FUZZY TOPOLOGICAL SPACES

A THESIS

SUBMITTED TO THE

NATIONAL INSTITUTE OF TECHNOLOGY, ROURKELA

IN THE PARTIAL FULFILMENT

FOR THE DEGREE OF

MASTER OF SCIENCE IN MATHEMATICS

BY

TAPATI DAS

UNDER THE SUPERVISION OF

Dr. DIVYA SINGH

DEPARTMENT OF MATHEMATICS

NATIONAL INSTITUTE OF TECHNOLOGY, ROURKELA

MAY, 2013

Abstract

The present thesis consisting of three chapters is devoted to the study of Fuzzy topological spaces. After giving the fundamental definitions we have discussed the concepts of fuzzy continuity, fuzzy compactness, and separation axioms, that is, fuzzy Hausdorff space, fuzzy regular space, fuzzy normal space etc.

Acknowledgements

I deem it a privilege and honor to have worked in association under Dr.Divya Singh, Assistant Professor in the Department of mathematics, National Institute of Technology, Rourkela. I express my deep sense of gratitude and indebtedness to him for guiding me throughout the project work.

I thank all faculty members of the Department of Mathematics who have always inspired me to work hard and helped me to learn new concepts during our stay at NIT Rourkela.

I would like to thanks my parents for their unconditional love and support. They have supported me in every situation. I am grateful for their support.

Finally I would like to thank all my friends for their support and the great almighty to shower his blessing on us.

Contents

Chapter 1

Preliminaries and Introduction

1.1. Fuzzy Set

Zadeh [11], 1965, introduced the concept of fuzzy sets by defining them in terms of mappings from a set into the unit interval on the real line. Fuzzy sets were introduced to provide means to describe situations mathematically which give rise to ill-defined classes, i.e., collections of objects for which there is no precise criteria for membership. Collections of this type have vague or "fuzzy" boundaries; there are objects for which it is impossible to determine whether or not they belong to the collection. The classical mathematical theories, by which certain types of certainty can be expressed, are the classical set theory and the probability theory. In terms of set theory, uncertainty is expressed by any given set of possible alternatives in situations where only one of the alternatives may actually happen. Uncertainty expressed in terms of sets of alternatives results from the nonspecificity inherent in each set. Probability theory expresses uncertainty in terms of a classical measure on subsets of a given set of alternatives. The set theory, introduced by Zadeh, presents the notion that membership in a given subset is a matter of degree rather than that of totally in or totally out. With fuzzy set theory, one obtains a logic in which statements may be true or false to different degrees rather than the bivalent situation of being true or false; consequently, certain laws of bivalent logic do not hold, e.g. the law of the excluded middle and the law of contradiction. This results in an enriched scientific methodology. Chang

[2], introduced the notion of a fuzzy topology of a set in 1968, and our work is based on the study of the properties of fuzzy topological spaces.

Definition 1.1.1 [11]. Let X be a non-empty set. A fuzzy set A in X is characterized by its membership function $\mu_A : X \to [0,1]$ and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy set A, for each $x \in X$. It is clear that A is completely determined by the set of tuples $A = \{(x, \mu_A(x)) : x \in X\}$.

1.2. Basic Operations on Fuzzy Sets

Definition 1.2.1 [11]: Let $A = \{(x, \mu_A(x)) : x \in X\}$ and $B = \{(x, \mu_B(x)) : x \in X\}$ be two fuzzy sets in X. Then their union $A \vee B$, intersection $A \wedge B$ and complement A^c are also fuzzy sets with the membership functions defined as follows:

(i) $\mu_{A\vee B}(x) = \max\,\{\mu_A(x), \mu_B(x)\}, \ \ \forall\, x \in X$.

(ii) $\mu_{A\wedge B}(x) = \min\,\{\mu_A(x), \mu_B(x)\}, \ \ \forall\, x \in X$.

(iii) $\mu_{A^c}(x) = 1 - \mu_A(x), \ \ \forall\, x \in X$.

Further,

(a) $A \subseteq B$ iff $\mu_A(x) \le \mu_B(x), \ \ \forall\, x \in X$.

(b) $A = B$ iff $\mu_A(x) = \mu_B(x), \ \ \forall\, x \in X$.

Lemma 1.2.2 [11]: The De Morgan's law are true for fuzzy sets. That is suppose $A = \{(x, \mu_A(x)) : x \in X\}$ and $B = \{(x, \mu_B(x)) : x \in X\}$ are fuzzy sets, then

$$(A \cup B)^c = A^c \cap B^c \cdots\cdots\cdots\cdots (1)$$

$$(A \cap B)^c = A^c \cup B^c \cdots\cdots\cdots\cdots (2)$$

Proof of equation (1). We know that the following identity is true.

$$1 - max[\mu_A, \mu_B] = min[1 - \mu_A, 1 - \mu_B] \cdots\cdots\cdots\cdots (3)$$

To show that we consider the two possible cases: $\mu_A \geq \mu_B$ and $\mu_A < \mu_B$. If $\mu_A \geq \mu_B$, then $1-\mu_A \leq 1-\mu_B$ and $1-max[\mu_A, \mu_B] = 1-\mu_A = min[1-\mu_A, 1-\mu_B]$, which is equation (3). If $\mu_A < \mu_B$, then $1-\mu_A > 1-\mu_B$ and $1-max[\mu_A, \mu_B] = 1-\mu_B = min[1-\mu_A, 1-\mu_B]$ which is again equation (3). Hence this equation (3) is true. Now, the membership function of $(A \cup B)^c$ is given by

$$\begin{aligned}
\mu_{(A\cup B)^c}(x) &= 1 - \mu_{A\cup B}(x) \\
&= 1 - max[\mu_A(x), \mu_B(x)] \\
&= min[1 - \mu_A(x), 1 - \mu_B(x)] \\
&= min[\mu_{A^c}(x), \mu_{B^c}(x)] \\
&= \mu_{A^c \cap B^c}(x)
\end{aligned}$$

This proves Equ. (1). Similarly, using (3) we can prove Equ. (2).

1.3 Images and Preimages of Fuzzy Sets

Definition 1.3.1 [6]: The symbol I will denote the unit interval [0,1]. Let X be a non-empty set. Now, for the sake of simplicity of notation we will not differentiate between A and μ_A. That is a fuzzy set A in X is a function with domain X and values in I, i.e. an element of I^X. Let $A, B \in I^X$ and let $f : X \to Y$ be a function. Then $f(A) \in I^Y$, i.e. $f(A)$ is a fuzzy set in Y, defined by

$$f(A)(y) = \begin{cases} \sup\{A(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi, \end{cases}$$

and $f^{-1}(B)$ is a fuzzy set in X, defined by $f^{-1}(B)(x) = B(f(x))$, $x \in X$.

