SOME STUDIES ON CONTROL SYSTEM USING LIE GROUPS

A PROJECT REPORT

submitted by

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for

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of

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under the supervision

of

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CERTIFICATE

This is to certify that the project report entitled **SOME STUDIES ON CONTROL SYSTEM USING LIE GROUPS** submitted by **Prasadini Mahapatra** to the National Institute of Technology Rourkela, Odisha for the partial fulfilment of requirements for the degree of master of science in Mathematics is a bonafide record of review work carried out by her under my supervision and guidance.

May, 2014

Prof. Kishor Chandra Pati
Department of Mathematics
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Contents

Abstract

This project related to control systems, Lie groups and Lie algebra. Firstly, we have discussed about control systems with its examples.Then various types of control systems and its applications. Control theory has continued to advance with advancing technology and has emerged in modern times as a highly developed discipline. Lie theory, the theory of Lie groups, Lie algebras and their applications is a fundamental part of mathematics. On this project review Lie groups controllability for left-invariant control systems on Lie groups are addressed.

SOME STUDIES ON CONTROL SYSTEM USING LIE GROUPS

Chapter-1

1 INTRODUCTION

Mathematical control theory is the area of application-oriented mathematics that treats the basic mathematical principles, theory, and problems underlying the analysis and design of control systems, principally those encountered in engineering. To control an object means to influence its behaviour so as to achieve a desired goal.

One major branch of control theory is optimization. One assumes that a good model of the control system is available and seeks to optimize its behaviour in some sense. Another major branch treats control systems for which there is uncertainty about the model or its environment. The central tool is the use of feedback in order to correct deviations from and stabilize the desired behaviour. Control theory has its roots in the classical calculus of variations, but came into its own with the advent of efforts to control and regulate machinery and to develop steering systems for ships and much later for planes, rockets, and satellites. During the 1930s, researchers at Bell Telephone Laboratories developed feedback amplifiers, motivated by the goal of assuring stability and appropriate response for electrical circuits. During the Second World War various military implementations and applications of control theory were developed. The rise of computers led to the implementation of controllers in the chemical and petroleum industries. In the 1950s control theory blossomed into a major field of study in both engineering and mathematics, and powerful techniques were developed for treating general multi variable, time-varying systems. Control theory has continued to advance with advancing technology, and has emerged in modern times as a highly developed discipline.

Lie theory, the theory of Lie groups, Lie algebras, and their applications, a fundamental part of mathematics that touches on a broad spectrum of mathematics, including geometry ordinary and partial differential equations, group, ring, and algebra theory, complex and harmonic analysis, number theory, and physics. It typically relies upon an array of substantial tools such as topology, differentiable manifolds and differential geometry, covering spaces, advanced linear algebra, measure theory, and group theory to name a few. However, we will considerably simplify the approach to Lie theory by restricting our attention to the most important class of examples, namely those Lie groups that can be concretely realized as (multiplicative) groups of matrices. Lie theory began in the late nineteenth century, primarily through the work of the Norwegian mathematician Sophus Lie, who called them continuous groups," in contrast to the usually finite permutation groups that had been principally studied up to that point. An early major success of the theory was to provide a viewpoint for a systematic understanding of the newer geometries such as hyperbolic, elliptic, and projective, that had arisen earlier in the century. This led Felix Klein in his Erlanger Programm to propose that geometry should be understood as the study of quantities or properties left invariant under an appropriate group of geometric

transformations.
In the early twentieth century Lie theory was widely incorporated into modern physics, beginning with Einstein's introduction of the Lorentz transformations as a basic feature of special relativity. Since these early beginnings research in Lie theory has burgeoned and now spans a vast and enormous literature. The essential feature of Lie theory is that one may associate with any Lie group G a Lie algebra g. The Lie algebra g is a vector space equipped with a bilinear nonasssociative anticommutative product, called the Lie bracket or commutator and usually denoted [.,.]. The crucial and rather surprising fact is that a Lie group is almost completely determined by its Lie algebra g. There is also a basic bridge between the two structures given by the exponential map $exp : g \to G$. For many purposes structure questions or problems concerning the highly complicated nonlinear structure G can be translated and reformulated via the exponential map in the Lie algebra g, where they often lend themselves to study via the tools of linear algebra (in short, nonlinear problems can often be linearized). This procedure is a major source of the power of Lie theory.
The two disciplines, control theory and Lie theory, come together in certain interesting classes of control problems that can be interpreted as problems on Lie groups or their coset spaces. In this case the states of the system are modeled by members of the Lie group and the controls by members of the Lie algebra, interpreted as invariant vector fields on the Lie group. There are significant advantages to interpreting problems in this framework whenever possible; these advantages include the availability of a rich arsenal of highly developed theoretical tools from Lie group and Lie algebra theory. In addition, Lie groups typically appear as matrix groups and one has available the concrete computational methods and tools of linear algebra and matrix theory.

Chapter-2

2 CONTROL SYSTEMS

A control system consists of
a.Inputs, which are things that we can not only measure, but to which we can assign chosen values (constants or functions of time).
Examples: Drug dosages and treatment regimens.
b.Outputs, which are things that we can measure, but to which we cannot assign values.
Examples: Concentrations of administered drug in urine, blood, etc.
c.States, which are things that effect the outputs, but which cannot even measure because we cannot directly access them.
Examples: Concentrations of drug in targeted organ.

2.1 Abstract Depiction of a Control System

Control systems may be thought of as having four functions, measure,compare, compute and correct. The main components of a control system are plant and a sensor which is used to measure the variable to be controlled.Control theory has two main roots, regulation and trajectory optimization.
(i)**Regulation**
Regulation is the more important and engineering oriented one.The problem of regulation is to design mechanisms that keep contain to be controlled variables at constant values against external disturbances.
(ii)**Trajectory Optimization**
This is the Mathematics based.The problem of Trajectory transfer is the question of determining the paths of a dynamical system that transfer the system from a given initial to a prescribed terminal state.A beautiful example of such a problem is the brachystochrone problem that was posed by Johann Bernoulli in 1696.
From the Greek word "brachystochrone" means shortest time.We suppose that a particle of mass m moves along some curve under the influence of gravity. we'll assume motion in two dimension here and that the particle moves, starting at rest, from fixed point a to fixed point b. we could imagine that the particle is a bead that that moves along a rigid wire without friction. The question is that what is the shape of the wire for which the time to get from a to b is minimized.

