BALANCING NUMBERS : SOME IDENTITIES

A report

submitted by

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of

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DECLARATION

I hereby declare that the topic " **Balancing numbers : some identities** " for completion for my master degree has not been submitted in any other institution or university for the award of any other Degree, Fellowship or any other similar titles.

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CERTIFICATE

This is to certify that the project report entitled **Balancing numbers : some identities** submitted by **Kaberi Parida** to the National Institute of Technology Rourkela, Odisha for the partial fulfilment of requirements for the degree of master of science in Mathematics is a bonafide record of review work carried out by her under my supervision and guidance. The contents of this project, in full or in parts, have not been submitted to any other institute or university for the award of any degree or diploma.

May, 2014

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Abstract

This paper studies a problem in the theory of figurate numbers identifying and investigating those numbers which are polygonal in two ways - triangular and square. In this report a study of Pell numbers, Associate Pell numbers, Balancing numbers, Lucas Balancing numbers is presented. These numbers can be better computed by means of recurrence relations through Pell's equation will play a central role.

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CHAPTER-1

1 Introduction

There is a quote by the famous mathematician Carl Friedrich Gauss $(1777-1855)$: Mathematics is the queen of all sciences, and number theory is the queen of mathematics. Number theory, or higher arithmetic is the study of those properties of integers and rational numbers, which go beyond the ordinary manipulations of everyday arithmetic.

In number theory, discovery of number sequences with certain specified properties has been a source of attraction since ancient times.The most beautiful and simplest of all number sequences is the Fibonacci sequence.This sequence was first invented by Leonardo of Pisa $(1180-1250)$, who was also known as Fibonacci, to describe the growth of a rabbit population.

Other interesting number sequences are the Pell sequence and the associated Pell sequence. In mathematics, the Pell numbers are infinite sequence of integers that have been known since ancient times.The denominators of the closest rational approximations to the square root of 2. This sequence of approximations begins with $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}$; so the sequence of Pell numbers begins with $1, 2, 5, 12, 29$. The numerators of the same sequence of approximations give the associated Pell sequence.

The concept of balancing numbers was first introduced by Behera and Panda in the year 1999 in connection with a Diophantine equation. It consists of finding a natural number n such that

$$1 + 2 + 3... + (n-1) = (n+1) + (n+2) + ... + (n+r)$$

for some natural number r, while they call r, the balancer corresponding to the balancing number n. If the nth triangular number $\frac{n(n+1)}{2}$ is denoted by T_n, then the above equation reduces to $T_{n-1} + T_n = T_{n+r}$ which is the problem of finding two consecutive triangular numbers whose sum is also a triangular number. Since

$$T_5 + T_6 = 15 + 21 = 36 = T_8$$

6 is a balancing number with balancer 2. Similarly,

$$T_{34} + T_{35} = 595 + 630 = 1225 = T_{49}$$

implies that, 35 is also a balancing number with balancer 14.

The balancing numbers, though obtained from a simple Diophantine equation, are very useful for the computation of square triangular numbers. An important result about balancing numbers is that,n is a balancing number if and only if $8n^2 + 1$ is a perfect square, and the number $\sqrt{8n^2+1}$ is called a Lucas balancing number. The most interesting fact about Lucas-balancing numbers is that, these numbers are associated with balancing numbers in the way Lucas numbers are associated with Fibonacci numbers.

The early investigators of Pell equation were the indian mathematicians Brahmagupta and Bhaskara. In particular Bhaskara studied Pell's equation for the values $d = 8, 11, 32, 61$.

Bhaskar found the solution $x = 1776319049, y = 2261590$ for $d = 61$. Fermat was also interested in the Pell's equation and worked out some of the basic theories regarding Pell's equation. A special type of Diophantine equation is the Pells equation $x^2 - dy^2 = 1$ where d is a natural number which is not a perfect square. Indeed, the english mathematician John Pell $(1610 - 1685)$ has nothing to do with this equation.In general Pell's equation is a Diophantine equation of the form $x^2 - dy^2 = 1$. where d is a positive non square integer.

Cobalancing number n are solutions of the Diophantine equation

$$1 + 2 + 3 ... + n = (n + 1) + (n + 2) + ... + (n + r)$$

Where r is called the cobalancer corresponding to the cobalancing number n.

Panda [13] generalized balancing and cobalancing numbers by introducing sequence balancing and cobalancing numbers, in which, the sequence of natural numbers, used in the definition of balancing and cobalancing numbers is replaced by an arbitrary sequence of real numbers. Thus if $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers, then a_k is called a sequence balancing number if

$$a_1 + a_2 + ... + a_{k-1} = a_{k+1} + a_{k+2} + ... + a_{k+r}$$

for some natural number r; a_k is called a sequence cobalancing number if

$$a_1 + a_2 + ... + a_k = a_{k+1} + a_{k+2} + ... + a_{k+r}$$

for some natural number r.

The Associated Pell sequence exhausts two sequences generated from balancing and cobalancing numbers,namely, the sequences of Lucas-balancing and the Lucas-cobalancing numbers. Pell and associated Pell numbers also appear as the greatest common divisors of two consecutive balancing numbers or cobalancing numbers or, a pair of balancing and cobalancing numbers of same order.Balancing and cobalancing numbers also arise in the partial sums of even ordered Pell numbers, odd order Pell numbers, even ordered associated Pell numbers, odd order associated Pell numbers, and in the partial sum of these numbers up to even and odd order.

CHAPTER - 2

2 Preliminaries

In this chapter we recall some definitions and known results on Pell numbers,associated Pell numbers, triangular numbers, recurrence relations, Binet formula, Diophantine equations including Pells equations. This chapter serves as base and background for the study of subsequent chapters. We shall keep on referring back to it as and when required.

2.1 Recurrence Relation

In mathematics, a recurrence relation is an equation that defines a sequence recursively; each term of the sequence is defined as a function of the preceding terms.

2.2 Generating function

The generating function of a sequence $\{x_n\}_{n=1}^{\infty}$ of real or complex numbers is given by $f(s) = \sum_{n=1}^{\infty} x_n s^n$. Hence, the nth term of the sequence is obtained as the coefficient of s^n in the power series expansion of $f(s)$.

2.3 Triangular numbers

A number of the form $\dfrac{n(n+1)}{2}$ where $n \in Z^+$ is called a triangular number.The justifications for the name triangular number are many.One such reason may be the fact that the triangular number $\dfrac{n(n+1)}{2}$ represents the area of a right angled triangle with base $n+1$ and perpendicular n. It is well known that $m \in Z^+$ is a triangular number if and only if $8m+1$ is a perfect square.

2.4 Diophantine Equation

In mathematics, a Diophantine equation is an indeterminate polynomial equation that allows the variables to be integers only. Diophantine problems have fewer equations than unknowns and involve finding integers that work correctly for all the equations.