Definition 1.3.2 [6]. The product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ of mapping $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ is defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$ for each $(x_1, x_2) \in X_1 \times X_2$. For a mapping $f : X \to Y$, the graph $g : X \to X \times Y$ of f is defined by $g(x) = (x, f(x))$, for each $x \in X$.

Definition 1.3.3 [6]. Let $A \in I^X$ and $B \in I^Y$. Then by $A \times B$ we denote the fuzzy set in $X \times Y$ for which $(A \times B)(x, y) = min(A(x), B(y))$, for every $(x, y) \in X \times Y$.

Proposition 1.3.4 [6]. $f^{-1}(B^c) = (f^{-1}(B))^c$, for any fuzzy set B in Y.

Proof. $f^{-1}(B^c)(x) = (B^c)f(x) = 1 - B(f(x)) = 1 - f^{-1}(B(x)) = (f^{-1}(B))^c(x)$, $\forall x \in X$.

Proposition 1.3.5 [6]. $f(f^{-1}(B)) \leq B$, for any fuzzy set B in Y.

Proof. The proof follows by noting that

$$f(f^{-1}(B)(y)) = \begin{cases} \sup\{f^{-1}(B)(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{if } f^{-1}(y) = \phi \end{cases}$$

$$= \begin{cases} \sup\{B(f(x)) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{if } f^{-1}(y) = \phi \end{cases}$$

$$= \begin{cases} B(y), & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{if } f^{-1}(y) = \phi \end{cases}$$

Proposition 1.3.6 [6]. Let $f : X \to Y$ be a mapping and A_j be a family of fuzzy sets of Y, then

(a) $f^{-1}(\vee A_j) = \vee f^{-1}(A_j)$

(b) $f^{-1}(\wedge A_j) = \wedge f^{-1}(A_j)$

Proof-(a).

$$\begin{aligned} f^{-1}(\vee A_j)(x) &= (\vee A_j)(f(x)) \\ &= (A_1 \vee A_2 \vee \cdots \vee A_j \cdots)(f(x)) \\ &= max\{A_1 f(x), A_2 f(x), \cdots, A_j f(x) \cdots\} \\ &= max\{f^{-1}(A_1)(x), f^{-1}(A_2)(x), \cdots, f^{-1}(A_j)(x), \cdots\} \\ &= (f^{-1}(A_1) \vee f^{-1}(A_2) \vee \cdots \vee f^{-1}(A_j) \cdots)(x) \\ &= \vee f^{-1}(A_j)(x) \end{aligned}$$

(b).

$$\begin{aligned} f^{-1}(\wedge A_j)(x) &= (\wedge A_j)(f(x)) \\ &= (A_1 \wedge A_2 \wedge \cdots \wedge A_j \cdots)(f(x)) \\ &= min\{A_1 f(x), A_2 f(x), \cdots, A_j f(x) \cdots\} \\ &= min\{f^{-1}(A_1)(x), f^{-1}(A_2)(x), \cdots, f^{-1}(A_j)(x), \cdots\} \\ &= (f^{-1}(A_1) \wedge f^{-1}(A_2) \wedge \cdots \wedge f^{-1}(A_j) \cdots)(x) \\ &= \wedge f^{-1}(A_j)(x) \end{aligned}$$

Proposition 1.3.7 [6]. If A is a fuzzy set of X and B is a fuzzy set of Y, then $1-(A\times B) = (A^c \times 1) \vee (1 \times B^c)$.

Proof. $(1-(A\times B))(x,y) = max(1-A(x), 1-B(y)) = max((A^c\times 1)(x,y), (1\times B^c)(x,y)) = ((A^c \times 1) \vee (1 \times B^c))(x,y)$ for each $(x,y) \in X \times Y$.

Proposition 1.3.8 [6]. Let $f_j : X_j \to Y_j$ be mappings and A_j be fuzzy sets of $Y_j, j = 1, 2$; then $(f_1 \times f_2)^{-1}(A_1 \times A_2) = f_1^{-1}(A_1) \times f_2^{-1}(A_2)$.

Proof. For each $(x_1, x_2) \in X_1 \times X_2$, we have $(f_1 \times f_2)^{-1}(A_1 \times A_2) = (A_1 \times A_2)(f_1(x_1), f_2(x_2)) = min(A_1 f_1(x_1), A_2 f_2(x_2)) = min(f_1^{-1}(A_1)(x_1), f_2^{-1}(A_2)(x_2)) = (f_1^{-1}(A_1) \times f_2^{-1}(A_2))(x_1, x_2)$.

Proposition 1.3.9 [6]. Let $g : X \to X \times Y$ be the graph of a mapping $f : X \to Y$. Let A be a fuzzy set of X and B be a fuzzy set of Y, then $g^{-1}(A \times B) = A \wedge f^{-1}(B)$.

Proof. For each $x \in X$, we have $g^{-1}(A \times B)(x) = (A \times B)g(x) = (A \times B)(x, f(x)) = min(A(x), B(f(x))) = (A \wedge f^{-1}(B))(x)$.

Chapter 2

Fuzzy Topological Space

2.1. Fuzzy Topological Space

Definition 2.1.1 [6]. A family $\tau \subseteq I^X$ of fuzzy sets is called a fuzzy topology for X if it satisfies the following three axioms:

(1) $\overline{0}, \overline{1} \in \tau$.

(2) $\forall A, B \in \tau \Rightarrow A \wedge B \in \tau$.

(3) $\forall (A_j)_{j\in J} \in \tau \Rightarrow \vee_{j\in J} A_j \in \tau$.

The pair (X, τ) is called a fuzzy topological space or fts, for short. The elements of τ are called fuzzy open sets. A fuzzy set K is called fuzzy closed if $K^c \in \tau$. We denote by τ^c the collection of all fuzzy closed sets in this fuzzy topological space. Obviously, we have:

(a) $\alpha^c \in \tau^c$,

(b) if $K, M \in \tau^c$, then $K \vee M \in \tau^c$ and

(c) if $\{K_j : j \in J\} \in \tau^c$, then $\wedge\{K_j : j \in J\} \in \tau^c$.

