B-Spline Wavelet for the Solution of Integral Equation

A THESIS

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of

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DEPARTMENT OF MATHEMATICS

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DECLARATION

I declare that the topic "**B-Spline Wavelet for the Solution of Integral Equation** " for completion for my master degree has not been submitted in any other institution or university for the award of any other degree or diploma.

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THESIS CERTIFICATE

This is to certify that the project report entitled **"B-Spline Wavelet for the Solution of Integral Equation"** submitted by **Alok Kumar Ranjan** to the National Institute of Technology Rourkela, Orissa for the partial fulfilment of requirements for the degree of master of science in Mathematics is a bonafide record of review work carried out by his under my supervision and guidance. The contents of this project, in full or in parts, have not been submitted to any other institute or university for the award of any degree or diploma.

May, 2014

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(Alok Kumar Ranjan)

ABSTRACT

In this thesis we deal with some problem related to Numerical analysis by uasing B-spline Wavelet Approximation. And we also study about “Maltiresolution Analysis” which is to discribe mathematically the process of studying singnal or images at different scale.

Compactly supported linear semiorthogonal B-spline wavelet together with their dual wavelet are devolepd to approximate the solution of Fredholm integral equation of the second type

An ordinary differential equation with a parameter in the boundary condition discribe the steady in an adiabatic tubular chemical reactor. We will solve this equation uasing B-spline wavelet mathod.

Contents

CHAPTER 1

Introduction

"Maltiresolution ananlysis provides a natural framwork for the undersatnding of wavelet bases, and for the cunstruction of new examples. The history of formulation of beautiful example of application stimulating theoretical development." Mathematically, the fandamental idea of multiresolution analysis is to represent a function (or singnal) f as a limit of successive approximation, each of which is a finer version of the function f. These successive approximation correspond to diffrent level of resolution. Thus, multiresolution analysis is a formal approch to contructing a orthogonal wavelet bases using a definite set of rule and procedure. The key feature of this analysis is to discribe mathematically the process of studying singnal or images at different scale. The basic princeple of MRA deal with decomposition of the whole function space in to individual subspace $V_n \subset V_{n+1}$ so that the space V_{n+1} consist of all rescaled function in V_n. This essentially means a decomposition of each function (or singnal) in to component of defferent scale (or frequency) so that an individual component of the orginal function f accure in each subspace these component can discribe finer and finer version of orginal function.

In genral, frames have many of the properties of bases, but these lack of very impotant property of orthogonality. If the condition of orthogoanlity

$$(\phi_{k,l}, \phi_{m,n}) = 0 \quad \text{for all} \quad (k, l) \neq (m, n) \tag{1.1}$$

is satisfied, the reconstruction of the function f $(f, \phi_{m,n})$ is much simpler and, for any $f \in L^2(\mathbb{R})$, we have the following represantation[1]

$$f = \sum_{m,n=-\infty}^{\infty} (f, \phi_{m,n})\phi_{m,n}, \tag{1.2}$$

where

$\phi_{m,n}(x) = 2^{-m/2}\phi(2^{-m}x - n).$

is an orthonromal basis of V_m

CHAPTER 2

Multiresolution Analysis

A multiresolution analysis (MRA)[1-3] consist of sequence $\{V_m | m \in \mathbb{Z}\}$ of embedded subspace of $L^2(\mathbb{R})$ that satisfied the following condition:

(i) $..., \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset, ..., \subset V_m \subset V_{m+1}, \cdots$

(ii) $\bigcup_{m=-\infty}^{\infty} V_m \mathbb{R}$ is dense in $L^2(\mathbb{R})$, that is $\overline{\bigcup_{m=-\infty}^{\infty} V_m} = L^2(\mathbb{R})$

(iii)$\bigcap_{m=-\infty}^{\infty} V_m = 0$,

(iv)$f(x) \in V_m$ iff $f(2x) \in V_{m+1}$ for all $m \in \mathbb{Z}$,

(v) There exist a function $\phi \in V_0$. such that $\{\phi_{0,n} = \phi(x-n)\}$, $n \in \mathbb{Z}\}$ is an orthonormal basis for V_0, that is $||f||^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |(f, \phi_{0,n})|^2$ for all $f \in V_0$.

The function ϕ is called the scaling function or father wavelet. If $\{V_m\}$ is multiresolution of $L^2(\mathbb{R})$ and of V_0 is the closed subspace genrated by the integer translates of the singnal function ϕ, then we say that ϕ generates the multiresolution.

Sometimes condition (v) is realesd by assumed that $\phi(x-n)$, $\{n \in \mathbb{Z}\}$ is a Riesz basis for V_0, that is for every $f \in V_0$, there exist a unique sequence $\{c_n\}_{n=-\infty}^{\infty} \in l^2\mathbb{Z}$ such that

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \phi(x-n) \tag{2.1}$$

with convergence in $L^2(\mathbb{R})$ and the exist two positive constants A and B independent of $f \in V_0$ such that

$$A \sum_{n=-\infty}^{\infty} |c_n|^2 \leq ||f|| \leq B \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{2.2}$$

where $0 < A < B < \infty$. In this case, we have a multiresolution analysis with a Riesz basis.

Note that condition (v) implies that $\phi(x-n)$, $n \in \mathbb{Z}\}$ is a Riesz basis for V_0 with $A = 1 = B$.

Since $\phi_{0,n}(x) \in V_0$ for all $n \in \mathbb{Z}$. Further, if $n \in \mathbb{Z}$, it follows from (iv) that

$$\phi_{m,n}(x) = 2^{m/2}\phi(2^m x - n), \quad m \in \mathbb{Z} \tag{2.3}$$

is an orthonormal basis for V_m

1. Consequences of Definition

A related application of condition (v) implies that $f \in V_m$ if and only if $f(2^k x) \in V_{m+k}$ for all $m, k \in \mathbb{Z}$. In other words, $f \in V_m$ if and only if $f(2^{-m}x)) \in V_0$ for all $m \in \mathbb{Z}$.

This shows that function in V_m are obtained from those in V_0 through a scaling 2^{-m}. If the scale $m = 0$ is associated with V_0, then the scale 2^{-m} is associated with V_m. Thus, subspace V_m are just called version of the central space V_0, which is invariant under translation by integer. That is

$$T_n V_0 = V_0 \text{ for all } n \in \mathbb{Z}.$$

It follows from difinition that a multiresolution analysis is completly determined by the scaling function ϕ but not conversly. For given $\phi \in V_0$. we define

$$V_0 = \{f(x) = \sum_{n=-\infty}^{\infty} c_n\phi_{0,n} = \sum_{n=-\infty}^{\infty} c_n\phi(x-n) : \{c_n\} \in l^2(\mathbb{Z})\} \tag{2.4}$$

Condition (iv) implies that V_0 has an orthonormal basis $\{\phi_{0,n}\} = \{\phi(x-n)\}$. Then V_0 cansist of all function $f(x) = \sum_{n=-\infty}^{\infty} c_n\phi(x-n)$ with finite energy.

$$||f||^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$$

Similarly, the space V_m has The orthogonal basis $\phi_{m,n}$ given by

$$\phi_{m,n}(x) = 2^{m/2}\phi(2^m x - n), m \in \mathbb{Z} \tag{2.5}$$

so that $f_m(x)$ is given by

$$f_m(x) = \sum_{n=-\infty}^{\infty} c_{mn}\phi_{m,n}(x)$$

with the finite energy.

$$||f||^2 = \sum_{n=-\infty}^{\infty} |c_{mn}|^2 < \infty$$

Thus f_m represent a typical functionin the space V_m. It builds in self invariance and scale invariance through the basis $\{\phi_{m,n}\}$.

Condition (ii) and (iii) can be expressed as an in term of orthogonal projection P_m on to V_m taht is, for all $f \in L^2(\mathbb{R})$.

$$lim_{m\to-\infty}P_mf = 0 \quad \text{and} \quad lim_{m\to+\infty}P_mf = f. \tag{2.6}$$

The projection P_mf can be considerd as an approximation of given function f at the scale 2^{-m}. Therefore, the successive approximation of a given function f are defined as orthogonal projection P_m on to V_m.

$$P_mf = \sum_{n=-\infty}^{\infty} (f, \phi_{m,n})\phi_{m,n}. \tag{2.7}$$

where $\phi_{m,n}(x) = 2^{m/2}\phi(2^m x - n) \;\; m \in \mathbb{Z}$ is an orthonormal basis for V_m.