2.5 Binet formula

While solving a recurrence relation as a difference equation,the n^{th} term of the sequence is obtained in closed form,which is a formula containing conjugate surds of irrational numbers is known as the Binet formula for the particular sequence. These surds are obtained from the auxiliary equation of the recurrence relation for the recursive sequence under consideration.

2.6 Pell numbers

The first two Pell numbers are $P_1 = 1, P_2 = 2$ and other terms of the sequence are obtained by means of the

Recurrence relation : $P_n = 2P_{n-1} + P_{n-2}$ for $P \geq 2$
Cassini formula : $P_{n-1}P_{n+1} - P_n^2 = (-1)^n$
Binet formula : $\frac{\alpha_1^n - \alpha_2^n}{2\sqrt{2}}$ and $\alpha_1 = 1 + \sqrt{2}$,$\alpha_2 = 1 - \sqrt{2}$
Where $\alpha_1 - \alpha_2 = 2\sqrt{2}, \alpha_1\alpha_2 = -1, \alpha_1 + \alpha_2 = 2$ and $P_n = 1, 2, 5, 12, 29, 70...$

2.7 Balancing numbers

The first two Balancing numbers are $B_1 = 1, B_2 = 6$ and other terms of the sequence are obtained by means of the
Recurrence relation : $B_{n+1} = 6B_n - B_{n-1}$
Cassini formula : $B_n^2 - B_{n+1}B_{n-1} = 1$
Binet formula : $\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}$ and $\alpha_1 = 3 + \sqrt{8}$, $\alpha_2 = 3 - \sqrt{8}$
Where $\alpha_1 - \alpha_2 = 2\sqrt{8}, \alpha_1\alpha_2 = 1, \alpha_1 + \alpha_2 = 6$ and $B_n = 1, 6, 35, 204, 1189...$

2.8 Associate Pell numbers

Recurrence relation : $Q_n = 2Q_{n-1} + Q_{n-2}$.
Cassini formula : $Q_{n-1}Q_{n+1} - Q_n^2 = (-1)^n$
Binet formula : $\frac{\alpha_1^n + \alpha_2^n}{2}$ and $\alpha_1 = 1 + \sqrt{2}$,$\alpha_2 = 1 - \sqrt{2}$
Where $\alpha_1\alpha_2 = -1, \alpha_1 + \alpha_2 = 2$ and $Q_n = 1, 3, 7, 17...$

2.9 Lucas Balancing numbers

Recurrence relation : $C_{n+1} = 6C_n - C_{n-1}$
Cassini formula : $C_n^2 - C_{n+1}C_{n-1} = -8$
Binet formula : $\frac{\alpha_1^n + \alpha_2^n}{2}$ and $\alpha_1 = 3 + \sqrt{8}$,$\alpha_2 = 3 - \sqrt{8}$
Where $\alpha_1 - \alpha_2 = 2\sqrt{8}, \alpha_1\alpha_2 = 1, \alpha_1 + \alpha_2 = 6$ and $C_n = 1, 3, 17, 99, 577...$

2.10 Co-Balancing numbers

Recurrence relation : $b_{n+1} = 6b_n - b_{n-1} + 2$
Binet formula : $b_n = \frac{\alpha_1^{2n-1} - \alpha_n^{2n-1}}{4\sqrt{2}}$ and $\alpha_1 = 1 + \sqrt{2}$, $\alpha_2 = 1 - \sqrt{2}$ and $b_n = 0, 2, 14, 84...$

2.11 Some identities of Pell numbers, Balancing numbers and Lucas balancing numbers

(a) **Relation between Pell and Associate Pell numbers**
(1)$P_1 + P_2 + ... + P_{2n-1} = B_n + b_n$.
(2)$P_1 + P_3 + ... + P_{2n-1} = b_n$.
(3)$P_2 + P_4 + ... + P_{2n} = b_{n+1}$.
(4)$Q_1 + Q_2 + ... + Q_{2n} = 2b_n + 1$.
(5)$Q_1 + Q_2 + ... + Q_{2n-1} = 2B_n - 1$.

$(6) P_{2n} + Q_{2n-1} = b_{2n}$.
$(7) P_{m+n} = P_m P_{n+1} + P_{m-1} P_n$.
$(8) P_{m+n} = 2P_m Q_n - (-1)^n P_{m-n}$.
$(9) P_1^2 + P_2^2 + ... + P_n^2 = \frac{P_{n+1} P_n}{2}$.
$(10) P_{2n+1} = \frac{1}{2} \left(P_n Q_{n+1} + Q_n P_{n+1} \right)$.
$(11) Q_{n+1} Q_{n-1} - Q_{2n} = 6(-1)^{n-1}$.

(b) **Relation between Balancing and Lucas Balancing numbers**
$(1) B_{2n-1} = B_n^2 - B_{n-1}^2$.
$(2) B_{2n} = B_n \left(B_{n+1} - B_{n-1} \right)$.
$(3) B_1 + B_3 + ... + B_{2n-1} = B_n^2$.
$(4) B_2 + B_4 + ... + B_n = B_n B_{n+1}$,
$(5) B_{n+1}^2 = 34B_n^2 + 2 \left(B_n^2 - B_{n+1} B_{n-1} \right) - B_{n-1}^2$.
$(6) B_{n+1} = 3B_n + (3B_n - B_{n-1})$.
$(7) B_{n+1} = 3B_n + \sqrt{8B_n^2 + 1}$.
$(8) B_{n-1} = 3B_n - \sqrt{8B_n^2 + 1}$.
$(9) C_{n+1}^2 = (3C_n + 8B_n)$.
$(10) B_{n-1} + B_n = C_n$.
$(11) C_{m-n} = B_{m+1} C_n - B_m C_{n+1}$.

CHAPTER - 3

3 Balancing numbers : some identities

The nth polygonal number of order g is the non negative integer such that $f_g(m) = \frac{n}{2}[\{(g-2)(n-1)\}+2], n = 0, 1, 2, ...$ Every triangular number taken 8 times and then increased by 1 gives a square i.e, $8\frac{n(n+1)}{2} + 1 = (2n+1)^2$. If a triangular number is polygonal in two ways

(i) Triangular $\frac{n(n+1)}{2}$.

(ii) Square m^2, then it is a Balancing number.

