REPRESENTATION OF LIE ALGEBRA WITH APPLICATION TO PARTICLES PHYSICS

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by

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CERTIFICATE

This is to certify that the project report entitled *Representation of the Lie algebra, with application to particle physics* is the bonafide work carried out by *Swornaprava Moharana*, student of M.Sc. Mathematics at National Institute Of Technology, Rourkela, during the year 2013, in partial fulfilment of the requirements for the award of the Degree of Master of Science In Mathematics under the guidance of *Prof. K.C Pati*, Professor, National Institute of Technology, Rourkela and that the project is a review work by the student through collection of papers by various sources.

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DECLARATION

I hereby declare that the project report entitled *Representation of the Lie algebras, with application to particle physics* submitted for the M.Sc. Degree is my original work and the project has not formed the basis for the award of any degree, associate ship, fellowship or any other similar titles.

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Rourkela,769008

May, 2013 **Swornaprava Moharana**

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INTRODUCTION

Symmetry has played an important role in various branches of physics, chemistry, Biology and other importants engineering application such as Robotics, computer vision etc. From outside symmetry of an underlying physical system may look fascinating. But in real worlds and many scientific applications symmetry breaking is also equally important. For example it is only when we discuss the symmetry breaking of some physical phenomena(details avioded) then only we can predict the exitance of Higgs boson, the physical discovery of which shook the scientific community recently. Similarly from the symmetry breaking of Lie group $Sp(6)$, now it is well understood why in nature we find only 21 amino acids responsible for explaining structure of DNA code. Similarly underlying symmetry of a given physical system also explains many physical structure phenomena observed in nature. For instance long before, mathematically it is proved that the zoo of particles observed inside the nucleus(universal) obeys some type of symmetry which is exactly same as that of symmetry of Special unitary group/algebra $SU(3)$. So in our work we have discussed this aspect in great detail explaining the representation of Lie group/Lie algebra $SU(3)$ and have shown how the symmetry of root of particles found in universe exactly fits in to this.

It has been observed that all type symmetries are closed related with some type of group structure either discrete group or continuous group. The continuous group which is also called Lie group and its algebra called Lie algebra is our subject of investigation. In this short project we have given introduction to Lie group Lie algebra more particularly the complex Lie algebra $sl(3,\mathbb{C})$ and real form $SU(3)$. We have studied in detail the representation of this lie algebra and we have shown how this is related with the symmetry of particles(particles observed inside the nucleus of an atom). This is not a calculation but a simple review to show one of the interesting application of Lie groups and Lie algebras in the realms of particle physics.

CHAPTER 1

Lie groups and Lie algebra

1.1 Lie algebra and Lie group

Definition 1.1.1. A Lie algebra L is a vector space with a binary operation $(x, y) \in L \times L \mapsto [x, y] \in L$ is called Lie bracket or commutator, which satisfies

- $[x, y] = -[y, x]$ $\forall$ $x, y \in L$ (antisymmetry).
- The binary operation is linear in each of this entries $[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$ and $[x, \alpha y + \beta z] = \alpha[x, y] + \beta[x, z]$ (bilinearity) $\forall$ $x, y \in L$ and $\alpha, \beta \in F$.
- $\forall$ $x, y, z \in L$ one has $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity)

A Lie algebra is called real or complex when the vector space is respectively real ($F = \mathbb{R}$) or complex ($F = \mathbb{C}$).

Definition 1.1.2. Let L and M be Lie algebras and $\phi : L \longrightarrow M$ a bijection such that $\forall$ $\alpha, \beta \in F$
$\phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y)$ and $\phi([x, y]) = [\phi(x), \phi(y)]$ then ϕ is called an isomorphism and the Lie algebra L and M are called isomorphic.

Definition 1.1.3. A Lie algebra L is called abelian if $[x, y] = 0$ $\forall$ $x, y \in L$. A subset K of a Lie algebra L is called a subalgebra of L if $\forall$ $x, y \in K$ and all $\alpha, \beta \in F$,

- $\alpha x + \beta y \in K$
- $[x, y] \in K$

Definition 1.1.4. Commutator of Lie algebra: Let M and N be subsets of L which are not necessarily subspaces. Then the commutator $[M, N]$ of M and N is defined to be the linear span of the set of elements of the form $[x, y]$ with $x \in M \ \ and \ \ y \in N$ that is

$$[M, N] = \{z \in L \mid z = \sum_{i,j} \alpha_{ij}[x_i, y_j]; \ \ x_i, y_j \in L, \alpha_{ij} \in F\}$$

Definition 1.1.5. An ideal I of a Lie algebra L is a subalgebra of L with the property $[I, L] \subset I$ i.e for all $x \in I$ and all $y \in I, [x, y] \in I$.

Every Lie algebra has at least two ideals, namely the Lie algebra L itself and the sub algebra 0 consisting of the zero element only $0 \equiv \{0\}$ both these ideals are called trivial. All non-trivial ideal are called proper.

Definition 1.1.6. Lie Group: A Lie group is a group G, equipped with a manifold structure such that the group operations MULT: $G \times G \rightarrow G, \quad (g_1, g_2) \mapsto g_1 g_2$, Inv : $G \rightarrow G, \quad g \mapsto g^{-1}$ are smooth. A morphism of Lie groups G, G' is a morphism of groups $\phi : G \rightarrow g'$ that is smooth.

Example 1.1.7.
The example of a Lie group is the general linear group. $GL(n, \mathbb{R}) = \{A \in M(n, \mathbb{R}) \mid det(A) \neq 0\}$ of invertible $n \times n$ matrices.