Since $V_0 \subset V_1$, the scaling function ϕ that lead to basis for V_0 is also V_1. Since $\phi \in V_1$ and $\phi_{1,n}(x) = \sqrt{2}\phi(2x - n)$ is an orthonrmal basis for V_1, ϕ can be expressed in the form

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n\phi_{1,n}(x) = \sqrt{2}\sum_{n=-\infty}^{\infty} c_n\phi(2x - n), \tag{2.8}$$

where $c_n = (\phi, \phi_{1,n})$ and $\sum_{n=-\infty}^{\infty} |c_n|^2 = 1$.

The above equation is called the dilation equation. It involve both x and $2x$ and is often refferd to as the two scale equation or refinment equation because it displays $\phi(x)$ in the refinment space V_1. That the space V_1 has the finer scale 2^{-1} and it contains $\phi(x)$ which has scale 1.

All the preceding fact reveal that multiresolution analysis can be discribed at least three ways so that we can spacify.

a) The subspace V_m.

b) The scaling function $\phi(x)$.

c) The coeficient c_n in the dilation equation.

The real importance of a multiresolution analysis lie in the simple fact that it enable us to construct orthonormal basis for $L^2(\mathbb{R})$. In order to prove this statement, we first assumed that $\{V_m\}$ is MRA.

Since $V_m \subset V_{m+1}$, we define W_m as the orthogonal component of V_m in V_{m+1} for every $m \in \mathbb{Z}$ so that we have,

$$\begin{aligned} V_{m+1} &= V_m \oplus W_m \\ &= (V_{m-1} \oplus W_{m-1}) \oplus W_m \\ &= ... \\ &= V_0 \oplus W_0 \oplus W_1 \oplus ... \oplus W_m \\ &= V_0 \oplus (\oplus_{m=0}^{m} W_m) \end{aligned}$$

and $V_n \perp W_m$ for $m \neq n$

Since $\bigcup_{m=-\infty}^{\infty} V_m$ is dence in $L^2(\mathbb{R})$, we may take limit as $m \to \infty$ to obatin

$$V_0 \oplus (\oplus_{m=0}^{\infty} W_m) = L^2(\mathbb{R}), \tag{2.9}$$

Similarly, we may go in the other direction to write

$$\begin{aligned} V_0 &= V_{-1} \oplus W_{-1} \\ &= (V_{-2} \oplus W_{-2}) \oplus W_{-1} \\ &= ... \\ &= V_{-m} \oplus W_{-m} \oplus W_{-(m-1)} \oplus ... \oplus W_{-1} \end{aligned}$$

we may take again limit as $m \to \infty$ since $\bigcap_{m\in\mathbb{Z}} W_m = \{0\}$ It follows that $V_{-m} = \{0\}$ consequently, it turn out that

$$\oplus_{m=0}^{\infty} W_m = L^2(\mathbb{R}) \tag{2.10}$$

Finally, the difference between the two successive apprpximation $P_m f$ and $P_{m+1} f$ is given by orthogonal projection $Q_m f$ of f on to the orthogonal complement W_m of V_m in V_{m+1} so that

$$Q_m f = P_{m+1} f - P_m f \tag{2.11}$$

It follows from condition (i) to (v) that the space W_m are also scaled version of W_0 and, for $f \in L^2\mathbb{R}$

$$f \in W_m \text{ iff } f(2^{-m}x) \in W_0 \text{ for all } m \in \mathbb{Z} \tag{2.12}$$

and the are translate invariant for discreat translation $n \in \mathbb{Z}$, that is

$$f \in W_0 \quad \text{iff} \quad f(x-n) \in W_0$$

and they are mutually orthogonal space gerating all of $L^2(\mathbb{R})$

$$W_m \perp W_k \quad for \quad m \neq k$$

$$\oplus_{m \in \mathbb{Z}} W_m = L^2(\mathbb{R})$$

Moreover, there exist a function $\Psi \in W_0$ such that $\Psi_{0,n}(x) = \Psi(x-n)$ constitutes an orthonormal basis for W_0. It follows from the equation (14) that

$$\Psi_{m,n}(x) = 2^{m/2}\Psi(2^m x - n) \quad \text{for} \quad n \in \mathbb{Z} \tag{2.13}$$

2. L^2 space

If $p \geq 1$ is any real number, vector space of all complex valued lebesgue integrable function f is defined on $\mathbb{R}$ is denoted by $L^p(\mathbb{R})$ with a norm

$$||f||_p = \left[\int_{-\infty}^{\infty} |f(x)|^p dx\right]^{1/p} \; < \; \infty$$

this inner product and norm for the space $L^2(\mathbb{R})$

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$$
$$||f||_2 = \langle f, f\rangle^{1/2},$$

where $f, g \in L^2(\mathbb{R})$. Note that any $j, k \in \mathbb{Z}$, we have

$$||f(2^j - k)||_2 = \left[\int_{-\infty}^{\infty} |f(2^j - k)|^2 dx\right]^{1/2}$$
$$= 2^{-j/2}||f||_2.$$

Hence if a function $\Psi \in l^2(\mathbb{R})$ has unit length, then all of the function $\Psi_{j,k}$ defined by

$$\Psi_{j,k}(x) = 2^{j/2}\Psi(2^j x - k), \quad j, k \in \mathbb{Z} \tag{2.14}$$

also have unit length ; that is,

$$||\Psi_{j,k}||_2 = ||\Psi||_2 = 1, \;\; j, k \in \mathbb{Z}$$

Kronceker Symbol

$$\delta_{j,k} = \begin{cases} 1, & for \;\; j = k, \\ 0, & \;\; j \neq k \end{cases}$$

defined on $\mathbb{Z} \times \mathbb{Z}$.

Definition

A function $\Psi \in L^2(\mathbb{R})$ is called orthogonal wavelet,if the family $\{\Psi_{j,k}\}$ as define above is orthonormal basis for $L^2(\mathbb{R})$ that is

$$\langle \Psi_{j,k}, \Psi_{l,m} \rangle = \delta_{j,l}.\delta_{k,m}, \quad j, k, l, m \in \mathbb{Z}$$

and every $f \in L^2(\mathbb{R})$ can be written as

$$f(x) = \sum_{j,k=-\infty}^{\infty} c_{j,k} \Psi_{j,k}(x),$$

The simplest example of an orthonormalwavelet is the haar function Ψ_H defined by

$$\Psi_H(x) = \begin{cases} 1, & for \ \ 0 \leq x < 1/2, \\ -1, & for \ \ 1/2 \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Definition

A wavelet $\psi \in L^2(\mathbb{R})$ is called a semiorthogonal wavelet if the Riesz basis $\{\Psi_{j,k}\}$ it genrates satisfies

$$\langle \Psi_{j,k}, \Psi_{l,m} \rangle = 0 \ \ j \neq l; \ \ j, k, l, m \in \mathbb{Z}$$

Obviously, every semiorthogonal wavelet genrate an orthonormal decomposition of $L^2(\mathbb{R})$ and every orthonormal wavelet is an semiorthonormal wavelet. A wavelet Ψ is called nonorthogonal wavelet if it is not an semiorthogonal wavelet.

CHAPTER 3

B-spline Scaling and wavelet functions

When semiorthogonal wavelet are constructed from B-spline[3] of order m, the lowest octave level $j = j_0$ is determined by setting by $2^{j_0} \geq 2m - 1$

$$\begin{aligned}
B_n(x) &= \int_{-\infty}^{\infty} B_{n-1}(x-t)B_1(t)dt \\
&= \int_0^1 B_{n-1}(x-t)dt \\
&= \int_{x-1}^{x} B_{n-1}(t)dt \\
B_n(x) &= B_1(x) * B_1(x) * ... * B_1(x) * B_1(x) \\
&= B_1(x) * B_{n-1}(x) \\
B_1(x) &= \chi_{[0,1]}(x)
\end{aligned}$$

For the calculatiuon of $B_2(x)$

$$\begin{aligned}
B_2(x) &= \int_{x-1}^{x} B_1(t)dt \quad = \int_{x-1}^{x} \chi_{[0,1]}(t)dt \\
B_2(x) &= 0 \quad \text{for} \quad x \leq 0 \\
B_2(x) &= \int_0^x dt = x \quad 0 \leq x \leq 1, \quad (x-1 \leq 0) \\
B_2(x) &= \int_{x-1}^{1} dt = 2 - x \quad 1 \leq x \leq 2, \quad (x-1 \leq 1 \leq x) \\
B_2(x) &= 0 \quad \text{for} \quad 2 \leq x
\end{aligned}$$

