A

Thesis

On

HIGHER MODES NATURAL FREQUENCIES OF STEPPED BEAM USING SPECTRAL FINITE ELEMENTS

Submitted By

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In partial fulfilment for the award of the Degree of

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CERTIFICATE

This is to certify that the thesis entitled **"Higher Modes Natural Frequencies Of Stepped Beam Using Spectral Finite Elements"** submitted by **Arshaghosh Ajayan (211CE2021)** in partial fulfilment of the requirement for the award of **Master of Technology** degree in **Civil Engineering** with specialization in **Structural Engineering** to the National Institute of Technology, Rourkela is an authentic record of research work carried out by her under my supervision. The contents of this thesis, in full or in part, have not been submitted to any other Institute or University for the award of any degree or diploma.

Project Guide

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PREFACE

The dynamic behaviour of a structure is of great importance in engineering for which it is necessary to accurately predict the dynamic characteristics of the structure. The finite element method (FEM) has been used extensively in structural dynamics. The finite element model may provide accurate dynamic characteristics of a structure if the wavelength is large compared to the mesh size. However, the finite element solutions become increasingly inaccurate as the frequency increases. Although the accuracy can be improved by refining the mesh, this is sometimes prohibitively expensive.

The conventional finite element (mass and stiffness) matrices are usually formulated from assumed frequency- independent polynomial shape functions. Because the vibrating shape of a structure varies with the frequency of vibration in reality, the FEM requires subdivision of the structure into finite elements (or a mesh) for accurate solutions. Alternatively, if the shape functions are frequency dependent, then the subdivision may not be necessary.

The spectral element method gives frequency dependent dynamic element stiffness matrix regard less of the length or size of the element. Once this Stiffness Matrix for an element is formulated the global Dynamic Stiffness Matrix is obtained by following the procedure similar to that of the Finite Element Method (FEM). The great advantage of such a matrix is that even higher frequencies of a structure can be obtained by considering only few elements thus minimizing the computational cost.

In this thesis the higher mode natural frequencies of a stepped beam are obtained. The natural frequency of a stepped beam was found up to the tenth mode by just considering two spectral elements.

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LIST OF SYMBOLS

English

A	Cross sectional area
E	Young’s modulus
EI	Bending rigidity
F	External end force
I	Moment of inertia
[K]	Element stiffness matrix
L	Length
M	bending moment
ρA	Mass per unit length
ω	Angular frequency
θ	Angle of rotation
k_n	Wavenumber
$kL = \xi$	Non dimensional wavenumber
$M_t(x, t)$	bending moment

$V_t(x,t)$	shear force
u (x,t)	longitudinal displacement
v (x,t)	transverse displacement
ρ	mass density
{u}	displacement
$\{\dot{u}\}$	velocity
$\{\ddot{u}\}$	acceleration

CHAPTER-1

INTRODUCTION

1.1. Introduction

The dynamic behaviour of a structure is of great importance in engineering which is necessary to accurately predict the dynamic characteristics of the structure. The finite element method (FEM) has been used extensively in structural dynamics. The finite element model may provide accurate dynamic characteristics of a structure if the wavelength is large compared to the mesh size. However, the finite element solutions become increasingly inaccurate as the frequency increases.

One of the fundamental characteristics of the wave propagation problem is that the incident pulse duration is very small (of the order of micro seconds) and hence the frequency content of pulse is very high (of the order of kHz). When such a pulse is applied to the structure, it will force all the higher order modes to participate in the response. At higher frequencies, the wave lengths are small. Hence, in order to capture all the higher order modes, the conventional finite element method requires very fine mesh to match the wavelengths. This makes the system size enormously large. The spectral element approach (SEA) could be the nice alternative for such problems .In SEA, first the governing equation is transformed in frequency domain using discrete Fourier transform (DFT). In doing so, for 1D waveguides, the governing partial differential equation (PDE) is reduced to a set of ordinary differential equations (ODE) with constant coefficients, with frequency as a parameter. The resulting ODEs are much easier to solve than the original PDE. The SEA begins with the use of exact solution to governing ODEs in the frequency domain as interpolating function. The use of exact solution results in exact mass distribution and hence the resulting dynamic stiffness matrix is exact. Hence, in the absence of any discontinuity, one single element is sufficient to handle a beam of any length. This substantially reduces the system size and they are many orders smaller than the sizes involved in the conventional FEM. First, the exact dynamic stiffness is used to determine the system transfer function (frequency response function). This is then convolved with load. Next, inverse fast Fourier transform (IFFT) is used to get the time history of the response.

1.2. Wave Propagation Problems

In the wave propagation problems, as the frequency of the input loading is very high, the short term effects are critical. To get the accurate mode shapes and natural frequencies, the wave length and mesh size should be small. Alternatively we can use the time marching schemes under the finite element environment. In this method, analysis is performed over a small time step, which is a fraction of total time for which response histories are required. For some time marching schemes, a constraint is placed on the time step, and this, coupled with very large mesh sizes, make the solution of wave propagation problem. Wave propagation deals with loading of very high frequency content and finite element (FE) formulation for such problems is computationally prohibitive as it requires large system size to capture all the higher modes. These problems are usually solved by assuming solution to the field variables say displacements such that the assumed solution satisfies the governing wave equation as closely as possible. It is very difficult to assume a solution in time domain to solve the governing wave equation. So the solution in frequency domain is assumed and the governing equations solve are transformed and solved exactly. This simplifies the problem by introducing the frequency as a parameter which removes the time variable from the governing equations by transforming to the frequency domain. Among these techniques, many methods are based on integral transforms which include Laplace transform, Fourier transform and most recently wavelet transform. The solution of these transformed equations is much easier than the original partial differential equations. The main advantage of this system is computational efficiency over the finite element solution. These solutions in transformed frequency domain contain information of several frequency dependent wave properties essential for the analysis. The time domain solution is then obtained through inverse transform. In the frequency domain Fourier methods can be used to achieve high accuracy in numerical differentiation. One such method is FFT based spectral finite element method. In FSFEM, first the governing PDEs are transformed to ODEs in spatial dimension using FFT in time. These ODEs are then usually solved exactly, which are used as interpolating functions for FSFE formulation.

The advantages of FSFEM are, they reduce the system size and the wave characteristics can be extracted directly from such formulation. The Fourier transform is a tool widely used for many scientific purposes, but it is well suited only to the study of stationary signals where all frequencies have an infinite coherence time. The Fourier analysis brings only global information which is not sufficient to detect compact patterns.

1.3. Spectral Analysis

Spectral analysis or Fourier synthesis is one of the important analytical techniques for treating wave propagation problems. In this method the behavior of the signal is viewed as a superposition of many infinitely long wave trains of different periods (or frequencies). The actual response is synthesized by judicious combination of these wave trains. Thus the problem of characterizing a signal is transformed in to determining the set of combination of coefficients. These coefficients are called the Fourier transform of the signal. The problem is invariably simplified when it is in the terms of Fourier transform and the last step in the analysis involves performing an inverse transform (reconstructing the signal) and this very difficult to do in exact manner. Another way to think of Fourier analysis is as a mathematical technique for transforming our view of the signal from a time-based one to a frequency-based one.