Example : $36 = \frac{8.9}{2} = 6^2$ is such a number represented in two ways. Let the n^{th} triangular number $T_n = \frac{n(n+1)}{2} = m^2 = s_m$ be the m^{th} square. T_n consists of the two consecutive numbers n and $n+1$, one is even and the other is odd. They are necessarily co-prime and so if n is even, then $gcd(\frac{n}{2}, n+1) = 1$, or if n is odd then $gcd(n, \frac{n+1}{2}) = 1$. Whenever of the two numbers are even, it contains only odd powers of 2. If n is even, then setting $n+1 = x^2$ i.e, $n = x^2 - 1$, and $\frac{n}{2} = y^2$ i.e, $n = 2y^2$ and equating n in terms of x and y.

$$x^2 - 2y^2 = 1 \tag{1}$$

Alternately, if n is odd then putting $n = x^2$ i.e, $\frac{n+1}{2} = y^2$, $n = 2y^2 - 1$ and equating n in terms of x and y

$$x^2 - 2y^2 = -1 \tag{2}$$

In the above cases, $T_n = (xy)^2 = m^2 = s_m = N_k$. Solutions of Pell equations are given the above two equation would give all N_k and $(3, 2)$ is the solution of (1) and $(1, 1)$ is the solution of (2) i.e,

$$1 = 1^k = (-1)^{2k} = \Big((3-2\sqrt{2})(3+2\sqrt{2})\Big)^k = \Big((1-2\sqrt{2})(1+2\sqrt{2})\Big)^{2k}$$

$$-1 = (-1)^{2k-1} = \Big((1-2\sqrt{2})(1+2\sqrt{2})\Big)^{2k-1}$$

Hence, the general solution of (1) and that of (2) respectively is:

$$x_k + y_k\sqrt{2} = (3+2\sqrt{2})^k = (1+\sqrt{2})^{2k} \tag{3}$$

$$x_k + y_k\sqrt{2} = (1+\sqrt{2})^{2k-1} \tag{4}$$

Combining the even and odd power solutions in (3) and (4), we get the single formula for all solutions of (1) and (2):

$$x_k + y_k\sqrt{2} = (1+\sqrt{2})^k \tag{5}$$

We also have the following explicit formula:

$$Q_k = x_k = \frac{(1+\sqrt{2})^k + (1-\sqrt{2})^k}{2} \tag{6}$$

$$P_k = y_k = \frac{(1+\sqrt{2})^k - (1-\sqrt{2})^k}{2\sqrt{2}} \tag{7}$$

The recurrence relations beginning with $x_0 = 1$, $y_0 = 0$ define them:

$$Q_{k+1} = Q_{k+1} = 2Q_k + Q_{k-1} \tag{8}$$

$$P_{k+1} = P_{k+1} = 2P_k + P_{k-1} \tag{9}$$

Recurrence relations can be very easily translated into generating functions and we have:

$$f(x) = \frac{1+x}{(1-2x-x^2)} = 1 + 3x + 7x^2 + ... \tag{10}$$

$$f(y) = \frac{1}{(1-2y-y^2)} = 1 + 2y + 5y^2 + ... \tag{11}$$

These Lucas-type formula involving binomial coefficients are adapted from Weisstein [6]:

$$Q_k = \sum_{r=0}^{[\frac{k}{2}]} 2^r \binom{k}{2r} \tag{12}$$

$$P_k = \sum_{r=1}^{[\frac{k+1}{2}]} 2^{r-1} \binom{k}{2r-1} \tag{13}$$

Having computed the values of Q and P, we can get the required values of n and m. For the even values of subscript k of n, we use $n = x^2 - 1 = 2y^2$, and for the odd, we take $n = x^2 = \sqrt{2}y^2 - 1$. For m, we simply need to multiply the corresponding values of x and y. These relations holds for $Q_k, B_n + b_n; P_k, B_n$:

$$\frac{((B_{n+2r-1} + b_{n+2r-1}) - (B_n + b_n))}{Q_{2r-1}} = \frac{(B_{k+2r-1} + B_k)}{P_{2r-1}} = Q_{2k+2r-1};$$
$$\frac{((B_{n+2r} + b_{n+2r}) - (B_k + b_k))}{2P_{2r}} = \frac{(B_{k+2r} + B_k)}{Q_{2r}} = P_{2k+2r}$$
$$\frac{(B_{k+2r-1} - B_k)}{Q_{2r-1}} = P_{2k+2r-1}; \frac{(B_{k+2r} - B_k)}{P_{2r}} = Q_{2k+2r}$$

The following explicit formula for $B_n + b_n$ and B_n are due to Euler:

$$(B_n + b_n) = n_k = \frac{(3+2\sqrt{2})^k + (3-2\sqrt{2})^k - 2}{4} \tag{14}$$

$$B_n = m_k = \frac{(3+2\sqrt{2})^k - (3-2\sqrt{2})^k}{4\sqrt{2}} \tag{15}$$

The following recurrence relations, with initial values 0 and 1, are also given by Euler:

$$(B_{k+1} + b_{k+1}) = n_{k+1} = 6(B_k + b_k) - (B_{k-1} + b_{k-1}) + 2 \tag{16}$$

$$B_{k+1} = m_k = 6B_k - B_{k-1}, k \geq 1 \tag{17}$$

These generating functions define them:

$$f(u) = \frac{1+u}{(1-u)(1-6u+u^2)} = 1 + 8u + 49u^2 + ... \tag{18}$$

$$f(v) = \frac{1}{(1-6v+v^2)} = 1 + 6v + 35v^2 + ... \tag{19}$$

These Lucas-type formula involving binomial coefficients from (12) and (13):

$$(B_n + b_n) = n_k = \sum_{r=1}^{n} 2^{r-1} \binom{2n}{2r} \tag{20}$$

$$B_n = m_k = \sum_{r=1}^{n} 2^{r-2} \binom{2n}{2r-1} \tag{21}$$

The following summation formula holds:

$$2\sum_{r=1}^{n}(B_r + br) = \left\{ \sum_{r=1}^{n} 2^{r-1} \binom{2n+1}{2r} \right\} - n \tag{22}$$

$$2\sum_{r=1}^{n} B_r = \left\{ \sum_{r=1}^{n} 2^{r-1} \binom{2n+1}{2r+1} \right\} + n \tag{23}$$

Now, the triangular square gives the formula:

$$\begin{aligned} N_k = m_k^2 = B_n^2 \qquad &= \left(\frac{(3+2\sqrt{2})^k - (3-2\sqrt{2})^k}{4\sqrt{2}} \right)^2 \\ &= \frac{(17+12\sqrt{2})^k - (17-12\sqrt{2})^k - 2}{32}. \end{aligned} \tag{24}$$

$$B_{n+1}^2 = N_{k+1} = 34B_n^2 - B_{n-1}^2 + 2 \tag{25}$$

Generating function for the square triangular numbers:

$$f(z) = \frac{1+z}{(1-z)(1-34z+z^2)} = 1 + 36z + 1225z^2 + ... \tag{26}$$

A product formula for the kth triangular square is recorded by Weisstein[6]:

$$B_n^2 = N_k = 2^{2n-5} \prod_{k=1}^{2n} \left(3 + cos\frac{k\pi}{k} \right) \tag{27}$$

