INTUITIONISTIC FUZZY TOPOLOGICAL SPACES

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Abstract

The present thesis consisting of three chapters is devoted to the study of Intuitionistic fuzzy topological spaces. After giving the fundamental definitions we have discussed the concepts of intuitionistic fuzzy continuity, intuitionistic fuzzy compactness, and separation axioms, that is, intuitionistic fuzzy Hausdorff space, intuitionistic fuzzy regular space, intuitionistic fuzzy normal space etc.

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Chapter 1

Preliminaries and Introduction

1.1. Intuitionistic Fuzzy Set

Fuzzy sets were introduced by Zadeh [11] in 1965 as follows: a fuzzy set A in a nonempty set X is a mapping from X to the unit interval $[0, 1]$, and $A(x)$ is interpreted as the degree of membership of x in A. Intuitionistic fuzzy sets [1] can be viewed as a generalization of fuzzy sets that may better model imperfect information which is in any conscious decision making. Intuitionistic fuzzy sets take into account both the degrees of membership and of nonmembership subject to the condition that their sum does not exceed 1. Let E be the set of all countries with elective governments. Assume that we know for every country $x \in E$ the percentage of the electorate that have voted for the corresponding government. Denote it by $M(x)$ and let $\mu(x) = M(x)/100$ (degree of membership, validity, etc.). Let $\nu(x) = 1 - \mu(x)$. This number corresponds to the part of electorate who have not voted for the government. By fuzzy set theory alone we cannot consider this value in more detail. However, if we define $\nu(x)$ (degree of non-membership, non-validity, etc.) as the number of votes given to parties or persons outside the government, then we can show the part of electorate who have not voted at all or who have given bad voting-paper and the corresponding number will be $\pi(x) = 1 - \mu(x) - \nu(x)$ (degree of indeterminacy, uncertainty, etc.). Thus we can construct the set $\{\langle x, \mu(x), \nu(x)\rangle : x \in E\}$. Intuitionistic fuzzy sets (IFS) are applied in different areas. The IF-approach to artificial intelligence includes treatment of decision making and machine learning, neural networks and pattern recognition, expert systems database, machine reasoning, logic programming etc. IFSs are used in medical diagnosis and in decision making in medicine. There are also IF generalized nets models of the gravitational field, in astronomy, sociology, biology, musicology, controllers, and others. Along with these IFS are also studied extensively in the topological framework introduced by D. Coker which is the basis of our work.

Definition 1.1.1 [1]: Let X be a non-empty fixed set. An intuitionistic fuzzy set (IFS for short) A is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for each $x \in X$.

Example 1.1.2: Every fuzzy set A on a non-empty set X is obviously an IFS having the form $A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$

1.2. Basic Operations on IFS

Definition 1.2.1 [1]: Let X be a non empty set, and the IFSs A and B be in the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle : x \in X\}$

1. $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$.

2. $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

3. $\overline{A} = \{\langle x, \gamma_A(x), \mu_A(x)\rangle : x \in X\}$.

4. $A \bigcap B = \{\langle x, \mu_A(x) \bigwedge \mu_B(x), \gamma_A(x) \bigvee \gamma_B(x)\rangle : x \in X\}$

5. $A \bigcup B = \{\langle x, \mu_A(x) \bigvee \mu_B(x), \gamma_A(x) \bigwedge \gamma_B(x)\rangle : x \in X\}$

6. $[\ ]A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$

7. $\langle\ \rangle A = \{\langle x, 1 - \gamma_A(x), \gamma_A(x)\rangle : x \in X\}$

Example 1.2.2 [4]: Let $X = \{a, b, c\}$

$A = \langle x, (\frac{a}{0.5}, \frac{b}{0.5}, \frac{c}{0.4}), (\frac{a}{0.2}, \frac{b}{0.4}, \frac{c}{0.4})\rangle, B = \langle x, (\frac{a}{0.4}, \frac{b}{0.6}, \frac{c}{0.2}), (\frac{a}{0.5}, \frac{b}{0.3}, \frac{c}{0.3})\rangle,$

$C = \langle x, (\frac{a}{0.5}, \frac{b}{0.6}, \frac{c}{0.4}), (\frac{a}{0.2}, \frac{b}{0.3}, \frac{c}{0.3})\rangle, D = \langle x, (\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.2}), (\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.4})\rangle,$

$E = \langle x, (\frac{a}{0.6}, \frac{b}{0.6}, \frac{c}{0.5}), (\frac{a}{0.1}, \frac{b}{0.2}, \frac{c}{0.2})\rangle$

Here $\bar{A} = \langle x, (\frac{a}{0.2}, \frac{b}{0.4}, \frac{c}{0.4}), (\frac{a}{0.5}, \frac{b}{0.5}, \frac{c}{0.4})\rangle$, $A \subseteq E$ because $\mu_A(x) \leq \mu_E(x)$ and $\gamma_A(x) \geq \gamma_E(x)$, for every $x \in X$. Further, $A \bigcup B = \{\langle x, \mu_A(x) \bigvee \mu_B(x), \gamma_A(x) \bigwedge \gamma_B(x)\rangle : x \in X\} = C$ and $A \bigcap B = \{\langle x, \mu_A(x) \bigwedge \mu_B(x), \gamma_A(x) \bigvee \gamma_B(x)\rangle : x \in X\} = D$.

Definition 1.2.3 [4]: Let $\{A_i : i \in J\}$ be an arbitrary family of IFS in X .Then

(a) $\bigcap A_i = \{\langle x, \bigwedge \mu_{A_i}(x), \bigvee \gamma_{A_i}(x)\rangle : x \in X\}$

(b) $\bigcup A_i = \{\langle x, \bigvee \mu_{A_i}(x), \bigwedge \gamma_{A_i}(x)\rangle : x \in X\}$

Definition 1.2.4 [4]: The IFS $0_\sim$ and $1_\sim$ in X are defined as

$0_\sim = \{\langle x, 0, 1\rangle : x \in X\}$

$1_\sim = \{\langle x, 1, 0\rangle : x \in X\}$,

where 1 and 0 represent the constant maps sending every element of X to 1 and 0, respectively.

Corollary 1.2.5 [4]: Let A ,B ,C be IFSs in X . Then

(a) $A \subseteq B$ and $C \subseteq D \Rightarrow A \bigcup C \subseteq B \bigcup D$ and $A \bigcap C \subseteq B \bigcap D$,

(b) $A \subseteq B$ and $A \subseteq C \Rightarrow A \subseteq B \bigcap C$,

(c) $A \subseteq C$ and $B \subseteq C \Rightarrow A \bigcup B \subseteq C$,

(d) $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$,

(e) $\overline{A \bigcup B} = \bar{A} \bigcap \bar{B}$,

(f) $\overline{A \bigcap B} = \bar{A} \bigcup \bar{B}$,

(g) $A \subseteq B \Rightarrow \bar{B} \subseteq \bar{A}$,

(h) $\overline{(\bar{A})} = A$,

(i) $\overline{1_\sim} = 0_\sim$ and $\overline{0_\sim} = 1_\sim$.

1.3 Images And Preimages of IFS

Definition 1.3.1 [4]: Let X and Y be two nonempty sets and $f : X \to Y$ be a function.

(a) If $B = \{\langle y, \mu_B(y), \gamma_B(y)\rangle : y \in Y\}$ is an IFS in Y ,then the preimage of B under f denoted by $f^{-1}(B)$ is the IFS in X defined by

$$f^{-1}(B) = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}(\gamma_B)(x)\rangle : x \in X\},$$

where $f^{-1}(\mu_B)(x) = \mu_B(f(x))$ and $f^{-1}(\gamma_B)(x) = \gamma_B(f(x))$.

(b) If $A = \{\langle x, \lambda_A(x), \nu_A(x)\rangle : x \in X\}$ is an IFS in X ,then the image of A under f ,denoted by $f(A)$ is the IFS in Y defined by

$$f(A) = \{\langle y, f(\lambda_A)(y), (1 - f(1 - \nu_A))(y)\rangle : y \in Y\}$$

$$f(\lambda_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \lambda_A(x) & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{otherwise,} \end{cases}$$

$$(1 - f(1-\nu_A)(y)) = \begin{cases} \inf_{x\in f^{-1}(y)} \nu_A(x) & \text{if } f^{-1}(y) \neq \phi \\ 1, & \text{otherwise,} \end{cases}$$

For the sake of simplicity, let us use the symbol $f_{-}\nu(A)$ for $1 - f(1-\nu_A)$.

Proposition 1.3.2 [4]: Let A, $A_i (i \in J)$ be IFSs in $X, B, B_j (j \in K)$ IFSs in Y and $f : X \to Y$ a function. Then

(a) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$,

(b) $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$,

(c) $A \subseteq f^{-1}(f(A))$ and if f is injective, then $A = f(f^{-1}(A))$,

(d) $f(f^{-1}(B)) \subseteq B$ and if f is surjective, then $f(f^{-1}(B)) = B$,

(e) $f^{-1}(\bigcup B_j) = \bigcup f^{-1}(B_j)$,

(f) $f^{-1}(\bigcap B_j) = \bigcap f^{-1}(B_j)$,

(g) $f(\bigcup A_i) = \bigcup f(A_i)$,

(h) $f(\bigcap A_i) \subseteq \bigcap f(A_i)$ [if f is injective, then $f(\bigcap A_i) = \bigcap f(A_i)$],

(i) $f^{-1}(1_\sim) = 1_\sim$ (j) $f^{-1}(0_\sim) = 0_\sim$,

(k) $f(1_\sim) = 1_\sim$, if f is surjective (l) $f(0_\sim) = 0_\sim$,

(m) $\overline{f(A)} \subseteq f(\bar{A})$, if f is surjective,

(n) $f^{-1}(\bar{B}) = \overline{f^{-1}(B)}$.