1.1.1 Relation between Lie groups and Lie algebra

Linear Lie groups the element of which are linear operators on some vector space. Our discussion is based on the complex general Lie group $GL(V)$, the group is bijective linear operators on a complex n-dimensional vector space V. Denoting the group element by capitals A and B,etc. We define the matrix representation of these operators by taking the basis in the vector space V and by considering the action of operators on the basis vectors. Let $\{e_1, e_2, \cdots, e_n\}$ be basis in a V. Then matrix (aij) representation the operator A is defined by $Ae_i = \sum_{j=1}^{n} e_j \alpha_{ji}, \ i = 1, 2, \cdots, n$. In this way we obtain the isomorphism $A \in GL(V) \mapsto (a_{ij}) \in GL(n, \mathbb{C})$ where $GL(n, \mathbb{C})$ is the group of all invertible complex $n \times n$ matrices. Our next aim is to show, $GL(V)$ and $GL(n, \mathbb{C})$ are n^2 dimensional complex Lie groups. Let $M(n, \mathbb{C})$ be the set of all complex $n \times n$ matrices then the map $k : a_{ij} \in M(n, \mathbb{C}) \mapsto (a_{11}, a_{12}, \cdots, a_{nn}) \in \mathbb{C}^{n^2}$ is a bijection. From the fact the map $det : (a_{ij}) \mapsto det(a_{ij}) \in \mathbb{C}$ is a continuous function of the matrix element it follows that $GL(n, \mathbb{C}) = \{A \in M(N, \mathbb{C}) \mid detA \neq 0\}$ is an open set in $\mathbb{C}^{n^2}$. This implies that the restriction $k|GL(n, \mathbb{C})$ maps the open set $GL(n, \mathbb{C})$ bijectively onto the open set $\mathbb{C}^{n^2} \backslash K$ where $K = \{(a_{11}, a_{12}, \cdots, a_{nn}) \in \mathbb{C}^{n^2} \mid det(a_{ij}) = 0\}$. In the general theory of Lie group of it is shown that the vector space structure of the Lie algebra of a Lie group is isomorphic with the tangent space at the unit element of the group manifold.For a linear lie group the tangent space is obtained. consider in $GL(n, \mathbb{C})$ a subset of operators $A(t)$ depending smoothly on a real parameter t and such that $A(0) = 1$, where 1 is the identity operator on V. Such a subset is called a curve through the unit element. The tangent vector at $t = 0$ is obtained by making the Taylor expansion of $A(t)$ up to the first order term $A(t) = A(o) + Mt + o(t^2)$ with M the derivative of $A(t)$ at $t = 0 : M = \dot{A}(0)$. The linear operators M obtained in this way are element of the Lie algebra of $GL(n, \mathbb{C})$. The basis $\{e_1, e_2, \cdots, e_n\}$ the operators M are represented by complex $n \times n$ matrices (m_{ij}). Considering all possible smooth curves through unit element of the group one obtains for the vector space structure of the Lie algebra the n^2 dimensional space of complex $n \times n$ matrices. This is the vector space $M(n, \mathbb{C})$. Now to obtain the Lie bracket of elements $M, N \in M(n, \mathbb{C})$. Let consider the group commutator $C(t) = A(t)B(t)A^{-1}(t)B^{-1}(t)$ of two smooth curves $A(t)$ and $B(t)$. If $\dot{A}(0) = M$ and $\dot{B}(0) = N$ then the Lie bracket of M and N is defined as the tangent vector at $t = 0$ of $\mathbb{C}(t)$. Then $\dot{C} = MN - NM$ that is the Lie bracket is the commutator $[M, N] = MN - NM$ of the matrices M and N. The general linear Lie algebra $gl(n, \mathbb{C})$ is the Lie algebra of the group $GL(n, \mathbb{C})$.

Definition 1.1.8. Lie Group and Lie Algebra with Exponential map: Let $M \in gl(n, \mathbb{C})$, then $A(t) = \exp Mt \;\; (t \in \mathbb{R})$ is a non singular linear operator that is $\exp Mt$ is a matrix in $GL(n, \mathbb{C})$. Clearly $\dot{A}(0) = M$ that is M is the tangent vector to the path $A(t)$. The map is $\exp : M \in gl(n, \mathbb{C}) \mapsto \exp M \in GL(n, \mathbb{C})$ is called the exponential map.

1.2 Types of Lie Algebra:

1.2.1 Nilpotent Lie algebra

Definition 1.2.1. We consider again the central descending series of ideals in $L \equiv L^0$. $L^0 \supset L^1 \supset \cdots \supset L^n \supset \cdots$ with $L^n = [L, L^{n-1}] \quad (n = 1, 2, \cdots)$ such that $L \neq o$ is nilpotent means that $L^r = 0$ for some r while $L^{r-1} \neq o$ $[L, L^{r-1}] = L^r = 0$. This means that L^{r-1} is a non trivial abelian ideal. The nilpotent Lie algebra is not Semisimple.

1.2.2 Solvable Lie algebra:

Definition 1.2.2. Let L be a Lie algebra. The sequence $L_0, L_1, L_2, \cdots, L_n, \cdots$ defined by $L_0 = L, \quad L_1 = [L_0, L_0], \cdots, \quad L_n = [L_{n-1}, L_{n-1}], \cdots$ is called the derived sequence. A Lie algebra L is called Solvable if $L_n = 0$ for some $n \in \mathbb{N}$.

1.2.3 Simple and Semisimple Lie algebra

Definition 1.2.3. A Lie algebra L is called simple if L is non-abelian and has no proper ideals. A Lie algebra L is called semisimple if $L \neq 0$ and L has no abelian ideals $\neq 0$.