Example 2.1.2 [6]. Let $X = \{a, b\}$. Let A be a fuzzy set on X defined as $A(a) = 0.5, A(b) = 0.4$. The $\tau = \{\overline{0}, A, \overline{1}\}$ is a fuzzy topology. (X, τ) is a fuzzy topological space. $\overline{0}(a) = 0, \forall a \in x, \overline{1}(a) = 1, \forall a \in x$.

Let τ_1 and τ_2 be two fuzzy topology for X. If the inclusion relation $\tau_1 \subset \tau_2$ holds, we say that τ_2 is finer than τ_1 and τ_1 is coarser than τ_2.

2.2 Base and Subbase for FTS

Definition 2.2.1 [1]. A base for a fuzzy topological space (X, τ) is a sub collection $\mathcal{B}$ of τ such that each member A of τ can be written as $A = \vee_{j\in\Lambda} A_j$, where each $A_j \in \mathcal{B}$.

Definition 2.2.2 [1]. A subbase for a fuzzy topological space(X, τ) is a subcollection $\mathcal{S}$ of τ such that the collection of infimum of finite subfamilies of $\mathcal{S}$ forms a base for (X, τ).

Definition 2.2.3. Let (X, τ) be an fts. Suppose A is any subset of X. Then (A, τ_A) is called a fuzzy subspace of (X, τ), where $\tau_A = \{B_A : B \in \tau\}$, $B = \{(x, \mu_B(x)) : x \in X\}$ and $B_A = \{(x, \mu_{B|A}(x)) : x \in A\}$.

Definition 2.2.4 [6]. A fuzzy point P in X is a special fuzzy set with membership function defined by

$$P(x) = \begin{cases} \lambda & \text{if } x = y, \\ 0 & \text{if } x \neq y; \end{cases}$$

where $0 < \lambda \leq 1$. P is said to have support y, value λ and is denoted by P_y^λ or $P(y, \lambda)$.

Let A be a fuzzy set in X, then $P_y^\alpha \subset A \Leftrightarrow \alpha \leq A(y)$. In particular, $P_y^\alpha \subset P_z^\beta \Leftrightarrow y = z, \alpha \leq \beta$. A fuzzy point P_y^α is said to be in A, denoted by $P_y^\alpha \in A \Leftrightarrow \alpha \leq A(y)$.

The complement of the fuzzy point P_x^λ is denoted either by $P_x^{1-\lambda}$ or by $(P_x^\lambda)^c$.

Definition 2.2.5 [6]. The fuzzy point P_x^λ is said to be contained in a fuzzy set A, or to belong to A, denoted by $P_x^\lambda \in A$ if and only if $\lambda < A(x)$.

Every fuzzy set A can be expressed as the union of all the fuzzy points which belong to A. That is, if $A(x)$ is not zero for $x \in X$, then $A(x) = sup\{\lambda : P_x^\lambda, 0 < \lambda \leq A(x)\}$.

Definition 2.2.6 [6]. Two fuzzy sets A, B in X are said to be intersecting if and only if there exists a point $x \in X$ such that $(A \wedge B)(x) \neq 0$. For such a case, we say that A and B intersect at x.

Let $A, B \in I^X$. Then $A = B$ if and only if $P \in A \Leftrightarrow P \in B$ for every fuzzy point P in X.

Proposition 2.2.7 [6]. Let $\{A_j : j \in J\}$ be a family of fuzzy sets in X, P_x^a and P_y^b be fuzzy points in X and f be a map of X into Y. Then we have the following:

1. $P_x^a \in \vee\{A_j : j \in J\}$ if and only if there exists $j \in J$ such that $P_x^a \in A_j$.

2. If $P_x^a \in \wedge\{A_j : j \in J\}$, then for every $j \in J$ we have $P_x^a \in A_j$.

3. $P_x^a \in P_y^b$ if and only if $x = y$ and $a < b$.

4. If $P_x^a \in P_y^b$ and for every $j \in J, P_y^b \in A_j$, then $P_x^a \in \wedge\{A_j : j \in J\}$.

5. If $P_x^a \in A$, where A is a fuzzy set in X, then there exists $a < b$ such that $P_x^b \in A$.

6. $f(P_x^a) = P_{f(x)}^a$

7. $f((P_x^a)^c) = (f(P_x^a))^c$

8. If $P_x^a \in A$, then $f(P_x^a) \in f(A)$

9. If $P_x^a \in f^{-1}(B)$, then $P_{f(x)}^a \in B$, where B is a fuzzy set in Y.

10. If $P_y^b \in f(A)$, then there exists $x \in X$ such that $f(x) = y$ and $P_x^a \in A$.

11. If $P_y^b \in B$ and $y \in f(X)$, then for every $x \in f^{-1}(y)$ we have $P_x^b \in f^{-1}(B)$.

Proof. (1) $P_x^a \in \vee\{A_j : j \in J\}$ if and only if there exists $j \in J$ such that $P_x^a \in A_j$.

Let $P_x^a \in A_j \Rightarrow a \leq A_j(x) \Rightarrow a \leq max\{A_j(x) : j \in J\} \Rightarrow a \leq (\vee A_j)(x)$.

Again $P_x^a \in \vee\{A_j : j \in J\} \Rightarrow a \leq (\vee_{j\in J} A_j)(x) \Rightarrow a \leq A_j(x) \Rightarrow P_x^a \in A_j, j \in J$.

(4) If $P_x^a \in P_y^b$ and for every $j \in J, P_y^b \in A_j$, then $P_x^a \in \wedge\{A_j : j \in J\}$.

$P_x^a \in P_y^b \in A_j \Rightarrow P_x^a \in A_j \Rightarrow a \leq A_j(x) \Rightarrow a \leq min\{A_j(x) : j \in J\} \Rightarrow a \leq (\wedge A_j)(x) \Rightarrow$

$P_x^a \in \wedge\{A_j : j \in J\}$.

(6) $f(P_x^a) = P_{f(x)}^a$

$$f(P_x^a)(y) = \begin{cases} \sup\{P_x^a(z) : z \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi, \end{cases}$$

$$= \begin{cases} a & \text{if } x \in f^{-1}(y), \\ 0 & \text{otherwise;} \end{cases}$$

$$= \begin{cases} a & \text{if } f(x) = y, \\ 0 & \text{otherwise;} \end{cases}$$

$= P_{f(x)}^a(y)\ \forall y \in Y \Rightarrow f(P_x^a) = P_{f(x)}^a$

(7) $f((P_x^a)^c) = (f(P_x^a))^c$

$f((P_x^a)^c)(y)$=

$$(P_{f(x)}^a)^c(y) = \begin{cases} 1 - a & \text{if } y = f(x), \\ 0 & \text{otherwise;} \quad \cdots (i) \end{cases}$$

Now

$$(P_x^a)^c(z) = \begin{cases} 1 - a & \text{if } z = x, \\ 0 & \text{if } z \neq x; \end{cases}$$

So

$$f((P_x^a)^c)(y) = \begin{cases} \sup\{(P_x^a)^c(z) : z \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi, \end{cases}$$

$$= \begin{cases} 1-a & \text{if } x \in f^{-1}(y), \\ 0 & \text{otherwise}; \end{cases}$$

$$= \begin{cases} 1-a & \text{if } f(x) = y, \\ 0 & \text{otherwise}; \quad \cdots (ii) \end{cases}$$

Thus $f((P_x^a)^c) = (f(P_x^a))^c$.

