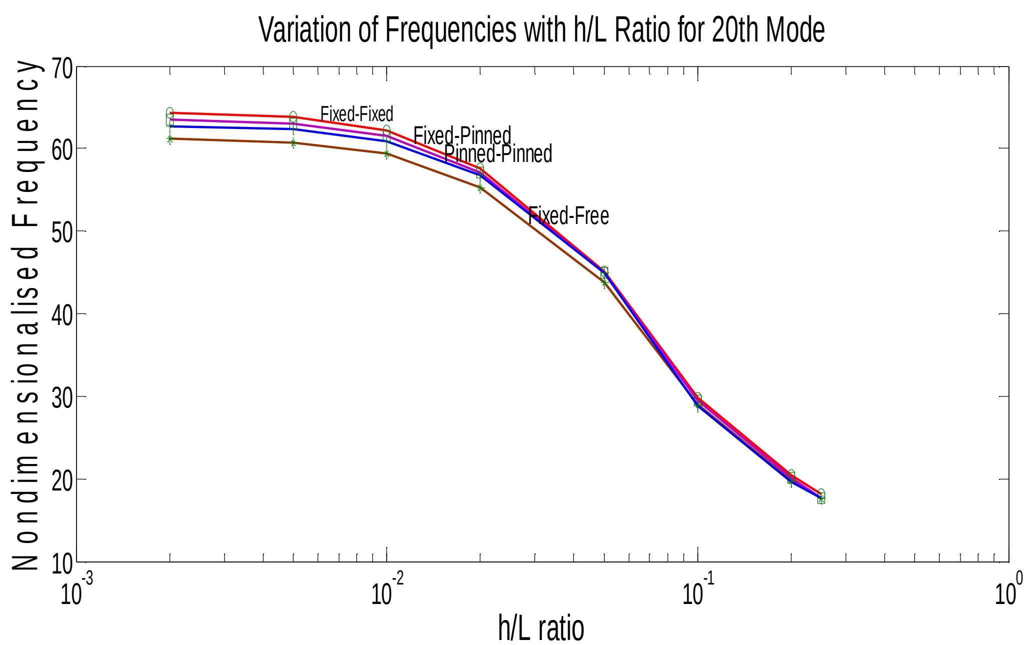

Fig5. Natural frequencies of twenty modes

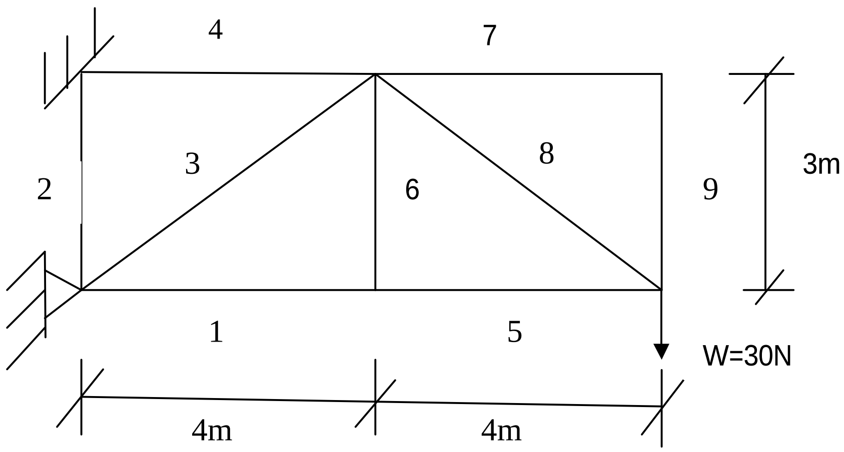

Fig.6 frame1

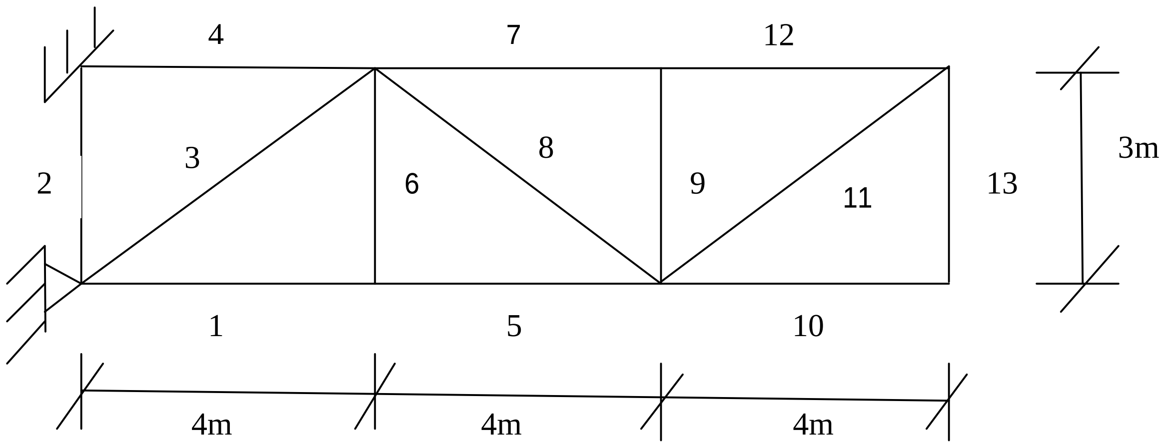

Fig.7 frame2

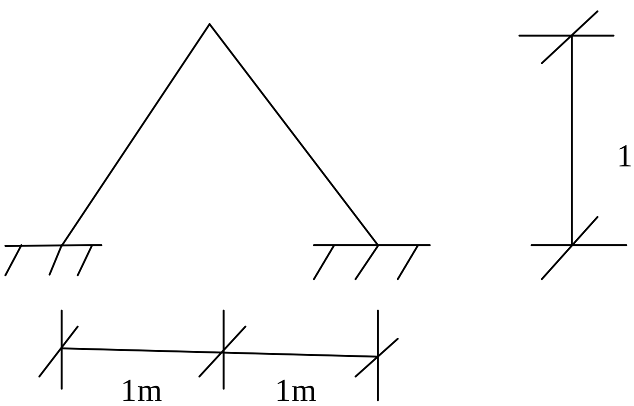

Fig. 8 frame 3

TABLE 11: Natural Frequencies of Frame1 & 2

Mode No.	Wittrick-Williams (Ref. Howson [32])	Spectral Element Method (frame 1)	New results (frame 2)
1	25.468	25.4676	15.0090
2	39.466	39.4660	38.3807
3	49.117	49.1172	40.3284
4		52.4718	48.0350
5		66.7489	50.1420
6		75.1757	54.3971
7		78.5016	62.3413
8		92.3116	69.5413
9		106.2880	74.4279
10		119.6163	77.8150
11		123.2461	79.7845
12		143.1479	97.8281
13		148.5353	110.7994
14		156.5590	116.7796
15		190.5314	119.5192
16		194.8439	125.8569
17		206.0644	130.6540
18		211.1034	141.1025
19		233.9603	147.1514
20	256.20	256.2040	155.7111

TABLE 12: Natural Frequencies of Frame 3:

Mode No.	(Ref. Petyt [41]) in Hz.	Spectral Element Method (frame 3)
1	88.9	88.86955
2	128.6	128.57911
3	286.9	286.82677
4	350.9	350.81938
5		591.14954
6		660.51226
7		856.74861
8		861.83153
9		1059.66856
10		1188.17071
11		1576.88004
12		1728.38640
13		2206.07712
14		2331.18429
15		2646.00809
16		2664.88082
17		3036.01831
18		3246.64220
19		3860.15273
20		4030.75653

CHAPTER 5

Conclusion

1. The Spectral Element Method (SEM) has the advantages of Spectral Analysis Method (SAM), Dynamic Stiffness Method (DSM) and Finite Element Method (FEM).
2. The Spectral Element Method is efficient to compute both the lower and higher modes natural frequencies even with a maximum number of two elements.
3. The natural frequencies of rods and beams up to the twenty numbers of modes have been presented by using two number of spectral element.
4. The natural frequencies of frames have been presented up to the twentieth mode by considering each member of the frame as a single element.
5. The finite element method couldn't converge to the exact frequencies even with 200 numbers of finite elements. In some cases with these many numbers of elements it couldn't converge even further first mode frequency.
6. The natural frequencies of rods, beams and frames for various boundary conditions are presented for the modes up to twenty where many higher modes results are not available in the literature.

SCOPE OF FUTURE WORK

Through this project work some studies for the frequencies of rods, beams and frames are attempted but still many areas are left which require further investigation. The possible extensions to the present study are as given below:

1. The Spectral Element Method may be used for buckling analysis of structures.
2. The Spectral Element Method may be extended for the dynamic study of composite beam, composite plates and structures of Functionally Graded materials.
3. The Spectral Element Method may be extended for response analysis of structures.
4. The Spectral Element Method may be used for dynamics of axially moving structures.
5. The Spectral Element Method may be applied to smart structures.