2.2 Examples of Control Systems

Traditional examples:
Thermostats for controlling room (or furnace) temperature Pilot control systems for fighter

planes (which are statically unstable)

Automotive: anti-skidding, fuel efficiency, etc.

Biological examples:
Gene regulatory networks Insulin delivery and blood glucose regulation system (natural and artificial)
Any orally administered drug passing through the body.

Some familiar examples of control systems
Autofocus mechanism in cameras
Cruise control system in cars
Thermostat temperature control systems

Classes of Control systems
We can classify control systems according to

Nature of time: continuous-time versus discrete time
Nature of quantities: continuous-state versus discrete state
Nature of behaviour: Deterministic versus stochastic

2.3 Different types of control system

All our tools and machines need appropriate control to work, otherwise it will be difficult to finish their designated tasks accurately. Therefore, we need control systems to guide, instruct and regulate our tools and machines. Common control systems include mechanical, electronic, pneumatic and computer aided. A system usually contains three main parts: input, process and output.

(a) Mechanical system
A mechanical system is a device made up of various mechanical parts. Its input is provided by an effort. Once the effort is applied, it can set off a motion to move a load. The force applied to the load is the output of the mechanical system. Examples of mechanical systems include levers, gears and shafts.Some examples of mechanical systems are Can opener and Corkscrew etc.

(b) Electronic system
An electronic system is a system that employs electronic signals to control devices, such as radios, calculators, video game machines, mobile phones, portable computers, etc. The input of an electronic system is provided by electronic signals. After they are processed, they can generate output signals, which control the operation of various devices, such as amplifiers and LCD. Electronic systems can carry out many different tasks, such as generat-

ing sound, transmitting information, displaying video, measuring, memorising, calculating, etc. Common examples of electronic devices include semi-conducting diode, transistors, capacitors that they are usually welded onto electronic circuit boards.
Examples of electronic systems are Mobile phone, Portable computer, Electronic circuit board.

(c) Computer control system
A computer control system uses a computer to control its output devices according to different input signals. Its function is similar to that of an electronic system. Yet a computer control system can use high speed calculation to process large volume of input signals within a very short time, and then generates appropriate outputs with the help of preset programs. Examples of computer control systems include computer numerical control press brakes, computer controlled home appliances, computer controlled underground railway systems, etc.
Examples of computer control systems are CNC press brake, a proposed computer controlled home appliances etc.

(d) Pneumatic system
A pneumatic system is a system that uses compressed air to transport and control energy. Air is first pressurized to give energy in the cylinder. Then signals are input into the system through the use of switches. Next, air is transferred through sealed pipes to the pneumatic parts for processing. Finally, the force produced by the pneumatic parts is utilized to finish the designated task. The use of pneumatic systems is very extensive, for example, in controlling the movement of train doors, the operation of automatic production lines and mechanical clamps, etc.
Examples of pneumatic systems are Production line of CD-ROM, Mechanical clamp etc.

(e) Other systems
There exist many other control systems apart from the ones mentioned above, for example, mail processing systems, commercial operation systems, etc. The input, process and output of different systems have different properties. In this chapter, we will discuss some of the most common control systems.

2.4 Open loop and closed loop control system

There are basically two types of control system: the open loop system and the closed loop system. They can both be represented by block diagrams.

1. **Open loop control system**

Fig. 1 Block diagram of an open loop control system

When an input signal directs the control element to respond, an output will be produced. Examples of the open loop control systems include washing machines, light switches, gas ovens, etc.
The operation of a washing machine does not depend on the cleanness of the clothes. Both the structure and the control process of an open loop control system are very simple, but the result of the output depends on whether the input signal is appropriate or not.

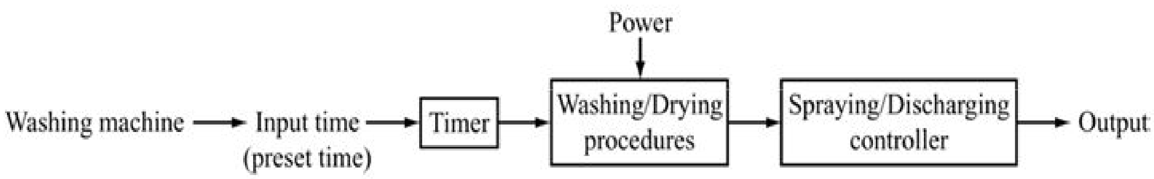

Fig. 2 Block diagram of an open loop control system (washing machine)

Another example of an open loop control system is the burglar alarm system. The function of the sensor is to collect data regarding the concerned house.When the electronic sensor is triggered off (for example, by the entry of an unauthorized person), it will send a signal to the receiver. The receiver will then activate the alarm, which will in turn generate an alarm signal.

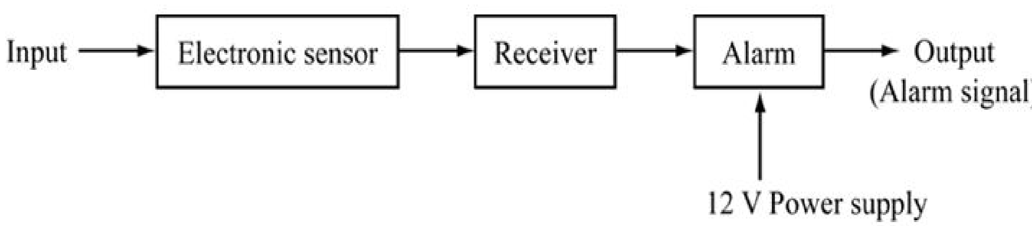

Fig. 3 Block diagram of an open loop control system (burglar alarm)

DRAWBACKS
The drawback of an open loop control system is that it is incapable of making automatic adjustments. Even when the magnitude of the output is too big or too small, the system will not make the appropriate adjustments. For this reason, an open loop control system is not suitable for use as a complex control system.