Theorem 2.2.8 [8]. $\mathcal{B}$ is a base for an fts (X, τ) iff $\forall A \in \tau$ and for every fuzzy point P in A, $\exists B \in \mathcal{B}$ such that $P \in B \subseteq A$.

Proof. Assume that $\mathcal{B}$ is a base for τ, that is, every $A \in \tau$ is a union of members of B. Let $A \in \tau$ and $P_x^\alpha \in A$. So $A \in \tau \Rightarrow A = \bigcup_{i \in I}\{B_i : B_i \in \mathcal{B}\} \Rightarrow P_x^\alpha \in A = \bigcup_{i \in I}\{B_i : B_i \in \mathcal{B}\} \Rightarrow P_x^\alpha \in \bigcup_{i \in I}\{B_i : B_i \in \mathcal{B}\} \Rightarrow P_x^\alpha \in B_x \subseteq A$ (for some B_x).

Conversely, assume that for each $A \in \tau$ and for each $P_x^\alpha \in A$, $\exists$ B_x such that $P_x^\alpha \in B_x \subset A$. Let $A \in \tau$. To prove that A can be written as a union of members of $\mathcal{B}$ consider any arbitrary $P_x^\alpha \in A$. So by hypothesis $\exists$ $B_x \in \mathcal{B}$ such that $P_x^\alpha \in B_x \subset A \Rightarrow A \subset \bigcup_{P_x^\alpha \in A} B_x$. Since $B_x \subset A$, for each $P_x^\alpha \in A$, therefore $A = \bigcup_{P_x^\alpha \in A} B$.

2.3 Closure and Interior of fuzzy sets

Definition 2.3.1 [6]. The closure $\overline{A}$ and the interior A^o of a fuzzy set A of X are defined as

$$\overline{A} = inf\{K : A \leq K, K^c \in \tau\}$$

$$A^o = sup\{O : O \leq A, O \in \tau\}$$

respectively.

Example 2.3.2 [6]. Let A, B and C be fuzzy sets of I defined as

$$A(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1; \end{cases}$$

$$B(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{4}, \\ -4x + 2 & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1; \end{cases}$$

$$C(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{4}, \\ \frac{4x-1}{3} & \text{if } \frac{1}{4} \leq x \leq 1; \end{cases}$$

Then $\tau = \{\overline{0}, A, B, A \vee B, \overline{1}\}$ is a fuzzy topology on I. It can be easily seen that $Cl(A) = B^c$, $Cl(B) = A^c$, $Cl(A \vee B) = \overline{1}$, $Int(A^c) = B$, $Int(B^c) = A$ and $Int(A \vee B)^c = \overline{0}$.

2.4 Neighborhood

Definition 2.4.1 [6]. A fuzzy point P_x^λ is said to be quasi-coincident with A, denoted by $P_x^\lambda qA$, if and only if $\lambda > A^c(x)$, or $\lambda + A(x) > 1$.

Proposition 2.4.2 [6]. Let f be a function from X to Y. Let P be a fuzzy point of X, A be a fuzzy set in X and B be a fuzzy set in Y. Then we have:

1 If $f(P)qB$, then $Pqf^{-1}(B)$.

2 If PqA, then $f(P)qf(A)$.

3 $P \in f^{-1}(B)$, if $f(P) \in B$.

4 $f(P) \in f(A)$, if $P \in A$.

Proof. (1) Let $P \equiv P_x^a$, then

$$f(P_x^a)(y) = \begin{cases} \sup\{P_x^a(z) : z \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi; \end{cases}$$

$$= \begin{cases} 0 & \text{if } f^{-1}(y) = \phi \\ a & \text{if } x \in f^{-1}(y), \text{if} f(x) = y \\ 0 & \text{if } x \notin f^{-1}(y); \end{cases}$$

Now, $f(P_x^a) \equiv P_{f(x)}^a \Rightarrow f(P_x^a)qB = P_{f(x)}^a qB$.

Note that $P_{f(x)}^a qB \Rightarrow a + B(f(x)) > 1 \Rightarrow a + f^{-1}B(x) > 1 \Rightarrow P_x^a q f^{-1}(B)$, which completes the proof.

Definition 2.4.3 [6]. A fuzzy set A in (X, τ) is called a neighborhood of fuzzy point P_x^λ if and only if there exists a $B \in \tau$ such that $P_x^\lambda \in B \leq A$; a neighborhood A is said to be open if and only if A is open. The family consisting of all the neighborhoods of P_x^λ is called the system of neighborhoods of P_x^λ.

Definition 2.4.4 [6]. A fuzzy set A in (X, τ) is called a Q-neighborhood of fuzzy point P_x^λ if and only if there exists a $B \in \tau$ such that $P_x^\lambda qB \leq A$. The family consisting of all the Q-neighborhoods of P_x^λ is called the system of Q-neighborhoods of P_x^λ.

Proposition 2.4.5 [6]. $A \leq B$ if and only if A and B^c are not quasi-coincident; particularly, $P_x^\lambda \in A$ if and only if P_x^λ is not quasi-coincident with A^c.

Proof. $A(x) \leq B(x) \Leftrightarrow A(x) + B^c(x) = A(x) + 1 - B(x) \leq 1$. In particular $P_x^\lambda \in A \Rightarrow \lambda \leq A(x) \Rightarrow \lambda + A^c(x) \leq A(x) + A^c(x) \Rightarrow \lambda + A^c(x) \leq 1$.

Theorem-2.4.6 [6]. A fuzzy point $e \in A^o$ if and only if e has a neighborhood contained in A.

Theorem 2.4.7 [6]. A fuzzy point $e = P_x^\lambda \in \overline{A}$ if and only if each Q-neighborhood of e is quasi-coincident with A.