$B_2 = x\chi_{[0,1]}(x) + (2-x)\chi_{[0,1]}(x)$

$$B_2(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Similarly; we can find

$$\begin{aligned}
B_3(x) &= \int_{x-1}^{x} B_2(t)dt \\
B_3(x) &= 0 \quad \text{for} \quad x \le 0 \\
B_3(x) &= \int_0^x tdt = \frac{x^2}{2} \quad 0 \le x \le 1,
\end{aligned}$$

$$\begin{aligned}
B_3(x) &= \int_{x-1}^{1} tdt + \int_1^x (2-t) = \tfrac{1}{2}(6x - 2x^2 - 3) \quad 1 \le x \le 2, \\
B_3(x) &= \int_{x-1}^{2} (2-t)dt = \tfrac{1}{2}(x-3)^2 \qquad 2 \le x \le 3 \\
B_3(x) &= 0 \quad \text{for} \quad 3 \le x
\end{aligned}$$

Thus we have

$$B_3(x) = \begin{cases} \frac{1}{2}x^2, & 0 \le x \le 1 \\ \frac{3}{4} - (x - \frac{3}{2})^2, & 1 \le x \le 2 \\ \frac{1}{2}(x-3)^2, & 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Now calculation of $B_4(x)$

$$\begin{aligned}
B_4(x) &= \int_{x-1}^{x} B_3(t)dt \\
B_4(x) &= 0 \quad \text{for} \quad x \le 0 \\
B_4(x) &= \int_0^x \tfrac{1}{2}t^2 dt = \frac{x^3}{6} \quad 0 \le x \le 1, \\
B_4(x) &= \int_{x-1}^{1} \tfrac{1}{2}t^2 dt + \int_1^x (-\tfrac{3}{2} + 3t - t^2) = (\tfrac{2}{3} - 2x - \tfrac{1}{3}x^3) \quad 1 \le x \le 2, \\
B_4(x) &= \int_{x-1}^{2} (-\tfrac{3}{2} + 3t - t^2)dt + \tfrac{1}{2}\int_2^x (3-t)^2 dt = \tfrac{1}{2}(x^3 - 2x^2 + 20x - 13) \qquad 2 \le x \le 3 \\
B_4(x) &= 0 \quad \text{for} \quad 3 \le x
\end{aligned}$$

Hence value of $B_4(x)$ is

$$B_4(x) = \begin{cases} \frac{1}{6}x^3, & 0 \le x \le 1 \\ \frac{2}{3} - 2x - \frac{1}{3}x^3), & 1 \le x \le 2 \\ \frac{1}{2}(x^3 - 2x^2 + 20x - 13), & 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Now the 2^{nd} order B-spline/ scaling function[4,5] are

$$B_2 = \Phi_j$$

or

$$\Phi_{j,k}(x) = \begin{cases} x_j - k, & 0 \le x_j - k \le 1 \\ 2 - x_j - k, & 1 \le x_j - k \le 2 \qquad k = 0, 1, 2, ...2^j - 2 \\ 0, & \text{otherwise} \end{cases}$$

We can write it as

$$\Phi_{j,k}(x) = \begin{cases} x_j - k, & k \le x_j \le 1 + k \\ 2 - x_j - k, & 1 + k \le x_j \le 2 + k \qquad k = 0, 1, 2, ...2^j - 2 \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

with respect to left and right-hand side boundary scaling function

$$\Phi_{j,k}(x) = \begin{cases} 2 - x_j - k, & 0 \le x_j \le -1 \qquad k = -1 \\ 0, & \text{otherwise} \end{cases}$$

$$\Phi_{j,k}(x) = \begin{cases} x_j - k, & k \le x_j \le k + 1, \quad k = 2_j - 1 \\ 0, & \text{otherwise} \end{cases}$$

The actual co-ordinate position x is related to x_j according to $x_j = 2^j x$, the second order B-spline wavelet are given by.

1. second order B-spline scaling function

$$\begin{aligned} \Psi_2(x) &= \textstyle\sum_{k=0}^{4} q_k \Phi_2(x_j - k) \\ q_k &= (-1)^k 2^{-1} \textstyle\sum_{l=0}^{2} \binom{3}{l} N_4(k+1) + 2N_4(k) + N_4(k-1) \\ N_4 &= \tfrac{k}{m-1} N_{m-1}(k) + \tfrac{m-k}{m-1} N_{m-1}(k-1) \end{aligned}$$

The matrix of $N_m(k)$ is given bilow

$$\left(\begin{array}{c|cccccc} N_m(k) & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 2 & 0 & 1 & 0 & ... & & \\ 3 & 0 & 1/2 & 1/2 & 0 & ... & \\ 4 & 0 & 1/6 & 2/3 & 1/6 & 0 & ... \\ 5 & 0 & 1/24 & 11/24 & 11/24 & 1/24 & 0 \\ 6 & 0 & 1/120 & 26/120 & 66/120 & 26/120 & 1/120 \end{array}\right)$$

$$\begin{aligned} q_0 &= (\tfrac{1}{2})\{N_4(1) + 2N_4(0) + N_4(-1)\} \\ &= (\tfrac{1}{2})\{1/6 + 0 + 0\} \\ &= \tfrac{1}{12} \\ q_1 &= (-\tfrac{1}{2})\{N_4(2) + 2N_4(1) + N_4(0)\} \\ &= (\tfrac{1}{2})\{2/3 + 2/6 + 0\} \\ &= -\tfrac{1}{2} \\ q_2 &= (\tfrac{1}{2})\{N_4(3) + 2N_4(2) + N_4(1)\} \\ &= (\tfrac{1}{2})\{1/6 + 4/3 + 1/6\} \\ &= \tfrac{5}{6} \\ q_3 &= (-\tfrac{1}{2})\{N_4(4) + 2N_4(3) + N_4(2)\} \\ &= (-\tfrac{1}{2})\{0 + 2/6 + 3/2\} \\ &= -\tfrac{1}{2} \\ q_4 &= (\tfrac{1}{2})\{N_4(5) + 2N_4(4) + N_4(3)\} \\ &= (\tfrac{1}{2})\{0 + 0 + 1/6\} \\ &= \tfrac{1}{12} \end{aligned}$$

Thus

$$\Psi_2(x) = \frac{1}{12}\Phi(x_j) - \frac{1}{2}\Phi(x_j - 1) + \frac{5}{6}\Phi(x_j - 2) - \frac{1}{2}\Phi(x_j - 3) + \frac{1}{12}\Phi(x_j - 4)$$

$$\Rightarrow \Psi_2(x) = \frac{1}{12}\Phi(2x) - \frac{1}{2}\Phi(2x - 1) + \frac{5}{6}\Phi(2x - 2) - \frac{1}{2}\Phi(2x - 3) + \frac{1}{12}\Phi(2x - 4)$$

Now from the equation (3.1) we have.

$$\Phi(2x-k) = \begin{cases} (2x-k), & \frac{k}{2} \leq x \leq \frac{k+1}{2} \\ 2-(2x-k), & \frac{k+1}{2} \leq x \leq \frac{k+2}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\Phi(2x) = \begin{cases} (2x), & 0 \leq x \leq 1/2 \\ 2-2x, & 1/2 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\Phi(2x-1) = \begin{cases} (2x-1), & \frac{1}{2} \leq x \leq 1 \\ 2-(2x-1), & 1 \leq x \leq \frac{3}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\Phi(2x-2) = \begin{cases} (2x-2), & 1 \leq x \leq \frac{3}{2} \\ 2-(2x-2), & \frac{3}{2} \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\Phi(2x-3) = \begin{cases} (2x-3), & \frac{3}{2} \leq x \leq 2 \\ 2-(2x-3), & 2 \leq x \leq \frac{5}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\Phi(2x-4) \begin{cases} (2x-4), & 2 \leq x \leq \frac{5}{2} \\ 2-(2x-4), & \frac{5}{2} \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