2. Closed loop control system

Sometimes, we may use the output of the control system to adjust the input signal. This is called feedback. Feedback is a special feature of a closed loop control system. A closed loop control system compares the output with the expected result or command status, then it takes appropriate control actions to adjust the input signal. Therefore, a closed loop system is always equipped with a sensor, which is used to monitor the output and compare it with the expected result.The output signal is fed back to the input to produce a new output.

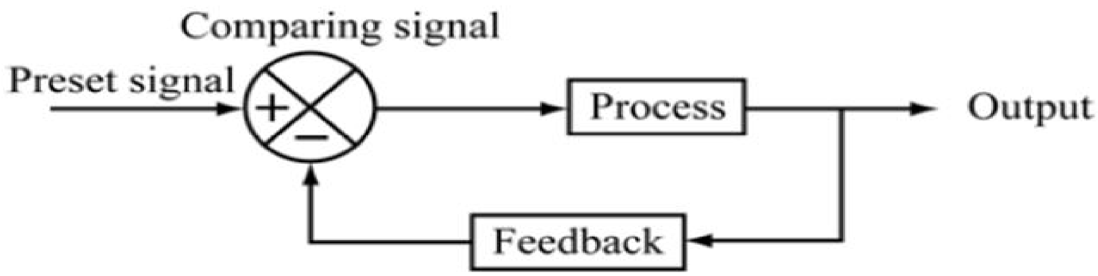

Fig. 4 Block diagram of a closed loop control system

Feedback can be divided into positive feedback and negative feedback. Positive feedback causes the new output to deviate from the present command status. For example, an amplifier is put next to a microphone, so the input volume will keep increasing, resulting in a very high output volume. Negative feedback directs the new output towards the present command status, so as to allow more sophisticated control. For example, a driver has to steer continuously to keep his car on the right track.

ADVANTAGES:
It is able to adjust its output automatically by feeding the output signal back to the input. When the load changes, the error signals generated by the system will adjust the output. However, closed loop control systems are generally more complicated and thus more expensive to make.

2.5 Applications of the control systems

There are many household and industrial application examples of the control systems, such as washing machine, air conditioner, security alarm system and automatic ticket selling machine, etc.

(i) Washing machine

Nowadays, many families use fully automatic washing machines. There are numerous preset washing procedures available for the users. When we have chosen the suitable washing procedures,the machine automatically starts to pour water, add washing powder, spin and wash clothes, discharge waste water, etc. After the completion of all the procedures, the washing machine will stop the operation. Fully automatic washing machine only requires the user to input a suitable procedure to complete the whole washing process, thus this saves much time for the users. However, this kind of machine only operates according to the preset time to complete the whole washing process. It ignores the cleanness of the clothes and does not generate feedback. Therefore, this kind of washing machine is of open loop control system indeed, and their block diagram of control system of the washing machine as shown in Figure.

Fig. 5 Block diagram of the open loop control system of the washing machine

(ii) Air conditioner

Nowadays, there are many families using automatic control system for the temperature of the air conditioner. Fig. 17 shows the interior structure of an air conditioner. The coolant circulated in the machine will absorb heat indoor, then it will be transported from the vaporization device to cooling device. The hot air is then blown to outdoor by a fan. There is an adjustable temperature device equipped in the air conditioner for the users to adjust the extent of cooling. When the temperature of the cool air is lower than the preset one, the controller of the air conditioner will stop the operation of the compressor to cease the circulation of the coolant. The temperature sensor installed near the vaporization device will continuously measure the indoor temperature, and send the results to the controller for further processing.

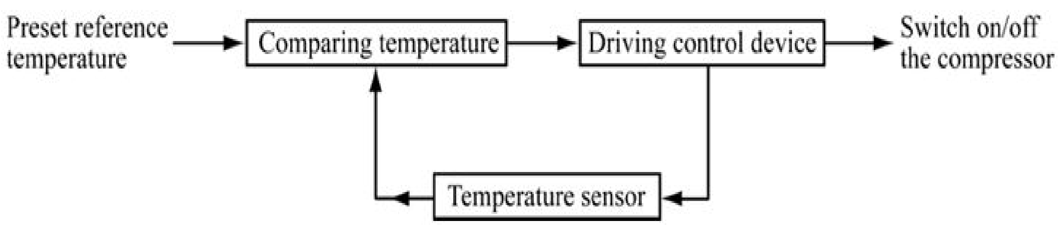

Fig. 6 Block diagram of the closed loop control system of the washing machine

Chapter-3

3 LIE GROUP AND LIE ALGEBRA

3.1 Lie Group

Lie groups are smooth manifolds and as such can be studied using differential calculus, in contrast with the case of more general topological groups. One of the key ideas in the theory of Lie groups is to replace the global object, the group, with its local or linearised version, which Lie himself called its "infinitesimal group" and which has since become known as its Lie algebra.

DEFINITION: A Lie group is defined as a topological group whose identity element has a neighbourhood that is homeomorphic to a subset of an r-dimensional Euclidean space, where r is then called the order or dimension of the Lie group.
On mathematically,

A set G is called a Lie group if:
(1) G is a smooth manifold,
(2) G is a group, and
(3) the group operations in G are smooth.
For example, $GL(n)$ is a Lie group.
The most important class of Lie groups is formed by linear Lie groups, i.e., groups of linear transformations of R^n.

3.2 Linear Lie Groups

A Lie group $G \subset M(n)$ is called a linear Lie group. The sufficient condition for a set of matrices to form a linear Lie group is G is a closed subgroup of GLn.