Proof. $P_x^\lambda \in \overline{A}$ if and only if, for every closed set $F \supset A, P_x^\lambda \in F$, or $F(x) \geq \lambda$. By taking complement, this fact can be stated as follows: $P_x^\lambda \in \overline{A}$ if and only if, for every open set $B \subset A^c, B(x) \leq 1 - \lambda$. In other words, for every open set B satisfying $B(x) > 1 - \lambda, B$ is not contained in A^c. From proposition, B is not contained in A^c if and only if B is quasi-coincident with $(A^c)^c = A$. We have thus proved that $P_x^\lambda \in \overline{A}$ if and only if, for every open Q-neighborhood B of P_x^λ is quasi-coincident with A, which is evidently equivalent to what we want to prove.

Definition 2.4.8 [6]. A fuzzy point e is called an adherence point of a fuzzy set A, if and only if, every Q-neighborhood of e is quasi-coincident with A.

Theorem 2.4.9 [6]. $(\overline{A})^c = (A^c)^o, \overline{(A^c)} = (A^o)^c$.

Definition 2.4.10 [6]. A fuzzy point e is called a boundary point of a fuzzy set A if and only if $e \in \overline{A} \wedge \overline{A^c}$. The union of all the boundary points of A is called a boundary of A, denoted by $b(A)$. It is clear that $b(A) = \overline{A} \wedge \overline{A^c}$.

Definition 2.4.11 [6]. A fuzzy point e is called an accumulation point of a fuzzy set A if and only if e is an adherence point of A and every Q-neighborhood of e and A are quasi-coincident at some point different from $supp(e)$, whenever $e \in A$. The union of all the accumulation points of A is called the derived set of A, denoted by A^d. It is evident that $A^d \subset \overline{A}$.

Theorem 2.4.12 [6]. $\overline{A} = A \vee A^d$, where A^d is the derived set of A.

Proof. Let $\Omega = \{e : e \text{ is an adherence point of } A\}$. Then, from Theorem (2.4.7) $\overline{A} = \vee\Omega$.

On the other hand, $e \in \Omega$ is either "$e \in A$"or "$e \notin A$"; from the Definition (2.4.11) we have $e \in A^d$, hence $\overline{A} = \vee\Omega < A \vee A^d$. The converse part follows directly.

Corollary 2.4.13 [6]. A fuzzy set A is closed if and only if A contains all the accumulation points of A.

Proof. We know that $\overline{A} = A \vee A^d$. A fuzzy set A is closed if $\overline{A} = A$ and since $A = \overline{A} = A \vee A^d$, therefore $A^d \leq A$. Conversely, if A contains all the accumulation point of A, then $A^d \leq A$ and hence, $\overline{A} = A \vee A^d \Rightarrow \overline{A} = A$.

2.5 Fuzzy continuous map

Definition 2.5.1 [1]. Given fuzzy topological space (X, τ) and (Y, γ), a function $f : X \to Y$ is fuzzy continuous if the inverse image under f of any open fuzzy set in Y is an open fuzzy set in X; that is if $f^{-1}(\nu) \in \tau$ whenever $\nu \in \gamma$.

Proposition 2.5.2 [1]. (a) The identity $id_X : (X, \tau) \to (X, \tau)$ on a fuzzy topological space (X, τ) is fuzzy continuous.

(b) A composition of fuzzy continuous functions is fuzzy continuous.

Proof. (a) For $\nu \in \tau$, $id_X^{-1}(\nu) = \nu \circ id_X = \nu$.

(b) Let $f : (X, \tau) \to (Y, \gamma)$ and $g : (Y, \gamma) \to (Z, \beta)$ be fuzzy continuous. For $\eta \in \beta$, $(g \circ f)^{-1}(\eta) = \eta \circ (g \circ f) = (\eta \circ g) \circ f = f^{-1}(\eta \circ g) = f^{-1}(g^{-1}(\eta))$. $g^{-1}(\eta) \in \gamma$ since g is fuzzy continuous, and so $(g \circ f)^{-1}(\eta) = f^{-1}(g^{-1}(\eta)) \in \tau$ since f is fuzzy continuous.

Proposition 2.5.3 [1]. Let (X, τ) be fuzzy topological space. Then every constant function from (X, τ) into another fuzzy topological space is fuzzy continuous if and only if τ contains all constant fuzzy sets in X.

Proof. Suppose that every constant function from (X, τ) into any fuzzy topological space is fuzzy continuous and consider the fuzzy topology γ on $[0, 1]$ defined by $\gamma = \{\bar{0}, \bar{1}, id_{[0,1]}\}$. Let k be a real number, $0 \leq k \leq 1$. The constant function $f : X \to [0, 1]$ defined by $f(x) = k$, for every $x \in X$, is fuzzy continuous, and so $f^{-1}(id_{[0,1]}) \in \tau$. But for $x \in X$, $f^{-1}(id_{[0,1]})(x) = id_{[0,1]}(f(x)) = id_{[0,1]}(k) = k$, whence the constant fuzzy set k in X belongs to τ.

Conversely, suppose that τ contains all constant fuzzy sets in X and consider a constant function $f : (X, \tau) \to (Y, \gamma)$ defined by $f(x) = y_0$. If $\nu \in \gamma$, then for any $x \in X$ we have $f^{-1}(\nu)(x) = \nu(f(x)) = \nu(y_0)$, so that $f^{-1}(\nu)$ is a constant fuzzy set in X and hence, a member of τ. Thus, f is fuzzy continuous.

Chapter 3

Compactness and Separation Axioms

3.1. Compact Fuzzy Topological Space

Definition 3.1.1 [1]. A fuzzy topological space (X, τ) is compact if every cover of X by members of τ contains a finite subcover, i.e. if $A_i \in \tau$, for every $i \in I$, and $\vee_{i \in I} A_i = \bar{1}$, then there are finitely many indices $i_1, i_2, \cdots, i_n \in I$ such that $\vee_{j=1}^{n} A_{i_j} = \bar{1}$.

Theorem 3.1.2. Let (X, τ) and (Y, γ) be fuzzy topological spaces with (X, τ) compact, and let $f : X \to Y$ be a fuzzy continuous surjection. Then (Y, γ) is also compact.