$$
\begin{aligned}
\Psi_2(x) &= \tfrac{1}{12}\Psi(x), \qquad 0 \le x \le 1/2 \\
&= \tfrac{1}{12}2x, \qquad 0 \le x \le 1/2 \\
&= \tfrac{1}{6}x, \qquad 0 \le x \le 1/2 \\
\Psi_2(x) &= \tfrac{1}{12}.\Phi(2x) - \tfrac{1}{2}\Phi(2x-1), \qquad 1/2 \le x \le 1 \\
&= \tfrac{1}{12}(2-2x) - \tfrac{1}{2}(2x-1), \qquad 1/2 \le x \le 1 \\
&= \tfrac{1}{6} - \tfrac{x}{2} - x + \tfrac{1}{2}, \qquad 1/2 \le x \le 1 \\
&= \tfrac{1}{6}(4-7x) \qquad 1/2 \le x \le 1 \\
\Psi_2(x) &= -\tfrac{1}{12}\Psi(2x-1) - \tfrac{5}{6}\Phi(2x-2), \qquad 1 \le x \le 3/2 \\
&= -\tfrac{1}{2} \times (2-(2x-1)) - \tfrac{5}{7}(2x-2), \qquad 1 \le x \le 3/2 \\
&= -1 + x - \tfrac{1}{2} - \tfrac{5}{3}x + \tfrac{5}{3}, \qquad 1 \le x \le 3/2 \\
&= -\tfrac{19}{6} - \tfrac{16}{6}x, \qquad 1 \le x \le 3/2 \\
\Psi_2(x) &= -\tfrac{5}{6}\Phi(2x-2) - \tfrac{1}{2}\Phi(2x-3), \qquad 3/2 \le x \le 1 \\
&= -\tfrac{5}{6}(2-(2x-2)) - \tfrac{1}{2}(2x-3), \qquad 3/2 \le x \le 1 \\
&= \tfrac{29}{6} + \tfrac{16}{6}x, \qquad 3/2 \le x \le 1 \\
\Psi_2(x) &= -\tfrac{1}{2}\Phi(2x-3) + \tfrac{1}{2}\Phi(2x-4), \qquad 2 \le x \le 5/2 \\
&= -\tfrac{1}{2}(2-(2x-3)) + \tfrac{1}{12}(2x-4), \qquad 2 \le x \le 5/2 \\
&= -\tfrac{17}{6} + 7x, \qquad 2 \le x \le 5/2 \\
\Psi_2(x) &= \tfrac{1}{12}\Phi(2x-4), \qquad 5/2 \le x \le 3 \\
&= \tfrac{1}{12}(2-(2x-4)) \\
&= \tfrac{3}{6} - \tfrac{x}{6}
\end{aligned}
$$

The actual co-ordinate position x is related to x_j according to $x_j = 2^j x$. The second order B-spline wavelet are given by.

$$\Psi_{j,k}(x) = \frac{1}{6} \begin{cases} (x_j - k), & k \leq x_j \leq k + \frac{1}{2} \\ (4 - 7(x_j - k)), & k + \frac{1}{2} \leq x_j \leq k + 1 \\ (-19 + 16(x_j - k)), & k + 1 \leq x_j \leq k + \frac{3}{2} \\ (29 - 16(x_j - k)), & k + \frac{3}{2} \leq x_j \leq k + 2 \\ (-17 + 7(x_j - k)), & k + 2 \leq x_j \leq k\frac{5}{2} \\ (3 - (x_j - k)), & k + \frac{5}{2} \leq x_j \leq k + 3 \\ 0, & \text{otherwise} \end{cases} \tag{3.2}$$

2. Third order B-spline scaling function

The third order B-spline scaling function is given by.

$$\Phi_{j,k}(x) = \begin{cases} \frac{1}{2}(x_j - k)^2, & k \leq x_j \leq k + 1 \\ \frac{3}{4} - ((x_j - k) - \frac{3}{2})^2, & k + 1 \leq x_j \leq k + 2 \quad k = 0, 1, ..., 2^j - 3 \\ \frac{1}{2}((x_j - k) - 3)^2, & k + 1 \leq x_j \leq k + \frac{3}{2} \\ 0, & \text{otherwise} \end{cases} \tag{3.3}$$

with respect to left and right hand side boundary scaling function are give as.

Left hand side boundary scaling function

$$\Phi_{j,k}(x) = \begin{cases} \frac{1}{2}(x_j - k)^2, & 0 \leq x_j \leq 1 \quad k = -2 \\ 0, & \text{otherwise} \end{cases}$$

$$\Phi_{j,k}(x) = \begin{cases} \frac{3}{4} - ((x_j - k) - \frac{3}{2})^2, & k+1 \leq x_j \leq k+2 \\ \frac{1}{2}((x_j - k) - 3)^2, & k+1 \leq x_j \leq k+\frac{3}{2} \qquad k = -1 \\ 0, & \text{otherwise} \end{cases}$$

Right hand side boundary scaling function

$$\Phi_{j,k}(x) = \begin{cases} \frac{1}{2}(x_j - k)^2, & k \leq x_j \leq k+1 \\ \frac{3}{4} - ((x_j - k) - \frac{3}{2})^2, & k+1 \leq x_j \leq k+2 \qquad k = 2^j - 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\Phi_{j,k}(x) = \begin{cases} \frac{1}{2}((x_j - k) - 3)^2, & k+1 \leq x_j \leq k+\frac{3}{2} \qquad k = 2^j - 1 \\ 0, & \text{otherwise} \end{cases}$$

The actuall co-ordinates position x is related to x_j according to $x_j = 2^j x$ The second order B-spline wavelet given by.

$$\begin{aligned} \Psi_3(x) &= \textstyle\sum_{k=0}^{7} q_k \Phi(x_j - k) \\ q_k &= (-1)^k 2^{-2} \textstyle\sum_{l=0}^{3} \binom{3}{l} N_6(k+1-l) \\ N_m(k) &= \frac{k}{m-1} N_{m-1}(k) + \frac{m-k}{m-1} N_{m-1}(k-1) \end{aligned}$$

$$\begin{aligned} q_0 &= (\tfrac{1}{4})\{N_6(1) + 3N_6(0) + 3N_6(-1) + N_6(-2)\} \\ &= (\tfrac{1}{4})\{\tfrac{1}{120} + 0 + 0 + 0\} \\ &= \tfrac{1}{480} \\ q_1 &= (-\tfrac{1}{4})\{N_6(2) + 3N_6(1) + 3N_6(0) + N_6(-1)\} \\ &= (\tfrac{1}{4})\{\tfrac{26}{120} + \tfrac{3}{120} + 0 + 0\} \\ &= -\tfrac{29}{480} \\ q_2 &= (\tfrac{1}{4})\{N_6(3) + 3N_6(2) + 3N_6(1) + N_6(0)\} \\ &= (\tfrac{1}{4})\{\tfrac{66}{120} + \tfrac{3\times 26}{120} + \tfrac{3}{120} + 0\} \\ &= \tfrac{147}{480} \end{aligned}$$

$$\begin{aligned}
q_3 &= (-\tfrac{1}{4})\{N_6(4) + 3N_6(3) + 3N_6(2) + N_6(1)\} \\
&= (\tfrac{1}{4})\{\tfrac{26}{120} + \tfrac{3\times 66}{120} + \tfrac{3\times 26}{120} + \tfrac{1}{120}\} \\
&= \tfrac{147}{480}
\end{aligned}$$

$$\begin{aligned}
q_4 &= (\tfrac{1}{4})\{N_6(5) + 3N_6(4) + 3N_6(3) + N_6(2)\} \\
&= (\tfrac{1}{4})\{\tfrac{1}{120} + \tfrac{3\times 26}{120} + \tfrac{3\times 66}{120} + \tfrac{26}{120}\} \\
&= \tfrac{303}{480} \\
q_5 &= (-\tfrac{1}{4})\{N_6(6) + 3N_6(5) + 3N_6(4) + N_6(3)\} \\
&= (\tfrac{1}{4})\{0 + \tfrac{3\times 1}{120} + \tfrac{3\times 26}{120} + \tfrac{66}{120}\} \\
&= \tfrac{147}{480}
\end{aligned}$$

similarly we can calculate $q_6 = \frac{29}{480}$ and $q_7 = -\frac{1}{480}$

Now we have

$$\begin{aligned}
\Psi(x) = \frac{1}{480}\Phi(2x) - \frac{29}{480}\Phi(2x-1) + \frac{147}{480}\Phi(2x-2) - \frac{303}{480}\Phi(2x-3) \\
+\frac{303}{480}\Phi(2x-4) - \frac{147}{480}\Phi(2x-5) + \frac{29}{480}\Phi(2x-6) - \frac{1}{480}\Phi(2x-7)
\end{aligned}$$