References

[1] Abramovich, H. Shear deformation and rotatory inertia effects of vibrating composites beams. *Composite structure*, **20**, 165-173, 1992.

[2] Abramovich, H. and Livshits, A. Free vibration of non-symmetric cross-ply laminated composite beams. *Journal of Sound and Vibration*, **176**, 597-612. 1994.

[3] Akesson, B. PFVIBAT- a computer programme for plane frame vibration analysis by an exact method. *International Journal for Numerical Methods in Engineering*.**10**, 1221-1231, 1976.

[4] Armstrong, I. D. The natural frequencies of grillages.*International Journal of Mechanical Science*, Pergamon Press, 10, 43-55, 1968.

[5] Banerjee, J. R. Free vibration of axially loaded composite Timoshenko beams using the dynamic stiffness matrix method. *Computers and Structures*,**69**, 197-208. 1998.

[6] Banerjee, J. R. Free vibration analysis of a twisted beam using the dynamic stiffness method. *International Journal of Solids and Structures*, **38**, 6703-6722. 2001.

[7] Banerjee, J. R. Free vibration of centrifugally stiffened uniform and tapered beams using the dynamic stiffness method. *Journal of Sound and Vibration*, **233**, 857-875, 2000.

[8] Banerjee, J. R. Development of an exact dynamic stiffness matrix for free vibration analysis of a twisted Timoshenko beam. *Journal of Sound and Vibration*, **270**, 379-401. 2004.

[9] Banerjee, J. R. Dynamic stiffness formulation for structural elements: A general approach. *Computers and Structures*, **63**(1), 101-103, 1997.

[10] Banerjee, J. R, Guo, S.and Howson, W. P. Exact dynamic stiffness matrix of a bending-torsion coupled beam including warping. *Computers & structures*.**59**, 613-621, 1996.

[11] Banerjee, J. R and Fisher, S. A. Coupled bending torsional dynamic stiffness matrix for axially loaded beam element. *International Journal numer.Meth.Engng,* **33**, 739-751, 1992.

[12] Banerjee, J. R and Williams, F.W. Clamped-Clamped natural frequencies of a bending-torsion coupled beam. *Journal of Sound and Vibration*.**176**, 301-306, 1994.

[13] Banerjee, J. R. Coupled bending-torsional dynamic stiffness matrix for beam elements. *International Journal for numerical Methods in Engineering,* **28**, 1283-1298, 1989.

[14] Banerjee, J. R and Williams, F.W. Vibration of composite beams-an exact method using symbolic computation. *Journal of Aircraft,* **32**, 636-642. 1995.

[15] Banerjee, J. R and Williams, F.W. Exact dynamic stiffness matrix for composite Timoshenko beams with applications.*Journal of Sound and Vibration,* **194**, 573-585. 1996.

[16] Banerjee, J. R and Williams, F.W. An exact dynamic stiffness matrix for coupled-extensional-torsional vibration of structural members.*Computers and Structures,* **50**, 161-166, 1994.

[17]] Banerjee, J. R and Williams, F.W. Coupled bending torsional dynamic stiffness matrix for Timoshenko beam elements.*Computers and Structures,* **42**,301-310, 1992.

[18] Banerjee, J. R and Williams, F.W. Exact Bernouli-Euler Dynamic Stiffness Matrix for a ranged of Tapered beams. *International Journal of numerical Methods in Engineering,* **21**, 2289-2302, 1985.

[19] Banerjee, J. R and Williams, F.W. Coupled bending torsional dynamic stiffness matrix of an axially loaded Timoshenko beam element. *International Journal of solids structure,* **31**, 749-762, 1994.

[20] Beskos, D. E. and Narayanan, G. V. Use of dynamic influence coefficients in forced vibration problems with the aid of fast Fourier transform. *Computer & structures*.**9**, 145-150, 1978.

[21] Capron, M. D. And Williams, F. W. Exact dynamic stiffness for an axially loaded uniform Timoshenko member embedded in an elastic medium. *Journal of Sound and Vibration*, **124**, 453-466, 1988.

[22] Chakraborty, A. And Gopalkrishinan, S. A spectrally formulated finite element for wave propagation analysis in functionally graded beams. *International Journal of solids and structures,***40** (10), 2421-2448, 2003.

[23] Chandrasekhar, K., Krishnamurthy, K. and Roy, S. Free vibration of composite beams including rotatory inertia and shear deformation. *Composites Structures*, **14**, 269-279, 1990.

[24] Cheng, F. Y. Vibration of Timoshenko beams and frameworks. *Journal of the structural division*, **96**, 551-571. 1970.