1.2.4 structure constants

Definition 1.2.4. One extremely important theorem about the structure of Lie groups is that the commutator of any two elements of Lie algebra can be written as a linear combination of the basis of the Lie algebra. Let g be a finite-dimensional real or complex Lie algebra, and $X_1, \cdots, X_n$ be a basis for g (as a vector space). Then for each i and j, $[X_i, X_J]$ can be written uniquely in the form

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k. \tag{1.1}$$

where the constants c_{ij}^k are called the structure constants of g. Clearly the structure constants determine the bracket operation on g. The structure constants satisfy the following two conditions:

- $c_{ij}^k + c_{ji}^k = 0$,
- $\sum_m (c_{ij}^m c'_{mk} + c_{jk}^m c'_{mi} + c_{ki}^m c'_{mj}) = 0$, (which can be visualise as follow $[[X_i, X_j], X_k] + [[X_j, X_k], X_i] + [[X_k, X_i], X_j] = 0$

The first of these conditions comes from the skew symmetry of the bracket, and the seconds comes from the jacobi identity.

1.3 Classical Lie algebra

The classical Lie groups are the special linear group $SL(n, \mathbb{C})$, the orthogonal group $O(n, \mathbb{C})$ and sympletic group $Sp(n, \mathbb{C})$. General linear Lie groups have corresponding general linear lie algebra. We will define these groups and Lie algebras.

1.3.1 The special linear group $SL(n, \mathbb{C})$ and its Liealgebra

The subset of linear operator $A \in GL(n, \mathbb{C})$ with $\det A = 1$ constitutes by definition the special linear group $SL(n, \mathbb{C})$. The condition on group element leads to a condition on the tangent vectors. Let $A(t) = \exp Mt$ be a smooth path in $GL(n, \mathbb{C})$. Imposing condition $\det A(t) = \det(\exp Mt) = \exp(TrMt)$ this give the restriction
$TrM = 0$ on the Lie algebra elements. Since the trace of a commutator is identically zero
$Tr[M, N] = 0$. The Lie algebra $sl(n, \mathbb{C})$ of the Lie group $SL(n, \mathbb{C})$ is given by the sub algebra of traceless matrices in $gl(n, \mathbb{C})$:
$sl(n, \mathbb{C}) = \{M \in gl(n, \mathbb{C}) \mid TrM = 0\}$.

1.3.2 The orthogonal group $O(n, \mathbb{C})$

The complex orthogonal group $O(n, \mathbb{C})$. With this group we consider the matrix representation with respect to the basis $\{e_1, e_2 \cdots e_n\}$ which is according to orthonormal with respect to the invariant form $(.,.)$. The condition of invariance of the bilinear form under the transformations of $O(n, \mathbb{C})$ leads for the matrices to the condition $A^T A = 1$. Matrices having this property are called orthogonal matrices.
From follows $1 = \det A^T \det A = \det A^2$. Hence the determinant of orthogonal matrices equal ± 1. The subset of matrices in $O(n, \mathbb{C})$ having determinant equal to one is easily seen to be a sub group of $O(n, \mathbb{C})$. This subgroup is called the special orthogonal group $SO(n, \mathbb{C})$. The lie algebra of the matrix representation of $SO(n, \mathbb{C})$. We consider again smooth curves through the unit element of the group. The tangent vectors M are restricted by the condition
$M^T + M = 0, M^T = -M$. Hence the tangent vectors are the orthogonal basis represented by antisymmetric matrices. Since the commuter of two antisymmetric matrices is again and an antisymmetric matrices we have a characterisation of lie algebra $SO(n, \mathbb{C})$ we have $SO(n, \mathbb{C}) = \{M \in gl(n, \mathbb{C}) \mid M^T = -M\}$.
$dim(SO(n, \mathbb{C})) = \frac{n(n-1)}{2}$.

- The Lie algebra $so(2k, \mathbb{C})$: Here we obtain another orthogonal algebra. The construction is identical to that for B_l, except that $dimV = 2l$ is even and s has the simpler form $\begin{pmatrix} 0 & l_t \\ l_t & 0 \end{pmatrix}$. To construct a basis and to verify that dim $SO(2l, F) = 2l^2 - l$. Let $t(n, F)$ be the set of upper triangular matrices $(a_{ij}, a_{ij}) = 0$ if $i > j$. Let $\eta(n, F)$ be the strictly upper triangular matrices $a_{ij} = 0$ if $i \geq j$. Finally, let $\delta(n, F)$ be the set of all diagonal matrices. Also that $t(n, F) = \delta(n, F) + \eta(n, F)$ with $[\delta(n, F), \eta(n, F)] = \eta(n, F)$, hence$[t(n, F), t(n, F)] = \eta(n, F)$, denotes the subspace of L spanned by commutators $[x, y], x \in H, y \in K$.

- The Lie algebra $so(2k + 1, \mathbb{C})$:

Let $dimV = 2l + 1$ be odd, and take f to be nondegenerate symmetric bilinear form on V whose matrix is $s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix}$. The orthogonal algebra $so(2l+1, F)$, consists of all endomorphisms of V satisfying $f(x(v), w) = -f(v, x(w))$.