Since $cos(\pi - \theta) = cos(\pi + \theta)$ and $cos\pi = -1$,$cos2\pi = 1$ then

$$\prod_{k=n+1}^{2n} \left(3 + cos\frac{k\pi}{k} \right) = 2\prod_{k=1}^{n} \left(3 + cos\frac{k\pi}{k} \right)$$

Hence,

$$B_n^2 = N_k = \left(2^{k-2} \prod_{k=1}^{n} \left(3 + cos\frac{k\pi}{k} \right)^2 \right)$$

$$B_n = 2^{k-2} \prod_{k=1}^{n} \left(3 + cos\frac{k\pi}{k} \right) \tag{28}$$

The above-noted formula give these values of Q_k, P_k, $T_{B_n+b_n}$, s_{B_k} and the associated square triangular numbers B_n^2:

K	Q_n	P_n	T_{n_k}	S_{B_n}	B_n^2
0	1	0	T_0	S_0	0
1	1	1	T_1	S_1	1
2	3	2	T_8	S_6	36
3	7	5	T_{49}	S_{35}	1225
4	17	12	T_{288}	S_{204}	41616
5	41	29	T_{1681}	S_{1189}	1413721
6	99	70	T_{9800}	S_{6930}	48024900
7	239	169	T_{57121}	S_{40391}	1631432881
8	577	408	T_{332928}	S_{235416}	55420693056
9	1393	985	$T_{1940449}$	$S_{1372105}$	1882672131025
10	3363	2378	$T_{11309768}$	$S_{7997214}$	63955431761796

The values assumed by P_k are known as Pell Numbers which are related to square triangular numbers:

$$(P_k(P_k + P_{k-1}))^2 = \frac{\{(P_k + P_{k-1})^2 - (-1)^k\} * (P_k + P_{k-1})^2}{2}, k \geq 1$$

There exist infinitely many primitive Pythagorean triples $< a, b, c >$ of positive integers satisfying $a^2 + b^2 = c^2$.These are given by $a = 2st$, $b = s^2 - t^2$, $c = s^2 + t^2$; $s, t \in \mathbb{Z}^+$. Hatch studied special triples with $| a - b |$ to uncover connection between them and triangular squares. Then, $s^2 - t^2 - 2st = \pm 1 \Rightarrow (s + t)^2 - 2s^2 = \mp 1$. One can verify that if $< g, g + 1, h >$ is a special primitive Pythagorean triple, then so is $< 3g + 2h + 1, 3g + 2h + 2, 4g + 3h + 2 >$. Beginning with the primitive triple $< 0, 1, 1 >$, we successively derive $< 3, 4, 5 >$,$< 20, 21, 29 >$, $< 119, 120, 169 >$,... This construction exhausts all such triples. They can also be obtained from any of the following four relations:

$$(Q_k Q_{k-1})^2 + \{Q_k Q_{k-1} + (-1)^k\}^2 = \left(\frac{Q_{2k-1} + Q_{2k-2}}{2}\right)^2, k \geq 1 \tag{29}$$

$$(2P_k P_{k-1})^2 + \{P_k P_{k-1} + (-1)^k\}^2 = (P_{2k-1})^2 \tag{30}$$

$$\left(\frac{((B_n + b_n) - (B_{n-1} + b_{n-1}) - 1)}{2}\right)^2 + \left(\frac{((B_n + b_n) - (B_{n-1} + b_{n-1}) + 1)}{2}\right)^2$$
$$= \left(\frac{((B_n + b_n) + (B_{n-1} + b_{n-1}) + 1)}{2}\right)^2 ; k \geq 1 \tag{31}$$

$$(B_n - B_{n-1})^2 = \left(\frac{7B_{n-1} - B_{n-2} - 1}{2}\right)^2 + \left(\frac{7B_{n-1} - B_{n-2} + 1}{2}\right)^2, k \geq 1, B_{-1} = -1 \tag{32}$$

It can also be shown that there exist infinitely many primitive Pythagorean triples having the property $a = T_k$,$b = T_{k+1}$ and $c = T_{(k+1)^2}$. We know T_{x^2} is a perfect square for infinitely many values of $n = x^2$. In fact, if $< g, g + 1, h >$ forms a Pythagorean triple, then so does $< T_{2g}, T_{2g+1}, (2g + 1)h >$ proving the infinity of triangular squares as on putting $n = hg1$, $m = g\frac{h-1}{2}$, we get $\frac{n(n+1)}{2} = m^2$.

3.1 Some identities : part A

Balancing numbers and the associated numbers discovered this relation between neighbouring Balancing numbers that are perfect squares:

$$(B_{n+1}+b_{n+1}) = 3(B_n+b_n)+2\sqrt{2(B_n+b_n)((B_n+b_n)+1)}+1 \tag{33}$$

$$(B_{n-1}+b_{n-1}) = 3(B_n+b_n)-2\sqrt{2(B_n+b_n)((B_n+b_n)+1)}+1 \tag{34}$$

The following relations holds:

$$2\left\{((B_n+b_n)-(B_{n-1}+b_{n-1}))^2+1\right\} = \{((B_n+b_n)+(B_{n-1}+b_{n-1})+1)\}^2 \tag{35}$$

The following relation holds between two consecutive squares that are Balancing numbers:

$$B_{n+1} = 3B_n+\sqrt{8B_n^2+1} \tag{36}$$

$$B_{n-1} = 3B_n-\sqrt{8B_n^2+1} \tag{37}$$

from (37) the following relation:

$$2\left(B_n-B_{n-1}\right)^2 = \left(B_n+B_{n-1}\right)^2+1 \tag{38}$$

The general recurrence relations regarding (B_n+b_n) and B_n:

$$(B_{n+d}+b_{n+d}) = \{(B_{d+1}-B_{d-1})(B_n+b_n)\}-\{(B_{n-d}+b_{n-d})-2(B_d+b_d)\}, d\geq 1 \tag{39}$$

$$B_{n+d} = \{(B_{d+1}-B_{d-1})B_n\}-B_{n-d}, d\geq 1 \tag{40}$$

The identities involving $2k+1$ number of consecutive values of (B_n+b_n) and B_n:

$$\sum_{j=0}^{2k}(B_{i+j}+b_{i+j})(-1)^j = Q_{k+1}\cdot P_k\left(2B_{i+k}+b_{i+k}\right)+(-1)^k(B_{i+k}+b_{i+k})-r \tag{41}$$

$r=1$ if $2k+1\equiv 3(mod4)$,$r=0$ if $2k+1\equiv 1(mod4)$;$i\geq 1, k\geq 1$.