Structures as waveguides

To make headway with a structural dynamic problem, it is necessary to approximate collection of waveguides appropriate connectivities at joints. A wave guide directs the wave energy along with its length and in its elementary form, can be viewed as a hydraulic or electric network analog.

1.4. Spectral analysis and FFT

Arbitrary time signal can be thought of as the superposition of many sinusoidal components. This is the basis of Fourier or spectral analysis. In wave analysis, the time domain for the disturbance is from minus infinity to plus infinity and thus components have continuous distribution (known as continuous Fourier transform). However the numerical evaluation of the transform requires discretizing the distribution. Here using the way of discrete Fourier transform (DFT.) it has two advantages. First, many ideas and methods of time series analysis can be used for the analysis. Second, it allows the use of efficient fast Fourier transform (FFT) computer algorithm

1.5. **FOURIER TRANSFORMS**

The Fourier transform, in essence, decomposes or separates a waveform or function into sinusoids of different frequency which sum to the original waveform. It identifies or distinguishes the different frequency sinusoids and their respective amplitudes. The main advantage of Fourier transform in structural dynamics and wave propagation problems is that several important characteristics of system can be obtained directly from the transformed frequency domain method. Fourier transform can be implemented analytically, semi analytically and numerically in the form of Continuous Fourier Transform (CFT), Fourier Series (FS) and Discrete Fourier Transform (DST) respectively.

Continuous Fourier transform (CFTs)

The continuous transform is convenient starting point for discussing spectral analysis because of its exact representation of functions. The continuous Fourier pair of a function F(t) defined on time domain from -∞ to +∞, given as:

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}\,(\omega) e^{i\omega t}\, d\omega$$

$$\hat{F}(\omega) = \int_{-\infty}^{\infty} F(t) e^{-i\omega t}\, dt$$

where $F(\omega)$ is continous fourier Transform (CFT), (ω) is the angular frequency i and is the complex √-1

Fast Fourier Transforms (FFTs)

Fourier transforms are one of the fundamental operations in signal processing. In digital computations, Discrete Fourier Transforms (DFT) are used to describe, represent, and analyze discrete-time signals. However, direct implementation of DFT is computationally very inefficient. Of the various available high speed algorithms to compute DFT, the Cooley-Tukey algorithm is the simplest and most commonly used. These efficient algorithms, used to compute DFTs, are called Fast Fourier Transforms (FFTs).

Spectral element method

The solution methods for the governing differential equations formulated in the time-domain can be categorized into two major groups. The first group consists of the time-domain methods, such as the numerical integration methods and the modal analysis method, which is commonly used for the vibration analysis. The second group consists of the frequency-domain methods. The spectral element method is one of the frequency–domain methods.

In SEM, the solutions to the governing differential equations are represented by the superposition of an infinite number of wave modes of different frequencies. This corresponds to the continuous Fourier transform of the solutions. This approach involves determining an infinite set of spectral components in the frequency domain and performing the inverse Fourier transform to reconstruct the time histories of the solutions. The continuous Fourier transform is feasible only when the function to be transformed is mathematically simple and inverse transform is biggest impediment to most practical cases. Instead of using the continuous Fourier transform, the discrete Fourier transform (DFT) is widely used.

The use of FFT algorithm make it possible to efficiently take into account as many spectral components as are needed up to the highest frequency of interest.

The dynamic responses of a structural element can be expressed by the spectral representations rather than by the simple harmonic solutions assumed to form the dynamic stiffness matrix. This spectral analysis procedure provides the dynamic stiffness matrix at each frequency component, which is often called the spectral element matrix or spectral dynamic stiffness matrix. The finite structure element corresponding to the spectral element matrix is called the spectral element, although it is often called the finite element in the FEM. If there are external forces given in the time domain, they are all transformed into the spectral representations by use of a forward FFT algorithm. Just as in the FEM, the spectral elements can be assembled to form a global spectral system matrix equation for the whole structure. The global spectral system matrix equation is solved for the spectral DOF, repeatedly at each frequency component, and the inverse FFT (IFFT) is applied to reconstruct the dynamic responses in the time domain. It is important to note that, as illustrated in Figure 1, the key features of the spectral analysis method and the FEM (i.e., the efficient use of the FFT and the IFFT computer algorithms in the spectral analysis method, and the refining and assembly of the finite elements in the FEM) are combined and used in this new solution method. This is why this new method is called the spectral element method

Advantages of Spectral Element Analysis

- The SEM combines the flexibility of a Finite Element Method (FEM) with the accuracy of a spectral method,
- The SEM is so useful that each element solved exactly for its dynamics irrespective of its length
- The effect of material damping and viscoelasticity can easily be incorporated simply by changing spectrum relation.
- Higher order beam or rod theories can be implanted without adding degree of freedom to the system to be solved
- Inverse problem can be performed. i.e. if response is known at some location then disturbance causing it can be determined

CHAPTER -2

LITERATURE REVIEW

Spectral element method

The spectral element method is a high-order finite element technique that combines the geometric flexibility of finite elements with the high accuracy of spectral methods. This method was pioneered in the mid 1980's by Anthony Patera at MIT [42] and Yvon Maday at Paris-VI. It exhibits several favorable computational properties, such as the use of tensor products, naturally diagonal mass matrices, and adequacy to implementations in a parallel computer system. Due to these advantages, the spectral element method is a viable alternative to currently popular methods such as finite volumes and finite elements, if accurate solutions of regular problems are sought.

Narayanan and Beskos [41] introduced the fundamental concept of SEM for the first time. He derived an exact dynamic stiffness matrix for the beam element and employed FFT for dynamic analysis of plane frame-works.

Spectral analysis was usually used in fluid dynamic problems, aerospace engineering etc. Abdelhmid and McConnell [1] introduced idea of spectral analysis for non-stationary field measurements.

Doyle [25] published his first work on the formulation of the spectral element for the longitudinal wave propagation of rods. The term spectral element method in structural dynamics is introduced by Doyle [45] in his work for the DFT/FFT-based spectral element analysis approach. A comprehensive list of the works by Doyle's research group and other researcher's up to 1997 can be found in the book by Doyle [26]. There he describes about spectral analysis of wave motion and presented an FFT-based Spectral Analysis Methodology, also explains about longitudinal waves and flexural waves in rods and beams respectively. Here spectral element formulation for bars beams and plates are also shown.

Doyle and Farris [22] presented spectral formulation of finite element for flexural wave propagation in beams.

Banerjee and Williams [9] presented an elegant and efficient alternative procedure for calculating the number of clamped-clamped natural frequencies of the bending-torsion coupled beam [7] exceeded by any trial frequency, thus enabling the Wittrick-Williams algorithm to be applied with ease when finding the natural frequencies of structure which incorporate such members.

Banerjee, Guo and Howson [7] have presented an exact dynamic stiffness matrix of a bending-torsion coupled beam including warping. The work presented in this paper extends the approach by recasting the equations in the form of a dynamic member stiffness matrix. A new procedure is presented, based on the Wittrick-Willliams algorithm [50] for converging with certainty upon any required natural frequency.