If we partition x in the same form as s, say $x = \begin{pmatrix} a & b_1 & b_2 \\ c_1 & m & n \\ c_2 & p & q \end{pmatrix}$, then the condition $sx = -x_t s$ translates into the following set of conditions: $a = 0, c_1 = -b_2^t, c_2 = -b_1^t, q = -m_t, n_t = -n, p_t = -p$. Add the $2l$ matrices involving only row one and column one $e_{1,l+i+1} - e_{i+1,1}$ and $e_{1,i+1} - e_{l+i+1,1} \quad (1 \leq i \leq l)$. Corresponding to n take $e_{i+1,l+j+1} - e_{l+j+1,l+i+1}(1 \leq i \neq j \leq l)$. For n take $e_{i+1,l+j+1} - e_{j+1,l+i+1}$, and for $p, e_{i+l+1,j+1} - e_{j+l+1,i+1}(1 \leq j < i \leq l)$.The number of basis elements is $2l_2 + l$.

1.3.3 The Symplectic group $Sp(k, \mathbb{C})$

$Sp(2l, \mathbb{C})$ the symplectic algebra, which consists of all endomorphisms x to V satisfying $f(x(v), w) = -f(v, x(w))$ denoted by $Sp(V)$, is close under the bracket operation. In matrix terms, the condition for $\begin{pmatrix} m & n \\ p & q \end{pmatrix}$ $(m, n, p, q) \in gl(l, f)$ to be symplectic is that $sx = -x_t s$. Take the diagonal matrices $e_{ii} - e_{l+i,l+i} \quad (1 \leq i \leq l)$,. Add to these all $e_{ij} - e_{l+j,l+i}(1 \leq i \neq j \leq l), l^2 - l$ in number. For n we use the matrices $e_{i,l+i}(1 \leq i \leq l)$ and $e_{i,l+j} + e_{j,l+i}(1 \leq i < j \leq l)$, a total of $l + l(l-1)$, and similarly for the position in p. Adding up, we find $dimSp(2l, F) = 2l^2 + l$.

1.3.4 Representation of Lie Algebra

For every $x \in L$, let us define a linear operator adx on the vector space L that is $ad_x : L \to L$, Such that $adx(y) = [x, y] \ \forall \ y \in L$.

The map $(x, y) \in L \times L \mapsto adx(y) \in L$.

1. $ad[x,y] = [adx, ady]$

2. Jacobi identity one obtain $\forall \ \ z \ \ \in L, \ \ ad[x,y](z) = [ad(x), ad(y)](z)$

Hence $ad : x \in L \mapsto adx \in gl(L)$ is a representation of L with representation space L. This representation is called the adjoint representation of L.

1.3.5 Finite-dimensional Representations of $sl(2,\mathbb{C})$

We construct the finite-dimensional representation of $sl(2,\mathbb{C})$ on a complex vector space V. To construct a representation for a basis in Lie algebra.
Basis for $sl(2,\mathbb{C})$
$\{e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}.$
Their commutation relation
$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$
Let V be a finite-dimensional $sl(2,\mathbb{C})$ module. Denoting the linear operators representing the action of e, f and h by $e\cdot, f\cdot$ and $h\cdot$, we have

- $[h,e]\cdot = h \cdot e \cdot - e \cdot h \cdot = 2e\cdot$

- $[h,f]\cdot = h \cdot f \cdot - f \cdot h \cdot = 2f\cdot$

- $[e,f]\cdot = h \cdot f \cdot - f \cdot e \cdot = h\cdot$.
  It has to be stressed that $e\cdot, f\cdot$ and $h\cdot$ are linear operators on the vector space V . Now a linear on a finite-dimensional complex vector space has at least one eigenvector. Let $v \in V$ be an eigenvector of the operator $h\cdot$ then $v \neq o$ and $h \cdot v = \lambda v \quad (\lambda \in \mathbb{C})$ we obtain $h \cdot (e \cdot v) = e \cdot h \cdot v + 2e \cdot v$ or equivalently $h \cdot (e \cdot v) = (\lambda + 2) e \cdot v$ Likewise one has $h \cdot (f \cdot v) = f \cdot h \cdot v - 2f \cdot v$ or $h \cdot (f \cdot v) = (\lambda - 2) f \cdot v$. When $e \cdot v \neq 0$, then it is an eigenvector of $h\cdot$ and its eigenvalue, compared to the eigenvalue of v, is raised by 2. Likewise if $f \cdot v \neq 0$ then it is an eigenvector of $h\cdot$ and its eigenvalue, compared to the eigenvalue of v, is lowered by 2. Hence $e\cdot$ acts as a raising operator and $f\cdot$ as a lowering operator when they act on the

eigenvectors of $h\cdot$. Repeating these actions we get a chain of independent eigenvectors of $h\cdot$. The commutation relation between $(e\cdot)^k, (f\cdot)^k$ and $h\cdot$.

1.4 Root space decomposition:

Definition 1.4.1. L be a finite dimensional complex semisimple Lie algebra. H be the maximal toral subalgebra of a L. The eigenvalues of the linear operator adh will be denoted by $\alpha(h)$ and define the subspace L_α of L by $L_\alpha = \{x \in L | \forall h \in H : adh(x) = \alpha(h)x\}$.
Then the Lie algebra L is a vector space direct sum of the subspaces L_α :

$$L = \bigoplus_\alpha L_\alpha. \tag{1.2}$$

This is called the root space decomposition of L with respect to H.

Definition 1.4.2. Root system
The subset Δ of H^* consisting of all roots of L is called the root system of L.

1.4.1 Root chain

Let α and $\beta \neq \pm\alpha$ be roots. The linear operator ade_α on e_β yields a sequence of non-zero vector $e_\beta, e_{\beta+\alpha}, \cdots, e_{\beta+q\alpha}$, after deleting the zero vectors. Likewise repeated application of the linear operator adf_α on e_β yields also a sequence of sequence of non-zero vectors $e_\beta, e_{\beta-2\alpha}, \cdots, e_{\beta-p\alpha}$. We obtain the sequence of roots
$\beta - p\alpha, \cdots, \beta - 2\alpha, \beta - \alpha, \beta + \alpha, \beta + 2\alpha, \cdots, \beta + q\alpha$.