$$\sum_{j=0}^{2k}(B_{i+j}+b_{i+j}) = (B_n+B_{n+1})(B_{i+k}+b_{i+k})+2\sum_{j=1}^{k}(B_j+bj), i\geq 1, k\geq 1 \tag{42}$$

$$\sum_{j=0}^{2k}B_{i+j}(-1)^j = 2Q_{k+1}\cdot P_k(B_{i+k})+(-1)^k(B_{i+k}), i\geq 1, k\geq 1 \tag{43}$$

$$\sum_{j=0}^{2k}B_{i+j} = (B_k+B_{k+1})(B_{i+k}), i\geq 1, k\geq 1 \tag{44}$$

Putting $i=1$ in (42) and (43),we get these sum formula:

$$\sum_{j=0}^{2k}(B_{1+j}+b_{1+j}) = (B_n+B_{n+1})(B_{1+k}+b_{1+k})+2\sum_{j=1}^{k}(B_j+bj), k\geq 1 \tag{45}$$

$$\sum_{j=0}^{2k}B_{1+j} = (B_k+B_{k+1})(B_{1+k}), k\geq 1 \tag{46}$$

The general formula for $2k$ number of consecutive values of $(B_n + b_n)$ and B_n:

$$\sum_{j=0}^{2k-1} (B_{i+j} + b_{i+j})(-1)^{j+1} = B_k\left((B_{i+k} + b_{i+k}) - (B_{i+k-1} + b_{i+k-1})\right), i \geq 1, k \geq 1 \quad (47)$$

$$\sum_{j=0}^{2k-1} (B_{i+j} + b_{i+j}) = B_k\left((B_{i+k-1} + b_{i+k-1}) + (B_{i+k} + b_{i+k}) + 1\right) - k, i \geq 1, k \geq 1 \quad (48)$$

$$\sum_{j=0}^{2k-1} B_{i+j}(-1)^{j+1} = B_k\left((B_{i+k}) - (B_{i+k-1})\right), i \geq 1, k \geq 1 \quad (49)$$

$$\sum_{j=0}^{2k-1} B_{i+j} = B_k\left((B_{i+k-1}) + (B_{i+k})\right), i \geq 1, k \geq 1 \quad (50)$$

Putting $i = 1$ in (48) and (49),we get these sum formula:

$$\sum_{j=0}^{2k-1} (B_{1+j} + b_{1+j}) = B_k\left((B_n + b_n) + (B_{n+1} + b_{n+1}) + 1\right) - k, k \geq 1 \quad (51)$$

$$\sum_{j=0}^{2k-1} B_{1+j} = B_k\left(B_k + B_{k+1}\right), k \geq 1 \quad (52)$$

We could get (52) by combining the next two results:
Identity: 1.

$$\sum_{j=1}^{k} B_{2j-1} = (B_k)^2, k \geq 1 \quad (53)$$

Proof. We can prove it by induction. The identity is obviously true for $n = 1$. Suppose the identity is true for $n = k$. Then, $\sum_{j=1}^{k} B_{2j-1} = (B_k)^2$ Hence

$$\begin{aligned}
\sum_{j=1}^{k+1} B_{2j-1} &= \sum_{j=1}^{k} B_{2j-1} + B_{2k+1} = (B_k)^2 + B_{2k+1} \\
&= \left(\frac{(3+2\sqrt{2})^k - (3-2\sqrt{2})^k}{4\sqrt{2}}\right)^2 + \frac{(3+2\sqrt{2})^{2k+1} - (3-2\sqrt{2})^{2k+1}}{4\sqrt{2}} \\
&= \frac{(3+2\sqrt{2})^{2k} - 2 + (3-2\sqrt{2})^{2k}}{4\sqrt{2}} + \frac{(3+2\sqrt{2})^{2k+1} - (3-2\sqrt{2})^{2k+1}}{4\sqrt{2}} \\
&= \frac{(3+2\sqrt{2})^{2k} + 4\sqrt{2}(3+2\sqrt{2})^{2k+1}}{32} + \frac{(3-2\sqrt{2})^{2k} - 4\sqrt{2}(3-2\sqrt{2})^{2k+1}}{32} - \frac{2}{32} \\
&= \frac{(3+2\sqrt{2})^{2k}\left[1 + 4\sqrt{2}(3+2\sqrt{2})\right]}{32} + \frac{(3-2\sqrt{2})^{2k}\left[1 - 4\sqrt{2}(3-2\sqrt{2})\right]}{32} - \frac{2}{32} \\
&= \frac{(17+12\sqrt{2})^k\left[17+12\sqrt{2}\right]}{32} + \frac{(17-12\sqrt{2})^k\left[17-12\sqrt{2}\right]}{32} - \frac{2}{32} \\
&\frac{(17+12\sqrt{2})^{k+1} + (17-12\sqrt{2})^{k+1} - 2}{32} = B_{k+1}^2 = (B_{k+1})^2
\end{aligned}$$

Thus if it is true for $n = k$ then it is true for $n = k+1$ also.Hence true for all n.
Identity: 2.

$$\sum_{j=1}^{k} B_{2j} = B_k B_{k+1}, k \geq 1 \tag{54}$$

Proof. We prove it by induction again.The result is true for $n = 1$. Suppose it is true for $n = k$. Then, $\sum_{j=1}^{k} B_{2j} = B_k B_{k+1}$. Hence

$$\begin{aligned}
\sum_{j=1}^{k+1} B_{2j} &= \sum_{j=1}^{k} B_{2j} + B_{2k+2} = B_k B_{k+1} + B_{2k+2} \\
&= \frac{(3+2\sqrt{2})^k - (3-2\sqrt{2})^k}{4\sqrt{2}} * \frac{(3+2\sqrt{2})^{k+1} - (3-2\sqrt{2})^{k+1}}{4\sqrt{2}} \\
&+ \frac{(3+2\sqrt{2})^{2k+2} - (3-2\sqrt{2})^{2k+2}}{4\sqrt{2}} \\
&= \frac{(3+2\sqrt{2})^{2k+1} - (3-2\sqrt{2}) - (3+2\sqrt{2}) + (3-2\sqrt{2})^{2k+1}}{32} \\
&+ \frac{(3+2\sqrt{2})^{2k+2} - (3-2\sqrt{2})^{2k+2}}{4\sqrt{2}} \\
& \frac{(3+2\sqrt{2})^{2k+1} - 6 + (3-2\sqrt{2})^{2k+1}}{32} + \frac{(3+2\sqrt{2})^{2k+2} - (3-2\sqrt{2})^{2k+2}}{4\sqrt{2}} \\
&= \frac{(3+2\sqrt{2})^{2k+1} + 4\sqrt{2}(3+2\sqrt{2})^{2k+2} - 6}{32} \\
&+ \frac{(3-2\sqrt{2})^{2k+1} - 4\sqrt{2}(3-2\sqrt{2})^{2k+2}}{32} \\
&= \frac{(3+2\sqrt{2})^{2k+1}\left[1 + 4\sqrt{2}(3+2\sqrt{2})\right] - 6}{32} \\
&+ \frac{(3-2\sqrt{2})^{2k+1}\left[1 - 4\sqrt{2}(3-2\sqrt{2})\right]}{32} \\
&= \frac{(3+2\sqrt{2})^{2k+1}\left[17 + 12\sqrt{2}\right] - 6 + (3-2\sqrt{2})^{2k+1}\left[17 - 12\sqrt{2}\right]}{32} \\
&= \frac{(3+2\sqrt{2})^{2k+3} - 6 + (3-2\sqrt{2})^{2k+3}}{32} \\
&= \frac{(3+2\sqrt{2})^{k+1} - (3-2\sqrt{2})^{k+1}}{4\sqrt{2}} * \frac{(3+2\sqrt{2})^{k+2} - (3-2\sqrt{2})^{k+2}}{4\sqrt{2}} \\
&= B_{k+1} B_{k+2}.
\end{aligned}$$