Gopalakrishnan, Martin and Doyle [30] have presented study in Timoshenko beam in which a matrix methodology for spectral analysis of wave propagation in multiply connected Timoshenko beam is developed and the analysis gives the exact frequency dependent response for the Timoshenko beam irrespective of the length of the element. The frequency domain response is converted to the time domain response using the Fast Fourier Algorithm (FFT). Methods of spectral analysis formulate an element which treats the distribution of mass and rotational inertia exactly. Only one spectral element need be placed between any two joints, substantially reducing the total number of degrees of freedom in the system. The spectral formulation requires that the assembled system of equations be solved in the frequency domain and utilizes the Fast Fourier Transform (FFT) to convert the frequency domain results back to the time domain.

Doyle and Gopalakrishnan [29] presented a paper in Wave propagation in connected waveguides of varying cross-section. There they formulated spectral element for rod of varying cross section.

Mahapatra Gopalakrishnan and Shankar [39] have their work in Spectral element based solution for wave propagation analysis of multiply connected unsymmetric laminated composite beams. In their paper a generalized 2-D beam element is derived and which can be used for wave propagation analysis of both symmetric and unsymmetric laminated composite multiply connected beams. They described about a methodology which is analogous to that of finite element method, that allow problems involving many connected elementary unsymmetric laminated composite waveguides to be handled in a convenient and straightforward manner. And it is mainly dealt with the behavior of elementary unsymmetric composite beams, without the effects of shear deformation and rotary inertia.

Doyle formulated spectral element method for reconstructing dynamic events from time limited spatially distributed data.

Chakraborty, Gopalakrishnan and Reddy [17] developed method for finite element analysis of functionally graded materials. They formulated exact beam finite element and static, free vibration and wave propagation studies are described using this element.

Chakraborty and Gopalakrishnan [17] presented their work in a spectrally formulated finite element for wave propagation analysis in functionally graded beams. Wave propagation analysis of FGM beam poses tremendous challenge due to the presence of material anisotropy. Because of this, an additional shear wave gets created beyond a certain high frequency, called the cut-off frequency. Due to this, there will be a three way (axial shear bending) coupling of modes .Tracking these individual waves is a very difficult task specially for a dispersive system such as a FGM beam. They devised an efficient methodology to capture such coupled waves. Here they also derived spectral element formulations for FGM beam .

Mahapatra, and Gopalakrishnan, have presented [40] their work in a spectral finite element model for analysis of axial flexural shear coupled wave propagation in laminated composite beams.

Lee and Kim [36] introduced spectral analysis for the transverse vibration of an axially moving Timoshenko beam.

Howson and Zare [33] formulated an exact dynamic stiffness matrix flexural vibration of 3-layered sandwich beams.

Exact dynamic stiffness matrices have been developed mostly for the 1-D structures including the Timoshenko beams with or without axial force [19, 32, 49], Rayleigh-Timoshenko beams [3, 36] and composite beams [29].

Cho and Lee [20] have presented an FFT-based spectral analysis method for linear discrete dynamic systems with non-proportional damping, shock and vibration. Lee has presented the

Spectral Element Method and its various applications like in the structural dynamics in his book [38]. He gives an excellent introduction about spectral element method, spectral analysis of signals, spectral element modeling and spectral element method in structural dynamics in which he incorporated all the research works of him and his students.

Lee and Lee [37] have presented spectral element modeling for extended Timoshenko beams. In their research they used extended Timoshenko beam theory to represent extension-transverse shear bending coupled vibrations of periodic lattice structures such as the large space lattice structures and carbon nanotubes by simply formulating spectral element model.

The exact solution for the fundamental natural frequencies of stepped beams for various boundary conditions have been developed up to 6 modes [35] and it is shown that frequency of the beam varies with step ratio.

CHAPTER-3

SPECTRAL ELEMENT FORMULATION

3.1. SPECTRAL ELEMENT FOR RODS

The major significance of this element is that it treats the mass distribution exactly and there for wave propagation within each element is treated exactly. It also means that the subdivision of the member into many small elements is no longer necessary.

3.1.1. The spectral formulation

Consider a rod of length L, where $u(x,t)$ the displacement in the x direction .where ρA is the mass density per unit length of volume.

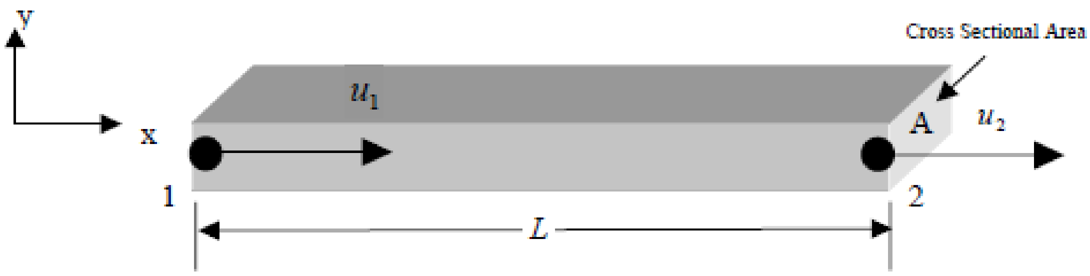

Figure 1. Schematic of Standard Bar Element with Nodal DOF

Consider the equations of motion of rod without neglecting the inertia. And assume that there are no applied loads between the rod ends

The general solution for the rod can be represented as

$$u(x,t) = \sum_n \hat{u}_n \ (x, \omega_n) e^{-i\omega_n t}$$

Assume that both modulus E and area A do not vary with position, and then the homogenous differential equation for the Fourier coefficients becomes

$$EA \frac{d^2\hat{u}}{dx^2} + \omega^2 \rho A \hat{u} = 0$$

Where the spectral displacement $\hat{u}_n$ have the simple solution

$$\hat{u}_n\ (x) = Ae^{-ik_nx} + Be^{-ik_n(L-x)} \quad , \ k_n = \ \omega_n\sqrt{\frac{\rho A}{EA}}$$

In finite element terms, this is called *shape function,* but obviously in this case it is dependent on frequencyω_n. That is, it is different at each frequency unlike the shape function in finite element terms. The nodal displacement can be related to the coefficient by imposing that

$$\hat{u}(0) = \hat{u}_1 = A + \ Be^{-ikL}, \quad \hat{u}(L) = \hat{u}_2 = Ae^{-ikL} + \ B$$

i.e. $\begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix} = \begin{bmatrix} 1 & e^{-ikL} \\ e^{-ikL} & 1 \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix}$

$$\begin{Bmatrix} A \\ B \end{Bmatrix} = \frac{1}{(1-e^{-i2kL})} \begin{bmatrix} 1 & -e^{-ikL} \\ -e^{-ikL} & 1 \end{bmatrix} \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix}$$

Allowing the displacement distribution to be written in terms of nodal values as

$$\hat{u}(x) = \frac{1}{(1-e^{-i2kL})} \left[\left(e^{-ikx} - e^{-ik(2L-x)} \right) \hat{u}_1 + \left(e^{-ik(L-x)} - e^{-ik(L+x)} \right) \hat{u}_2 \right]$$