Proof. Let $B_j = \left\{\langle y, \mu_{B_j}, \gamma_{B_j}\rangle : y \in Y\right\}, A_i = \{\langle x, \lambda_{A_i}, \vartheta_{A_i}\rangle : x \in X\}$, where $(i \in J, j \in K)$ and $B = \{\langle y, \mu_B, \gamma_B\rangle : y \in Y\}$, $A = \{\langle x, \lambda_A, \vartheta_A\rangle : x \in X\}$.

(a) Let $A_1 \subseteq A_2$. Since $\lambda_{A_1} \leq \lambda_{A_2}$ and $\vartheta_{A_1} \geq \vartheta_{A_2}$, we obtain $f(\lambda_{A_1}) \leq f(\lambda_{A_2})$ and $1 - \vartheta_{A_1} \leq 1 - \vartheta_{A_2} \Rightarrow f(1-\vartheta_{A_1}) \leq f(1-\vartheta_{A_2}) \Rightarrow 1 - f(1-\vartheta_{A_1}) \geq 1 - f(1-\vartheta_{A_2})$ from which it follows that $f(A_1) \subseteq f(A_2)$.

(c) $f^{-1}(f(A)) = f^{-1}(f(\langle x, \lambda_A, \vartheta_A \rangle)) = f^{-1}(\langle y, f(\lambda_A), f_-(\vartheta_A) \rangle) = \langle x, f^{-1}(f(\lambda_A)), f^{-1}(f_-(\vartheta_A)) \rangle \supseteq \langle x, \lambda_A, \vartheta_A \rangle = A$. [Notice that $f^{-1}(f(\lambda_A)) \geq \lambda_A$ and $f^{-1}(f_-(\vartheta_A)) = f^{-1}(1 - f(1 - \vartheta_A)) = 1 - f^{-1}(f(1 - \vartheta_A)) \leq 1 - (1 - \vartheta_A) = \vartheta_A$].

(h) $f(\bigcap A_i) = f(\langle x, \bigwedge \lambda_{A_i}, \bigvee \vartheta_{A_i} \rangle) = \langle y, f(\bigwedge \lambda_{A_i}), f_-(\bigvee \vartheta_{A_i}) \rangle \subseteq \langle y, \bigwedge f(\lambda_{A_i}), \bigvee f_-(\vartheta_{A_i}) \rangle = \bigcap f(A_i)$. [Notice that $f(\bigwedge A_i) \leq \bigwedge f(A_i)$ and $f_-(\bigvee \vartheta_{A_i}) = 1 - f(1 - \bigvee \vartheta_{A_i}) = 1 - f(\bigwedge (1 - \vartheta_{A_i})) \geq 1 - \bigwedge f(1 - \vartheta_{A_i}) = \bigvee (1 - f(1 - \vartheta_{A_i})) = \bigvee f_-(\vartheta_{A_i})$.]

Chapter 2

Intuitionistic Fuzzy Topological Space

2.1. Intuitionistic fuzzy topological space

Definition 2.1.1 [4]: An intuitionistic fuzzy topology (IFT) on a nonempty set X is a family τ of IFS in X satisfying the following axioms

(T_1) $0_\sim, 1_\sim \in \tau$

(T_2) $G_1 \bigcap G_2 \in \tau$, for any $G_1, G_2 \in \tau$

(T_3) $\bigcup G_i \in \tau$, for any arbitrary family $\{G_i : G_i \in \tau,\ i \in I\}$.

In this case the pair (X, τ) is called an intuitionistic fuzzy topological space and any IFS in τ is known as intuitionistic fuzzy open set in X .

Example 2.1.2 [4]: Let $X = \{a, b, c\}$

$A = \langle x, (\frac{a}{0.5}, \frac{b}{0.5}, \frac{c}{0.4}), (\frac{a}{0.2}, \frac{b}{0.4}, \frac{c}{0.4})\rangle$, $B = \langle x, (\frac{a}{0.4}, \frac{b}{0.6}, \frac{c}{0.2}), (\frac{a}{0.5}, \frac{b}{0.3}, \frac{c}{0.3})\rangle$,

$C = \langle x, (\frac{a}{0.5}, \frac{b}{0.6}, \frac{c}{0.4}), (\frac{a}{0.2}, \frac{b}{0.3}, \frac{c}{0.3})\rangle$, $D = \langle x, (\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.2}), (\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.4})\rangle$.

Then the family $\tau = \{0_\sim, 1_\sim, A, B, C, D\}$ of IFSs in X is an IFT on X .

Proposition 2.1.3 [4]: Let (X, τ) be an IFTS on X . Then we can also construct several IFT on X in the following way

(a) $\tau_{0,1} = \{[\ \ ]G : G \in \tau\}$

(b) $\tau_{0,2} = \{\langle\ \ \rangle G : G \in \tau\}$.

Proof: (a) (T_1) $0_\sim, 1_\sim \in \tau_{0,1}$ is obvious.

(T_2) Let $[\ \ ]G_1, [\ \ ]G_2 \in \tau_{0,1}$.

Since $G_1, G_2 \in \tau$, therefore $G_1 \bigcap G_2 = \langle x, \mu_1 \bigwedge \mu_2, \gamma_1 \bigvee \gamma_2 \rangle \in \tau$. This implies that

$$([\ ]G_1) \bigcap ([\ ]G_2) = \langle x, \mu_{G_1} \bigwedge \mu_{G_2}, (1-\mu_{G_1}) \bigvee (1-\mu_{G_2}) \rangle$$
$$= \left\langle x, \mu_{G_1} \bigwedge \mu_{G_2}, 1 - (\mu_{G_1} \bigwedge \mu_{G_2}) \right\rangle \in \tau_{0,1}.$$

(T_3) Let $\{[\ ]G_i, i \in J, G_i \in \tau\} \subseteq \tau_{0,1}$. Since $\bigcup G_i = \langle x, \bigvee \mu_{G_i}, \bigwedge \gamma_{G_i} \rangle \in \tau$, we have

$$\bigcup ([\ ]G_i) = \langle x, \bigvee \mu_{G_i}, \bigwedge (1-\mu_{G_i}) \rangle$$
$$= \langle x, \bigvee \mu_{G_i}, 1 - \bigvee \mu_{G_i} \rangle \in \tau_{0,1}.$$

(b) (T_1) It is obvious that $0_\sim$ and $1_\sim \in \tau_{0,2}$.

(T_2) Let $\langle\ \rangle G_1, \langle\ \rangle G_2 \in \tau_{0,2}$.

Since $G_1, G_2 \in \tau$, therefore $G_1 \bigcap G_2 = \langle x, \mu_1 \bigwedge \mu_2, \gamma_1 \bigvee \gamma_2 \rangle \in \tau$.

Thus, $(\langle\ \rangle G_1) \bigcap (\langle\ \rangle G_2) = \langle x, (1-\gamma_1) \bigwedge (1-\gamma_2), \gamma_1 \bigvee \gamma_2 \rangle = \langle x, 1-(\gamma_1 \bigvee \gamma_2), \gamma_1 \bigvee \gamma_2 \rangle \in \tau_{0,2}$

(T_3) Let $\{\langle\ \rangle G_i, i \in J, G_i \in \tau\} \subseteq \tau_{0,2}$. Since $\bigcup G_i = \langle x, \bigvee \mu_{G_i}, \bigwedge \gamma_{G_i} \rangle \in \tau$, we have $\bigcup (\langle\ \rangle G_i) = \langle x, \bigvee (1-\gamma_{G_i}), \bigwedge \gamma_{G_i} \rangle = \langle x, 1 - (\bigwedge \gamma_{G_i}), \bigwedge \gamma_{G_i} \rangle \in \tau_{0,2}$.

Definition 2.1.4 [4]: Let (X, τ_1) ,(X, τ_2) be two IFTSs on X. Then τ_1 is said to be contained in τ_2 if $G \in \tau_2$ for each $G \in \tau_1$. In this case, we also say that τ_1 is coarser than τ_2.

Proposition 2.1.5 [4]: Let $\{\tau_i : i \in J\}$ be a family of IFTS on X . Then $\cap\tau_i$ is also an IFT on X. Furthermore, $\cap\tau_i$ is the coarsest IFT on X containing all $\tau_i's$.

Proof: Let $\{\tau_i : i \in J\}$ be a family of IFTS on X. We have to show that $\cap\tau_i, i \in J$ is an IFT on X .

(i) $0_\sim \in \tau_i$, for every $i \in J$. From this it follows that $0_\sim \in \cap\tau_i$. Similarly, $1_\sim \in \cap\tau_i$

(ii) Let $G_1, G_2 \in \cap\tau_i$. Then $G_1, G_2 \in \tau_i$, for every $i \in J$ and hence, $G_1 \cap G_2 \in \tau_i$, $\forall i \in J$. Thus, $G_1 \cap G_2 \in \cap\tau_i$.

(iii) Let $\{G_j : j \in K\} \subseteq \cap\tau_i$. Then $\{G_j : j \in K\} \subseteq \tau_i$, for every $i \in J$ and hence, $\bigcup_{j\in K} G_j \in \tau_i$, $\forall i \in J$. Thus, $\bigcup_{j\in K} G_j \in \cap\tau_i$.

Clearly, it is the coarsest topology on X containing all $\tau_i's$. Since if τ' is any other IFT on X which contains every τ_i, then obviously it will also contain $\cap\tau_i$.