1.4.2 Cartan matrix

Definition 1.4.3. A_{ij} for $i, j = 1 \cdots k$ of a Semisimple Lie algebra is defined by means of the dual contraction between $\Pi = \{\alpha_1, \alpha_2 \cdots, \alpha_k\}$ and

$\Pi^v = \{\alpha_1^v \cdots, \alpha_k^v\}$:

$$A_{ij} =< \alpha_j, \alpha_i^v >= \frac{2(\alpha_j|\alpha_i)}{(\alpha_i|\alpha_i)}. \tag{1.3}$$

The matrix element of the cartan matrix are the cartan integers of simple roots. Relation between the ratio of lengths of simple roots and matrix elements of the cartan matrix

$$\frac{A_{ij}}{A_{ji}} = \frac{\|\alpha_j\|^2}{\|\alpha_i\|^2}. \tag{1.4}$$

1.4.3 Dynkin diagrams

Definition 1.4.4. Let $A(L)$ be a $k \times k$ cartan matrix of a semisimple Lie algebra L. The Dynkin diagram $D(A)$ of the Lie algebra L is constructed following rules:

- Draw k vertices, one for each simple root, vertex i corresponding to the simple root α_i.
- Connect the vertices i and j with n_{ij} lines,
  where $n_{ij} = A_{ij} \times A_{ji} \quad (i, j = 1, \cdots, k)$.
- If $|A_{ij}| > 1$ draw an arrow pointing from vertex j to vertex i.

A diagram which is obtained from a cartan matrix A only by means of the rules is called a Coxeter diagram.This diagram is denoted by $C(A)$.

1.4.4 Root system of $sl(3, \mathbb{C})$:

The simple root system of $sl(3, \mathbb{C})$ is $\Pi = \{\alpha_1, \alpha_2\}$.
The Cartan matrix is $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

Here length of α_1, α_2 are 1.
Now consider α_2 string through α_1:

$$\alpha_1 - 2\frac{(\alpha_1, \alpha_2)}{(\alpha_2, \alpha_2)}\alpha_2 = (\alpha_1 + \alpha_2)$$

$(\alpha_1 + \alpha_2)$ string through α_2:

$$\alpha_1 + \alpha_2 - 2\frac{(\alpha_1 + \alpha_2, \alpha_2)}{(\alpha_2, \alpha_2)}\alpha_2 = \alpha_1 + \alpha_2 - [2\frac{(\alpha_1, \alpha_2)}{(\alpha_2, \alpha 2)} + 2\frac{(\alpha_2, \alpha_2)}{(\alpha_2, \alpha_2)}]\alpha_2$$
$$= (\alpha_1 + \alpha_2) - (-1 + 2) \times \alpha_2 = \alpha_1$$

α_1 string through α_2:

$$\alpha_2 - 2\frac{(\alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)}\alpha_1 = (\alpha_1 + \alpha_2)$$

$(\alpha_2 + \alpha_1)$ string through α_1:

$$\alpha_2 + \alpha_1 - 2\frac{(\alpha_2 + \alpha_1), \alpha_1}{\alpha_1, \alpha_1}\alpha_1 = \alpha_2$$

The length of $(\alpha_1 + \alpha_2)$ is,

$$\|\alpha_1 + \alpha_2\| = \sqrt{(\alpha_1 + \alpha_2, \alpha_1 + \alpha_2)} = \sqrt{(\alpha_1, \alpha_1) + (\alpha_2, \alpha 2) + (\alpha_1, \alpha_2) + (\alpha_2, \alpha_1)}$$
$$= \sqrt{1 + 1 - \frac{1}{2} - \frac{1}{2}} = 1$$

Now we have to calculate angle between α_1 and α_2. We have $\|\alpha_1\| = \|\alpha_2\|$.

$$2\frac{(\alpha_1, \alpha_2)}{(\alpha_2, \alpha_2)} = -1 \Rightarrow 2\frac{\|\alpha_1\|\|\alpha_2\| \cos\theta_{\alpha_1\alpha 2}}{\|\alpha_2\|\|\alpha_2\|} = -1$$
$$\Rightarrow \cos\theta_{\alpha_1\alpha_2} = -\frac{1}{2}$$
$$\Rightarrow \theta_{\alpha_1\alpha_2} = \frac{2\pi}{3}.$$

Angle between α_1 and $(\alpha_1 + \alpha_2)$:

$$(\alpha_1, \alpha_1 + \alpha_2) = \|\alpha_1\|\|\alpha_1 + \alpha_2\| \cos\theta_{\alpha_1, \alpha_1 + \alpha_2}$$
$$\Rightarrow (\alpha_1, \alpha_1) + (\alpha_1, \alpha_2) = 1 \times 1 \cos\theta_{\alpha_1, \alpha_1 + \alpha_2}$$
$$\Rightarrow \theta = \frac{\pi}{3}$$