Thus if it is true for $n = k$ then it is true for $n = k+1$ also. Hence true for all n. The interesting sum formula holds:

$$6 \sum_{j=0}^{[\frac{k-1}{2}]} B_{2k-4j-1} = B_k B_{k+1}, k \geq 1 \tag{55}$$

[] is the greatest integer function.

$$6 \sum_{r=1}^{k} B_{2r} = 36 \sum_{r=1}^{[\frac{k+1}{2}]} B_{2k-4r+3} = B_k^2 + B_{k+1}^2 - 1 \tag{56}$$

The identities expressing n and m in terms of all preceding values:

$$(B_i + b_i) = 5(B_{i-1} + b_{i-1}) + 4\sum_{j=1}^{i-2}(B_j + b_j) + (2i-1), i \geq 1. \quad (57)$$

$$B_i = 5B_{i-1} + 4\sum_{j=1}^{i-2} B_j + 1, i \geq 1 \quad (58)$$

This identity is more interesting.
Identity: 3.

$$B_{2r-1} = (B_{2r})^2 - (B_{r-1})^2 \quad (59)$$

Proof. We can establish it by simply manipulating the defining formula.

$$\begin{aligned}
R.H.S &= \left(\frac{(3+2\sqrt{2})^r - (3-2\sqrt{2})^r}{4\sqrt{2}}\right)^2 - \left(\frac{(3+2\sqrt{2})^{r-1} - (3-2\sqrt{2})^{r-1}}{4\sqrt{2}}\right)^2 \\
&= \frac{(3+2\sqrt{2})^{2r} - 2 + (3-2\sqrt{2})^{2r}}{32} - \frac{(3+2\sqrt{2})^{2r-2} - 2 + (3-2\sqrt{2})^{2r-2}}{32} \\
&= \frac{(3+2\sqrt{2})^{2r} - (3+2\sqrt{2})^{2r-2} + (3-2\sqrt{2})^{2r} - (3-2\sqrt{2})^{2r-2}}{32} \\
&= \frac{(3+2\sqrt{2})^{2r-2}\left[(3+2\sqrt{2})^2 - 1\right] + (3-2\sqrt{2})^{2r-2}\left[(3-2\sqrt{2})^2 - 1\right]}{32} \\
&= \frac{(3+2\sqrt{2})^{2r-2}(16+12\sqrt{2}) + (3-2\sqrt{2})^{2r-2}(16-12\sqrt{2})}{32} \\
&= \frac{(3+2\sqrt{2})^{2r-2}(4+3\sqrt{2}) + (3-2\sqrt{2})^{2r-2}(4-3\sqrt{2})}{8} \\
&= \frac{(3+2\sqrt{2})^{2r-2}(2\sqrt{2}+3) + (3-2\sqrt{2})^{2r-2}(2\sqrt{2}-3)}{4\sqrt{2}} \\
&= \frac{(3+2\sqrt{2})^{2r-1} - (3-2\sqrt{2})^{2r-1}}{4\sqrt{2}} = B_{2r-1} \\
&= L.H.S
\end{aligned}$$

Identity: 4.

$$B_{2r} = B_r\left(B_{r+1} - B_{r-1}\right) \quad (60)$$

Proof. We establish it with the help of the method employed above.

$$\begin{aligned}
R.H.S &= \frac{(3+2\sqrt{2})^r - (3-2\sqrt{2})^r}{4\sqrt{2}}\left(\frac{(3+2\sqrt{2})^{r+1} - (3-2\sqrt{2})^{r+1}}{4\sqrt{2}} - \frac{(3+2\sqrt{2})^{r-1} - (3-2\sqrt{2})^{r-1}}{4\sqrt{2}}\right) \\
&= \frac{(3+2\sqrt{2})^r - (3-2\sqrt{2})^r}{4\sqrt{2}}\frac{(3+2\sqrt{2})^{r-1}\left[(3+2\sqrt{2})^2 - 1\right]}{4\sqrt{2}} - \frac{(3-2\sqrt{2})^{r-1}\left[(3-2\sqrt{2})^2 - 1\right]}{4\sqrt{2}} \\
&= \frac{(3+2\sqrt{2})^r - (3-2\sqrt{2})^r}{4\sqrt{2}}\frac{(3+2\sqrt{2})^{r-1}(4+3\sqrt{2})}{\sqrt{2}} - \frac{(3-2\sqrt{2})^{r-1}(4-3\sqrt{2})}{\sqrt{2}} \\
&= \frac{(3+2\sqrt{2})^r - (3-2\sqrt{2})^r}{4\sqrt{2}}[(3+2\sqrt{2})^{r-1}(2\sqrt{2}+3) - (3-2\sqrt{2})^{r-1}(2\sqrt{2}-3)] \\
&= \frac{(3+2\sqrt{2})^{2r} - (3-2\sqrt{2})^{2r}}{4\sqrt{2}} = B_{2r} \\
&= L.H.S
\end{aligned}$$

Identity: 5.

$$(B_{2r-1}+b_{2r-1}) = ((B_r+b_r)-(B_{r-1}+b_{r-1}))^2 \tag{61}$$

Proof.