2.2. Basis and Subbasis for IFTS

Definition 2.2.1 [9]: Let $\alpha, \beta \in (0,1)$ and $\alpha + \beta \leq 1$. An intuitionistic fuzzy point (IFP for short) $p^x_{(\alpha,\beta)}$ of X is an IFS of X defined by $p^x_{(\alpha,\beta)} = \langle x, \mu_p, \gamma_p \rangle$, where for $y \in X$

$$\mu_p(y) = \begin{cases} \alpha & \text{if } y = x \\ 0 & \text{if } y \neq x, \end{cases}$$

$$\gamma_p(y) = \begin{cases} \beta & \text{if } y = x \\ 1 & \text{if } y \neq x, \end{cases}$$

In this case, x is called the support of $p^x_{(\alpha,\beta)}$. An IFP $p^x_{(\alpha,\beta)}$ is said to belong to an IFS $A = \langle x, \mu_A, \gamma_A \rangle$ of X, denoted by $p^x_{(\alpha,\beta)} \in A$, if $\alpha \leq \mu_A(x)$ and $\beta \geq \gamma_A(x)$.

Proposition 2.2.2 [9]: An IFS A in X is the union of all IFP belonging to A.

Definition 2.2.3: A collection $\mathcal{B}$ of IFS on a set X is said to be basis (or base) for an IFT on X, if

(i) For every $p^x_{(\alpha,\beta)}$ in X, there exists $B \in \mathcal{B}$ such that $p^x_{(\alpha,\beta)} \in B$.

(ii) If $p^x_{(\alpha,\beta)} \in B_1 \cap B_2$, where $B_1, B_2 \in \mathcal{B}$, then $\exists B_3 \in \mathcal{B}$ such that $P^x_{(\alpha,\beta)} \in B_3 \subseteq B_1 \cap B_2$.

Proposition 2.2.4: Let $\mathcal{B}$ be a basis for an IFT on X. Let τ contains those IFS G of X for which corresponding to each $p^x_{(\alpha,\beta)} \in G$, $\exists B \in \mathcal{B}$ such that $p^x_{(\alpha,\beta)} \in B \subseteq G$. Then τ is an IFT on X.

Proof:

(i) Since $0_\sim$ does not contain any IFP, therefore for it the condition is vacuously true. Further, $1_\sim$ contains every IFP and for it the condition follows from the definition of the basis.

(ii) Let $G_i = \langle x, \mu_{G_i}, \nu_{G_i} \rangle$, where $i \in I$, be a family of members of τ. We have to prove that $\bigcup_{i\in I} G_i \in \tau$. That is $\bigcup_{i\in I} G_i = \{\langle x, \vee\mu_{G_i}(x), \wedge\nu_{G_i}(x)\rangle : x \in X\} \in \tau$. Let $p^x_{(\alpha,\beta)} \in \bigcup_{i\in I} G_i$. Then, $p^x_{(\alpha,\beta)} \in G_j$ for some $j \in I$. Therefore $\exists B_j \in \mathcal{B}$ such that $p^x_{(\alpha,\beta)} \in B_j \subseteq G_j \subseteq \bigcup_{i\in I} G_i \in \tau$.

(iii) Let $G_1, G_2 \in \tau$. If $G_1 \cap G_2 = 0_\sim$ then obviously $G_1 \cap G_2 \in \tau$. Now, suppose that $p^x_{(\alpha,\beta)} \in G_1 \cap G_2$. Then there exist B_1, $B_2 \in \mathcal{B}$ such that $p^x_{(\alpha,\beta)} \in B_1 \subseteq G_1$ and $p^x_{(\alpha,\beta)} \in B_2 \subseteq G_2$. That is, $p^x_{(\alpha,\beta)} \in B_1 \cap B_2 \subseteq G_1 \cap G_2$. By the definition of the basis there exists $B_3 \in \mathcal{B}$ such that $p^x_{(\alpha,\beta)} \in B_3 \subseteq B_1 \cap B_2$. Thus $p^x_{(\alpha,\beta)} \in B_3 \subseteq G_1 \cap G_2$. Hence $G_1 \cap G_2 \in \tau$.

Proposition 2.2.5: Let τ be an IFT on a set X, generated by a basis $\mathcal{B}$. Then members of τ are precisely the union of members of $\mathcal{B}$, that is, $G \in \tau$ iff $G = \bigcup_{\alpha\in\mathcal{A}} B_\alpha$, where $B_\alpha \in \mathcal{B}$, $\forall \alpha \in \mathcal{A}$.

Proof: Clearly $\mathcal{B} \subseteq \tau$. Since τ is a topology on X, therefore any arbitrary union of members of $\mathcal{B}$ belongs to τ. That is, $\bigcup_{\alpha \in \mathcal{A}} B_\alpha \in \tau$ as $B_\alpha \in \mathcal{B}$. Conversely suppose that $G \in \tau$. Then for each $p^x_{(\alpha,\beta)} \in G$, $\exists\ B_x \in \mathcal{B}$ such that $p^x_{(\alpha,\beta)} \in B_x \subseteq G$. Thus $G = \bigcup_{p^x_{(\alpha,\beta)} \in G} B_x$.

Definition 2.2.6 [9]: Let (X, τ) be an IFTS. Then a subfamily $\mathcal{S} \subseteq \tau$ is called a subbasis for τ if the family of finite intersections of members of $\mathcal{S}$ forms a base for τ.

Definition 2.2.7 [4]: The complement $\bar{A}$ of an IFOS A in an IFTS (X, τ) is called an intuitionstic fuzzy closed set (IFCS) in X.

2.3. Closure and Interior of IFS

Definition 2.3.1 [4]: Let (X, τ) be an IFTS and $A = \langle x, \mu_A, \gamma_A \rangle$ be an IFS in X. Then the fuzzy interior and fuzzy closure of A are defined by

$cl(A) = \bigcap\{K : K \text{ is an IFCS in } X \text{ and } A \subseteq K\}$,

$int(A) = \bigcup\{G : G \text{ is an IFOS in } X \text{ and } G \subseteq A\}$.

Note that $cl(A)$ is an IFCS and $int(A)$ is an IFOS in X. Further,

(a) A is an IFCS in X iff $cl(A) = A$;

(b) A is an IFOS in X iff $int(A) = A$.

Example 2.3.2 [4]: Let $X = \{a, b, c\}$

$A = \langle x, (\frac{a}{0.5}, \frac{b}{0.5}, \frac{c}{0.4}), (\frac{a}{0.2}, \frac{b}{0.4}, \frac{c}{0.4})\rangle, B = \langle x, (\frac{a}{0.4}, \frac{b}{0.6}, \frac{c}{0.2}), (\frac{a}{0.5}, \frac{b}{0.3}, \frac{c}{0.3})\rangle$,

$C = \langle x, (\frac{a}{0.5}, \frac{b}{0.6}, \frac{c}{0.4}), (\frac{a}{0.2}, \frac{b}{0.3}, \frac{c}{0.3})\rangle, D = \langle x, (\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.2}), (\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.4})\rangle$.

Then the family $\tau = \{0_\sim, 1_\sim, A, B, C, D\}$ of IFSs in X is an IFT on X .

If $F = \langle x, (\frac{a}{0.55}, \frac{b}{0.55}, \frac{c}{0.45}), (\frac{a}{0.3}, \frac{b}{0.4}, \frac{c}{0.3})\rangle$, then

$int(F) = \bigcup\{G : G \text{ is an IFOS in } X \text{ and } G \subseteq F\} = D$, and

$cl(F) = \bigcap\{K : K \text{ is an IFCS in } X \text{ and } F \subseteq K\} = 1_\sim$.

Proposition 2.3.3 [4]: For any IFS A in (X, τ) we have

(a) $cl(\bar{A}) = \overline{int(A)}$

(b) $int(\bar{A}) = \overline{cl(A)}$

Proof: (a) Let $A = \langle x, \mu_A, \gamma_A \rangle$ and suppose that the IFOS's contained in A are indexed by the family $\{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle : i \in J\}$. Then, $int(A) = \langle x, \vee\mu_{G_i}, \wedge\gamma_{G_i} \rangle$ and hence

$$\overline{int(A)} = \langle x, \wedge\gamma_{G_i}, \vee\mu_{G_i} \rangle. \cdots\cdots\cdots (1)$$

Since $\bar{A} = \langle x, \gamma_A, \mu_A \rangle$ and $\mu_{G_i} \leq \mu_A, \gamma_{G_i} \geq \gamma_A$, for every $i \in J$ we obtain that $\{\langle x, \gamma_{G_i}, \mu_{G_i} \rangle : i \in J\}$ is the family of IFCS's containing $\bar{A}$, that is,

$$cl(\bar{A}) = \langle x, \wedge\gamma_{G_i}, \vee\mu_{G_i} \rangle. \cdots\cdots\cdots (2)$$

Hence from equation (1) and (2) we get $cl(\bar{A}) = \overline{int(A)}$.

(b) Let $A = \langle x, \mu_A, \gamma_A \rangle$ and suppose that the family of IFCS's containing A is given by $\{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle : i \in J\}$. Then we have that $cl(A) = \langle x, \wedge\mu_{G_i}, \vee\gamma_{G_i} \rangle$ and hence,

$$\overline{cl(A)} = \langle x, \vee\gamma_{G_i}, \wedge\mu_{G_i} \rangle. \cdots\cdots\cdots (3)$$

Since $\bar{A} = \langle x, \gamma_A, \mu_A \rangle$ and $\mu_A \leq \mu_{G_i}, \gamma_A \geq \gamma_{G_i}$, for each $i \in J$, we obtain that $\{\langle x, \gamma_{G_i}, \mu_{G_i} \rangle : i \in J\}$ is the family of IFOS's contained in $\bar{A}$, that is,

$$int(\bar{A}) = \langle x, \vee\gamma_{G_i}, \wedge\mu_{G_i} \rangle. \cdots\cdots\cdots (4)$$

Hence, from equation (3) and (4) we get $int(\bar{A}) = \overline{cl(A)}$.