Now we can compute,

$$\Phi(2x-k) = \begin{cases} \frac{1}{2}(2x-k)^2, & \frac{k}{2} \le x \le \frac{k+1}{2} \\ \frac{3}{4} - (2x-k-\frac{3}{2}), & \frac{k+1}{2} \le x \le \frac{k+2}{2} \\ \frac{1}{2}(2x-k-3)^2, & \frac{k+2}{2} \le x \le \frac{k+3}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\Phi(2x) = \begin{cases} \frac{1}{2}(2x)^2, & 0 \le x \le 1/2 \\ \frac{3}{4} - (2x-\frac{3}{2})^2, & 1/2 \le x \le 1 \\ \frac{1}{2}(2x-3)^2, & 1 \le x \le 3/2 \\ 0, & \text{otherwise} \end{cases}$$

$$\Phi(2x-1) = \begin{cases} \frac{1}{2}(2x-1)^2, & \frac{1}{2} \le x \le 1 \\ \frac{3}{4} - ((2x-1) - \frac{3}{2})^2, & 1 \le x \le \frac{3}{2} \\ \frac{1}{2}((2x-1) - 3)^2, & \frac{3}{2} \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\Phi(2x-2) = \begin{cases} \frac{1}{2}(2x-2)^2, & 1 \le x \le 3/2 \\ \frac{3}{4} - ((2x-2) - \frac{3}{2})^2, & 3/2 \le x \le 2 \\ \frac{1}{2}((2x-2) - 3)^2, & 2 \le x \le 5/2 \\ 0, & \text{otherwise} \end{cases}$$

$$\Phi(2x-3) = \begin{cases} \frac{1}{2}(2x-3)^2, & 3/2 \le x \le 2 \\ \frac{3}{4} - ((2x-3) - \frac{3}{2})^2, & 2 \le x \le 5/2 \\ \frac{1}{2}((2x-3) - 3)^2, & 5/2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

$$\Phi(2x-4) = \begin{cases} \frac{1}{2}(2x-4)^2, & 2 \le x \le 5/2 \\ \frac{3}{4} - ((2x-4) - \frac{3}{2})^2, & 5/2 \le x \le 3 \\ \frac{1}{2}((2x-4) - 3)^2, & 3 \le x \le 7/2 \\ 0, & \text{otherwise} \end{cases}$$

$$\Phi(2x-5) = \begin{cases} \frac{1}{2}(2x-5)^2, & 5/2 \le x \le 3 \\ \frac{3}{4} - ((2x-5) - \frac{3}{2})^2, & 3 \le x \le 7/2 \\ \frac{1}{2}((2x-5) - 3)^2, & 7/2 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$$

$$\Phi(2x-6) = \begin{cases} \frac{1}{2}(2x-6)^2, & 3 \leq x \leq 7/2 \\ \frac{3}{4} - ((2x-6)-\frac{3}{2})^2, & 7/2 \leq x \leq 4 \\ \frac{1}{2}((2x-6)-3)^2, & 4 \leq x \leq 9/2 \\ 0, & \text{otherwise} \end{cases}$$

$$\Phi(2x-7) = \begin{cases} \frac{1}{2}(2x-7)^2, & 7/2 \leq x \leq 4 \\ \frac{3}{4} - ((2x-7)-\frac{3}{2})^2, & 4 \leq x \leq 9/2 \\ \frac{1}{2}((2x-7)-3)^2, & 9/2 \leq x \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

Now we can calulate the value of Ψ functions

$$\begin{aligned} \Psi_3(x) &= \tfrac{1}{480}\Phi(x), \quad 0 \leq x \leq 1/2 \\ &= \tfrac{1}{480}2x^2, \quad 0 \leq x \leq 1/2 \end{aligned}$$

$$\begin{aligned} \Psi_3(x) &= \tfrac{1}{480}\Phi(2x) - \tfrac{29}{480}\Phi(2x-1), \quad 1/2 \leq x \leq 1 \\ &= \tfrac{1}{480}\left[\tfrac{3}{4} - (2x-\tfrac{3}{2})^2 - 29(\tfrac{1}{2}(2x-1)^2)\right] \\ &= \tfrac{1}{480}\left[\tfrac{3}{4} - 4x^2 - \tfrac{9}{4} - 6x - \tfrac{29}{2}(4x^2+1-4x)\right] \\ &= \tfrac{1}{480}\left[-16 + 64x - 62x^2\right], \quad 1/2 \leq x \leq 1 \end{aligned}$$

$$\begin{aligned} \Psi_3(x) &= \tfrac{1}{480}\Phi(2x) - \tfrac{29}{480}\Phi(2x-1) + \tfrac{147}{480}, \quad 1 \leq x \leq 3/2 \\ &= \tfrac{1}{480}\left[\tfrac{1}{2}(2x-3)^2 - 29(\tfrac{3}{2} - (2x-\tfrac{5}{2})^2) + 147(\tfrac{1}{2}(2x-2)^2)\right] \\ &= \tfrac{1}{480}\left[\tfrac{1}{2}(4x^2+9-12x) - \tfrac{29}{4} + 29(4x^2-10x+\tfrac{25}{4}) + 147(4x^2-8x-9)\right] \\ &= \tfrac{1}{480}\left[458 - 884x - 412x^2\right] \quad 1 \leq x \leq 3/2 \end{aligned}$$

$$\begin{aligned} \Psi_3(x) &= -\tfrac{29}{480}\Phi(2x-1) + \tfrac{147}{480}\Phi(2x-2) - \tfrac{303}{480}\Phi(2x-3), \quad 3/2 \leq x \leq 2 \\ &= \tfrac{1}{480}\left[-29(\tfrac{1}{2}((2x-1)-3)^2) + 147(\tfrac{3}{4} - (2x-2) - \tfrac{3}{2})^2 - 303\tfrac{1}{2}(2x-3)^2\right] \\ &= \tfrac{1}{480}\left[\tfrac{-29}{2}(4x^2+16-16x) + 147(\tfrac{3}{4} - (4x^2-14x+\tfrac{49}{4})) - \tfrac{303}{2}(4x^2-12x+9)\right] \\ &= \tfrac{1}{480}\left[-3286 + 4108x - 1252x^2\right] \quad 3/2 \leq x \leq 2 \end{aligned}$$

$$
\begin{aligned}
\Psi_3(x) &= \tfrac{147}{480}\Phi(2x-2) - \tfrac{303}{480}\Phi(2x-3) + \tfrac{303}{480}\Phi(2x-4)+, \qquad 2 \le x \le 5/2 \\
&= \tfrac{1}{480}\left[147(\tfrac{1}{2}((2x-5)^2) - 303(\tfrac{3}{4} - (2x-\tfrac{9}{2})^2) - \tfrac{3}{2})^2 + 303\tfrac{1}{2}(2x-4)^2\right] \\
&= \tfrac{1}{480}\left[\tfrac{147}{2}(4x^2 - 25 - 20x) - 303(\tfrac{3}{4} - (4x^2 - 18x + \tfrac{81}{4})) + \tfrac{303}{2}(4x^2 - 16x - 16)\right] \\
&= \tfrac{1}{480}\left[6(1695 - 1553x + 352x^2)\right] \qquad 2 \le x \le 5/2
\end{aligned}
$$

$$
\begin{aligned}
\Psi_3(x) &= -\tfrac{303}{480}\Phi(2x-3) + \tfrac{303}{480}\Phi(2x-4) - \tfrac{147}{480}\Phi(2x-2), \qquad 5/2 \le x \le 3 \\
&= \tfrac{1}{480}\left[-303(\tfrac{1}{2}((2x-6)^2) + 303(\tfrac{3}{4} - (2x-\tfrac{11}{2})^2) - 147\tfrac{1}{2}(2x-5)^2\right] \\
&= \tfrac{1}{480}\left[-\tfrac{303}{2}(4x^2 + 36 - 24x) + 303(\tfrac{3}{4} - (4x^2 - 22x + \tfrac{121}{4})) - \tfrac{147}{2}(4x^2 - 20x + 25)\right] \\
&= \tfrac{1}{480}\left[-6(2705 - 1962x + 352x^2\right] \qquad 5/2 \le x \le 3
\end{aligned}
$$

$$
\begin{aligned}
\Psi_3(x) &= \tfrac{303}{480}\Phi(2x-4) - \tfrac{147}{480}\Phi(2x-5) + \tfrac{29}{480}\Phi(2x-6), \qquad 3 \le x \le 7/2 \\
&= \tfrac{1}{480}\left[303(\tfrac{1}{2}((2x-7)^2) - 147(\tfrac{3}{4} - (2x-\tfrac{13}{2})^2) + 29\tfrac{1}{2}(2x-6)^2\right] \\
&= \tfrac{1}{480}\left[\tfrac{303}{2}(4x^2 - 49 - 23x) - 147(\tfrac{3}{4} - (4x^2 - 26x + \tfrac{169}{4})) + \tfrac{29}{2}(4x^2 - 24x + 36)\right] \\
&= \tfrac{1}{480}\left[2(7023 - 4206x + 626x^2\right] \qquad 3 \le x \le 7/2
\end{aligned}
$$