The axial force at arbitrary position is related to nodal displacements by

$$\hat{F}(x) = EA\frac{\partial u}{\partial x}$$

$$\hat{F}(x) = \frac{EA}{L} \frac{iLk}{(1-e^{-i2kL})} \left[\left(-e^{-ikx} - e^{-ik(2L-x)} \right) \hat{u}_1 + \left(e^{-ik(L-x)} + e^{-ik(L+x)} \right) \hat{u}_2 \right]$$

Since nodal forces are related to member forces by $\hat{F}_1 = -\hat{F}(0), \hat{F}_2 = \hat{F}(L)$, then , in matrix notation it can be expressed in the form

$$\begin{Bmatrix} \hat{F}_1 \\ \hat{F}_2 \end{Bmatrix} = \frac{EA}{L} \frac{iLk}{(1-e^{-i2kL})} \begin{bmatrix} 1+e^{-i2kL} & -2e^{-ikL} \\ -2e^{-ikL} & 1+e^{-i2kL} \end{bmatrix} \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix}$$

This can be written in the familiar form of $\hat{F} = [\hat{k}]\{\hat{u}\}$ where $[\hat{k}]$ is the frequency dependent dynamic element stiffness for the rod. It is symmetric and real. This can be confirmed by expanding above to trigonometric expression

$$\begin{Bmatrix} \hat{F}_1 \\ \hat{F}_2 \end{Bmatrix} = \frac{EA}{L} \frac{kL}{\sin kL} \begin{bmatrix} coskL & -1 \\ -1 & coskL \end{bmatrix} \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix}$$

3.1.2. Efficiency of a Spectral Element method over Conventional Method

In *Finite element method*

Element stiffness matrix for rods is as given below

$$k = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix}$$

Mass matrix for rods

The wave motion in structures occurs because of the inertia terms in the equation of motion. The concentrated mass at the end of the each element effectively approximate the distributed mass along the member. The only difference in dynamic formulation is the effect of distributed mass. For convenience this will be assumed to act as additional force (inertia force) acting at each node.

The way of evaluating the equivalent inertia forces is by estimating the effect of the mass is to establish an equivalence between the energies of original and equivalent system .the kinetic energy of a general system of concentrated masses are related to the mass and velocities by

$T = \frac{1}{2}\sum_i \sum_j m_{ij}\ \dot{u}_i \dot{u}_j$ Or $m_{ij} = \frac{\partial^2 T}{\partial \dot{u}_i \partial \dot{u}_j}$

The actual kinetic energy is obtained by using function of displacement and is

$$T = \frac{1}{2}\int_0^L \rho A[\dot{u}]^2\, dx = \frac{1}{2}\rho A \int_0^L \left[\left(1 - \frac{x}{L}\right)\dot{u}_1 + \frac{x}{L}\dot{u}_2\right] dx$$

$$= \frac{1}{2}\rho AL \left\{\frac{1}{3}\dot{u}_1{}^2 + \frac{1}{3}\dot{u}_1\dot{u}_2 + \frac{1}{3}\dot{u}_2{}^2\right\}$$

Using this in above relation gives the equivalent mass matrix by differentiation as

$[m] \equiv \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$The equivalent inertia forces can therefore be written in matrix form as

$\begin{Bmatrix} \acute{F}_1 \\ \acute{F}_2 \end{Bmatrix} = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}$

Assembling both elastic and inertia $forces\ gives$

$$\{F\} = \{F\} + \{\acute{F}\}$$

$$[k]\{u\} + [m]\{\ddot{u}\} = \{F\}$$

For a dynamic problem when excitation force is harmonic

$$\{F\} = \{\hat{F}\}e^{i\omega t}$$

Then the response is also harmonic, given by

$$\{u\} = \{\hat{u}\}e^{i\omega t}$$

Substituting this into differential equation gives

$$[k]\{\hat{u}\}e^{i\omega t} - \omega^2[m]\{\hat{u}\}e^{i\omega t} = \{\hat{F}\}e^{i\omega t}$$

$$\big[[k] - \omega^2[m]\big]\{\hat{u}\} = \{\hat{F}\}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \left(\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix}$$

$$= \frac{EA}{L}\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho AL^2}{EA}\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}\right)\begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix}$$

Put $\xi = \omega\sqrt{\frac{\rho A}{EA}} = kL$

Then

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 - 1/3\,\xi^2 & -(1 + 1/6\,\xi^2) \\ -(1 + 1/6\,\xi^2) & 1 - 1/3\,\xi^2 \end{bmatrix}\begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix}, \ \xi = kL$$

Spectral element

$$\begin{Bmatrix} \hat{F}_1 \\ \hat{F}_2 \end{Bmatrix} = \frac{EA}{L}\,\frac{kL}{\sin kL}\begin{bmatrix} coskL & -1 \\ -1 & coskL \end{bmatrix}\begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix}$$

When$k_{11} = 0$,

$\cos kL = 0, \ \ kL = \frac{\pi}{2} = 1.57$

Conventional element

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 - {}^1/_3\,\xi^2 & -(1 + {}^1/_6\,\xi^2) \\ -(1 + {}^1/_6\,\xi^2) & 1 - {}^1/_3\,\xi^2 \end{bmatrix} \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix}, \xi = kL$$

When $k_{11} = 0$,

$1 - {}^1/_3\,\xi^2 = 0,\ \xi = \sqrt{3}\ = kL = 1.73$

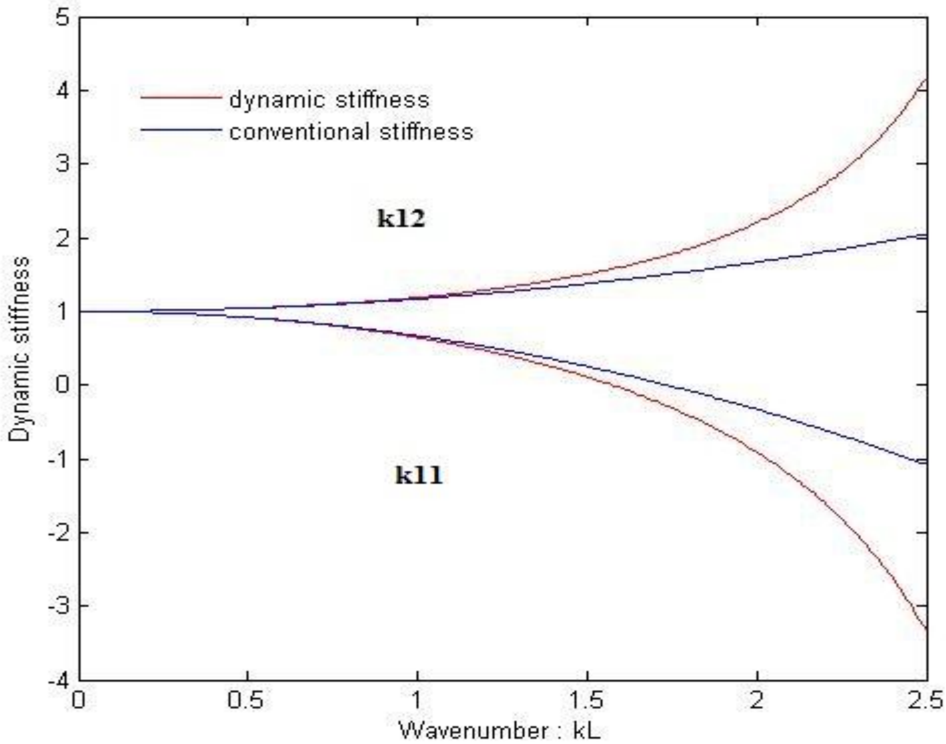

Figure 1.1: Comparison of conventional and spectral element stiffnesses.