3.3 Examples of Lie groups

a. General Linear Group($GL(n)$)
The general linear group consists of all $n \times n$ invertible matrices:

$$GL(n, R) = GL(n) = \{X \in M(n) | detX \neq 0\}$$

To show $GL(n)$ is a Lie group
1. $GL(n)$ is a manifold
$det : M(n) \rightarrow R$ is continuous, by this continuity the set GLn is an open domain thus a smooth sub manifold in the linear space $M(n)$.

2. GLn is a group with respect to matrix product.If $X, Y \in GL(n)$, then the product $XY \in GL(n)$. Further, the identity matrix $Id = \delta_{ij}$ is contained in $GL(n)$.
where, $\delta_{ij} = \begin{Bmatrix} 1, if i = j \\ 0, if i \neq j \end{Bmatrix}$.
3. The group operations in $GL(n)$ are smooth.
$(X, Y) \to XY$, $(XY)_{ij}$ are polynomials in X_{ij},Y_{ij}
$X \to X^{-1}$, $(X^{-1})_{ij}$ are rational functions in X_{ij}.

b. Special Linear group($SL(n)$)

The special linear group consists of all $n \times n$ unimodular matrices.

$$SL(n, R) = SL(n) = \{X \in M(n) | detX = 1\}$$

Such matrices correspond to linear operators $v \to Xv$, preserving the standard volume in R^n .

c. Orthogonal Group

The orthogonal group is formed by $n \times n$ orthogonal matrices.

$$O(n) = \left\{X \in M(n) | XX^T = Id\right\}$$

Orthogonal transformations $v \to Xv$ preserve the Euclidean structure in R^n.
Orthogonal matrices have determinant ± 1.

d. Special Orthogonal Group

Orthogonal unimodular matrices form the special orthogonal group

$$SO(n, R) = \left\{X \in M(n) | XX^T = Id, detX = 1\right\}$$

Special orthogonal transformations $v \to Xv$ preserve both the Euclidean structure and orientation in R^n.

e. Affine group

The afine group is defined as follows:

$$Aff(n) = \left\{X = \left(\begin{smallmatrix} Yb \\ 01 \end{smallmatrix} \right) \in M(n+1) | Y \in GL(n), b \in R^n\right\} \subset GL(n+1).$$

Such matrices correspond to invertible affine transformations in R^n of the form $v \to Yv + b$.

f. Euclidean Group

The Euclidean group is the subgroup of the affine group.

$$E(n) = \left\{ X = \left(\begin{smallmatrix} Y b \\ 0 1 \end{smallmatrix} \right) \in M(n+1) | Y \in SO(n), b \in R^n \right\} \subset GL(n+1).$$

Such matrices parametrize orientation-preserving affine isometrics $v \to Yv + b$.

g. Triangular Group

A group formed by real matrices, the triangular group consists of all invertible triangular matrices.

$$T(n) = \{X = x_{ij} \in M(n) | x_{ij} = 0, i > j, x_{ii} \neq 0\}$$

These are matrices of invertible linear operators $v \to Xv$ preserving the flag of subspaces

$$Re_1 \subset span(e_1, e_2) \subset ... \subset span(e_1, e_2, ..., e_{n-1}) \subset R^n.$$

h. Complex General Linear Group

The complex general linear group consists of all complex $n \times n$ invertible matrices

$$\begin{aligned} GL(n, C) \quad &= \{z \in M(n, C) | det z \neq 0\} \\ &= \left\{ \left(\begin{smallmatrix} X & -Y \\ Y & X \end{smallmatrix} \right) \in M(2n, R) | det^2 X + det^2 Y \neq 0 \right\} \end{aligned}$$

i. Complex Special Linear Group

The complex special linear group consists of all complex $n \times n$ unimodular matrices

$$\begin{aligned} SL(n, C) \quad &= \{z \in M(n, C) | det z = 1\} \\ &= \left\{ \left(\begin{smallmatrix} X & -Y \\ Y & X \end{smallmatrix} \right) \in M(2n, R) | det(X + iY) = 1 \right\} \end{aligned}$$

j. Unitary Group

An important example of a linear Lie group is the unitary group consisting of all $n \times n$ unitary matrices.

$$U(n) = \{z \in M(n, C) | \bar{z}^T z = Id\}$$

Such matrices correspond to linear transformations that preserve the unitary structure in C^n.

k. Special Unitary Group

$$
\begin{aligned}
Su(n) \quad &= U(n) \bigcap SL(n, C) \\
&= \{z \in M(n, C) | \bar{z}^T z = Id, detz = 1\} \\
&= \{\left(\begin{smallmatrix} X & -Y \\ Y & X \end{smallmatrix}\right) \in M(2n, R) | X^T X + Y^T Y = Id, X^T Y - Y^T X = 0, det(X + iY) = 1\}
\end{aligned}
$$

3.4 The algebra of Lie groups

Consider the tangent space to the general linear group at the identity.

$$T_{Id}GL(n) = \{X'(0) | X(t) \in GL(n), X(0) = Id\}$$

Since the velocity vector $X'(t) = (x'_{ij}(i))$ is an $n \times n$ matrix, we obtain

$$X'(0) = (x'_{ij}(0) = A \in M(n).$$

Thus,

$$T_{Id}GL(n) \bigcap M(n).$$

choose an arbitrary matrix $A \in M(n)$. The curve $X(t) = Id + tA$ belongs to $GL(n)$ for $|t| < \varepsilon$ and small $\varepsilon > 0$.
$X(0) = Id$ and $X'(0) = A$

$$\Rightarrow T_{Id}GL(n) = M(n)$$

The tangent space $T_{Id}GL(n)$ is a linear space.

$$[A, B] = AB - BA \in M(n), A, B \in M(n).$$

This operation satisfies the following properties:
(1) bilinearity,
(2) skew-symmetry
(3) Jacobi identity

3.5 Lie Algebra

A linear space L endowed with a binary operation [.,.] which is
(1) bilinear
(2) skew-symmetric
(3) satisfies Jacobi identity
is called a Lie algebra.
For example, The space M(n) with the matrix commutator is a Lie algebra.