$$
\begin{aligned}
\Psi_3(x) &= -\tfrac{147}{480}\Phi(2x-5) + \tfrac{29}{480}\Phi(2x-6) - \tfrac{1}{480}\Phi(2x-7), \qquad 7/2 \le x \le 4 \\
&= \tfrac{1}{480}\left[-147(\tfrac{1}{2}((2x-8)^2) + 29(\tfrac{3}{4} - (2x-\tfrac{15}{2})^2) - \tfrac{1}{2}(2x-7)^2\right] \\
&= \tfrac{1}{480}\left[\tfrac{-147}{2}(4x^2 + 64 - 32x) + 29(\tfrac{3}{4} - (4x^2 - 30x + \tfrac{225}{4})) - \tfrac{1}{2}(4x^2 - 28x + 49)\right] \\
&= \tfrac{1}{480}\left[-6338 + 3236x - 412x^2\right] \qquad 7/2 \le x \le 4
\end{aligned}
$$

$$
\begin{aligned}
\Psi_3(x) &= \tfrac{29}{480}\Phi(2x-6) - \tfrac{1}{480}\Phi(2x-7), \qquad 4 \le x \le 9/2 \\
&= \tfrac{1}{480}\left[29(\tfrac{1}{2}((2x-9)^2) - (\tfrac{3}{4} - (2x-\tfrac{17}{2})^2)\right] \\
&= \tfrac{1}{480}\left[\tfrac{29}{2}(4x^2 + 64 - 32x) - (\tfrac{3}{4} - (4x^2 - 30x + \tfrac{289}{4}))\right] \\
&= \tfrac{1}{480}\left[1246 - 556x + 62x^2\right] \qquad 4 \le x \le 9/2
\end{aligned}
$$

$$\begin{aligned}
\Psi_3(x) &= -\tfrac{1}{480}\Phi(2x-7), \qquad 9/2 \le x \le 5 \\
&= \tfrac{1}{480}\left[-(\tfrac{1}{2}((2x-10)^2\right] \\
&= \tfrac{1}{480}\left[\tfrac{1}{2}4(x-5)^2\right] \\
&= \tfrac{1}{480}\left[-2(x-5)^2\right] \qquad 9/2 \le x \le 5
\end{aligned}$$

Thus third order B-spline wavelet are given by[4,5].

$$\Psi_{j,k}(x) = \frac{1}{480}\begin{cases}
2(x_j-k)^2, \quad k \le x_j \le k+\frac{1}{2} \\
-16+64(x_j-k)-62(x_j-k)^2, \qquad k+\frac{1}{2} \le x \le k+1 \\
458-884(x_j-k)-412(x_j-k)^2 \qquad k+1 \le x \le k+\frac{3}{2} \\
-3286+4108(x_j-k)-1252(x_j-k)^2 \qquad k+\frac{3}{2} \le x \le k+2 \\
6(1695-1553(x_j-k)+352(x_j-k)^2) \qquad k+2 \le x \le k+\frac{5}{2} \\
-6(2705-1962(x_j-k)+352(x_j-k)^2) \qquad k+\frac{5}{2} \le x \le k+3 \\
2(7023-4206(x_j-k)+626(x_j-k)^2) \qquad k+3 \le x \le k+\frac{7}{2} \\
-6338+3236(x_j-k)-412(x_j-k)^2) \qquad k+\frac{7}{2} \le x \le k+4 \\
1246-556(x_j-k)+62(x_j-k)^2) \qquad k+4 \le x \le k+\frac{9}{2} \\
-2((x_j-k)-5)^2) \qquad k+\frac{9}{2} \le x \le k+5
\end{cases} \tag{3.4}$$

CHAPTER 4

Solution for Hammerstein Integral Equation by using Semiorthogonal Spline Waveles

There are several numerical method for approximating the solution of Hammerstein integral equation. And in the present paper, we apply compactly supported linear semiorthoganal B-spline wavelet specially constructed for the bounded interval to solve the nonlinear Fredholm-Hammerstein integral equation of the form

$$u(x) = f(x) + \int_0^1 K(x,y)g[y,u(y)]dy \tag{4.1}$$

where $x \in [0,1]$ and f, g and K are given continuous functions, with $g(y,u)$ nonlinear in u.

1. MRA and wavelets

In this section we shall briefly summarize the essentials of the theory of wavelet expansion and MRA. These concept have been introduced by mallat. A set of subspaces $(V_j)_{j\in z}$ is said to be MRA of $L^2(\mathbb{R})$ if it possess the following properties:

$(i) V_j \subset V_{j+1}, \quad \forall j \in \mathbb{Z}$

$(ii) \bigcup_{j\in z} V_j \; is \; dense \; in \; L^2(\mathbb{R}),$

$(iii) \bigcap_{j\in z} V_j = \phi,$

$(iv) f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}, \quad \forall j \in \mathbb{Z}$

where Z denotes set of integer. Above properties state that $(V_j)_{j\in z}$ is nasted sequence of subspace that effectivly covers $L^2(\mathbb{R})$. That is, every squre integrable function can be approximated as closely as desired by a function that belong to atleast one of the subspace V_j. A function $\phi \in L^2(\mathbb{R})$ (finite energy function) is called a scalling function if it generates the nasted sequence of the subspace V_j and satisfied the dilation(refinement) equation, namely

$$\phi(x) = \sum_k p_k \phi(ax - k) \tag{4.2}$$

with $p_k \in l^2$(finite energy sequence) and a being any rational number. With $a = 2$ (this value that correspond to the octave scale will be assumed for the rest of this paper), the subspace V_j are genrated by $\phi_{j,k} = \phi(2^j x - k), \quad k \in \mathbb{Z}$.

For each scale j, since $V_j \subset V_{j+1}$, there exist a unique orthogonal complementry subspace W_j of V_j in V_{j+1}.This subspace is called wavlet subspace and genrated by $\psi_{j,k} = \psi(2^j x - k)$, where $\psi \in L^2$ is called the wavelet. From the above discussion, these result follow easily.

$(i) V_{j_1} \bigcap V_{j_2} = V_{j_2}, \quad j_1 > j_2,$

$(ii) W_{j_1} \bigcap W_{j_2} = 0, \quad j_1 \neq j_2,$

$(iii) V_{j_1} \bigcap W_{j_2} = 0, \quad j_1 \leq j_2$

1)**Vanishing moment:** A wavelet is said to have a vanishing moment of order m if

$$\int_{-\infty}^{\infty} x^p \psi(x) = 0; \quad p = 0, ..., m - 1 \tag{4.3}$$

All wavelet must satisfy the above condtion for $p = 0$.

2)**Semiorthogonality:** The wavelet $\psi_{j,k}$ form a semiorthogonal basis if

$$\langle \psi_{j,k}, \psi_{s,i} \rangle = 0; \quad j \neq s; \quad \forall\, j, k, s, i \in Z. \tag{4.4}$$

2. Wavelet and scaling function on bounded interval

2.1. DEFINITION. Let m and n be two positive integer and

$$a = x_{-m+1} = ... = x_0 < x_1 < ... < x_n = x_{n+1} = ... = x_{n+m-1} = b, \tag{4.5}$$

be an equally spaced knots sequnce. The functions

$$B_{m,j,X}(x) = \frac{x - x_j}{x_{j+m-1} - x_j} B_{m-1,j,X}(x) + \frac{x_{j+m} - x}{x_{j+m} - x_{j+1}} B_{m-1,j+1,X}(x) \quad j = -m+1, ..., n-1. \tag{4.6}$$

$$B_{1,j,X}(x) = \begin{cases} 1, & \text{if } \; x \in [x_j, x_{j+1}), \\ 0, & \text{otherwise} \end{cases}$$

are called cardinal B- spline function of order$m \geq 2$ for the knot sequence $X = (x_i)_{i=-m+1}^{n+m-1}$, and supp $B_{m,j,X}(x) = [x_j, x_{j+m}] \bigcap [a, b]$.

For the sake of simplicity, suppose $[a, b] = [0, n]$ and $x_k = k, k = 0, ..., n$. The $B_{m,j,X} = B_m(x - j), j = 0, ...n - m$, are interior B-spline function, while the remaing $B_{m,j,X}, j = -m + 1, ..., -1 and j = n - m, ..., n - 1$, are baundary B-spline function for the baunded interval $[0, n]$. Since the boundary B-spline function at 0 are symmetric reflection of those at n,it is sufficient to construct only the first half function by simply replacing x with $n - x$.