The above conventional stiffness is for a single element of length L. It can be replaced by two elements of length $\frac{1}{2}L$ and by assembling them the resulting stiffness relation becomes

$$\begin{Bmatrix} F_1 \\ 0 \\ F_2 \end{Bmatrix} = \frac{2EA}{L} \begin{bmatrix} 1 - {}^1\!/_{12}\,\xi^2 & -(1 + {}^1\!/_{24}\,\xi^2) & 0 \\ -(1 + {}^1\!/_{24}\,\xi^2) & 2(1 - {}^1\!/_{12}\,\xi^2) & -(1 + {}^1\!/_{24}\,\xi^2) \\ 0 & -(1 + {}^1\!/_{24}\,\xi^2) & 1 - {}^1\!/_{12}\,\xi^2 \end{bmatrix} \begin{Bmatrix} \hat{u}_1 \\ \hat{u}^* \\ \hat{u}_2 \end{Bmatrix}$$

The middle force is zero because there are no applied loads there. Now solving for $\hat{u}^*$ in terms of the other displacement and removing it from the system to give

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{2EA}{L} \frac{1}{2(1 - {}^1\!/_{12}\,\xi^2)} \begin{bmatrix} 1 - \frac{5}{12}\xi^2 + \frac{1}{192}\xi^4 & -(1 + {}^1\!/_{24}\,\xi^2)^2 \\ -(1 + {}^1\!/_{24}\,\xi^2)^2 & 1 - \frac{5}{12}\xi^2 + \frac{1}{192}\xi^4 \end{bmatrix} \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix}$$

In this case behavior of k_{11} can be known. For instance when it crosses zero,

i.e.,when $k_{11} = 0$ $,1 - \frac{5}{12}\xi^2 + \frac{1}{192}\xi^4 1 - \frac{5}{12}\xi^2 + \frac{1}{192}\xi^4 = 0$.,$\xi = kL = 1.573$

which is near to the spectral value. It also goes through an infinity at $kL = \sqrt{12} = 3.46$. thus the apparently odd behavior of figure 1.2 (spectral stiffness behavior at higher frequencies) is also implied in conventional formulation, if a sufficient number of elements are used.

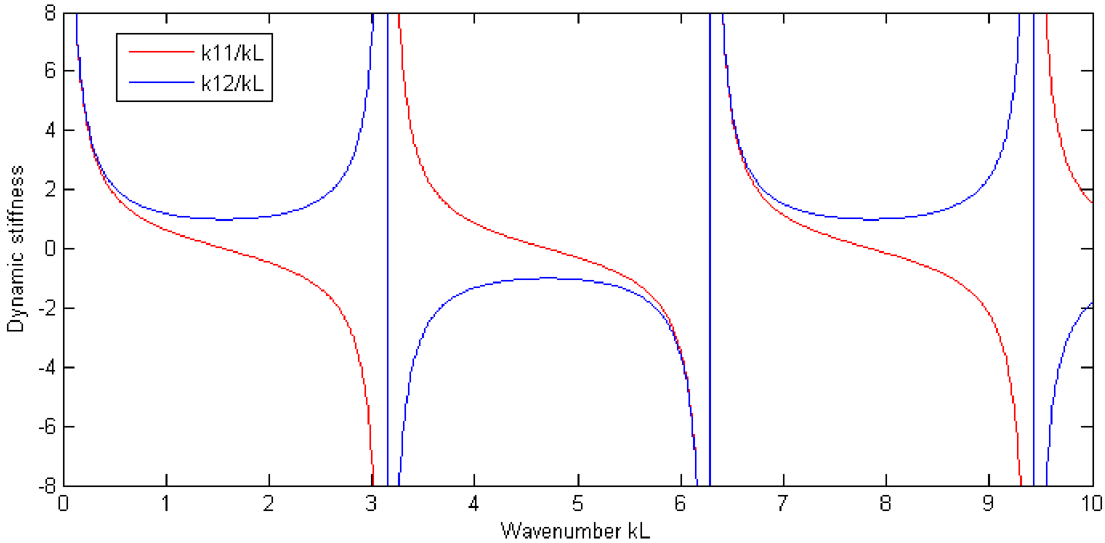

Figure 1.2: Spectral stiffness behavior at higher frequencies

The spectral approach is equivalent to an infinite number of conventional elements. In SEM only one element is used per uniform element which can result in enormous reduction in the size of matrix to be solved.

3.2. SPECTRAL ELEMENT FOR BEAMS

Consider a Euler-Bournelli beam of length L, where $u(x,t)$ the displacement in the x direction .where ρA is the mass density per unit length of volume. E is the Young's modulus, A is the cross-sectional area, I is the area moment of inertia about the neutral axis. The free bending vibration of a Bernoulli-beam is represented by

$$\mathrm{EI}w''''+\rho\mathrm{A}\ddot{w}=0 \tag{1}$$

The spectral representation of displacement of the beam is given as

$$v(x,t) = \sum_n \hat{v}_n \ (x,\omega_n) e^{-i\omega_n t} \tag{2}$$

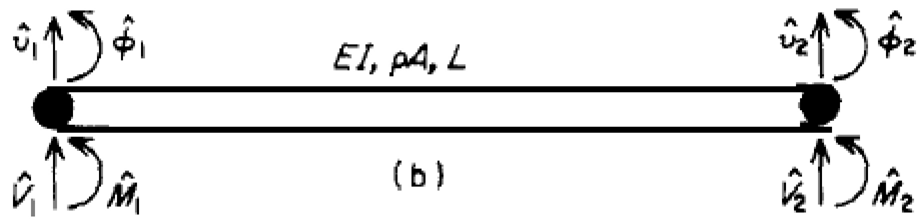

Figure 3.4 Nodal loads and degrees of freedom for element formulation

The spectral components $\hat{v}_n$ have been shown to have a simple solution

$$\hat{u}_n \ (x) = Ae^{-ik_n x} + Be^{-ik_n x} + Ce^{-ik_n(L-x)} + De^{-ik_n(L-x)}, \tag{3}$$

Where

$$k_n = \sqrt{\omega}\ \left[\frac{\rho A}{EI}\right]^{1/4} \tag{4}$$

Equation (3) gives two pure real roots and two pure imaginary roots as

$$k_1 = k_n = -k_2 \quad , \quad k_3 = ik_n = -k_4 \tag{5}$$

Where the first two are appropriate to wave moving in the plus direction and the second two are backward –moving waves. Both sets are necessary since the element is finite.