Sub algebra
A subset K of a Lie algebra L is called a subalgebra of L if for all $x, y \in K$ and all $\alpha, \beta \in F$, $\alpha x + \beta y \in K$,$[x, y] \in K$.

Ideals
An Ideal I of a Lie algebra L is a sub algebra of L with the property $[I, L] \subset I$.

Some results of ideals
1. If $L' = [L, L]$ then this is an ideal in L.
2. If $L' = [L, L] = 0$ then this is an Abelian Lie algebra.
3. Ideals in a Lie algebra can be combined in various ways to give new ideals.
4. Let I and K be ideals in L. Then the sum $I + K$, the intersection $I \bigcap K$ and the commutator $[I, K]$ are ideals in L.
5. If the central sequence is $L^0 \supset L^1 \supset L^2 \supset ... \supset L^n \supset ...$ and that $L^0, L^1, L^2, ...$ and ideals in L.
6. If the central sequence is $L_0 \supset L_1 \supset L_2 \supset ... \supset L_n \supset ...$ and that $L_0, L_1, L_2, ...$ are ideals in L.

Lie algebras have two special types.
1.Solvable Lie algebra and
2. Nilpotent Lie algebra

Solvable Lie algebra
Definition.
The derived series of a Lie algebra L is the sequence of ideals

$$L_0 = L, \quad L_1 = [L, L], \quad L_2 = [L_1, L_1], ..., \quad L_i = [L_{i-1}, L_{i-1}].$$

Further, we call a Lie algebra solvable whenever $L_n = 0$ for some n.

Nilpotent Lie algebra
Definition.
The lower central series of a Lie algebra L is the sequence of ideals

$$L^0 = L, \quad L^1 = [L, L], \quad L^2 = [L, L^1], ..., \quad L^i = [L, L^{i-1}].$$

Further, L is called nilpotent if $L^n = 0$ for some n.
Commutator
Let M and N be subsets of Lie algebra L which are not necessary subspaces. Then the Commutator $[M, N]$ of M and N is defined to be the Linear span of the set of elements of the form $[x, y]$ with $x \in M$ and $y \in N$.

$$[M, N] = \left\{ z \in L | z = \sum_{i,j} (\alpha_{ij})[x_i, y_j]; x_i, y_j \in L; \alpha_{ij} \in F \right\}$$

3.6 Relation of this Lie algebra with the Lie group

The tangent space to a Lie group G at the identity element is called the Lie algebra of the Lie group G.
i.e

$$L = T_{Id}G$$

.

Examples
a. The Lie algebra of a Lie group $GL(n)$ is $gl(n)$.

$$T_{Id}GL(n) = gl(n).$$

b. The Lie Algebra of $SL(n)$
The Lie algebra of the special linear group is denoted by $sl(n)$.
$sl(n) = T_{Id}SL(n) = \{X'(0)|X(t) \in SL(n), X(0) = Id\}$
Let us take a curve,

$$X(t) = Id + tX'(0) + O(t) \in SL(n)$$

then

$$\begin{aligned} 1 = detX(t) \quad &= det(Id + tX'(0) + O(t)) = 1 + t(detX'(0)) + O(t) \\ &= 1 + ttrX'(0) + O(t), t \to 0 \end{aligned}$$

So

$$trX'(0) = 0.$$

Thus $sl(n) = \{A \in M(n)|traceA = 0\}$, the traceless matrices.

c. The Lie Algebra of $SO(n)$
The Lie algebra of $SO(n)$ is denoted as $so(n)$.

$$so(n) = T_{Id}SO(n) = \{X'(0)|X(t) \in SO(n), X(0) = Id\}.$$

We have $X(t)X^T(t) \equiv Id$, thus

$$0 = X'(0)X^T(0) + X(0)X'^T(0) = X'(0) + X'^T(0).$$

Donating $A = X'(0)$, we obtain $A + A^T = 0$.
Thus, $So(n) = \left\{A \in M(n) | A + A^T = 0\right\}$, the skew-symmetric matrices.

d. Lie Algebra of $Aff(n)$
The Lie Algebra of $Aff(n)$ is defined as

$$\begin{aligned} aff(n) &= T_{Id}Aff(n) \\ &= \left\{\left(\begin{smallmatrix} A & b \\ 0 & 0 \end{smallmatrix}\right) | A \in gl(n), b \in R^n\right\} \end{aligned}$$

As,

$$aff(n) = T_{Id}Aff(n) = \{X'(0) | X(t) \in Aff(n), X(0) = Id\}$$

We have, $\left(\begin{smallmatrix} A & b \\ 0 & 0 \end{smallmatrix}\right) \in Aff(n)$, where the transformation $v \to Yv + b$
We have,

$$\left(\begin{smallmatrix} Y(0) & b \\ 0 & 1 \end{smallmatrix}\right)' = \left(\begin{smallmatrix} Y(0) & b \\ 0 & 0 \end{smallmatrix}\right)$$

Let $Y'(0) = A$ then $\left(\begin{smallmatrix} Y(0) & b \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} A & b \\ 0 & 0 \end{smallmatrix}\right) \in gl(n), b \in R^n$.

Similarly,
The Lie Algebra of $E(n)$ is denoted by $e(n)$.such that

$$e(n) = T_{Id}E(n) = \left\{\left(\begin{smallmatrix} A & b \\ 0 & 0 \end{smallmatrix}\right) | A \in so(n), b \in R^n\right\}$$

The Lie Algebra of $T(n)$ is denoted by $t(n)$ such that

$$t(n) = \{A = (a_{ij}) \in M(n) | a_{ij} = 0, i > j\}$$

The Lie Algebra of U(n) is denoted by u(n) such that

$$u(n) = \left\{A \in M(n, C) | A + \bar{A}^T = 0\right\}$$

The Lie Algebra of SU(n) is denoted by su(n) such that

$$su(n) = T_{Id}SU(n) = \left\{A \in M(n, C) | A + \bar{A}^T = 0, trace A = 0\right\}$$

We can find Lie group from Lie Algebra by reverse passage via matrix exponential.