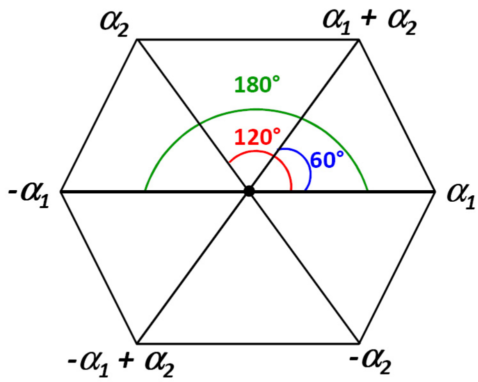

Figure 1.1: Root diagram of $sl(3, \mathbb{C})$

1.5 Real form of Lie algebra

Definition 1.5.1. In the special case that $k = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ and V is a real vector space, the complex vector space V^C is called the complexification of V. If W is complex, then $W^{\mathbb{C}}$ is W regarded as a real vector space. The operation $(.)^C$ and $(.)^R$ are not inverse to each other: $(V^{\mathbb{C}})^{\mathbb{R}}$ has twice the real dimension of V, and $(W^{\mathbb{R}})^{\mathbb{C}}$ has twice the complex dimension of W.

$$(V^{\mathbb{C}})^{\mathbb{R}} = V \bigoplus iV \tag{1.5}$$

as real vector spaces, where V means $V \bigotimes 1$ in $V \bigotimes_{\mathbb{R}} \mathbb{C}$ and the i refers to the real linear transformation of multiplication by i.

$$V^{\mathbb{C}} = V \bigotimes iV. \tag{1.6}$$

When a complex vector space W and a real vector space V are related by

$$W^{\mathbb{R}} = V \bigoplus iV, \tag{1.7}$$

we say that V is a real form of the complex vector space W. In the (R) linear map that is 1 on V and -1 on iV is called the conjugation of the complex vector space $V^{\mathbb{C}}$ with respect to the real form V. Suppose that g_0 is a Lie algebra over K vector space $g = (g_0)^k$, we introduce the 4-linear map $g_0 \times K \times K \longrightarrow g_0 \bigotimes_k K$ given by, $(X, a, Y, b) \mapsto [X, Y] \bigotimes ab \in g_0 \bigotimes_k K$. This 4-linear map extends to a K linear map on $g_0 \bigotimes_k K \bigotimes_k g_0 \bigotimes_k K$ that we can restrict to a k bilinear map $(g_0 \bigotimes_k K) \times (g_0 \bigotimes_k K) \longrightarrow g_0 \bigotimes_k K$. The result is the bracket product on $g = (g_0)_k = g_0 \bigotimes_k K$. K is bilinear and extend the bracket product in g_0. Using bases, it has the property $[X, X] = 0$ and satisfy the jacobi identity. Hence g is a Lie algebra over K. Now consider g_0 is a real Lie algebra, the complex Lie algebra $(g_0)^{(C)}$ is called the complexification of g_0. If we have $g^{(R)} = g_0 \bigoplus ig_0$, g_0 is a real form of the complex Lie algebra g.

1.5.1 classification of real Lie algebra

Definition 1.5.2. Split real form of real Lie algebra: A real form of g that contains h_0 for some Cartan subalgebra h is called a split real form of g.

$$h_0 = \{H \in h \mid \alpha(H) \in \mathbb{R} \, \forall \, \alpha \in \Delta\}, \tag{1.8}$$

$$g_0 = h_0 \oplus \bigoplus_{\alpha \in \Delta} \mathbb{R} X_\alpha \tag{1.9}$$

Result: Any complex semisimple Lie algebra contains a split real form.

Definition 1.5.3. Compact real form of real Lie algebra: A real form of the complex semisimple Lie algebra g that is a compact Lie algebra is called a compact real form of g.

Result: If g is a complex semisimple Lie algebra, then g has a compact real form u_0.

$$u_0 = \sum_{\alpha \in \Delta} \mathbb{R}(iH_\alpha) + \sum_{\alpha \in \Delta} \mathbb{R}(X_\alpha - X_{-\alpha}) + \sum_{\alpha \in \Delta} \mathbb{R}(X_\alpha + X_{-\alpha}) \tag{1.10}$$

1.5.2 Compact form of A_2

The chevelley generators are

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, e_2 =$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Compact form is generated by $\{e_1 - f_1, i(e_1 + f_1), (e_2 - f_2), i(e_2 + f_2), ih_1, ih_2\}$

$$a_1\ (e_1 + f_1) + a_2\ (e_2 + f_2) + a_3\ i(e_1 - f_1) + a_4\ i(e_2 - f_2) + a_5\ (ih_1) + a_6\ (ih_2)$$

$$= a_1 \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_4 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} +$$

$$a_5 \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_6 \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} = \begin{pmatrix} ia_5 & a_1 + ia_3 & 0 \\ -a_1 + ia_3 & -ia_5 + ia_6 & a_2 + ia_4 \\ 0 & -a_2 + ia_4 & -ia_6 \end{pmatrix}$$

$$= \begin{pmatrix} -ia_5 & a_1 + ia_3 & 0 \\ a_1 - ia_3 & ia_5 - ia_6 & a_2 + ia_4 \\ 0 & a_2 - ia_4 & ia_6 \end{pmatrix}$$

$= A \in\ SU(3)$, since $A^* = -A$

Definition 1.5.4. $SU(n) = \{A \mid A^* = -A \text{ and } TrA = 0\}$

1.6 Symmetry

In this section we are going to discuss about symmetry and will learn about two major types of symmetry namely Discrete and Continuous symmetry and we will see, how continuous symmetry and we will see, how continuous symmetry give rise to the Lie groups.

1.6.1 What is Symmetry ?

- For mathematician, symmetries are special invertible functions. The concept comes from the idea of symmetry of a geometric object, for example a polygon made out of paper lying on a flat surface. In the informal language if we say then symmetry of a polygon is a motion that corresponds to picking up the paper polygon and replacing it so that it appear unmoved. Then we say the motion preserves the polygon.

- To be specific, consider symmetries of a square. Imagine a paper square placed with its centre at the origin of the the plane $\mathbb{R}^2$ and with its side parallel to the co-ordinate axes.