By considering the interval $[a, b] = [0, 1]$,at any level$j \in Z_+$, the discretization step is 2^{-j}, and this genrates $n = 2^j$ number of segments in $[0, 1]$ with knot sequence.

$$X^{(j)} = \begin{cases} x_{-m+1}^{(j)} = ... = x_0^{(j)}, \\ x_k^{(j)} = \frac{k}{2^{(j)}} \quad k = 1, ..., n - 1, \\ x_n^{(j)} = ... = x_{n+m-1}^{(j)} = 1. \end{cases} \tag{6.1}$$

Let j_0 be the level for which $2^{j_0} \geq 2m - 1$; for each level, $j \geq j_0$ the scaling function of order m can be definfe as follows in [6,7]:

$$\varphi_{m,j,i}(x) = \begin{cases} B_{m,j_0,i}(2^{j-j_0}x) & i = -m + 1, ..., -1 \\ B_{m,j_0,2^j-m-i}(1 - 2^{j-j_0}x) & i = 2^j - m + 1, ..., 2^j - 1 \\ B_{m,j_0,0}(2^{j-j_0}x - 2^{-j_0 i}) & i = 0, ..., 2^j - m \end{cases} \tag{6.2}$$

And the two scale relation for m-order semiorthogonal compactly supported B-wavelet function are defined as follows:

$$\psi_{m,j,i-m} = \sum_{k=i}^{2i+2m-2} q_{i,k} B_{m,j,k-m}, \qquad i = 1, ..., m - 1, \tag{4.7}$$

$$\psi_{m,j,i-m} = \sum_{k=2i-m}^{2i+2m-2} q_{i,k} B_{m,j,k-m}, \qquad i = m, ..., n-m+1, \tag{4.8}$$

$$\psi_{m,j,i-m} = \sum_{k=2i-m}^{n+i+m-1} q_{i,k} B_{m,j,k-m}, \qquad i = n-m+2, ..., n, \tag{4.9}$$

where $q_{i,k} = q_{k-2i}$.

Hence, there are $2(m-1)-$boundary wavelets and $(n-2m+2)$ inner wavelet in the bounde interval $[a,b]$. Finally by considering the level j with $j \geq j_0$, the B-wavelet function in $[0,1]$ can be expressed as follows:

$$\psi_{m,j,i}(x) = \begin{cases} \psi_{m,j_0,i}(2^{j-j_0}x) & i = -m+1, ..., -1 \\ \psi_{m,2^j-2m+1-i,i}(1-2^{j-j_0}x) & i = 2^j-2m+2, ..., 2^j-m \\ \psi_{m,j_0,0}(2^{j-j_0}x - 2^{-j_0}i) & i = 0, ..., 2^j-2m+1 \end{cases} \tag{9.1}$$

the scaling function $\varphi_{m,j,i}(x)$ occupy m segments and the wavelet function $\psi_{m,j,i}(x)$ occupy $2m-1$ segments.

Therefore, the condition $2^j \geq 2m-1$ must be satisfied in order to have at least one inner wavelet. In the following, the scaling function and wavelet function used in this paper, for $j_0 = j = 2$ and $m = 2$, are reported in [4,8].

3. Function approximation

A function $f(x)$ defined over $[0,1]$ may be approximated by B-spline as [1,2]

$$f(x) = \sum_{i=-1}^{2^{j_0}-1} c_{j_0,i}\phi_{j_0,i}(x) + \sum_{k=j_0}^{\infty}\sum_{j=-1}^{2^k-2} d_{k,j}\psi_{k,j}(x) \tag{4.10}$$

where $\phi_{j_0,i}$ and $\psi_{k,j}$ are scaling and wavelet function, respectively. In particular, for $j_0 = 2$, if the infinite series in equation (10) is truncated at M, then Eq. (10) can be written as [4,9]

$$f(x) \approx \sum_{i=-1}^{2^{j_0}-1} c_{j_0,i}\phi_{j_0,i}(x) + \sum_{k=j_0}^{M}\sum_{j=-1}^{2^k-2} d_{k,j}\psi_{k,j}(x) = C^T\Psi(x) \tag{4.11}$$

where C and Ψ are $(2^{M+1}+1)\times 1$ vector given by

$$C = [c_{-1}, c_0, ..., c_3, d_{2,-1}, ..., d_{2,2}, d_{3,-1}, ..., d_{3,6}, ..., d_{M,-1}, ..., d_{M,2^M-2}]^T \tag{4.12}$$

$$\Psi = [\phi_{2,-1}, \phi_{2,0}, ..., \phi_{2,3}, \psi_{2,-1}, ..., \psi_{2,2}, \psi_{3,-1}, ..., \psi_{3,6}, ..., \psi_{M,-1}, ..., \psi_{M,2^M-2}]^T \tag{4.13}$$

with

$$c_k = \int_0^1 f(x)\tilde{\phi_{2,k}}(x)dx, \qquad k = -1, 0, ..., 3, \tag{4.14}$$

$$d_{j,k} = \int_0^1 f(x)\tilde{\psi_{j,k}}(x)dx, \qquad j = 2, 3..., M, \quad k = -1, 0, ..., 2^j - 2, \tag{4.15}$$

where $\tilde{\phi_{2,k}}(x)$ and $\tilde{\psi_{j,k}}(x)$ are dual function of $\phi_{2,k}(x)$ and $\psi_{j,k}(x)$ respectively. These can be obtained by linear combination of $\phi_{2,k}(x)$, $k = -1, 0, ..., 3$ and $\psi_{j,k}(x,)$ $j = 2, 3, ..., M$, $k = -1, 0, ..., 2^j - 2$ as follows.

Let

$$\Phi = [\phi_{2,-1}(x), \phi_{2,0}(x), \phi_{2,1}(x), \phi_{2,2}(x), \phi_{2,3}(x)]^T \tag{4.16}$$

$$\bar{\Psi} = [\psi_{2,-1}(x), \psi_{2,0}(x), ..., \psi_{M,2^{M+1}}(x)]^T \tag{4.17}$$

using (6.2) and (16) we get,

$$\int_0^1 \Phi\Phi^T dx = P_1 = \begin{pmatrix} \frac{1}{12} & \frac{1}{24} & 0 & 0 & 0 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & 0 & 0 \\ 0 & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & 0 \\ 0 & 0 & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} \\ 0 & 0 & 0 & \frac{1}{24} & \frac{1}{12} \end{pmatrix}$$

and from the equation (9.1) and (17) we have,

$$\int_0^1 \bar{\Psi}\bar{\Psi}^T dx = P_2 = \begin{pmatrix} N_{4\times 4} & & & & & \\ & \frac{1}{2}N_{8\times 8} & & & & \\ & & . & & & \\ & & & . & & \\ & & & & . & \\ & & & & & \frac{1}{2^{M-2}}N_{2^M\times 2^M} \end{pmatrix}$$

where P_1 and P_2 are 5×5 and $(2^{M+1} - 4) \times (2^{M+1} - 4)$ matrices, respectively, and N is a five-diagonal matrix given by

$$N = \begin{pmatrix} \frac{2}{27} & \frac{1}{96} & -\frac{1}{864} & 0 & 0 & . & . & . & 0 \\ \frac{1}{96} & \frac{1}{16} & \frac{5}{432} & -\frac{1}{864} & 0 & . & . & . & 0 \\ -\frac{1}{864} & \frac{5}{423} & \frac{1}{16} & \frac{1}{96} & -\frac{1}{864} & . & . & . & 0 \\ . & . & . & . & . & . & & & . \\ . & & . & . & . & . & . & & . \\ . & & & . & . & . & . & . & . \\ 0 & . & . & . & -\frac{1}{864} & \frac{5}{432} & \frac{1}{16} & \frac{5}{432} & -\frac{1}{864} \\ 0 & . & . & . & 0 & -\frac{1}{864} & \frac{5}{432} & \frac{1}{16} & \frac{1}{96} \\ 0 & . & . & . & 0 & 0 & -\frac{1}{864} & \frac{1}{96} & \frac{2}{27} \end{pmatrix}$$