Proposition 2.3.4 [4]: Let (X, τ) be an IFTS and A, B be IFSs in X. Then the following properties holds

(a) $int(A) \subseteq A$

(b) $A \subseteq cl(A)$

(c) $A \subseteq B \Rightarrow int(A) \subseteq int(B)$

(d) $A \subseteq B \Rightarrow cl(A) \subseteq cl(B)$

(e) $int(int(A)) = int(A)$

(f) $cl(cl(A)) = cl(A)$

(g) $int(A \cap B) = int(A) \cap int(B)$

(h) $cl(A \cup B) = cl(A) \cup cl(B)$

(i) $int(1_\sim) = 1_\sim$

(j) $cl(0_\sim) = 0_\sim$.

Proposition 2.3.5 [4]: Let (X, τ) be an IFTS. If $A = \langle x, \mu_A, \gamma_A \rangle$ is an IFS in X,then we have

(i) $int(A) \subseteq \langle x, int_{\tau_1}(\mu_A), cl_{\tau_2}(\gamma_A) \rangle \subseteq A$

(ii) $A \subseteq \langle x, cl_{\tau_2}(\mu_A), int_{\tau_1}(\gamma_A) \rangle \subseteq cl(A)$,

where τ_1 and τ_2 are fuzzy topological spaces on X defined by

$$\tau_1 = \{\mu_G : G \in \tau\} \qquad \tau_2 = \{1 - \gamma_G : G \in \tau\} .$$

Proof: (i) Let $A = \langle x, \mu_A, \gamma_A \rangle$ and suppose that the family of IFOSs contained in A are indexed by the family $\{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle : i \in J\}$. Then $int(A) = \langle x, \vee\mu_{G_i}, \wedge\gamma_{G_i} \rangle$. Each member of the family of fuzzy open sets $\{\mu_{G_i} : i \in J\} \in \tau_1$ is contained in μ_A and hence $\bigvee\{\mu_{G_i} : i \in J\} \leq int_{\tau_1}(\mu_A)$. Again each member of the family of fuzzy closed sets $\{\gamma_{G_i} : i \in J\} \in \tau_2$ contains γ_A and hence $\bigwedge\{\gamma_{G_i} : i \in J\} \geq cl_{\tau_2}(\gamma_A)$. Thus we get $int(A) \subseteq \langle x, int_{\tau_1}(\mu_A), cl_{\tau_2}(\gamma_A)\rangle \subseteq A$.

(ii) Let $B = \langle x, \mu_B, \gamma_B\rangle$. Then from (i), we get $int(B) \subseteq \langle x, int_{\tau_1}(\mu_B), cl_{\tau_2}(\gamma_B)\rangle \subseteq B$, or

$\bar{B} \subseteq \langle x, cl_{\tau_2}(\gamma_B), int_{\tau_1}(\mu_B)\rangle \subseteq \overline{int(B)} = cl(\bar{B})$. $\cdots\cdots\cdots$ (1)

Now suppose that $A = \bar{B}$, i.e. $\langle x, \mu_A, \gamma_A\rangle = \langle x, \gamma_B, \mu_B\rangle$. Then, from (1) we get $A \subseteq \langle x, cl_{\tau_2}(\mu_A), int_{\tau_1}(\gamma_A)\rangle \subseteq cl(A)$.

Corollary 2.3.6 [4]: Let $A = \langle x, \mu_A, \gamma_A\rangle$ be an IFS in (X, τ).

(a) If A is an IFCS, then μ_A is fuzzy closed in (X, τ_2) and γ_A is fuzzy open in (X, τ_1).

(b) If A is an IFOS, then μ_A is fuzzy open in (X, τ_1) and γ_A is fuzzy closed in (X, τ_2).

Proof: (a) Let $A = \langle x, \mu_A, \gamma_A\rangle$ be an IFS in (X, τ). If A is an IFCS, then it means that $cl(A) = A$, and hence from part (ii) of the previous result, we get $\langle x, \mu_A, \gamma_A\rangle = \langle x, cl_{\tau_2}(\mu_A), int_{\tau_1}(\gamma_A)\rangle$. This implies that $\mu_A = cl_{\tau_2}(\mu_A)$ and $\gamma_A = int_{\tau_1}(\gamma_A)$. Hence, μ_A is fuzzy closed in (X, τ_2) and γ_A is fuzzy open in (X, τ_1).

(b) Let $A = \langle x, \mu_A, \gamma_A\rangle$ be an IFS in (X, τ). If A is an IFOS, then $A = int(A)$. From part (i) of the previous result, we get $\langle x, \mu_A, \gamma_A\rangle = \langle x, int_{\tau_1}(\mu_A), cl_{\tau_2}(\gamma_A)\rangle$. Thus, $\mu_A = int_{\tau_1}(\mu_A)$ and $\gamma_A = cl_{\tau_2}(\gamma_A)$ and hence μ_A is fuzzy open in (X, τ_1) and γ_A is fuzzy closed in (X, τ_2).

Example 2.3.7 [4]: Consider the IFTS (X, τ), where $X = \{a, b, c\}$,

$A = \langle x, (\frac{a}{0.5}, \frac{b}{0.5}, \frac{c}{0.4}), (\frac{a}{0.2}, \frac{b}{0.4}, \frac{c}{0.4})\rangle, B = \langle x, (\frac{a}{0.4}, \frac{b}{0.6}, \frac{c}{0.2}), (\frac{a}{0.5}, \frac{b}{0.3}, \frac{c}{0.3})\rangle$,

$C = \langle x, (\frac{a}{0.5}, \frac{b}{0.6}, \frac{c}{0.4}), (\frac{a}{0.2}, \frac{b}{0.3}, \frac{c}{0.3})\rangle, D = \langle x, (\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.2}), (\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.4})\rangle$, and

$\tau = \{0_\sim, 1_\sim, A, B, C, D\}$. Let $F = \langle x, (\frac{a}{0.55}, \frac{b}{0.55}, \frac{c}{0.45}), (\frac{a}{0.3}, \frac{b}{0.4}, \frac{c}{0.3})\rangle$, then

$int_{\tau_1}(\mu_F) = \sup\{O : O \leq F, O \in \tau_1\} = (\frac{a}{0.5}, \frac{b}{0.5}, \frac{c}{0.4})$ and $cl_{\tau_2}(\gamma_F) = \inf\{K : F \leq K, K^c \in \tau_2\} = (\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.4})$.

2.4. Intuitionistic Fuzzy Neighbourhood

Definition 2.4.1 [6]: Let $p^x_{(\alpha,\beta)}$ be an IFP of an IFTS (X, τ). An IFS A of X is called an intuitionistic fuzzy neighborhood (IFN for short) of $p^x_{(\alpha,\beta)}$ if there is an IFOS B in X such that $p^x_{(\alpha,\beta)} \in B \subseteq A$.

Theorem 2.4.2 [6]: Let (X, τ) be an IFTS. Then an IFS A of X is an IFOS if and only if A is an IFN of $p^x_{(\alpha,\beta)}$ for every IFP $p^x_{(\alpha,\beta)} \in A$.

Proof: Let A be an IFOS of X. Clearly, A is an IFN of every $p^x_{(\alpha,\beta)} \in A$. Conversely, suppose that A is an IFN of every IFP belonging to A. Let $p^x_{(\alpha,\beta)} \in A$. Since A is an IFN of $p^x_{(\alpha,\beta)}$, there is an IFOS $B_{p^x_{(\alpha,\beta)}}$ in X such that $p^x_{(\alpha,\beta)} \in B_{p^x_{(\alpha,\beta)}} \subseteq A$. So we have $A = \bigcup\{p^x_{(\alpha,\beta)} : p^x_{(\alpha,\beta)} \in A\} \subseteq \bigcup\{B_{p^x_{(\alpha,\beta)}} : p^x_{(\alpha,\beta)} \in A\} \subseteq A$ and hence $A = \bigcup\{B_{p^x_{(\alpha,\beta)}} : p^x_{(\alpha,\beta)} \in A\}$. Since each $B_{p^x_{(\alpha,\beta)}}$ is an IFOS, A is also an IFOS in X.

2.5. Intuitionistic Fuzzy Continuity

Definition 2.5.1 [4]: Let (X, τ) and (Y, ϕ) be two IFTSs and let $f : X \to Y$ be a function. Then f is said to be fuzzy continuous iff the preimage of each IFS in ϕ is an IFS in τ.

Definition 2.5.2 [4]: Let (X, τ) and (Y, ϕ) be two IFTSs and let $f : X \to Y$ be a function. Then f is said to be fuzzy open iff the image of each IFS in τ is an IFS in ϕ.

Example 2.5.3 [4]: Let (X, τ_0) and (Y, ϕ_0) be two fuzzy topological space in the sense of Chang.