- If the square is rotated counter clockwise around its origin through an angle of $\frac{\Pi}{2}$ is symmetry of the square. We say this symmetry of a square because, if we rotate by angle $\frac{\Pi}{2}$, we will see that it is unmoved, that is no change in the orientation of the square. Similarly there exit other three distinct rotation symmetry of the square which are the rotation by angle $0, \Pi$, and $\frac{3\Pi}{2}$ counter clockwise. Other symmetry of the square is the integral multiple of the $\frac{\Pi}{2}$ but they are repetition of the square about the X-axis we will see that there is no change in the orientation of the square and we conclude that it is also the symmetry of the square. But still there exist three more distinct symmetry of the square of the square by reflection through Y-axis and two of its diagonal. It turns out that these four reflection together with the four rotation described earlier constitute of the symmetries of the paper square.

CHAPTER 2

Application to particle Physics

2.1 Representation of Lie algebras, with application to particle physics

2.1.1 The symmetry of various intraction

According to modern views, there are four main type of forces in nature

- strong(nuclear)
- electromagnetic
- Weak
- gravitational

Definition 2.1.1. Nuclear forces strongly bind the neutrons and the protons in atomic nuclei. They are responsible for a wide variety of nuclear reactions.Those reaction that release energy in the core of a nuclear reactor at an atomic power station. Hadrons are responsible for strong interactions, while leptons do not participate in them.

Definition 2.1.2. Electromagnetic interactions, when studying electric and magnetic phenomena and properties of matter and electromagnetic radiation. Electromagnetic interaction determine the structure and properties of atoms and molecules.This interaction contain Coulomb forces, the forces acting on a current-bearing conductor, the forces of friction of all the elementary particles, except for both neutrino and antineutrino participate in electromagnetic interaction.

Definition 2.1.3. Weak interaction are predominant in the realm of subatomic particle. They are responsible for the interactions of particles involving neutrino and antineutrino.In this process include the decays of kaons and hyperons.They are involved in neutrinoless decays that are characterized by a relatively long time of the decaying particles-about 10^{-10} or more.

Definition 2.1.4. Gravitational interactions are inherent in all particles, without exception, but they are of no significance for elementary particles. These interactions only manifest themselves on a sufficiently large scale when the masses involved are rather large.

The strong interaction is about 100 times higher than the electromagnetic one and 10^{14} higher than the weak one. The stronger the interaction, the faster it carries out its task. So the particles called resonances, whose decay occurs through nuclear interactions, have a lifetime of about 10^{-23}s; The neutral pions, which decay through an electromagnetic interaction ($\Pi^0 \to \gamma + \gamma$), have a life time of 10^{-16}s; the decays through a weak interaction have a lifetime of $10^{-8} - 10^{-10}$s.The strong interaction produces fast processes,the weak interaction slow processes. The duration of a process is defined as a quantity that is reciprocal of the probability of the process per unit time.The smaller the probability, slower the process. The electromagnetic interaction, strong and weak interactions manifest themselves over extremely short distances, have a small. The strong interaction between two baryons shows when the particles approach each other and come within a distance of only 10^{-15}m. The range of the weak interaction is shorter, it is known to be within 10^{-19}m. The types of interaction is associated with symmetry. All the interactions of particles are controlled by the absolute conservation laws. So the laws of conservation of spatial and charge parity hold for both the electromagnetic and strong interaction, but they do not hold for the weak interaction. The rule is "The stronger an interaction the more symmetrical it is". The weaker an interaction it is controlled by conservayion laws,"weaker interactions turn in to infringer of the law, and the weaker an interaction the more lawlessness".

2.1.2 Isotopic invariance of strong interaction(isospin)

Suppose that all the protons in the atomic nucleus are replaced by neutrons, and all the neutrons by protons. The resultant nucleus is called the mirror nucleus of the initial nucleus. The mirror nuclei reflects a measure of symmetry of nuclear forces. This symmetry is a special case of the so called isotopic invariance.The nuclear force are independent of the electric charge of particles. Associated with the isotopic invariance of the strong interaction is the concept of isotopic spin. The connection of the proton and the neutron can be viewed as two charge conditions of one particle-the nucleon. It is said that the proton and the neutron form an isotopic doublet. Isotopic doublets are also formed by two xi-hyperons (Ξ^{-1}, Ξ^0) and two kaons (K^0, K^+). It appeared that pions are better combined into a triplet, by adding to η^0, to each of them an isotopic singlet. Isotopic multiplets of known elementary particles come in three types-triplets, doublets and singlets. This multiplets is formed by Δ particles which belongs to short-lived baryons, called the resonances. The magnitude of isospin I of a particle is related to the number of charge states n in the multiplet by the relationship $n = 2 \times I + 1$. The electron's spin is $s =$, its projection in a given direction in conventional space takes on the values $s_z = +$ and $s_z = -$. The isospin of the nucleon $I =$, its projection in some direction in the isospin space assumes the values $I_\zeta =$ and $I_\zeta = -$. When the dealing with the strong interaction of particles,the isospin vectors of particles must be combined combined by the same rules as the spin vectors. The isospin projection of several particles is the algebraic sum of the isospin projections for individual particles. The isotopic invariance of the strong interaction lies at the foundation of the physics of isospin formalism. This invariance means that the laws of nature are invariant under rotations in isospin space.This find its expression in the law of conservation isospin. The conservation of isospin using two processes: $p+p \to \Pi^+ + D$ and $n + p \to \Pi^0 + D$, where D is the deuteron of heavy hydrogen, the deuteron's isospin is zero; therefore, the products of reactions in the general case have the the total isospin that is equal to the isospin of pions, that is unity.