Proof. Let $B_i \in \gamma$, for each $i \in I$, and assume that $\vee_{i \in I} B_i = \bar{1}_Y$. For each $x \in X, \vee_{i \in I} f^{-1}(B_i)(x) = \vee_{i \in I} B_i(f(x)) = \bar{1}_X$. So the τ-open fuzzy sets $f^{-1}(B_i)(i \in I)$ cover X. Thus, for finitely many indices $i_1, i_2, \cdots, i_n \in I, \vee_{j=1}^{n} f^{-1}(B_{i_j}) = \bar{1}_X$. If B is any fuzzy set in Y, the fact that f is a surjection mapping onto Y implies that, for any $y \in Y$, $f(f^{-1}(B))(y) = sup\{f^{-1}(B)(z) : z \in f^{-1}(y)\} = sup\{(B)f(z) : f(z) = y\} = B(y) \Rightarrow f(f^{-1}(B)) = B$. Thus, $\bar{1}_Y = f(\bar{1}_X) = f(\vee_{j=1}^{n}(f^{-1}(B_{i_j}))) = \vee_{j=1}^{n} f(f^{-1}(B_{i_j})) = \vee_{j=1}^{n} B_{i_j}$. Therefore, (Y, γ) is also compact.

Lemma 3.1.3 (Alexander Subbase Lemma) [1]. If $\mathcal{S}$ is a subbase for a fuzzy topological space (X, τ), then (X, τ) is compact iff every cover of X by members of $\mathcal{S}$ has a finite sub cover (i.e. if $A_\alpha \in \mathcal{S}$ for each $\alpha \in \Lambda$ and $\vee_{\alpha \in \Lambda} A_\alpha = \bar{1}$, then there are finitely many indices $\alpha_i, (i = 1, 2, \cdots, n)$ such that $\vee_{i=1}^{n} A_{\alpha_i} = \bar{1}$).

Definition 3.1.4 [1]. Let (X_i, τ_i) be a fuzzy topological space, for each index $i \in I$. The product fuzzy topology $\tau = \prod_{i\in I} \tau_i$ on the set $X = \prod_{i\in I} X_i$ is the coarsest fuzzy topology on X making all the projection mappings $\pi_i : X \to X_i$ fuzzy continuous.

Theorem 3.1.5 (Fuzzy Tychonoff Theorem) [1]. Let n be a positive integer and for each $i = 1, 2, \cdots, n$, let (X_i, τ_i) be a compact fuzzy topological space. Then $(X, \tau) = (\prod_{i=1}^n X_i, \prod_{i=1}^n \tau_i)$ is compact.

Proof. We will say that a collection of open fuzzy sets of a fuzzy topological space has the finite union property (FUP) if none of its finite sub collections cover the space (i.e. none of its finite sub collections have supremum identically equal to $\bar{1}$). Since $\mathcal{S} = \{\pi_i^{-1}(A_i) : A_i \in \tau_i, i = 1, 2, \cdots, n\}$ is a subbase for (X, τ), by the Lemma it suffices to show that no subcollection of $\mathcal{S}$ with FUP covers X. Let $\mathcal{C}$ be a sub collection of $\mathcal{S}$ with FUP. For each $i = 1, 2, \cdots, n$ let $C_i = \{A \in \tau_i : \pi_i^{-1}(A) \in \mathcal{C}\}$. Then C_i is a collection of open fuzzy sets in (X_i, τ_i) with FUP. Indeed, if $A_{i,1}, A_{i,2}, \cdots, A_{i,k} \in C_i$ satisfy $\bigvee_{j=1}^k A_{i,j} = \bar{1}_{X_i}$, then $\bigvee_{j=1}^k \pi_i^{-1}(A_{i,j}) = \pi_i^{-1}(\bigvee_{j=1}^k A_{i,j}) = \pi_i^{-1}(\bar{1}_{X_i}) = \bar{1}_X$, and this would contradict the fact that $\mathcal{C}$ has FUP. Therefore, by the compactness of (X_i, τ_i), the collection C_i cannot cover X_i, and we can select a point $x_i \in X_i$ such that $(\bigvee C_i)(x_i) = a_i < 1$. Now if we consider the point $x = (x_1, x_2, \cdots, x_n) \in X$ and the collection $C_i' = \{\pi_i^{-1}(A) : A \in \tau_i\} \bigcap C$, then it follows that $(\bigvee C_i')(x) = \bigvee\{\pi_i^{-1}(A)(x) : A \in \tau_i \text{ and } \pi_i^{-1}(A) \in C\} = \bigvee\{A(x_i) : A \in \tau_i \text{ and } \pi_i^{-1}(A) \in C\} = (\bigvee C_i)(x_i) = a_i$.

Further noting that $C = \bigcup_{i=1}^n C_i'$, we obtain $(\bigvee C)(x) = \bigvee_{i=1}^n (\bigvee C_i')(x) = \bigvee_{i=1}^n (\bigvee C_i)(x_i) = \bigvee_{i=1}^n a_i$ which is strictly less than 1 since each of the finitely many real numbers a_i is strictly less than 1. Thus $\bigvee C \neq \bar{1}$, as desired.

3.2. Fuzzy Regular Space

Definition 3.2.1. An fts (X, τ) will be called regular if for each fuzzy point P and each fuzzy closed set C such that $P \wedge C = \bar{0}$ there exist fuzzy open sets U and V such that $P \in U$ and $C \subseteq V$.

Proposition 3.2.2. Every subspace of regular space is also regular.

Proof. Let X be a fuzzy regular space and A be a subspace of X. We have to prove that A is regular. Recall that $\tau_A = \{G_A : G \in \tau\}$, where $G = \{(x, \mu_G(x)) : x \in X\}$ and $G_A = \{(x, \mu_{G|A}(x)) : x \in A\}$. Let P_x^α be fuzzy point in A and F_A is closed set of A such that $P_x^\alpha \notin F_A$. Since A is a subspace of X, therefore $P_x^\alpha \in X$ and there is a closed set F in X, which generated the closed subset F_A of A. Since X is regular space and $P_x^\alpha \wedge F = \bar{0}$ there exist open sets U and V such that $P_x^\alpha \subseteq U = (x, \mu_U)$ and $F \subseteq V = (x, \mu_V)$. Thus $U_A = (x, \mu_{U|A}), V_A = (x, \mu_{V|A})$ are open sets in A such that $P_x^\alpha \subseteq U_A$ and $F_A \subseteq V_A$. Hence A is a regular subspace of X.

Proposition 3.2.3. If a space X is a regular space, then for any open set U and a fuzzy point $P \in X$ such that $P \cap U' = \bar{0}$, there exists an open set V such that $P \in V \subseteq \overline{V} \subseteq U$.