$$\begin{aligned}
R.H.S &= \left(\frac{(3+2\sqrt{2})^r+(3-2\sqrt{2})^r-2}{4}-\frac{(3+2\sqrt{2})^{r-1}+(3-2\sqrt{2})^{r-1}-2}{4}\right)^2 \\
&= \left(\frac{(3+2\sqrt{2})^r-(3+2\sqrt{2})^{r-1}}{4}+\frac{(3-2\sqrt{2})^r-(3-2\sqrt{2})^{r-1}}{4}\right)^2 \\
&= \left(\frac{(3+2\sqrt{2})^{r-1}(3+2\sqrt{2}-1)}{4}+\frac{(3-2\sqrt{2})^{r-1}(3-2\sqrt{2}-1)}{4}\right)^2 \\
&= \left(\frac{(3+2\sqrt{2})^{r-1}(1+\sqrt{2})}{2}+\frac{(3-2\sqrt{2})^{r-1}(1-\sqrt{2})}{2}\right)^2 \\
&= \frac{(3+2\sqrt{2})^{2r-2}(1+\sqrt{2})^2}{4}+\frac{(3-2\sqrt{2})^{2r-2}(1-\sqrt{2})^2}{4} \\
&+2*\frac{(3+2\sqrt{2})^{r-1}(1+\sqrt{2})}{2}*\frac{(3-2\sqrt{2})^{r-1}(1+\sqrt{2})}{2} \\
&= \frac{(3+2\sqrt{2})^{2r-1}+(3-2\sqrt{2})^{2r-1}}{4} = (B_{2r-1}+b_{2r-1}) \\
&= L.H.S
\end{aligned}$$

Identity: 6.

$$(B_{2r}+b_{2r}) = (2(B_r+b_r)+1)^2-1, r \geq 1 \tag{62}$$

Proof.

$$\begin{aligned}
R.H.S &= (2(B_r+b_r)+1)^2-1 \\
&= \left(2\frac{(3+2\sqrt{2})^r+(3-2\sqrt{2})^r-2}{4}+1\right)^2-1 \\
&= \left(\frac{(3+2\sqrt{2})^r+(3-2\sqrt{2})^r-2+2}{2}\right)^2-1 \\
&= \frac{(3+2\sqrt{2})^{2r}+(3-2\sqrt{2})^{2r}-2}{4} = (B_{2r}+b_{2r}) = L.H.S
\end{aligned}$$

$$(B_{3r}+b_{3r}) = (B_r+b_r)\left(4\left(B_r+b_r\right)+3\right)^2 \tag{63}$$

Identity: 7.

$$B_{3r} = B_r\left(2^5{B_r}^2+3\right) \tag{64}$$

Proof.

$$\begin{aligned}
R.H.S &= \frac{(3+2\sqrt{2})^r-(3-2\sqrt{2})^r}{4\sqrt{2}}\left\{32\left(\frac{(3+2\sqrt{2})^r-(3-2\sqrt{2})^r}{4\sqrt{2}}\right)^2+3\right\} \\
&= \frac{1}{4\sqrt{2}}\left\{(3+2\sqrt{2})^r-(3-2\sqrt{2})^r\right\}\left\{(3+2\sqrt{2})^{2r}+(3-2\sqrt{2})^{2r}+1\right\} \\
&= \frac{1}{4\sqrt{2}}\left\{(3+2\sqrt{2})^{3r}-(3-2\sqrt{2})^{3r}\right\} = B_{3r} = L.H.S
\end{aligned}$$

The general identity for $B_{(2n+1)r}$ is

$$\begin{aligned} B_{(2n+1)r} &= 2^5 B_r^{2n+1} + 2^{5(n-1)}(2n+1)B_r^{2n-1} \\ &+ \left[(2n+1)\left\{\sum_{k=1}^{n-1} 2^{5(n-k-1)} \frac{P_k}{(k+1)!} B_r^{2(n-k)-1}\right\}\right] \end{aligned}$$

where the product

$$P_k = \prod_{j=1}^{k} 2(n-k)+j-1 \tag{65}$$

some interesting relations between n and m:

$$(B_i + b_i) + (B_{i-2j+1} + b_{i-2j+1}) = 2\left(B_j - B_{j-1}\right)\left(B_{i-j+1} - B_{i-j}\right) - 1, j \geq 1, i \geq 2j \tag{66}$$

$$B_i + B_{i-2j+1} = \left(B_j - B_{j-1}\right)\left((B_{i-j+1} + b_{i-j+1}) - (B_{i-j} + b_{i-j})\right) j \geq 0, i \geq 2j \tag{67}$$

$$((B_{i+j} + b_{i+j}) + (B_{i-j} + b_{i-j})) = (2(B_j + b_j) + 1)\left(3B_i - B_{i-1}\right) - 1 \tag{68}$$

$$((B_{i+j} + b_{i+j}) - (B_{i-j} + b_{i-j})) = 8B_iB_j; i > j > 0 \tag{69}$$

$$B_{i+2j-1} - B_{i-2j+1} = 2B_{2j-1}\left(2(B_i + b_i) + 1\right), j \geq 0, i \geq 2j \tag{70}$$

$$(B_i + b_i) = B_i + 2\sum_{j=1}^{i-1} B_i, i \geq 1 \tag{71}$$

3.2 Identities: part B

We consider the triangular squares. Neighbouring triangular squares can be expressed in terms of each other:

$$B_{n+1}^2 = 6\sqrt{B_n^2(8B_n^2+1)} + 17B_n^2 + 1 \tag{72}$$

$$B_{n-1}^2 = 17B_n^2 - 6\sqrt{B_n^2(8B_n^2+1)} + 1 \tag{73}$$

As B_n^2 is both a square and triangular number, the quantity under the square root above is an integer.equation (72) and equation (73), obtained from (2), can be proved otherwise. The relation from (72):

Identity: 1.

$$\left\{9\left\{8\left(B_n^2 - B_{n-1}^2\right)\right\}^2 + 1\right\} = \left\{8(B_n^2 + B_{n-1}^2) + 1\right\}^2 \tag{74}$$

Proof. We establish it by using definition and the identity to come later with proof.

$$
\begin{aligned}
R.H.S \quad &= \left(8\frac{(17+12\sqrt{2})^k+(17-12\sqrt{2})^k-2}{32}+8\frac{(17+12\sqrt{2})^{k-1}+(17-12\sqrt{2})^{k-1}-2}{32}+1\right)\\
&= \left(\frac{(17+12\sqrt{2})^{k-1}(17+12\sqrt{2}+1)}{4}+\frac{(17-12\sqrt{2})^{k-1}(17-12\sqrt{2}+1)^{k-1}}{4}\right)^2\\
&= 9\left(\frac{(17+12\sqrt{2})^{k-1}(3+2\sqrt{2})}{2}+\frac{(17-12\sqrt{2})^{k-1}(3-2\sqrt{2})}{2}\right)^2\\
&= 9\left(\frac{(3+2\sqrt{2})^{2k-1}}{2}+\frac{(3-2\sqrt{2})^{2k-1}}{2}\right)^2\\
&= 9\left\{\frac{(3+2\sqrt{2})^{2(2k-1)}+2+(3-2\sqrt{2})^{2(2k-1)}}{4}\right\}\\
&= 9\left\{8\frac{(3+2\sqrt{2})^{2(2k-1)}-2+(3-2\sqrt{2})^{2(2k-1)}}{32}+1\right\}\\
&= 9\left\{8\frac{(17+12\sqrt{2})^{2k-1}+(17-12\sqrt{2})^{2k-1}-2}{32}+1\right\}\\
&= 9\left\{8B_{2k-1}^2+1\right\} = 9\left\{8\left(B_k^2-B_{k-1}^2\right)^2+1\right\} = L.H.S
\end{aligned}
$$