2.1.3 Strangeness conservation in strong and Electromagnetic interaction

Particles came in pair-a kaon paired with a hyperon. $\Pi^- + p \to K^0 + \Lambda^0, \Pi^- + P \to K^+ + \Sigma^-, \Pi^+ + p \to K^+ + \Sigma^+$. Second the life time of new particles produced by without leptons ($K+ \to \Pi^+ + \Pi^0, \Lambda^0 \to p + \Pi^-, \Lambda^0 \to n + \Pi^0, \Sigma^+ \to p + \Pi^0, \Sigma^+ \to n + \Pi^+, \Sigma^- \to n + \Pi^-$). The fact that the decay schemes included no leptons suggested that these decays are associated with the strong interaction in which case the lifetime of the particles must be about $10^{-22} - 10^{-23}$. The long life time of kaons and hyperons is associated with the conservation of some hitherto-unknown physical quantity. A new conservation law was established that is valid for strong and electromagnetic interactions:the total strangeness of the mesons and the baryons involved in the process is conserved. The long lifetime of kaons is accounted for by the fact that the kaon is the lightest particle with nonzero strangeness. It cannot decay due to the strong interaction, or due to the electromagnetic interaction since there is no particle to which it could transfer its strangeness. The kaon has one possibility to decay by weak interaction, since in such interactions strangeness is not conserved. The long lifetime of the lambda hyperon stems from the fact that this hyperon is the lightest baryon with nonzero strangeness. The decay of the lambda hyperons into kaons is absolutely prohibited by the law of conservation of baryon number, and the decay into nucleons is prohibited by strangeness conservation. The charged sigma hyperons Σ^- and Σ^+ can only decay through the weak interaction. Since the mass difference of a sigma and a lambda hyperon is smaller than the pion mass.In this case of the neutral sigma hyperon, a decay is possible that conserves strangeness. $\Sigma^0 \to \Lambda^0 + \gamma$. Therefore, the life time of a Σ^0-hyperon is shorter than 10^{-14}s.

2.1.4 Interaction and conservation

The highest symmetry is inherent in processes occurring due to the strong interaction.For them we have ten conservation laws: energy, momentum, angular momentum electric charge, baryon number, space, charge and time parity, strangeness, isospin. Turning to electromagnetic interactions, symmetry becomes lower-isospin conservation is no longer valid. Yet more marked reduction is observed when we go over the weak interaction.

2.1.5 A curious Formula

Gell-Mann and Nishijima turned their attention to a rather curious fact. It turns out that the electric charge Q of a particles , the isospin projection I_ζ, the baryon number B and the strangeness S are related by the following simple relationship: $Q = I_\zeta + \frac{B+S}{2}$.

2.1.6 The Unitary symmetry of strong Interaction

Consider a system of coordinates in which the abscissa axis is the projection of isospin I_ζ, and the ordinate system is $Y = B + S$, a quantity called hypercharge. On this plane we will position all the baryons with $s =\ \ : p, n, \Lambda^0, \Sigma^-, \Sigma^0, \Sigma+, \Xi^-, \Xi^0$. The eight baryons with spin form a hexagon in the plane I_ζ, Y. At each vertex of the hexagon there lies one baryon, in the center two baryon. The arrangement of the baryons in the plane allows the Q axis to be introduced. Where all the eight baryons with spin appear to be combined within a geometrically symmetric closed figure. This is an example of some concealed symmetry in nature. If we place other strong interacting particles on the plane I_ζ, Y by combining them into group with the same spin s. It appears the eight particles with $s = 0$, which include all the mesons and antimeasons form exactly the same hexagon as the eight baryons. Among these particles, which refer to baryons, we know nine particles with $s = 3/2 : \Delta^-, \Delta^0, \Delta^+, \Delta{++}, Y_1^{*0}, Y_1^{*+}, \Xi^{*0}, \Xi^{*-}$ in the plane I_ζ, Y they form the rectangle. However one place is vacant-the vertex A. It is clear that the missing particles must be included in the isotopic singlet and have negative charge and strangeness $S = -3$. The missing particle actually found. So the hyperon Ω^- was added to the list of elementary particles. The eight baryons, the eight mesons, the ten baryons are called supermultiplets. Each supermultiplets have several isotopic multiplets and different values of strangeness. The symmetry that manifests its self through a union of mesons and baryons in to several supermultiplets is the so called unitary symmetry. The unitary symmetry lies beyond the internal relationship

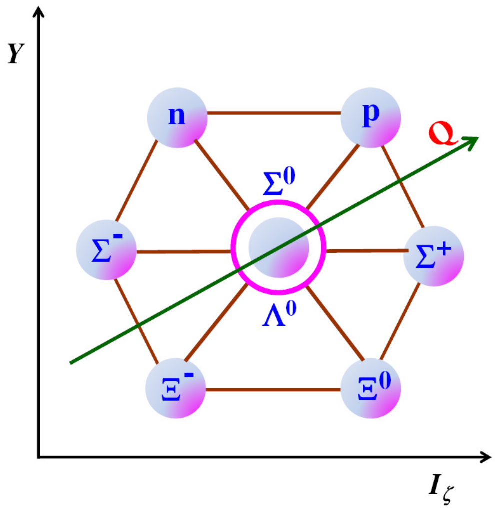

Figure 2.1: The Unitary Symmetry of Strong Interactions

between the particles belonging to various isotopic multiplets and having different strangeness. The set of mesons and baryons can be compressed in to small number of eight fold and ten-fold supermultiplets suggests that in the world of strongly interacting particles there exit a general order.

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