Suppose $\tilde{\Phi}$ and $\tilde{\bar{\Psi}}$ are the dual function of Φ and $\bar{\Psi}$, respectively, given by

$$\tilde{\Phi} = [\tilde{\phi}_{2,-1}(x), \tilde{\phi}_{2,0}(x), \tilde{\phi}_{2,1}(x), \tilde{\phi}_{2,2}(x), \tilde{\phi}_{2,3}(x)]^T \tag{4.18}$$

$$\tilde{\bar{\Psi}} = [\tilde{\psi}_{2,-1}(x), \tilde{\psi}_{2,0}(x), ..., \tilde{\psi}_{M,2^{M+1}}(x)]^T \tag{4.19}$$

combining both equations we will get

$$\tilde{\Psi} = [\tilde{\phi}_{2,-1}(x), \tilde{\phi}_{2,0}(x), \tilde{\phi}_{2,1}(x), \tilde{\phi}_{2,2}(x), \tilde{\phi}_{2,3}(x), \tilde{\psi}_{2,-1}(x), \tilde{\psi}_{2,0}(x), ..., \tilde{\psi}_{M,2^{M+1}}(x)]^T \tag{4.20}$$

using (4.16),(4.17) and (4.18),(4.19) we have

$$\int_0^1 \tilde{\Phi}\Phi^T dx = I_1, \qquad \int_0^1 \tilde{\bar{\Psi}}\bar{\Psi}^T dx = I_2, \tag{4.21}$$

where I_1 and I_2 are 5×5 and $(2_{(M+1)} - 4) \times (2_{(M+1)} - 4)$ identity matrices respectively. Then from matice P_1 and P_2 we get,

$$\tilde{\Phi} = P_1^{-1}\Phi, \qquad \tilde{\bar{\Psi}} = P_2^{-1}\bar{\Psi} \tag{4.22}$$

4. Application of B-spline wavelet method to the Hammerstein integral equations

In this section, we have solved the system of nonlinear Fredholm integral equations using B-spline wavelets[8]. First, we assum

$$F_{i,j}(x, y_j(x)) = u_{i,j}(x) \quad 0 \leq x \leq 1 \tag{4.23}$$

Now from Eq. (4.11), we can approximate the functions $u_{i,j}(x)$ and $y_j(x)$ as

$$u_{i,j}(x) = A_{i,j}^T \Psi(x) \tag{4.24}$$

$$y_j(x) = B_j^T \Psi(x) \tag{4.25}$$

Where $A_{i,j}$ and B_j are unknown column vector of $(2^{M+1} + 1) \times 1$ similar to C as in Eq. (4.12)

Again using the dual of wavelet function, we can approximate $f_i(x)$ and $K_{i,j}(x,t)$ as follows.

$$f_i(x) = C_i^T \tilde{\Psi}(x) \tag{4.26}$$

$$K_{i,j}(x,t) = \tilde{\Psi}^T(x) \Theta \tilde{\Psi}(x) \tag{4.27}$$

where $\Theta_{i,j} = \int_0^1 [\int_0^1 K_{i,j}(x)\Psi(t)dt]\Psi(x)dx$

And C_j^T can be calculated as

$$C_j^T = \int_0^1 f_i(x)\Psi(x)dx$$

From equation (4.23)-(4.27) we will get

$$\begin{aligned}\int_0^1 K_{i,j}(x,t)F_{i,j}(t, y_j(t))dt &= \int_0^1 A_{i,j}^T \Psi(t)\tilde{\Psi}^T(t)\Theta_{i,j}\tilde{\Psi}(x)dt \\ &= A_{i,j}^T[\int_0^1 \Psi(t)\tilde{\Psi}^T(t)dt]\Theta_{i,j}\tilde{\Psi}(x) \\ &= A_{i,j}^T \Theta_{i,j}\tilde{\Psi}(x) \qquad (4.27a)\end{aligned}$$

since $\int_0^1 \Psi(t)\tilde{\Psi}^T(t)dt = 1$

Appling equation (4.23)-(4.27a) in the equation main equation we will get

$$\sum_{j=1}^{n} g_{i,j} B_j^T \Psi(x) = C_j^T \tilde{\Psi}(x) + \sum_{j=1}^{n} A_{i,j} \Theta_{i,j} \tilde{\Psi}(x) \tag{4.28}$$

Multiplying both side $\Psi(x)^T$ from the right side and integrating with respect to x from 0 to 1.

$$\sum_{j=1}^{n} g_{i,j} B_j^T P = C_j^T + \sum_{j=1}^{n} A_{i,j} \Theta_{i,j} \qquad i = 1(1)n \tag{4.29}$$

where P is a $(2^{M+1}+1) \times (2^{M+1}+1)$ squre matrix given by

$$P = \int_0^1 \Psi(x)\Psi(x)^T dx = \begin{pmatrix} P_1 & \\ & P_2 \end{pmatrix}$$

and $\int_0^1 \tilde{\Psi}(x)\Psi(x)^T = I$

Equation (4.29) gives a system of $n(2^{(M+1)}+1)$ algebric equation with $(n^2+n)(2^{(M+1)}+1)$ unknown in $A_{i,j}$ and B_j for $i, j = 1(1)n$, given in (4.24) and (4.25) to find the solution of y_j in the equation (4.25), we first utilize the following equation

$$F_{i,j}(x, B_j^T \Psi(x)) = A_{i,j} \Psi(x), \tag{4.30}$$

with the collocation point

$$x_s = \frac{s-1}{2^{M+1}} \qquad s = 1, 2, ...2^{M+1} + 1; \tag{4.31}$$

equation (4.30) gives a system of $n^2(2^{(M+1)}+1)$ algebric equation with $(n^2+n)(2^{(M+1)}+1)$ unknowns in $A_{i,j}$ andB_j for $i, j = 1(1)n$.

Combining equation (4.29) and (4.30), we have total number of $(n^2+n)(2^{(M+1)}+1)$ algebric equation with $(n^2+n)(2^{(M+1)}+1)$ unknowns in $A_{i,j}$ and B_j for $i, j = 1(2)n$. slov-ing those equation for the unknown coeficient in the vector $A_{i,j}$ and B_j for $i, j = 1(2)n$, we can obtain the solution $y_j(x) = B_j^T \Psi(x)$ for $j = 1, 2, ..., n$.

CHAPTER 5

Application of B-spline method to Hammerstein Integral Equation Arising from Chemical Reactor Theory

1. Solution of Hammerstein Integral Equation Arising from Chemical Reactor Theory

We consider the mathematical model for an adiabatic turbular chemical reactor which is prosesses an nonreversible exothermic chemical reaction. For steady state solution we reduce the model to the ordinary differential equation

$$u" - \lambda u' + F(\lambda, \mu, \beta, u) = 0 \tag{5.1}$$

with boundary condition

$$u'(0) = \lambda u(0) \quad u'(1) = 0 \tag{5.2}$$

where

$$F(\lambda, \mu, \beta, u) = \lambda\mu(\beta - u)exp(u)$$

(see [10], [11]). And here

u- Unknown steady state temparature of the reaction,

λ - The Peclet Number,

μ- Damkohlar Number,

β- Dimensionless adiabatic temperature.

This problem has been studied by numerous auther (*e.g.* [10], [12], [13]).To develop result concerning the solution of equation (5.1) and (5.2) the problem can be converted, using Green's function technique, into a Hammerstein integral equation

$$u(x) = \int_0^1 K(x,y)f(y,u(y))dy, \qquad 0 \le x \le 1 \tag{5.3}$$

wehere $K(x,y)$ and $f(y,u)$ are defined by

$$K(x,y) = \begin{cases} e^{\lambda(x-y)} & for\ 0 \le x \le y, \\ 1, & y \le x \le 1 \end{cases} \tag{5.4}$$

$$f(y,u) = \mu(\beta - u)exp(u). \tag{5.5}$$

We have been solved This nonlinear Fredholm- Hammerstein integral equation by using B-Spline wavelet method and compared with Contraction Principle Method(CPM), Shooting Mehtod(SM)and Adomian'sMethod (AM), result is shown below.

n	x	$B-Spline(M=2)$	$B-Spline(M=4)$	$CPM[11]$	$SM[9]$	$AM[9]$
1.	0.0	0.006045	0.006048	0.006079	0.006048	0.006048
2.	0.2	0.018194	0.018193	0.018224	0.018192	0.018192
3.	0.4	0.030424	0.030424	0.030456	0.030424	0.030424
4.	0.6	0.042675	0.042669	0.042701	0.042669	0.042669
5.	0.8	0.054332	0.054368	0.054401	0.054371	0.054371
6.	1.0	0.06203	0.061505	0.061459	0.061458	0.061458

CHAPTER 6

Conclusion

In this thesis, the semi-orthogonal compactly supported linear B-Spline Wavelets have been applied to solve the nonlinear Hammerstein integral equation. We have solved a model for an adiabatic tubular chemical reactor theory which forms a nonlinear Hammerstein integral equation. Using this method, the integral equation has been reduced to a system of algebraic equations. The numerical results obtained by present method have been compared with the results obtained by Contract Mapping Princeple, the Shooting Method, and Adomians Method and this comparision justify that the present method gives more accurate results than other methods if we increase the value of M.

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