[25] Cheng, F. Y. and Tseng, W. H. Dynamic matrix of Timoshenko beam column. *Journal of the structural division*.**99**, 527-549. 1973.

[26] Cho, J. and Lee, U. An FFT-based spectral analysis method for linear discrete dynamic systems with non-proportional damping, *Shock and Vibration*, **13** (6), 595-606, 2006.

[27] Doyle, J. F. A Spectrally formulated finite elements for longitudinal wave propagation. *International Journal of Analytical and Experimental Modal Analysis*, **3**, 1-5, 1988.

[28] Doyle, J. F. *Wave propagation in structures*: *Spectral Analysis Using Fast Discrete Fourier Transforms*, 2ndedn,Springer-Verlag, New York. 1997.

[29] Eisenberger, M. Abramovich, H. and Shulepov, O. Dynamic stiffness analysis of laminated beams using a first order shear deformation theory. *Composites Structures*, **31**, 265-271, 1995.

[30] Friberg, P. O. Coupled vibration of beams-an exact dynamic element stiffness matrix. *International Journal of numer. Meth.Engng*, **19**, 479-493, 1983.

[31] Gopalakrishna, S. And Doyle, J. F. Wave propagation in connected waveguides of varying cross-section. *Journal of Sound and Vibration*, **175** (3), 347-363, 1994.

[32] Howson, W. P. A compact method for computing the eigenvalues and eigenvectors of plane frames.

[33] Howson, W.P. and Williams, F. W. Natural frequencies of frames with axially loaded Timoshenko members. *Journal of Sound and vibration*, **26**, 503-515. 1973.

[34] Howson, W.P. and Zare, A. Exact dynamic stiffness matrix for flexural vibration of the three-layered sandwich beams. I. Forward calculation. *Journal of Sound and Vibration*, **282**, (3-5), 753-767, 2005.

[35] Kolousek, V. Anwendung des gesetzes der virtuellenverschiebungen und des reziprozitatssatzes in der stabwerksdynamic. *IngenieurArchiv*, **12**, 363-370.1941.

[36] Lunden, R. and Akesson, B. Damped second-order Rayleigh-Timoshenko beam vibration in space- an exact complex dynamic member stiffness matrix. *International Journal for Numerical Methods in Engineering.***19**, 431-449, 1983.

[37] Lee, J. and Schultz, W. W. Eigenvalue analysis of Timoshenko beams and axisymmetric Mindlin plates by the pseudospectral method. *Journal of Sound and Vibration*, **269**, 609-621, 2004.

[38] Lee, U. Spectral Element Method in Structural Dynamics. 2009.

[39] Narayanam, G.V. and Beskos,D.E. Use the dynamic influence coefficients in forced vibration problems with the aid of fast fourier transform. *Computer & structures,* 9 (2),145-150,1978.

[40] Przemieniecki, J.S. *Theory of Matrix Structural Analysis*, McGrow-Hill Inc, New York.1968.

[41] Petyt, M. Introduction to Finite Element Vibration Analysis . Cambridge University Press, 1990.

[42] Rizzi, S. A. And Doyle, J. F.A spectral element approach to wave motion in layered solids, Journal of Vibration and Acoustics.114 (4), 569-577, 1992.

[43] Rosen, A. Structural and dynamics behaviour of pretwisted rods and beams. *Applied Mechanics Reviews*, **44**, 483-515, 1991.

[44] Teh, K. K. and Huang, C. C., The effects of fibre orientation on free vibration of composite beams. *Journal of Sound and Vibration,* **69**, 327-337, 1980.

[45] Teoh, S. L. and Huang, C. C., The vibration of beams of fibre reinforced material.*Journal of Sound and Vibration*, **51**, 467-473,1977.

[46] Wang, T. M. and Kinsman, T. A. Vibration of frame structures according to the Timoshenko theory. *Journal of Sound and Vibration*, **14**, 215-227, 1971.

[47] Wittrick, W. H. and Williams F. W. A general algorithm for computing natural frequencies of elastic sttructures.*Quarterly Journal of Mechanics and Applied Mathematics,* **24**, 263-284,1971.

[48] Wittrick, W. H. and Williams, F. W. Exact buckling and frequency calculation surveyed. *Journal Structures*. Engng ASCE. **109**, 169-187, 1983.

[49] Wittrick, W. H. and Williams, F. W. An Automatic computational procedure for calculating natural frequencies of skeletal Structures.*Intenational Journal. Mech. Sci.*Pergamon Press. **12**, 781-791, 1970.

[50] Yang, Bingen. Stress Strain and Structural Dynamics. Elsevier Academic Press, 2005.