The dynamic stiffness is set up before by first relating the coefficients to the nodal displacement as

$$\{A, B, C, D\} = [G]\{\ \hat{v}_1\ \hat{\phi}_1\ \hat{v}_2\ \hat{\phi}_2\ \} \tag{6}$$

The spectral nodal displacements and slope of finite B-beam element can be related to the displacement field by

$$\hat{v}_1 = v(0), \hat{\phi}_1 = v'(0)$$

$$\hat{v}_2 = v(L), \hat{\phi}_2 = v'(L)$$

$$\begin{Bmatrix} \hat{v}_1 \\ \hat{\phi}_1 \\ \hat{v}_2 \\ \hat{\phi}_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & e^{-ikL} & e^{-kL} \\ e^{-ikL} & e^{-kL} & 1 & 1 \\ -ik & -k & ike^{-ikL} & ke^{-kL} \\ -ike^{-ikL} & -ke^{-kL} & ik & k \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} \tag{7}$$

$$\begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = \begin{bmatrix} 1 & 1 & e^{-ikL} & e^{-kL} \\ e^{-ikL} & e^{-kL} & 1 & 1 \\ -ik & -k & ike^{-ikL} & ke^{-kL} \\ -ike^{-ikL} & -ke^{-kL} & ik & k \end{bmatrix}^{-1} \begin{Bmatrix} \hat{v}_1 \\ \hat{\phi}_1 \\ \hat{v}_2 \\ \hat{\phi}_2 \end{Bmatrix} \tag{8}$$

$$a = [G]d$$

The matrix $[G]$ is fairly complicated .The nodal loads obtained by differentiation can be rearranged,

$$v(x,t) = g_1(x)v_1 + g_2(x)\phi_1 + g_3(x)v_2 + g_4(x)\phi_2 \quad (9)$$

$$M(x) = EI\frac{d^2v(x,t)}{dx^2} \quad V(x) = -EI\frac{d^3v(x,t)}{dx^3} \quad (10)$$

$$m_1 = M(0), \quad m_2 = M(L)$$

$$V_1 = -V(0), \quad V_2 = V(L) \quad (11)$$

It gives $\{F\} = \frac{EI}{L^3}\left[\hat{k}\right]\{\hat{u}\}$

Where [k] is the frequency depended dynamic element stiffness for a beam. The individual stiffness terms are a little more complicated than for the rod such as

$$\hat{k}_{11} = \frac{i(1+i)(z_{11}z_{22} + iz_{12}z_{21})}{{z_{11}}^2 + {z_{12}}^2}$$

$$\hat{k}_{22} = \frac{i(1+i)(z_{11}z_{22} - iz_{12}z_{21})}{{z_{11}}^2 + {z_{12}}^2}$$

Where

$$z_{11} = 1 - e^{-ikL}e^{-kL}, \quad z_{22} = 1 + e^{-ikL}e^{-kL}$$

$$z_{12} = e^{-ikL} - e^{-kL}, \quad z_{21} = e^{-ikL} + e^{-kL} \quad (12)$$

The matrix is real and symmetric. So it can be expanded as

$$\begin{Bmatrix} \hat{V}_1 \\ \hat{m}_1 \\ \hat{V}_2 \\ \hat{m}_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} \alpha & \bar{\gamma}L & -\bar{\alpha} & \gamma L \\ \bar{\gamma}L & \beta L^2 & -\gamma L & \bar{\beta}L^2 \\ -\bar{\alpha} & -\gamma L & \alpha & -\bar{\gamma}L \\ \gamma L & \bar{\beta}L^2 & -\bar{\gamma}L & \beta L^2 \end{bmatrix} \begin{Bmatrix} \hat{v}_1 \\ \hat{\phi}_1 \\ \hat{v}_2 \\ \hat{\phi}_2 \end{Bmatrix}$$

$$\alpha = \frac{(CSh+SCh)(kL)^3}{det}, \ \bar{\alpha} = \frac{(S+Sh)(kL)^3}{det}$$

$$\beta = \frac{(-CSh+SCh)(kL)}{det}, \quad \bar{\beta} = \frac{(-S+Sh)(kL)}{det}$$

$$\gamma = \frac{(-C+Ch)(kL)^2}{det}, \qquad \bar{\gamma} = \frac{(SSh)(kL)^2}{det}$$

$$det = 1 - CCh, \ \ C = \cos kL, S = \sin kL, Ch = \cosh kL, Sh = \sinh kL. \qquad (13)$$

3.2: Efficiency Spectral Element method over conventional method for a beam

In *Finite element method*

Element stiffness matrix for beam is as given below

$$\begin{Bmatrix} V_1 \\ m_1 \\ V_2 \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \hat{v}_1 \\ \hat{\phi}_1 \\ \hat{v}_2 \\ \hat{\phi}_2 \end{Bmatrix} \quad (14)$$

$$\{F\} = [k]\{u\} \qquad (14)$$

Element mass matrix for beam is as given below

$$\begin{Bmatrix} V_1'' \\ m_1'' \\ V_2'' \\ m_2'' \end{Bmatrix} = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}\begin{Bmatrix} \ddot{v}_1 \\ \ddot{\phi}_1 \\ \ddot{v}_2 \\ \ddot{\phi}_2 \end{Bmatrix}$$

$$\{F''\} = [m]\{\ddot{u}\}$$

Assembling both elastic and inertia $forces\ gives$

$$\{F\} = \{F\} + \{\acute{F}\}$$

$$[k]\{u\} + [m]\{\ddot{u}\} = \{F\}$$

For a dynamic problem when excitation force is harmonic

$$\{F\} = \{\hat{F}\}e^{i\omega t}$$

Then the response is also harmonic motion, given by

$$\{u\} = \{\hat{u}\}e^{i\omega t}$$

Substituting this into differential equation gives

$$[k]\{\hat{u}\}e^{i\omega t} - \omega^2[m]\{\hat{u}\}e^{i\omega t} = \{\hat{F}\}e^{i\omega t}$$

$$\big[[k] - \omega^2[m]\big]\{\hat{u}\} = \{\hat{F}\}$$

For comparison, the form for the conventional element with the consistent mass matrix can be written as

$$\begin{Bmatrix} \hat{V}_1 \\ \hat{m}_1 \\ \hat{V}_2 \\ \hat{m}_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} \alpha & \bar{\gamma}L & -\bar{\alpha} & \gamma L \\ \bar{\gamma}L & \beta L^2 & -\gamma L & \bar{\beta}L^2 \\ -\bar{\alpha} & -\gamma L & \alpha & -\bar{\gamma}L \\ \gamma L & \bar{\beta}L^2 & -\bar{\gamma}L & \beta L^2 \end{bmatrix}\begin{Bmatrix} \hat{v}_1 \\ \hat{\phi}_1 \\ \hat{v}_2 \\ \hat{\phi}_2 \end{Bmatrix}$$

But with the associations

$\alpha = 12 - (kL)^4\frac{13}{35}$, $\qquad \bar{\alpha} = 12 + (kL)^4\frac{9}{70}$

$\beta = 4 - (kL)^4\frac{1}{105}$, $\qquad \bar{\beta} = 2 + (kL)^4\frac{1}{140}$

$\gamma = 6 + (kL)^4\frac{13}{420}$, $\qquad \bar{\gamma} = 6 - (kL)^4\frac{11}{210}$