(a) If $f : X \to Y$ is fuzzy continuous in the usual sense, then in this case, f is fuzzy continuous iff the preimage of each IFS in ϕ_0 is an IFS in τ_0. Consider the IFTs on X and Y, respectively, as follows:

$$\tau = \{\langle x, \mu_G, 1-\mu_G\rangle : \mu_G \in \tau_0\} \text{ and } \phi = \{\langle y, \lambda_H, 1-\lambda_H\rangle : \lambda_H \in \phi_0\}.$$

In this case we have for each $\langle y, \lambda_H, 1-\lambda_H\rangle \in \phi, \mu_H \in \phi_0$. $f^{-1}(\langle y, \lambda_H, 1-\lambda_H\rangle) = \langle x, f^{-1}(\lambda_H), f^{-1}(1-\lambda_H)\rangle = \langle x, f^{-1}(\lambda_H), 1-f^{-1}(\lambda_H)\rangle \in \tau$.

(b) Let $f : X \to Y$ be a fuzzy open function in the usual sense. Then f is fuzzy open according to definition (2.5.2). In this case we have, for each $\langle x, \mu_G, 1-\mu_G\rangle \in \tau$, $\mu_G \in \tau_0$ and hence, $f(\langle x, \mu_G, 1-\mu_G\rangle) = \langle y, f(\mu_G), f_{-}(1-\mu_G)\rangle = \langle y, f(\mu_G), 1-f(\mu_G) \in \phi$.

Proposition 2.5.4 [4]: $f : (X,\tau) \to (Y,\phi)$ is fuzzy continuous iff the preimage of each IFCS in ϕ is an IFCS in τ.

Proof: Let $f : (X,\tau) \to (Y,\phi)$ is fuzzy continuous. Let $B = \langle y, \mu_B, \gamma_B\rangle$ is an IFS in ϕ, $\bar{B} = \langle y, \gamma_B, \mu_B\rangle$ is IFCS in ϕ. $f^{-1}(\bar{B}) = \langle x, f^{-1}(\gamma_B), f^{-1}(\mu_B)\rangle = \overline{f^{-1}(B)}$. since f is continuous, so by definition of continuous $f^{-1}\bar{B} = \overline{f^{-1}(B)} \in \tau$.

conversely given $f : (X,\tau) \to (Y,\phi)$ and the preimage of each IFCS in ϕ is an IFCS in τ. We have to show f is fuzzy continuous. Let $B = \langle y, \mu_B, \gamma_B\rangle$ is IFS in ϕ, $\bar{B} = \langle y, \gamma_B, \mu_B\rangle$ is IFCS in ϕ. $f^{-1}(\bar{B}) = \langle x, f^{-1}(\gamma_B), f^{-1}(\mu_B)\rangle = \overline{f^{-1}(B)}$. Since f is a function from X, τ to Y, ϕ.So f^{-1} is a function from Y, ϕ to (X,τ).$\bar{B}$ is IFCS in ϕ,So $f^{-1}(B) = \overline{f^{-1}(B)}$ is an IFCS in X. $\Rightarrow f^{-1}(B) \in \tau$. Hence f is fuzzy continuous.

Proposition 2.5.5 [4]: The following are equivalent to each other.

(a) $f : (X,\tau) \to (Y,\phi)$ is fuzzy continuous.

(b) $f^{-1}(int(B)) \subseteq int(f^{-1}(B))$ for each IFS B in Y.

(c) $cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$ for each IFS B in Y.

Proof: (a) ⇒ (b) Given $f : (X, \tau) \to (Y, \phi)$ is fuzzy continuous. Then we have to show that $f^{-1}(int(B)) \subseteq int(f^{-1}(B))$, for each IFS B in Y. Let $B = \langle y, \mu_B, \gamma_B \rangle$ be an IFS in Y. Let $int(B) = \{\langle y, \vee\mu_{H_i}, \wedge\gamma_{H_i}\rangle : i \in I\}$, where $\mu_{H_i} \leq \mu_B$ and $\gamma_{H_i} \geq \gamma_B$ for each $i \in I$. By the definition of continuity $f^{-1}(int(B))$ is an IFS in τ. Now, $f^{-1}(int(B)) = f^{-1}(\langle y, \vee\mu_{H_i}, \wedge\gamma_{H_i}\rangle) = \langle x, f^{-1}(\vee\mu_{H_i}), f^{-1}(\wedge\gamma_{H_i})\rangle = \langle x, \vee(f^{-1}(\mu_{H_i})), \wedge(f^{-1}(\gamma_{H_i}))\rangle \subseteq int(f^{-1}(B))$, since $f^{-1}(\mu_{H_i}) \leq f^{-1}(\mu_B)$ and $f^{-1}(\gamma_{H_i}) \geq f^{-1}(\gamma_B)$, for every $i \in I$.

(b) ⇒ (a) Given $f^{-1}(int(B)) \subseteq int(f^{-1}(B))$, for each IFS B in Y. To show that f is fuzzy continuous. Let $B = \langle y, \mu_B, \gamma_B \rangle$ be an IFS in ϕ. We have to show that $f^{-1}(B)$ is an IFS in τ. We know that B is open in Y iff $int(B) = B$ and hence, $f^{-1}(int(B)) = f^{-1}(B)$. But according to our assumption $f^{-1}(int(B)) \subseteq int(f^{-1}(B))$, therefore we get $f^{-1}(B) \subseteq int(f^{-1}(B))$. Hence, $f^{-1}(B) = int(f^{-1}(B))$, i.e., $f^{-1}(B)$ is an IFS in τ and this proves that f is fuzzy continuous.

(a) ⇒ (c) Given $f : (X, \tau) \to (Y, \phi)$ is fuzzy continuous. We have to show that $cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$, for each IFS B in Y. Let $B = \langle y, \mu_B, \gamma_B \rangle$ be an IFS in Y. Let $cl(B) = \{\langle y, \wedge\mu_{F_i}, \vee\gamma_{F_i}\rangle : i \in I\}$, where $\mu_{F_i} \geq \mu_B$ and $\gamma_{F_i} \leq \gamma_B$, for each $i \in I$. Since f is fuzzy continuous iff the inverse image of each IFCS in Y is an IFCS in X, therefore $f^{-1}(cl(B))$ is an IFCS in X. Now, $f^{-1}(cl(B)) = f^{-1}(\langle y, \wedge\mu_{F_i}, \vee\gamma_{F_i}\rangle) = \langle x, f^{-1}(\wedge\mu_{F_i}), f^{-1}(\vee\gamma_{F_i})\rangle = \langle x, \wedge(f^{-1}(\mu_{F_i})), \vee(f^{-1}(\gamma_{F_i}))\rangle \supseteq cl(f^{-1}(B))$, since $f^{-1}(\mu_{F_i}) \geq f^{-1}(\mu_B)$ and $f^{-1}(\gamma_{F_i}) \leq f^{-1}(\gamma_B)$, for every $i \in I$.

(c) ⇒ (a) Given that $cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$, for each IFS B in Y. We have to prove that f is fuzzy continuous, that is, we have to show that the inverse image of each IFCS in Y is an IFCS in X. Let $B = \langle y, \mu_B, \gamma_B \rangle$ be an IFCS in Y. We have to show that $f^{-1}(B)$ is an IFCS in X. Since $B = cl(B)$, therefore $f^{-1}(B) = f^{-1}(cl(B))$ but it is given that $cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$, hence $cl(f^{-1}(B)) \subseteq f^{-1}(B) = f^{-1}(cl(B))$. So from this we conclude that $f^{-1}(B) = cl(f^{-1}(B))$, i.e., $f^{-1}(B)$ an IFCS in X. This proves that f is fuzzy continuous.

Chapter 3

Compactness and Separation Axioms

3.1. Intuitionistic Fuzzy Compactness

Definition 3.1.1 [4]: Let (X, τ) be an IFTS.

(a) If a family $\{\langle x, \mu_{G_i}, \gamma_{G_i}\rangle : i \in J\}$ of IFOS in X satisfy the condition $\bigcup\{\langle x, \mu_{G_i}, \gamma_{G_i}\rangle : i \in J\} = 1_\sim$ then it is called a fuzzy open cover of X. A finite subfamily of fuzzy open cover $\{\langle x, \mu_{G_i}, \gamma_{G_i}\rangle : i \in J\}$ of X, which is also a fuzzy open cover of X is called a finite subcover of $\{\langle x, \mu_{G_i}, \gamma_{G_i}\rangle : i \in J\}$.

(b) A family $\{\langle x, \mu_{K_i}, \gamma_{K_i}\rangle : i \in J\}$ of IFCSs in X satisfies the finite intersection property iff every finite subfamily $\{\langle x, \mu_{K_i}, \gamma_{K_i}\rangle : i = 1, 2, \cdots, n\}$ of the family satisfies the condition $\bigcap_{i=1}^{n}\{\langle x, \mu_{K_i}, \gamma_{K_i}\rangle\} \neq 0_\sim$.

Definition 3.1.2 [4]: An IFTS (X, τ) is called fuzzy compact iff every fuzzy open cover of X has a finite subcover.

Example 3.1.3 [4]: Consider the IFTS (X, τ), where $X = \{1, 2\}$, $G_n = \langle x, (\frac{1}{\frac{n}{n+1}}, \frac{2}{\frac{n+1}{n+2}}), (\frac{1}{\frac{1}{n+2}}, \frac{2}{\frac{1}{n+3}})\rangle$ and $\tau = \{0_\sim, 1_\sim\} \cup \{G_n : n \in \mathbb{N}\}$. Note that $\bigcup_{n\in\mathbb{N}} G_n$ is an open cover for X, but this cover has no finite subcover. Consider

$$G_1 = \langle x, (\frac{1}{0.5}, \frac{2}{0.6}), (\frac{1}{0.3}, \frac{2}{0.25})\rangle$$

$$G_2 = \langle x, (\frac{1}{0.6}, \frac{2}{0.75}), (\frac{1}{0.25}, \frac{2}{0.2})\rangle$$

$$G_3 = \langle x, (\frac{1}{0.75}, \frac{2}{0.8}), (\frac{1}{0.2}, \frac{2}{0.16})\rangle$$

and observe that $G_1 \cup G_2 \cup G_3 = G_3$. So, for any finite subcollection $\{G_{n_i} : i \in I$, where I is a finite subset of $\mathbb{N}\}$, $\bigcup_{n_i \in I} G_{n_i} = G_m \neq 1_\sim$, where $m = max\{n_i : n_i \in I\}$. Therefore the IFTS (X, τ) is not compact.