3.7 Matrix exponential

Consider a matrix ODE, $X' = XA$ (1)
where $A \in M(n)$ is a given matrix. In the case $n = 1$, solutions to the ODE $x' = xa$ are given by the exponential $x(t) = x(0)e^{at}$,
where

$$e^a = 1 + a + \frac{a^2}{2!} + ... + \frac{a^n}{n!} + ...$$

For arbitrary natural no n, Let us consider a matrix $A \in M(n)$,

$$exp(A) = e^A = Id + A + \frac{A^2}{2!} + ... + \frac{A^n}{n!} + ...$$

This matrix series converges absolutely thus it can be differentiated termwise,

$$(e^{At})' = \left(Id + At + \frac{A^2t^2}{2!} + ... + \frac{A^nt^n}{n!} + ...\right)' = A + \frac{A^2t}{1!} + ... + \frac{A^nt^{n-1}}{(n-1)!} + ... = e^{At}.A$$

Thus the matrix exponential

$$X(t) = e^{At} \qquad (2)$$

is the solution to the Cauchy problem $X' = XA, X(0) = Id$ and all solutions of the matrix equation (2) have the form,

$$X(t) = X(0)e^{At}$$

For an arbitrary matrix $A \in gl(n)$,then its exponential $exp(A) \in GL(n)$.
Since $detexp(A) = exp(trA) \neq 0$. So we constructed a (smooth) mapping
$exp : gl(n) \to GL(n)$.

Chapter-4

4 Control systems and Lie groups

For an arbitrary matrix $A \in gl(n)$, the Cauchy problem $X' = XA$, $X(0) = X_0$, $X \in GL(n)$ has a unique solution of the form $X(t) = X_0 exp(tA)$.
For Lie group G, the solution of $X' = XA$, $X \in G$ is $X(t) = X_0 exp(tA)$. Similarly, for the special linear group the cauchy problem $X' = XA$, $X \in SL(n)$ has the solution $X(t) = exp(tA)$.

4.1 Left invariant vector fields for Lie Group

Vector fields of the form $V(X) = XA$, $X \in G$, $A \in L$ are called left-invariant vector fields on the linear Lie group G.
Let V, W be two left invariant vector fields on a Lie group G then $[V, W]$ is also a left invariant vector fields.
Left invariant vector fields on a Lie group G from a Lie algebra isomorphic to the Lie algebra $L = T_{Id}G$.
A left invariant vector field $XA \in vecG$ has two representation of Lie algebra of a Lie group G.
(1) $L = T_{Id}G$
(2) L=Left invariant vector fields on G

Control system can be divided into two parts as considering vector fields.
(1)Left invariant control system
(2)Right invariant control system.

4.2 Left invariant control system

Let G be a Lie group and L its Lie algebra. A left-invariant control system Γ on a Lie group G is an arbitrary set of left-invariant vector fields on G.

Example:

a.Control-affine left-invariant systems

A particular class of left-invariant systems, which is important for applications is formed by control affine systems.

$$\Gamma = \left\{ A + \sum_{i=1}^{m} (u_i b_i) | u = (u_1, ... u_m) \in U \subset R^m \right\}$$

where $A, B_1, B_2, ..., B_m$ are some elements of L. If the control set U coincides with R^m, then the system is an affine subspace of L.

b.Trajectory

A trajectory of a left-invariant system Γ on G is a continuous curve $X(t)$ in G defined on an interval $[t_0, T] \subset R$ so that there exists a partition.
$t_0 < t_1 < ... < t_N = T$ and left-invariant vector fields $A_1, ..., A_N \in \Gamma$ such that the restriction of $X(t)$ to each open interval (t_{i-1}, t_i) is differentiable and

$$X'(t) = X(t)A_i \quad fort \in (t_{i-1}, t_i), i = 1, 2, ..., N.$$

c. Reachable set for time T

For any $T \geq 0$ and any X in G, the reachable set for time T of a left-invariant system $\Gamma \subset L$ from the point X is the set $A_\Gamma(X, T)$ of all points that can be reached from X in exactly T units of time. $A_\Gamma(X, T) = \{X(T) \mid X(.)a \quad trajectory \quad of \quad \Gamma, X(0) = X, T \geq 0\} = \cup_{t \geq 0} A_\Gamma(X, \leq T)$ The reachable set for time not greater than $T \geq 0$ is defined as $A_\Gamma(X, \leq T) = A_\Gamma(X, \leq T) = \bigcup_{0 \leq t \leq T} A_\Gamma(X, t)$.

d. Controllable set:

A system $\Gamma \subset L$ is called controllable if, given any pair of points X_0 and X_1 in G, the point X_1 can be reached from X_0 along a trajectory of Γ for a non negative time, $X_1 \in A(X_0)$, for any $X_0, X_1 \in G$ $A(X) = G$, for any $X \in G$.
This corresponds to global controllability, or complete controllability.

4.3 Right-invariant control systems:

It is Similar to left-invariant vector fields$X' = XA$
Consider right-invariant vector fields of the form $Y' = BY$. The inversion $i : G \to G, i(X) = X^{-1} = Y$ transforms left-invariant vector fields to right-invariant ones. Indeed, let $X(t)$ be a trajectory of a left-invariant ODE $X' = XA$. Since $Y(t)X(t) = Id$, we have $Y'(t)X(t) + Y(t)X'(t) = 0$
thus,

$$Y'(t) = -Y(t)X'(t)X^{-1}(t) = -Y(t)X(t)AY(t) = -AY(t)$$

Consequently, $X' = XA \Leftrightarrow Y' = -AY$ and $Y = X^{-1}$.
Since $X(t) = X_0e^{tA}$ then

$$Y(t) = e^{-tA}Y_0.$$

The Lie algebra $L = T_{Id}G$ of a Lie group G can be identified with the Lie algebra of right-invariant vector fields $\{AX|A \in L\}$ on G.