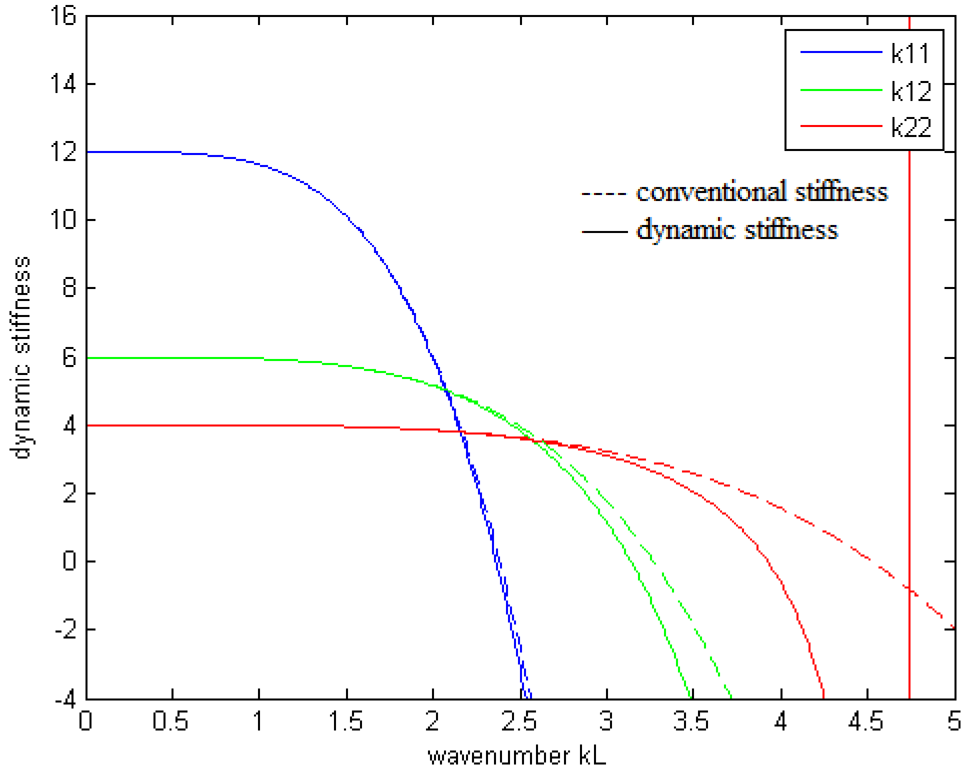

Figure 3.5: Comparison of conventional and spectral stiffness of a beam

The figure 3.5 shows a comparison between the present and conventional dynamic stiffness . it is apparent that they both have the same limiting behavior for small frequency .in fact when the consistent mass matrix is used , they agree to order $\omega^2(or\ (kL)^4)$ terms. As can be seen from the figure3.6 however beyond $kL \simeq 2$, there are significant differences because the conventional form behaves monotonically while the present form exhibits multiple zeros. For the conventional element to have these zeros it is necessary to piece many of them together.

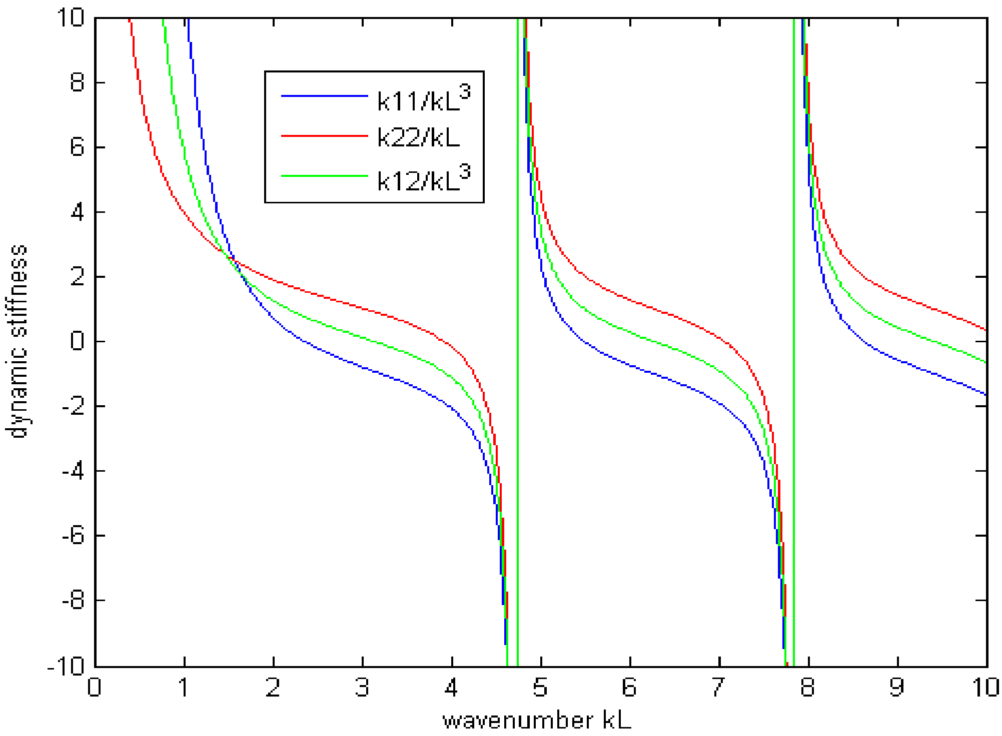

Figure 3.6: spectral stiffness at higher frequencies for a beam

3.3. SPECTRAL ELEMENT FOR STEPPED BEAM

In stepped beam same spectral elements formulation for uniform beam is considered where two elements are taken with different cross sectional area.

CHAPTER 4

RESULTS AND DISCUSSIONS

A Euler-Bournelli Fixed-Free Beam is considered, and obtained natural frequencies up to first 10 modes using spectral element formulation and compared with the exact solution given by Kwan and Bang[35] and Petyt[44] First three natural frequencies are matching exactly and SEM can easily catch higher modes of frequencies as shown in Tables 1 and 2. Then a stepped beam is considered and its natural frequencies are obtained up to first 10 modes with just 2 elements where SFEM is able to retrieve all the frequencies. The results are shown in the tables 3 to 7.

TABLE.1. Euler-Bournelli Fixed-Free Beam (Natural Frequencies by Dynamic Stiffness)

ρ=1000Kg/m^3 ,E=100GPa ,b=.02m , d=.02m, L=1m (Ref. p.285 Kwan and Bang[35])

No of elements=2

Mode No	Frequency (rad/s) {SEM}	Freq. (Exact)
1	202.99	203
2	1272.16	1272
3	3562.09	3562
4	6980.27	
5	11538.89	
6	17237.11	
7	24074.97	
8	32052.48	
9	41169.63	
10	51426.42	

TABLE.2. Euler-Bournelli Cantilever Beam (Natural Frequencies by Dynamic Stiffness)

Non dimensional value $\overline{\omega} = \left(\frac{\omega}{L^2}\right)\sqrt{\frac{EI_1}{\rho A_1}}$ (Ref: p 63. M. Petyt)

No of elements=2

Mode No	Frequency (rad/s) {SEM}	Non dimensional value	Exact value[44]
1	12430.99	3.516	3.516
2	77903.69	22.034	22.035
3	218132.59	61.697	
4	427452.82	120.901	
5	706610.14	199.859	
6	1055553.20	298.555	