Proposition 3.1.4 [4]: Let (X, τ) be an IFTS on X. Then (X, τ) is fuzzy compact iff the IFTS $(X, \tau_{0,1})$ is fuzzy compact.

Proof: Let (X, τ) be fuzzy compact and consider a fuzzy open cover $\{[\ \ ]G_j : j \in K\}$ of X in $(X, \tau_{0,1})$. Since $\bigcup([\ \ ]G_j) = 1_\sim$ we obtain $\bigvee \mu_G = 1$, and hence, by $\gamma_{G_j} \leq 1 - \mu_{G_j} \Rightarrow \bigwedge \gamma_{G_j} \leq 1 - \bigvee \mu_{G_j} = 1 - 1 = 0 \Rightarrow \bigwedge \gamma_{G_j} = 0$, we deduce $\bigcup G_j = 1_\sim$. Since (X, τ) is fuzzy compact there exist $G_1, G_2, \cdots G_n$ such that $\bigcup_{i=1}^n G_i = 1_\sim$ from which we obtain $\bigvee_{i=1}^n \mu_{G_i} = 1$ and $\bigwedge_{i=1}^n (1 - \mu_{G_i}) = 0$, that is, $(X, \tau_{0,1})$ is fuzzy compact.

Suppose that $(X, \tau_{0,1})$ is fuzzy compact and consider a fuzzy open cover $G_j : j \in K$ of X in (X, τ). Since $\bigcup G_j = 1_\sim$, we obtain $\bigvee \mu_{G_j} = 1$ and $\bigwedge(1 - \mu_{G_j}) = 0$. Since $(X, \tau_{0,1})$ is fuzzy compact there exist $G_1, G_2, \cdots G_n$ such that $\bigcup_{i=1}^n([\ \ ]G_i) = 1_\sim$, that is, $\bigvee_{i=1}^n \mu_{G_i} = 1$ and $\bigwedge_{i=1}^n(1 - \mu_{G_i}) = 0$. Hence $\mu_{G_i} \leq 1 - \gamma_{G_i} \Rightarrow 1 = \bigvee_{i=1}^n \mu_{G_i} \leq 1 - \bigwedge_{i=1}^n \gamma_{G_i} \Rightarrow \bigwedge_{i=1}^n \gamma_{G_i} = 0$. Hence $\bigcup_{i=1}^n G_i = 1_\sim$. Therefore (X, τ) is fuzzy compact.

Corollary 3.1.5 [4]: Let (X, τ), (Y, ϕ) be IFTSs and $f : X \to Y$ a fuzzy continuous surjection. If (X, τ) is fuzzy compact, then so is (Y, ϕ).

Proof: Given that f is continuous and onto and (X, τ) is fuzzy compact. To show that $f(X) = Y$ is also fuzzy compact. Let us consider an open cover $\{G_j : j \in K\}$ of Y, then $\bigcup_{j \in K} G_j = 1_\sim^Y$. Let $G_j = \langle y, \mu_{G_j}, \gamma_{G_j} \rangle$. Now, $f^{-1}(\bigcup_{j \in K} G_j) = f^{-1}(1_\sim^Y) \Rightarrow \bigcup_{j \in K} f^{-1}(G_j) = 1_\sim^X$. Since G_j is open in Y, for every $j \in K$, therefore $f^{-1}(G_j)$ is open in X, for every $j \in K$ as the map f is fuzzy continuous. Thus the family $\{f^{-1}(G_j) : j \in K\}$

is an open cover for X and since X is compact this family has a finite subcover, say, $\{f^{-1}(G_1), f^{-1}(G_2), \cdots, f^{-1}(G_n)\}$. Thus, $\bigcup_{i=1}^{n} f^{-1}(G_j) = 1_{\sim}^{X}$. Now, $f(\bigcup_{i=1}^{n} f^{-1}(G_j)) = f(1_{\sim}^{X}) \Rightarrow \bigcup_{i=1}^{n} f(f^{-1}(G_j))) = f(1_{\sim}^{X}) \Rightarrow \bigcup_{j=1}^{n}(G_j) = 1_{\sim}^{Y}$, (as the map f is surjective). This proves that Y is fuzzy compact.

Corollary 3.1.6 [4]: An IFTS (X, τ) is fuzzy compact iff every family $\{\langle x, \mu_{K_i}, \gamma_{K_i}\rangle : i \in J\}$ of IFCSs in X having the FIP has a nonempty intersection.

Proof: Assume that X is fuzzy compact i.e every open cover of X has a finite subcover. Let $\{K_i = \langle x, \mu_{K_i}, \gamma_{K_i}\rangle : i \in J\}$ be a family of IFCS of X. Also assume that this family has finite intersection property. We have to show that $\bigcap_{i\in J} K_i = \bigcap_{i\in J}\{\langle x, \mu_{K_i}, \gamma_{K_i}\rangle : i \in J\} \neq 0_{\sim}$. On the contrary suppose that

$$\bigcap_{i\in J} K_i = 0_{\sim} \Rightarrow \overline{\bigcap_{i\in J} K_i} = \overline{0_{\sim}} \Rightarrow \bigcup_{i\in J} \overline{K_i} = \bigcup_{i\in J}\langle x, \gamma_{K_i}, \mu_{K_i}\rangle = 1_{\sim}$$

Since for every $i \in J$, K_i is an IFCS of X, therefore $\overline{K_i}$ will be an IFOS of X. Thus, $\{\bar{K}_i = \langle x, \gamma_{K_i}, \mu_{K_i}\rangle : i \in J\}$ is an open cover for X. Since X is fuzzy compact therefore this cover has a finite subcover, say, $\bigcup_{i=1}^{n} \bar{K}_i = \bigcup_{i=1}^{n}\{\langle x, \gamma_{K_i}, \mu_{K_i}\rangle : i \in J\} = 1_{\sim}$. Then,

$$\overline{\bigcup_{i=1}^{n} \overline{K_i}} = \overline{1_{\sim}} \Rightarrow \bigcap_{i=1}^{n} K_i = 0_{\sim}.$$

Thus, the above considered family does not satisfy the FIP which is a contradiction. Therefore, $\bigcap_{i\in J} K_i \neq 0_{\sim}$.

Conversely, assume that the family of IFCS of X having FIP has nonempty intersection. To show that X is compact let $\{G_i = \langle x, \mu_{G_i}, \gamma_{G_i}\rangle : i \in J\}$ be an open cover of X. Suppose that this open cover has no finite subcover, i.e. for every finite subcollection of the given cover, say,

$$\bigcup_{i=1}^{n} G_i \neq 1_\sim \Rightarrow \overline{(\bigcup_{i=1}^{n} G_i)} \neq \overline{1_\sim} \Rightarrow \bigcap_{i=1}^{n} \overline{G_i} \neq 0_\sim.$$

As each G_i is an IFOS of X therefore, each $\overline{G_i}$ is an IFCS of X. Thus, $\{\bar{G}_i = \langle x, \gamma_{G_i}, \mu_{G_i} \rangle : i \in J\}$ is a family of IFCS of X having FIP. So by the hypothesis it has nonempty intersection, i.e.,

$$\bigcap_{i \in J} \overline{G_i} \neq 0_\sim \Rightarrow \overline{(\bigcap_{i \in J} \overline{G_i})} \neq \overline{0_\sim} \Rightarrow \bigcup_{i \in J} G_i \neq 1_\sim.$$

This shows that the family $\{G_i = \langle x, \mu_{G_i}, \gamma_{G_i} \rangle : i \in J\}$ is not a cover for X, which is a contradiction. Therefore, the given family must have a finite subcover and this shows that X is fuzzy compact.

Definition 3.1.7 [4]: (a) Let (X, τ) be an IFTS and A an IFS in X. If a family $\{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle : i \in J\}$ of IFOSs in X satisfies the condition $A \subseteq \bigcup \{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle : i \in J\}$, then it is called a fuzzy open cover of A. A finite subfamily of the fuzzy open cover $\{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle : i \in J\}$ of A, which is also a fuzzy open cover of A, is called a finite subcover of $\{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle : i \in J\}$.

(b) An IFS $A = \langle x, \mu_A, \gamma_A \rangle$ in an IFTS (X, τ) is called fuzzy compact iff every fuzzy open cover of A has a finite subcover.

Corollary 3.1.8 [4]: An IFS $A = \langle x, \mu_A, \gamma_A \rangle$ in an IFTS (X, τ) is fuzzy compact iff for each family $\mathcal{G} = \{G_i : i \in J\}$, where $G_i = \langle x, \mu_{G_i}, \gamma_{G_i} \rangle (i \in J)$, of IFOSs in X with properties $\mu_A \leq \bigvee_{i \in J} \mu_{G_i}$ and $1 - \gamma_A \leq \bigvee_{i \in J} (1 - \gamma_{G_i})$ there exists a finite subfamily $\{G_i : i = 1, 2, \cdots, n\}$ of $\mathcal{G}$ such that $\mu_A \leq \bigvee_{i=1}^{n} \mu_{G_i}$ and $1 - \gamma_A \leq \bigvee_{i=1}^{n} (1 - \gamma_{G_i})$.