4.4 Proposition

Proposition 1.

The Lie bracket of right invariant vector fields computes as follows,
$[AX, BX] = [B, A]X$.

Proof.

The flows of right invariant vector fields are given by the matrix exponential

$$e^{tv}(X) = exp(-tA)X, e^{tw}(X) = exp(-tB)X$$

Now,

$$\begin{aligned}
\gamma(t) &= e^{-tw}oe^{-tv}oe^{tw}oe^{tv}(X) \\
&= ((Id - tB + \frac{t^2}{2}B^2 - ...)(Id - tA + \frac{t^2}{2}A^2 - ...)(Id - tB + \frac{t^2}{2}B^2 - ...)(Id - tA + \frac{t^2}{2}A^2 - ...))X \\
&= ((Id - t(B + A) + \frac{t^2}{2}(B^2 + 2BA + A^2) - ...)(Id + t(B + A) + \frac{t^2}{2}(B^2 + 2BA + A^2)...))X \\
&= (Id + t(B + A) + \frac{t^2}{2}(B^2 + 2BA + A^2) + ... - t(B + A) - \frac{t^2}{2}(B + A)^2 - ... + \frac{t^2}{2}(B^2 + 2BA + A^2 \\
&= (Id + t^2(BA - AB) + ...)X \\
&= (Id + t^2[B, A] + ...)X
\end{aligned}$$

Thus,

$$\gamma(\sqrt{t}) = (Id + t[B, A] + ...)X$$

It is a smooth curve at $t = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t} |_{t=0} \gamma(\sqrt{t}) = X[A, B] = [BX, AX] = [AX, BX]$.
A right invariant control system on a Lie group G is an arbitrary set of right invariant vector fields on G.

For example

A control-affine right-invariant control system on a Lie group G has the form

$$Y' = AY + \sum_{i=1}^{m}(u_i)B_iY, u \in U \subset R^m, Y \in G.$$

The inversion $X = Y^{-1}$ transforms right-invariant system to the left-invariant system

$$X' = -XA - \sum_{i=1}^{m} u_i X_{B_i} Y, u \in U, X \in G.$$

Proposition 2.

Let $X(t)$,$t \in [0,T]$, be a trajectory of a left-invariant system $\Gamma \subset L$ with $X(0) = X_0$. Then there exist $N \in \mathbb{N}$ and $T_1, ..., T_N > 0$, $A_1, ..., A_N \in \Gamma$, such that $X(T) = X_0 exp(T_1 A_1)...exp(T_N A_N)$, $T_1 + ... + T_N = T$.

Proof. By the definition of a trajectory, there exist $N \in \mathbb{N}$ and

$$0 = t_0 < t_1 < ... < t_N = T, \quad A_1, ..., A_N \in \Gamma$$

such that X(t) is continuous and

$$t \in (t_{i-1}, t_i) \Rightarrow X'(t) = X(t) A_i.$$

Consider the first interval:

$$t \in (0, t_1) \Rightarrow X' = X(t) A_1, \quad X(0) = X_0$$

Thus,

$$X(t) = X_0 exp(A_1 t), \quad X(t) = X_0 exp(A_1 t_1)$$

Further,

$$t \in (t_1, t_2) \Rightarrow X' = X(t) A_2, \quad X(t_1) = X_0 exp(A_1 t_1)$$

So,

$$X(t) = X_0 exp(t_0 A_1) exp((t - t_1) A_2)$$

$$\begin{aligned} X(t_2) \quad &= X_0 exp(t_1 A_1) exp((t_2 - t_1) A_2) \\ &= X_0 exp(A_1 T_1) exp(T_2 A_2) \end{aligned}$$

where

$$T_1 = t_1, \quad T_2 = t_2 - t_1.$$

continuing this process, we get our required representation,

$$X(t_N) = X(T) = X_0 exp(T_1 A_1)...exp(T_N A_N)$$

$$T_N = t_N - t_{N-1}, ..., T_2 = t_2 - t_1, \quad T_1 = t_1$$
$$T_N + T_{N-1} + ... + T_1 = T_N = T$$

Proposition 3.

Let $\Gamma \subset L$ be a left-invariant system, and let X be an arbitrary point of G. Then
(1)$A_\Gamma(X) = \{X exp(t_1 A_1)...exp(t_N A_N) | A_i \in \Gamma, t_i > 0, N \geq 0\}$
(2)$A_\Gamma(X) = X A_\Gamma(Id)$
(3)$A_\Gamma(Id)$ is a sub semigroup of G.
(4)$A_\Gamma(X)$ is an arcwise-connected subset of G.

Proof. Items (1) and (2) follow immediately from Proposition (2) and (2) follows from(1). Now, (3) is true, since

$$A_\Gamma(Id) = \{exp(t_1 A_1)...exp(t_N A_N) | A_i \in \Gamma, t_i > 0, N \geq 0\}$$

then for any $X_1, X_2 \in A_\Gamma(Id)$, the product $X_1, X_2 \in A_\Gamma(Id)$.
(4) is true as any point in $A_\Gamma(X)$ is connected with the initial point X by a trajectory X(t). □

4.5 Orbit of a control system

The orbit of a system Γ through a point $X \in G$ is the following subset of the Lie group G.

$$O_\Gamma(X) = \{X exp(t_1 A_1)...exp(t_N A_N) | A_i \in \Gamma, t_i \in R, N \geq 0\}$$

From previous proposition, we get $A_\Gamma(X) \subset O_\Gamma(X)$, In the orbit, one is allowed to move both forward and backward in time, while in the attainable set only the forward motion is allowed.