Proof. Suppose that X is a fuzzy regular space. Let $U = \{(x, \mu_U) : x \in X\}$ be a fuzzy open set of X such that $P_x^\alpha \cap U' = 0$. Then $U' = (x, 1 - \mu_U)$ is fuzzy closed set of X such that $P_x^\alpha \notin U' = (x, 1 - \mu_U)$ and hence, $P_x^\alpha \in U$. Since X is regular, therefore there exist two disjoint fuzzy open set V and W such that $P_x^\alpha \in V$ and $U' \subseteq W$. Now W' is a closed set of X such that $V \subseteq W' \subseteq U$. Thus, $P_x^\alpha \in V \subseteq \overline{V}$ and $\overline{V} \subseteq W' \subseteq U$ and hence, $\overline{V} \subseteq U$. This proves that $P_x^\alpha \in V \subseteq \overline{V} \subseteq U$.

3.3. Fuzzy Normal Space

Definition 3.3.1. A fuzzy topological space (X, τ) will be called normal if for each pair of fuzzy closed sets C_1 and C_2 such that $C_1 \wedge C_2 = 0$ there exist fuzzy open sets M_1 and M_2 such that $C_i \subseteq M_i (i = 1, 2)$ and $M_1 \wedge M_2 = 0$.

Proposition-3.3.2. If a space X is a normal space, then for each closed set F of X and any open set G of X such that $F \wedge G' = 0$ there exists an open set G_F such that $F \subseteq G_F \subseteq \overline{G_F} \subseteq G$.

Proof. Let X be a normal space. Let F be a fuzzy closed set in X and G be an fuzzy open set in X such that $F \wedge G' = 0$, then $F \subseteq G$. Let $G = (x, \mu_G)$ and $F = (x, \mu_F)$, then F and G' are two disjoint fuzzy closed sets of X. Since X is fuzzy normal, so $\exists$ two disjoint fuzzy open sets G_F and $G_{G'}$ such that $F \subseteq G_F$ and $G' \subseteq G_{G'}$ and $G_F \wedge G_{G'} = 0$. Thus, $G_F \subseteq G'_{G'}$, but $G'_{G'}$ is a fuzzy closed set and hence $\overline{G_F} \subseteq G'_{G'}$. Thus from the above we have $F \subseteq G_F \subseteq \overline{G_F} \subseteq G$.

3.4. Other Separation Axioms

Definition 3.4.1 [9]. An fts (X, τ) is said to be fuzzy T_0 iff $\forall x, y \in X, x \neq y,\ \exists\, U \in \tau$ such that either $U(x) = 1$ and $U(y) = 0$ or $U(y) = 1$ and $U(x) = 0$.

Definition 3.4.2 (a) [9]. An fts (X, τ) is said to be fuzzy T_1- topological space iff $\forall x, y \in X, x \neq y,\ \exists\, U, V \in \tau$ such that $U(x) = 1$, $U(y) = 0$ and $V(y) = 1$, $V(x) = 0$.

Definition 3.4.2(b) [7]. An fts (X, τ) is $F - T_1$ iff singletons are closed.

Definition 3.4.3(a) [7]. An fts (X, τ) is said to be Hausdorff or fuzzy T_2 iff the following conditions hold:

If p, q are any two disjoint fuzzy points in X then

(i) if $x_p \neq x_q$, $\exists$ open sets V_p and V_q, such that $p \in V_p,\ q \notin \overline{V}_p,\ q \in V_q,\ p \notin \overline{V}_q$;

(ii) if $x_p = x_q$, and $\mu_p(x_p) < \mu_q(x_p)$, then $\exists$ an open set V_p such that $p \in V_p$, but $q \notin \overline{V}_p$.

Definition 3.4.3(b) [8]. A fts (X, τ) is said to be fuzzy Hausdorff iff for any two distinct fuzzy points $p, q \in X$, there exist disjoint $U, V \in \tau$ with $p \in U$ and $q \in V$.

Definition 3.4.4 [7]. An fts (X, τ) is $F - T_3$ iff it is T_1, or $F - T_1$ and regular.

Definition 3.4.5 [7]. An fts (X, τ) is $F - T_4$ iff it is T_1, or $F - T_1$ and normal.

Proposition 3.4.6. Every subspace of T_1-space is T_1.

Proof. Let X be a T_1 fuzzy topological space and A be a subspace of X. So $\tau_A = \{G_A \mid G_A = (x, \mu_{G|A}), G \in \tau\}$. Let $x, y \in A$ such that $x \neq y$. Then $x, y \in X$ are two distinct points and as X is T_1, there exist $U, V \in \tau$ such that $U(x) = 1$, $U(y) = 0$ and $V(y) = 1$, $V(x) = 0$. Then, U_A and V_A are fuzzy open sets of A such that $U_A(x) = 1$, $U_A(y) = 0$ and $V_A(y) = 1$, $V_A(x) = 0$. This shows that A is also T_1.

Theorem 3.4.7 [8]. A fuzzy subspace of a fuzzy Hausdorff topological space is fuzzy Hausdorff.

Proof. Let X be a fuzzy Hausdorff topological space and A be a subspace of X. Let $P_x^\alpha,\ P_y^\beta$ be any two arbitrary points in A with $P_x^\alpha \neq P_y^\beta$. Then, we have $P_x^\alpha,\ P_y^\beta \in X$, with $P_x^\alpha \neq P_y^\beta$. Since, X is a Hausdorff space therefore $\exists\, U, V \in \tau$ such that $P_x^\alpha \in U$, $P_y^\beta \in V$ and $U \cap V = 0$. Since $U,\ V$ are fuzzy open subsets of X and $\mu_U(z) \wedge \mu_V(z) = 0$, for every $z \in X$, therefore $U_A = (x, \mu_{U_{|A}})$ and $V_A = (x, \mu_{V_{|A}})$ are fuzzy open subsets of A such that $P_x^\alpha \in U_A$, $P_y^\beta \in V_A$ and $U_A \cap V_A = 0$, . Thus (A, τ_A) is also a fuzzy Hausdorff topological space.

Proposition 3.4.8 [7]. No subset of a Hausdorff fts can be compact.

Corollary 3.4.9 [7]. Singletons in an Hausdorff fts are not compact.

Theorem 3.4.10 [8]. If $\{(X_i, \tau_i) : i \in I\}$ is a family of fuzzy Hausdorff topological spaces, their product(X, τ) is also fuzzy Hausdorff.

Proof. Let $\{X_i : i \in I\}$ be a family of fuzzy Haudorff spaces and $X = \prod_{i \in I} X_i$. We have to show that X is fuzzy Hausdorff.