Example 3.1.9 [4]: Let $X = I$ and consider the IFSs $(G_n)_{n \in \mathbb{Z}_2}$, where $G_n = \langle x, \mu_{G_n}, \gamma_{G_n} \rangle$

$,n=2,3,\cdots$ and $G=\langle x,\mu_G,\gamma_G\rangle$ defined by

$$\mu_{G_n}(x)=\begin{cases}0.8, & \text{if } x=0,\\ nx, & \text{if } 0<x\le\frac{1}{n},\\ 1, & \text{if } \frac{1}{n}<x\le 1.\end{cases}$$

$$\gamma_{G_n}(x)=\begin{cases}0.1, & \text{if } x=0,\\ 1-nx, & \text{if } 0<x\le\frac{1}{n},\\ 0, & \text{if } \frac{1}{n}<x\le 1.\end{cases}$$

$$\mu_G(x)=\begin{cases}0.8, & \text{if } x=0,\\ 1, & \text{otherwise.}\end{cases}$$

$$\gamma_G(x)=\begin{cases}0.1, & \text{if } x=0,\\ 0, & \text{otherwise.}\end{cases}$$

Then $\tau=\{0_\sim,1_\sim,G\}\cup\{G_n:n\in\mathbb{Z}_2\}$ is an IFT on X, and consider the IFSs $C_{\alpha,\beta}$ in (X,τ) defined by $C_{\alpha,\beta}=\{\langle x,\alpha,\beta\rangle : x\in X\}$, where $\alpha,\beta\in I$ are arbitrary and $\alpha+\beta\le 1$. Then the IFSs $C_{0.85,0.05}$, $C_{0.85,0.15}$, $C_{0.75,0.05}$ are all fuzzy compact, but the IFS $C_{0.75,0.15}$ is not fuzzy compact.

Corollary 3.1.10 [4]: Let $(X,\tau),(Y,\phi)$ be IFTSs and $f:X\to Y$ a fuzzy continuous function. If A is fuzzy compact in (X,τ), then so is $f(A)$ in (Y,ϕ).

Proof: Let $\mathcal{B}=\{G_i:i\in J\}$, where $G_i=\langle y,\mu_{G_i},\gamma_{G_i}\rangle$, $i\in J$ be a fuzzy open cover of $f(A)$. Then, by the definition of fuzzy continuity $\mathcal{A}=\{f^{-1}(G_i):i\in J\}$ is a fuzzy open cover of A, too. Since A is fuzzy compact, there exists a finite subcover of $\mathcal{A}$, i.e., there exists $G_i (i = 1, 2, \cdots, n)$ such that $A \subseteq \bigcup_{i=1}^{n} f^{-1}(G_i)$. Hence $f(A) \subseteq f(\bigcup_{i=1}^{n} f^{-1}(G_i)) = \bigcup_{i=1}^{n} f(f^{-1}(G_i)) \subseteq \bigcup_{i=1}^{n} G_i$. Therefore, $f(A)$ is also fuzzy compact.

Lemma 3.1.11 (The Alexander subbase Lemma) [5]: Let δ be a subbase of an IFTS (X, τ). Then (X, τ) is fuzzy compact iff for each family of IFCSs chosen from $\delta^c = \{K : K \in \delta\}$ having the FIP there is a nonzero intersection.

Definition 3.1.12 [5]: The product set X equipped with the IFT generated on X by the family $\mathcal{S}$ is called the product of the IFTSs $\{(X_i, \tau_i) : i \in J\}$. For each $i \in J$ and for each $S_i \in \tau_i$, we have $\pi_i^{-1}(S_i) \in \tau$. So π_i is indeed a fuzzy continuous function from the product IFTS onto (X_i, τ_i), $\forall i \in J$. The product IFT τ is the coarsest IFT on X having this property.

Theorem 3.1.13 (Tychonoff Theorem) [5]: Let the IFTSs (X_1, τ_1) and (X_2, τ_2) be fuzzy compact. Then the product IFTS on $X = X_1 \times X_2$ is fuzzy compact.

Proof: Here we will make use of the Alexander subbase lemma. Suppose, on the contrary that there exists a family

$$P = \{\pi_1^{-1}(P_{i_1}) : i_1 \in J_1\} \cup \{\pi_2^{-1}(P_{i_2}) : i_2 \in J_2\} \cdots\cdots\cdots (1)$$

consisting of some of the IFCSs obtained from the subbase

$$\delta = \{\pi_1^{-1}(T_1), \pi_2^{-1}(T_2) : T_1 \in \tau_1, T_2 \in \tau_2\} \cdots\cdots\cdots (2)$$

of the product IFT on X such that P has FIP and $\cap P = 0$. Now, it can be shown easily that the families

$$P_1 = \{P_{i_1} : i_1 \in J_1\}, P_2 = \{P_{i_2} : i_2 \in J_2\} \cdots\cdots\cdots (3)$$

have the FIP, and since (X_i, τ_i)'s are fuzzy compact we have $\cap P_1 \neq 0$ and $\cap P_2 \neq 0$ which means that

$$(\wedge \mu_{P_{i_1}} \neq 0 \text{ or } \vee \gamma_{P_{i_1}} \neq 1), \qquad (\wedge \mu_{P_{i_2}} \neq 0 \text{ or } \vee \gamma_{P_{i_2}} \neq 1) \cdots\cdots\cdots (4)$$

But from $\cap P = 0$, we obtain

$$(\wedge \mu_{P_{i_1}} \circ \pi_1) \wedge (\wedge \mu_{P_{i_2}} \circ \pi_2) = 0, \qquad (\vee \gamma_{P_{i_1}} \circ \pi_1) \vee (\vee \gamma_{P_{i_2}} \circ \pi_2)) = 1. \cdots\cdots\cdots (5)$$

Hence there exist four cases.

Case-I If $\wedge \mu_{P_{i_1}} \neq 0$ and $\wedge \mu_{P_{i_2}} \neq 0$, then there exists $x_1 \in X_1, x_2 \in X_2$ such that $\wedge \mu_{P_{i_1}}(x_1) \neq 0$ and $\wedge \mu_{P_{i_2}}(X_2) \neq 0$ from which we obtain a contradiction to equation (5), if it is evaluated in (x_1, x_2)

Case-II If $\vee \gamma_{P_{i_1}} \neq 1$ and $\vee \gamma_{P_{i_2}} \neq 1$, then we get a similar contradiction as in the first case.

Case-III If $\wedge \mu_{P_{i_1}} \neq 0$ and $\vee \gamma_{P_{i_1}} \neq 1$, then there exist $x_1 \in X_1, x_2 \in X_2$ such that $\wedge \mu_{P_{i_1}}(x_1) \neq 0$ and $\vee \gamma_{P_{i_2}}(x_2) \neq 1$ from which we obtain $\wedge \mu_{P_{i_2}}(x_2) = 0$ and $\vee \mu_{P_{i_1}}(x_1) = 1$ and then, since $\gamma_{P_{i_1}} \leq 1 - \mu_{P_{i_1}}$ for each P_{i_1},

$$\vee \gamma_{P_{i_1}} \leq \vee(1 - \mu_{P_{i_1}}) = 1 - \wedge \mu_{P_{i_1}} \Rightarrow 1 = \vee \gamma_{P_{i_1}}(x_1) \leq 1 - \wedge \mu_{P_{i_1}}(x_1) \Rightarrow \wedge \mu_{P_{i_1}}(x_1) = 0,$$

which is contradiction because $\wedge \mu_{P_{i_1}}(x_1) \neq 0$.

Case-IV If $\vee \gamma_{P_{i_1}} \neq 1$ and $\wedge \mu_{P_{i_2}} \neq 0$, then we obtain a similar contradiction as in the third case.

Hence by the Alexander subbase lemma, (X, τ) is also fuzzy compact.

3.2. Intuitionistic Fuzzy Regular Spaces

Definition 3.2.1 [7]: An IFTS (X, τ) will be called regular if for each IFP $p^x_{(\alpha,\beta)}$ and each IFCS C such that $p^x_{(\alpha,\beta)} \cap C = 0_\sim$ there exists IFOS M and N such that $p^x_{(\alpha,\beta)} \in M$ and $C \subseteq N$.

Note: For the simplification of the notation we will write the IFP $p^x_{(\alpha,\beta)}$ as $x_{(\alpha,\beta)}$.

Proposition 3.2.2: If a space X is a regular space then for any open set U and intuitionistic fuzzy point $x_{(\alpha,\beta)}$ such that $x_{(\alpha,\beta)} \cap U' = 0_\sim$, $\exists$ an open set V such that $x_{(\alpha,\beta)} \in V \subseteq \overline{V} \subseteq U$.

Proof: Suppose that X is a IFRS. Let U be an IFOS of X such that $x_{(\alpha,\beta)} \cap U' = 0_\sim$ and $U = \langle y, \mu_U, \gamma_U \rangle$. Then $U' = \langle y, \nu_U, \mu_U \rangle$ is an IFCS in X. Since X is regular, therefore $\exists$ two IFOSs V and W such that $x_{(\alpha,\beta)} \in V$, $U' \subseteq W$ and $V \cap W = 0_\sim$. Now, W' is an IFCS of X such that $V \subseteq W' \subseteq U$. Thus, $x_{(\alpha,\beta)} \in V \subseteq \overline{V}$ and $\overline{V} \subseteq W' \subseteq U$, so $\overline{V} \subseteq U$. Hence, $x \in V \subseteq \overline{V} \subseteq U$.