Proposition 4.

Let $\Gamma \subset L$ be a left-invariant system, and let X be an arbitrary point of G. Then
(1)$O_\Gamma(X) = X O_\Gamma(Id)$
(2)$O_\Gamma(Id)$ is the connected Lie subgroup of G with the Lie algebra $Lie(\Gamma)$. where $Lie(\Gamma)$ the Lie algebra generated by Γ, i.e., the smallest Lie subalgebra of L containing Γ.

Proof. Item (1) is obvious.

(2) First of all, the orbit $O_\Gamma(Id)$ is connected since any point in it is connected with the identity by a continuous curve provided by the definition of an orbit.
Clearly, $O_\Gamma(Id)$ is a subgroup of G. If $X, Y \in O_\Gamma(Id)$ then $XY \in O_\Gamma(Id)$ as a product of exponentials.
If

$$X = exp(t_1 A_1)...exp(t_N A_N) \in O_\Gamma(Id)$$

then

$$X^{-1} = exp(-t_N A_N)...exp(-t_1 A_1) \in O_\Gamma(Id) \Rightarrow Id \in O_\Gamma(Id)$$

From the general Orbit Theorem $O_\Gamma(Id) \subset G$ is a smooth submanifold with the tangent space

$$T_{Id}(O_\Gamma(Id)) = Lie(\Gamma)$$

Then the orbit $O_\Gamma(Id)$ is a Lie subgroup of G with the Lie algebra $Lie(\Gamma)$. □

Proposition 5.

A left-invariant system Γ is controllable iff $A_\Gamma(Id) = G$.

Proof. By definition, Γ is controllable iff $A_\Gamma(X) = G$ for any $X \in G$. Since $A_\Gamma(X) = XA_\Gamma(Id)$, controllability is equivalent to the identity $A_\Gamma(X) = G$. That is why in the sequel we use the following short notation for the attainable set and orbit from the identity.

$$A_\Gamma(Id) = A_\Gamma = A, \quad A_\Gamma(Id) = A_\Gamma = A. \qquad \square$$

4.6 Theorem 3.1 (Connectedness Condition)

If $\Gamma \subset L$ is controllable on G, then the Lie group G is connected.

Proof. The attainable set A is a connected subset of G. □

Example 3.1. The Lie group $GL(n)$ is not connected since it consists of two connected components $GL_+(n)$ and $GL_-(n)$, where

$$GL_\pm(n) = \{X \in M(n) | sign(detX) = \pm 1\}.$$

Thus, there are no controllable systems on $GL(n)$.

Example 3.2 Similarly, the orthogonal group $O_-(n) = SO(n) \bigcup O_-(n)$ is disconnected, where $O_-(n) = \{X \in O(n) | detX = -1\}$. So there no controllable systems on $O(n)$.

4.7 Theorem 3.2 (Rank Condition).

Let $\Gamma \subset L$
(1) If Γ is controllable, then $Lie(\Gamma) = L$.
(2) int $A \neq \phi$ if and only if $Lie(\Gamma) = L$.

Proof.(1) If Γ is controllable, then $A = G$, the more so $O = G$, thus

$$Lie(\Gamma) = L.$$

(2) By Krener's theorem, if $Lie(\Gamma) = L$, then int $A \neq \phi$.
Conversely, let

$$Lie(\Gamma) \neq L.$$

Then

$$dimO = dimLie(\Gamma) < dimL = dimG.$$

Thus int $O = \phi$, then int $A = \phi$. □

4.8 Theorem 3.3 (Local Controllability Test).

A system $\Gamma \subset L$ is controllable on a Lie group G iff the following conditions hold:
(1) G is connected,
(2) Γ is locally controllable at the identity
Note that identity element is always contained in the attainable set, and there may be two cases: either $Id \in intA$, or $Id \in \partial A$.
In the first case the system is controllable, while in the second case not.
Now we prove Theorem 3.3.

4.9 Theorem 3.4 (Closure Test).

A system $\Gamma \subset L$ is controllable on a Lie group G iff the following conditions hold:
(1) $Lie(\Gamma) = L$,
(2) $clA = G$.

Proof. The necessary part is obvious. Let us prove the sufficiency. Consider the time-reversed system

$$-\Gamma = \{-A | A \in \Gamma\}$$

Trajectories of the system $-\Gamma$ are trajectories of the initial system Γ passed in the opposite direction, thus

$$\begin{aligned} A_{-\Gamma} &= \{Xexp(-t_1A_1)...exp(-t_NA_N) | A_i \in \Gamma, t_i \geq 0\} \\ &= \left\{exp(t_NA_N)^{-1}...exp(t_1A_1)^{-1} | A_i \in \Gamma, t_i \geq 0\right\} \\ &= A_\Gamma^{-1} \end{aligned}$$

Since $Lie(-\Gamma) = Lie(\Gamma) = L$, then $intA_{-\Gamma} \neq \phi$, Thus there exists an open subset $V \subset A_{-\Gamma}$. Further, by the hypothesis of this theorem, $clA_{\Gamma} = G$, thus there exists a point $X \in A_{\Gamma} \bigcap V \neq \phi$. We have $X \in V \subset A_{-\Gamma} = A_{\Gamma}^{-1}$, thus the open set $V^{-1} \subset A_{\Gamma}$ is a neighbourhood of the inverse X^{-1}. Consequently, the open set $V^{-1}X \subset A_{\Gamma}$. But $Id = X^{-1}X \in V^{-1}X \subset A_{\Gamma}$, thus $Id \in intA_{\Gamma}$ and the system Γ is controllable by Theorem 3.3.

5 Conclusion

In this project we have received some aspects of application of Lie group to the geometrical control theory. Our work is heavily drawn from the work of Yu.L.Sachkov "Control theory on Lie groups" available from the website having ref. no. Journal of Mathematical Science.

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