TABLE.3. Euler-Bournelli Stepped Beam (Natural Frequencies by Dynamic Stiffness)

Non dimensional results for $\overline{\omega} = \left(\frac{\omega}{L^2}\right)\sqrt{\frac{EI_1}{\rho A_1}}$ of first 10 modes for Clamped-Clamped boundary condition No of elements=2

Mode no	I_2/I_1=5		I_2/I_1=10		I_2/I_1=20		I_2/I_1=40	
	Present	[34]	Present	[34]	Present	[34]	Present	[34]
1	25.9591	25.959	27.680	27.680	30.321	30.3213	34.325	34.3252
2	78.1518	78.151	85.365	85.365	89.493		89.493	
3	89.4931		89.493		90.209	90.209	92.550	92.5507
4	133.823		154.494	154.495	173.279	173.279	198.276	198.276
5	142.087	142.088	246.691		189.254		225.063	
6	245.591	245.592	259.252	259.252	246.691		246.691	
7	246.691		398.972		266.839	266.839	272.912	272.912
8	359.097	359.097	398.972	398.972	444.350	444.351	474.506	474.506
9	368.889		438.686		483.613		483.613	
10	483.613		483.613		512.688		617.522	617.523

TABLE.4. Euler-Bournelli Stepped Beam (Natural Frequencies by Dynamic Stiffness)

Non dimensional results for $\overline{\omega} = \left(\frac{\omega}{L^2}\right)\sqrt{\frac{EI_1}{\rho A_1}}$ of first 10 modes for Clamped- Free boundary condition No of elements=2

Mode no	I_2/I_1=5		I_2/I_1=10		I_2/I_1=20		I_2/I_1=40	
	Present	[34]	Present	[34]	Present	[34]	Present	[34]
1	2.4373	2.4373	2.06292	2.0629	1.7417	1.7417	1.4684	1.4685
2	22.333	22.335	21.0943	21.094	19.366	19.367	17.385	17.385
3	78.559	78.559	85.6244	85.625	89.493		89.493	
4	89.493		89.4931		90.1427	90.143	92.129	92.129
5	133.82	142.572	155.515	155.515	174.940	174.940	200.361	200.362
6	142.571		159.143		189.254		225.063	
7	245.589	245.589	246.691		246.691		246.691	
8	246.691		259.312	259.312	267.044	267.045	273.521	273.521
9	359.051	359.051	398.888	398.889	444.265	444.266	474.472	474.473
10	368.889		438.686		483.613		483.613	

TABLE.5. Euler-Bournelli Stepped Beam (Natural Frequencies by Dynamic Stiffness)

Non dimensional results for $\overline{\omega} = \left(\frac{\omega}{L^2}\right)\sqrt{\frac{EI_1}{\rho A_1}}$ of first 10 modes for Pinned-Pinned boundary condition No of elements=2

Mode no	I_2/I_1=5		I_2/I_1=10		I_2/I_1=20		I_2/I_1=40	
	Present	[34]	Present	[34]	Present	[34]	Present	[34]
1	10.4129	10.412	9.8780	9.8781	9.07466	9.0747	8.1369	8.1369
2	50.6565	50.656	56.089	56.089	60.1464	60.146	62.354	62.354
3	89.4931		89.493		89.4931		89.493	
4	103.711	103.71	111.79	111.79	124.360	124.36	142.406	142.407
5	133.823		159.143		189.254		216.754	216.754
6	195.126	195.12	207.036	207.036	213.376	213.37	225.063	
7	246.691		246.691		246.691		246.691	
8	295.691	295.50	327.590	327.590	367.833	367.83	400.704	400.704
9	368.889		438.686		472.467	472.07	483.613	
10	431.289	431.28	452.454		483.613		512.044	512.044

TABLE.6. Euler-Bournelli Stepped Beam (Natural Frequencies by Dynamic Stiffness)

Non dimensional results for $\bar{\omega} = \left(\frac{\omega}{L^2}\right)\sqrt{\frac{EI_1}{\rho A_1}}$ of first 10 modes for Sliding-Sliding boundary condition No of elements=2

Mode no	I_2/I_1=5		I_2/I_1=10		I_2/I_1=20		I_2/I_1=40	
	Present	[34]	Present	[34]	Present	[34]	Present	[34]
1	13.512	13.5124	15.906	15.906	18.294	18.2949	20.195	20.195
2	45.002	45.0027	47.045	47.045	50.322	50.3222	55.814	55.814
3	89.493		89.045		89.493		89.493	
4	111.345	111.345	119.996	119.99	125.062	125.062	127.10	127.109
5	133.823		159.143		189.254		225.063	
6	187.132	187.132	205.449	205.44	231.822	231.822	246.691	
7	246.691		246.691		246.691		262.736	262.737
8	301.794	301.794	317.520	317.52	327.487	327.487	340.030	340.030
9	368.889		438.686		483.613		483.613	
10	428.901	428.902	476.768	476.76	521.688		552.308	552.309

TABLE.7. Euler-Bournelli Stepped Beam (Natural Frequencies by Dynamic Stiffness)

Non dimensional results for $\overline{\omega} = \left(\frac{\omega}{L^2}\right)\sqrt{\frac{EI_1}{\rho A_1}}$ of first 10 modes for Free-Free boundary condition No of elements=2

Mode no	I_2/I_1=5		I_2/I_1=10		I_2/I_1=20		I_2/I_1=40	
	Present	[34]	Present	[34]	Present	[34]	Present	[34]
1	24.1649	24.1650	23.5459	23.5459	22.4725	22.4725	21.9069	21.1907
2	78.0078	78.0079	84.8859	84.8860	89.2592	89.2592	89.4931	
3	89.4931		89.4931		89.4931		91.1384	91.1384
4	133.823		155.527	155.527	174.980	174.981	200.418	200.418
5	142.545	142.572	159.143		189.254		225.063	
6	245.622	245.623	246.691		246.691		246.691	
7	246.691		259.351	259.352	267.085	267.085	273.554	273.555
8	359.050	359.050	398.885	398.886	444.261	444.262	474.468	474.468
9	368.889		438.686		483.613		483.613	
10	483.613		483.613		521.688		617.363	617.364

CHAPTER 5

CONCLUSION

A methodology (SEM), analogous to that of the finite element method is presented that allows problems involving many connected beams and rods to be handled in a convenient and straight forward manner. Unlike conventional finite elements, the length of the spectral element is not a limiting factor; each element is formulated exactly irrespective of its length. Hence, structural connections and discontinuities are the factors which govern the length of the element. This leads to a substantial reduction in the number of equations that are to be solved.

1. The Spectral Element Method is efficient to compute both the lower and higher modes natural frequencies even with a maximum number of two elements.
2. The natural frequencies of uniform and stepped beams up to the ten numbers of modes have been presented by using two number of spectral finite element for various boundary conditions
3. The natural frequencies of uniform beams up to the ten numbers of modes have been presented by using two number of spectral element.
4. The natural frequencies of stepped beam have been presented up to the tenth mode by considering two elements with different cross sectional area
5. The natural frequencies of uniform and stepped beams for various boundary conditions are presented for the modes up to ten..
6. Spectral finite element is very efficient method to obtain higher mode natural frequencies with less computational cost.

CHAPTER 6

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