Proposition 3.2.3: Every subspace of regular space is also regular.

Proof: Let X be a IFRS and Y is a subspace of X. To prove that Y is regular. We know that $\tau_Y = \{G_Y = \langle x, \mu_{G|Y}, \nu_{G|Y} \rangle : x \in Y, G \in \tau\}$, where $G = \langle x, \mu_G, \nu_G \rangle$. Let $x_{(\alpha,\beta)}$ be an IFP in Y and F_Y is an IFCS of Y such that $x_{(\alpha,\beta)} \cap F_Y = 0_\sim$. Since Y is a subspace of X, so $x_{(\alpha,\beta)} \in X$ and there exists an IFCS F in X such that the closed set generated by it for Y is F_Y. Since X is regular space and $x_{(\alpha,\beta)} \cap F = 0_\sim$, there exist two IFOSs M and N such that $x_{(\alpha,\beta)} \in M = \langle x, \mu_M, \nu_M \rangle$ and $F \subseteq N = \langle x, \mu_N, \nu_N \rangle$. Thus $M_Y = \langle x, \mu_{M|Y}, \nu_{M|Y} \rangle$, and $N_Y = \langle x, \mu_{N|Y}, \nu_{N|Y} \rangle$ are open sets in Y such that $x_{(\alpha,\beta)} \in M_Y$ and $F_Y \subseteq N_Y$. Hence, Y is a regular subspace of X.

3.3. Intuitionistic Fuzzy Normal Spaces

Definition 3.3.1 [7]: An IFTS (X, τ) will be called normal if for each pair of IFCSs C_1 and C_2 such that $C_1 \cap C_2 = 0_\sim$ there exists IFOSs M_1 and M_2 such that $C_i \subseteq M_i (i = 1, 2)$ and $M_1 \cap M_2 = 0_\sim$.

Proposition 3.3.2: If a space X is a normal space, then for each closed set F of X and any open set G of X such that $F \cap G' = 0_\sim$ there exists an open set G_F such that $F \subseteq G_F \subseteq \overline{G_F} \subseteq G$.

Proof: Let X be a normal space. Let F be a closed set in X and G be an open set in X such that $F \cap G' = 0_\sim$, then $F \subseteq G$. Let $G = \langle x, \mu_G, \nu_G \rangle$ and $F = \langle x, \nu_F, \mu_F \rangle$. Since X is normal and G' is an IFCS in X, therefore there exist two disjoint IFOSs G_F and $G_{G'}$, such that $F \subseteq G_F$, $G' \subseteq G_{G'}$ and $G_F \cap G_{G'} = 0_\sim$ This implies that $G'_{G'} \subseteq G$ and $G_F \subseteq G'_{G'}$. But $G'_{G'}$ is a closed set, therefore $\overline{G_F} \subseteq G'_{G'}$. Thus we have $F \subseteq G_F \subseteq \overline{G_F} \subseteq G$.

3.4. Other Separation Axioms in IFTS

Definition 3.4.1 [9]: An IFTS (X, τ) is called

(a) T_0 if for all $x, y \in X, x \neq y \; \exists \; U = (\mu_U, \nu_U), V = (\mu_V, \nu_V) \in \tau$ such that $(\mu_U, \nu_U)(x) = (1, 0), \quad (\mu_U, \nu_U)(y) = (0, 1)$ or $(\mu_V, \nu_V)(x) = (0, 1), \quad (\mu_V, \nu_V)(y) = (1, 0)$.

(b) T_1 if for all $x, y \in X, x \neq y \; \exists \; U = (\mu_U, \nu_U), V = (\mu_V, \nu_V) \in \tau$ such that $(\mu_U, \nu_U)(x) = (1, 0), (\mu_U, \nu_U)(y) = (0, 1), (\mu_V, \nu_V)(x) = (0, 1) \text{and} (\mu_V, \nu_V)(y) = (1, 0)$.

(c) T_2 (or Hausdorff) if for all pair of distinct intuitionistic fuzzy points $x_{(\alpha,\beta)}, y_{(\gamma,\delta)}$ in $X, \exists U, V \in \tau$ such that $x_{(\alpha,\beta)} \in U, y_{(\gamma,\delta)} \in V$ and $U \cap V = 0_\sim$.

Example 3.4.2 [9]: Let $X = \{a, b\}$ and let $\tau = \{0_\sim, A, B, 1_\sim\}$ where $A = \langle x, (\frac{a}{1}, \frac{b}{0}), (\frac{a}{0}, \frac{b}{1})\rangle$ and $B = \langle x, (\frac{a}{1}, \frac{b}{0}), (\frac{a}{0}, \frac{b}{1})\rangle$ then (X, τ) is an IFTS and it is T_0, T_1, T_2.

Proposition 3.4.3 [9]: The following statement are equivalent in an IFTS (X, τ)

(1) (X, τ) is T_1

(2) $(\{x\}, \{x'\})$ is IFC in (X, τ) $\forall\, x \in X$.

Proposition 3.4.4 [9]: Every subspace of T_1 space is T_1.

Proof: Let X be a T_1 IFTS and Z be subspace of X. So $\tau_Z = \{G_Z = \langle x, \mu_{G|Z}, \gamma_{G|Z}\rangle : x \in Z, G \in \tau\}$, where $G = \langle x, \mu_G, \gamma_G\rangle$. Let $x, y \in Z$ such that $x \neq y$. Then, as $Z \subseteq X$, we have $x, y \in X$ such that $x \neq y$. Since X is T_1, therefore $\exists\; U = (\mu_U, \nu_U), V = (\mu_V, \nu_V) \in \tau$ such that $(\mu_U, \nu_U)(x) = (1, 0), (\mu_U, \nu_U)(y) = (0, 1), (\mu_V, \nu_V)(x) = (0, 1)$ and $(\mu_V, \nu_V)(y) = (1, 0)$. Thus, there exist $\exists\; U_Z = (\mu_{U_{|Z}}, \nu_{U_{|Z}}), V_Z = (\mu_{V_{|Z}}, \nu_{V_{|Z}}) \in \tau_Z$ such that $(\mu_{U_{|Z}}, \nu_{U_{|Z}})(x) = (1, 0), (\mu_{U_{|Z}}, \nu_{U_{|Z}})(y) = (0, 1), (\mu_{V_{|Z}}, \nu_{V_{|Z}})(x) = (0, 1)$ and $(\mu_{V_{|Z}}, \nu_{V_{|Z}})(y) = (1, 0)$. This proves that the subspace Z is also T_1.

Proposition 3.4.5 [9]: Every subspace of T_2 space is T_2.

Proof: Let (X, τ) be a IF T_2 space and A be subspace of X, where $\tau_A = \{G_A = \langle x, \mu_{G|A}, \nu_{G|A}\rangle : x \in A, G \in \tau\}$ and $G = \langle x, \mu_G, \nu_G\rangle$. Let $x_{(\alpha,\beta)}$ and $y_{(\gamma,\delta)}$ be two distinct IFP in A, i.e., they have distinct supports. Then, clearly $x_{(\alpha,\beta)}$ and $y_{(\gamma,\delta)}$ are also distinct IFPs in X and as X is T_2, therefore $\exists U, V \in \tau$ such that $x_{(\alpha,\beta)} \in U, y_{(\gamma,\delta)} \in V$ and $U \cap V = 0_\sim$. Thus, $\exists U_A,\; V_A \in \tau_A$ such that $x_{(\alpha,\beta)} \in U_A, y_{(\gamma,\delta)} \in V_A$ and $U_A \cap V_A = 0_\sim$.

Theorem 3.4.6: An IFP $x_{(\alpha,\beta)}$ and a compact set K such that $x_{(\alpha,\beta)} \cap K = 0_\sim$, in a Hausdorff (HDF) space can be separated by disjoint open sets.

Proof: Let (X, τ) be an IFTS. Let K be a IF compact set in (X, τ). Since, $x_{(\alpha,\beta)} \cap K = 0_\sim$, therefore $x_{(\alpha,\beta)} \notin K$ and $\mu_K(x) = 0; \quad \gamma_K(x) = 1$. Let $y_{(\nu,\delta)} \in K$, then clearly $x \neq y$ and thus, $x_{(\alpha,\beta)}$ and $y_{(\nu,\delta)}$ are distinct IFPs. Since X is HDF, therefore there exist two IFOSs $G_x^{y_{(\nu,\delta)}}$ and $G_{y_{(\nu,\delta)}}$ such that $G_x^{y_{(\nu,\delta)}} \cap G_{y_{(\nu,\delta)}} = 0_\sim$. Thus, corresponding to each IFP in K there exist two disjoint open sets separating that point with $x_{(\alpha,\beta)}$. Clearly, $K \subseteq \bigcup_{y_{(\nu,\delta)} \in K} G_x^{y_{(\nu,\delta)}}$. Since K is compact, therefore there exist finitely many open sets such that K is contained into there union. Suppose that the union of these finitely many open sets be represented by H and the intersection of corresponding IFOS containing $x_{(\alpha,\beta)}$ be given by G. Now, we want to show that $G \cap H = 0_\sim$. On the contrary suppose that $G \cap H \neq 0_\sim$. Then there will exist an IFP, say, $z_{(\alpha',\beta')}$ which will belong to the intersection of G and H, but this will contradict the existence of the IFOSs of the type $G_x^{y_{(\nu,\delta)}} \cap G_{y_{(\nu,\delta)}} = 0_\sim$. Hence, $G \cap H = 0_\sim$.

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