This relation can be verified easily using (55) and (56):

$$B_n^2+B_{n-1}^2 = 6*\sqrt{B_n^2B_{n-1}^2}+1 \tag{75}$$

The general recurrence relation:

$$B_{n+d}^2 = \left\{\left(B_{2d+1}-B_{2d+1}\right)B_n^2\right\}-B_{n-d}^2+2B_d^2, d\geq 1 \tag{76}$$

The following relation holds between four consecutive triangular squares:

$$B_{n+1}^2-B_{n-2}^2 = 35\left(B_n^2-B_{n-1}^2\right) \tag{77}$$

We now have identities involving many more consecutive triangular squares:

$$\sum_{j=0}^{4k-2} B_{i+j}^2(-1)^j = B_n\left(16B_{i+k}^2+1\right)+B_{i+k}^2; i,k\geq 1 \tag{78}$$

$B_n^{'} = 2\sum_{i=1}^{k} B_{4i-3}$; alternatively, $B_n^{'} == 2\left(B_{2n}^2-\sum_{i=1}^{k} B_{4i-3}\right)$.

$$\sum_{j=0}^{2k-1} B_{i+j}^2(-1)^{j+1} = B_{2n}\left(B_{i+k}^2-B_{i+k-1}^2\right), i\geq 1, k\geq 2 \tag{79}$$

$B_{2n} = B_n^2-2\sum_{j=0}^{[\frac{k-2}{2}]} B_{2k-4j-3}$;[ ] denotes the greatest integer function;$k\geq 2$. Alternatively, $B_{2n} = \left|\sum_{j=1}^{k} B_{2j-1}(-1)^j\right|; k\geq 2$.

We next calculate the even order Balancing numbers. We shall prove $B_{2k} : B_{2k+2} = B_{2k-2} + 32B_n^2 + 2; k \geq 1$. Observe that $B_{4k-2} - 1 = B_{2k-2}\left(B_{2k+1} - B_{2k-1}\right); k \geq 2; B_{4k} = B_{2k} * \left(B_{2k+1} - B_{2k-1}\right) 2; k \geq 2$

Putting $i = 1$ in (79) gives the difference of the first $2k$ triangular squares:

$$\sum_{j=0}^{2k-1} B_{1+j}^2(-1)^{j+1} = B_{2n}\left(B_{1+k}^2 - B_k^2\right), i \geq 1, k \geq 2 \tag{80}$$

We have these general sum formula:

$$\sum_{j=0}^{2k} B_{i+j}^2 = B_{2k+1}B_{i+k}^2 + 2\sum_{j=1}^{k} B_j^2, i \geq 1, k \geq 1 \tag{81}$$

$$\sum_{j=0}^{2k-1} B_{i+j}^2 = B_{2k}\left(B_{i+k-1}^2 + B_{i+k}\right) + 4B_{k-1}^2, i \geq 1, k \geq 2 \tag{82}$$

Putting $i = 1$ in the preceding identities gives the sum of first $2k+1/2k$ triangular squares:

$$\sum_{j=0}^{2k} B_{1+j}^2 = B_{2k+1}B_{1+k}^2 + 2\sum_{j=1}^{k} B_j^2, k \geq 1 \tag{83}$$

$$\sum_{j=0}^{2k-1} B_{1+j}^2 = \left\{B_n^2 - 2\sum_{j=0}^{[\frac{k-2}{2}]} B_{2k-4j-3}\right\}\left(B_n^2 + B_{n+1}^2\right) + 4B_{n-1}^2, k \geq 2 \tag{84}$$

$$\sum_{j=0}^{2k-1} B_{1+j}^2 = \left|\sum_{j=1}^{k} B_{2j-1}(-1)^j\right|\left(B_n^2 + B_{n+1}^2\right) + 4B_{n-1}^2, k \geq 2 \tag{85}$$

I found this relation expressing N in terms of all preceding values:

$$B_n^2 = 33B_{n-1}^2 + 32\sum_{j=1}^{k-2} B_j^2 + (2k-1), k \geq 1. \tag{86}$$

Identity:2.

$$B_{2r-1}^2 = \left(B_r^2 - B_{r-1}^2\right)^2 \tag{87}$$

Proof.

$$\begin{aligned}
R.H.S &= \left(\frac{(17+12\sqrt{2})^r + (17-12\sqrt{2})^r - 2}{32} - \frac{(17+12\sqrt{2})^{r-1} + (17-12\sqrt{2})^{r-1} - 2}{32}\right)^2 \\
&= \left(\frac{(17+12\sqrt{2})^{r-1}(17+12\sqrt{2}-1)}{32} - \frac{(17-12\sqrt{2})^{r-1}(17-12\sqrt{2}-1)}{32}\right)^2 \\
&= \left(\frac{(17+12\sqrt{2})^{r-1}(4+3\sqrt{2})}{8} - \frac{(17-12\sqrt{2})^{r-1}(4-3\sqrt{2})}{8}\right)^2 \\
&= \frac{(17+12\sqrt{2})^{2r-1} + (17-12\sqrt{2})^{2r-1} - 2}{32} = B_{2r-1}^2 = L.H.S.
\end{aligned}$$

Identity:3.

$$B_{2r}^2 = 4B_r^2\left(8B_r^2+1\right), r \leq 1. \tag{88}$$

Proof.

$$\begin{aligned}
R.H.S \quad &= 32B_r^2 + 4B_r^2 \\
&= 32\left(\frac{(17+12\sqrt{2})^r-(17-12\sqrt{2})^r-2}{32}\right)^2 + 4\left(\frac{(17+12\sqrt{2})^r-(17-12\sqrt{2})^r-2}{32}\right) \\
&= \frac{(17+12\sqrt{2})^{2r}+(17-12\sqrt{2})^{2r}+4+2(17+12\sqrt{2})^r(17-12\sqrt{2})^r}{32} \\
&-\frac{4(17-12\sqrt{2})^r-4(17+12\sqrt{2})^r}{32} + \left(\frac{(17+12\sqrt{2})^r+(17-12\sqrt{2})^r-2}{8}\right) \\
&= \left(\frac{(17+12\sqrt{2})^{2r}+(17-12\sqrt{2})^{2r}-2}{32}\right) - \left(\frac{(17+12\sqrt{2})^r+(17-12\sqrt{2})^r-2}{8}\right) \\
&+\left(\frac{(17+12\sqrt{2})^r+(17-12\sqrt{2})^r-2}{8}\right) = B_{2r}^2 = L.H.S
\end{aligned}$$

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