Let $P_x^\alpha, P_y^\beta \in X$ with $P_x^\alpha \neq P_y^\beta$. We know that the projection $P_i : X \to X_i,\ i \in I$ is fuzzy continuous. $P_x^\alpha \neq P_y^\beta \Rightarrow$ there exists some $i_0 \in I$ such that say $P_x^\alpha = m$ and $P_y^\beta = n, \quad P_{i_0}(m) = P_{i_0}(n) \Rightarrow m_{i_0} \neq n_{i_0}$ and we have $P_{i_0} : X \to X_{i_0}$ and here $m_{i_0}, n_{i_0} \in X_{i_0}$ with $m_{i_0} \neq n_{i_0}$. X_{i_0} is fuzzy Hausdorff $\Rightarrow$ there exists open sets U and V in X_{i_0} such that $m_{i_0} \in U$ and $n_{i_0} \in V$ and $U \cap V = \phi$. $P_{i_0}^{-1}(U)\ open\ \subset X$ and $P_{i_0}^{-1}(V)\ open\ \subset X$. Since P_{i_0} is continuous $m_{i_0} \in U \Rightarrow P_{i_0}(m) \in U \Rightarrow m \in P_{i_0}^{-1}(U)$ again $n_{i_0} \in V \Rightarrow P_{i_0}(n) \in V \Rightarrow n \in P_{i_0}^{-1}(V)$.

Claim. $P_{i_0}^{-1}(U) \cap P_{i_0}^{-1}(V) = \phi$. Suppose to the contrary $P_{i_0}^{-1}(U) \cap P_{i_0}^{-1}(V) \neq \phi$. This $\Rightarrow$ some $q \in P_{i_0}^{-1}(U) \cap P_{i_0}^{-1}(V) \Rightarrow q \in P_{i_0}^{-1}(U)$ and $q \in P_{i_0}^{-1}(V) \Rightarrow P_{i_0}(q) \in U$ and $P_{i_0}(q) \in V \Rightarrow q_{i_0} \in U$ and $q_{i_0} \in V \Rightarrow U \cap V = \phi$ which is a contradiction. Therefore (X, τ) is also fuzzy Hausdorff.

Proposition 3.4.11. Every subspace of T_3-space is T_3.

Proof. We know that $T_3 = T_1$+Regular. The proof follows by noting that every subspace of T_1-space is T_1 and every subspace of regular space is regular.

Proposition 3.4.12. Every subspace of T_4-space is T_4

Proof. We know that $T_4 = T_1$+Normal. Since every subspace of T_1-space is T_1 and every subspace of normal space is normal, therefore every subspace of a T_4-space is T_4.

Theorem 3.4.13 [7]. An $F-T_2$-space is an $F-T_1$-space.

Proof. Let p be a fuzzy point in X. Then any point $q \in \{p\}'$ belongs to an open set V_q such that $\mu_{\{p\}'}(x_p) \geq \mu_{\overline{V}_q}(x_p)$. So $V_q \subset \{p\}'$. If, on the other hand, p is crisp, let $x_q \in X - \{x_p\}$ be arbitrary. If $\{q_n, n \in N\}$ be a sequence of fuzzy points, where $x_{q_n} = x_q, \forall n \in N$ and the sequence $\{\mu_{q_n}(x_q), n \in N\}$ is decreasing and converges to zero, then there exists a sequence of open sets $\{V_{pq_n}, n \in N\}$, such that $p \in V_{pq_n}$ and $q_n \notin \overline{V}_{pq_n}, \forall n \in N$, as (X, τ) is Hausdorff. Therefore, if $P = \bigcap_{n \in N} \overline{V}_{pq_n}$, then P is a closed set, where $\mu_p(x_q) = 0$ and $\mu_p(x_p) = 1$. So P' is an open set contained in $\{p\}'$ and containing the crisp point q.

Theorem 3.4.14 [7]. An $F-T_3$-space is an $F-T_2$-space.

Proof. Let p, q be two fuzzy points, where $x_p \neq x_q$ and let w be a third fuzzy point, where $x_w = x_p$ and $\mu_w(x_p) > 1 - \mu_p(x_p)$. Then $\{w\}'$ is open and

$$\mu_{\{w\}'}(x) = \begin{cases} 1 - \mu_w(x_p) < \mu_p(x_p) & \text{for } x = x_p, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore $q \in \{w\}'$, but $p \notin \{w\}'$. Now since (X, T) is regular, there exists $V_q \in \tau$ such that $q \in V_q \subset \overline{V}_q \subset \{w\}'$. Obviously then, $p \notin \overline{V}_q$. Similarly, an open set V_p can be determined such that $p \in V_p$ and $q \notin \overline{V}_p$.

Theorem 3.4.15 [7]. An $F-T_4$-space is an $F-T_3$-space.

Proof. Let (X, τ) be a regular space. Let $p \in X$ and $V \in \tau$. Since X is $F-T_4$ it is $F-T_1$ and normal. Since X is $F-T_1$. $\{p\}$ is a closed set in X. Since X is normal. There exists $G \in \tau$ such that $\{p\} \subset G \subset \overline{G} \subset V \Rightarrow p \in G \subset \overline{G} \subset V$.

References

[1] S. Carlson, *Fuzzy topological spaces*, Part 1: Fuzzy sets and fuzzy topologies, Early ideas and obstacles, Rose-Hulman Institute of Technology.

[2] C. L. Chang, *Fuzzy topological spaces*, Journal of Mathematical Analysis and Applications, **24** (1968), 182–190.

[3] R. Lowen, *Fuzzy topological spaces and fuzzy compactness*, Journal of Mathematical Analysis and Applications, **56** (1976), 621–633.

[4] J. N. Mordeson, *Zadehs influence on mathematics*, Scientia Iranica D,**18**(3) (2011) , 596-601.

[5] J. R. Munkres, *Topology*, Prentice Hall Inc., New Jersey, 2000.

[6] N. Palaniappan, *Fuzzy topology*, Narosa Publications, 2002.

[7] M. Sarkar, *On fuzzy topological spaces*, Journal of Mathematical Analysis and Applications, **79** (1981), 384–394.

[8] R. Srivastava, S. N. Lal and A. K. Srivastava, *Fuzzy Hausdorff topological spaces*, Journal of Mathematical Analysis and Applications, **81** (1981), 497–506.

[9] R. Srivastava, S. N. Lal and A. K. Srivastava, *On fuzzy T_0 and R_0 topological spaces*, Journal of Mathematical Analysis and Applications, **136** (1988), 66–73.

[10] S. Willard, *General Topology*, Dover Publications, New York, 2004.

[11] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-353.