SOME PERTURBATION METHODS TO SOLVE LINEAR AND NON-LINEAR DIFFERENTIAL EQUATION

A PROJECT REPORT

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Declaration

I declare that the topic "**SOME PERTURBATION METHODS TO SOLVE LINEAR AND NON-LINEAR DIFFERENTIAL EQUATION** " for completion for my master degree has not been submitted in any other institution or university for the award of any other degree or diploma.

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Certificate

This is to certify that the project report entitled **SOME PERTURBATION METHODS TO SOLVE LINEAR AND NON-LINEAR DIFFERENTIAL EQUATION** submitted by **Sashi Kanta Sahoo** to the National Institute of Technology Rourkela, Odisha for the partial fulfilment of requirements for the degree of master of science in Mathematics and the review work is carried out by him under my supervision and guidance. It has fulfilled all the guidelines required for the submission of his research project paper for M.Sc. degree. In my opinion, the contents of this project submitted by him is worthy of consideration for M.Sc. degree and in my knowledge this work has not been submitted to any other institute or university for the award of any degree.

May, 2014

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Abstract

In this research project paper, our aim to solve linear and non-linear differential equation by the general perturbation theory such as regular perturbation theory and singular perturbation theory as well as by homotopy perturbation method. The problem of an incompressible viscous flow i.e. Blasius equation over a flat plate is presented in this research project. This is a non-linear differential equation. So, the homotopy perturbation method (HPM) is employed to solve the well-known Blasius non-linear differential equation. The obtained result have been compared with the exact solution of Blasius equation.

Contents

CHAPTER 1

1 Introduction

In this research project report, we plan to focus on perturbation method and Homotopy Perturbation method and to solve linear and non-linear differential Equation.

At first,almost all perturbation methods are based on an assumption that a small parameter must exist in the equation. This is so called small parameter assumption greatly restrict application of perturbation techniques. On Secondly, the determination of small parameter seems to be a special art requiring special techniques. An appropriate choice of small parameter leads to ideal result. However an unsuitable choice of small parameter results badly. The Homotopy Perturbation method does not depend upon a small parameter in the equation. This method, which is a combination of homotopy and perturbation techniques, provides us with a convenient way to obtain analytic or approximate solution to a wide variety of problems arising in different field. So, this was introduced as a powerful tool to solve various kinds of non-linear problems.

In Chapter 2, we discuss classical perturbation techniques . In the beginning of chapter 3, we focus on some basic idea about homotopy perturbation method In chapter 4, we plan to study about Blasius equation and solution of this equation by HPM.

CHAPTER 2

2 Perturbation Theory

In this chapter, we wish to revise perturbation theory. We also focus on *Singular perturbation theory* and *regular perturbation theory.* Perturbation theory leads to an expression for the desired solution in terms of a formal power series in small parameter (ϵ), known as perturbation series that quantifies the deviation from the exactly solvable problem. The leading term in this power series is the solution of the exactly solvable problem and further terms describe the deviation in the solution. Consider,

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...$$

Here, x_0 be the known solution to the exactly solvable initial problem and $x_1, x_2 ...$ are the higher order terms. For small ϵ these higher order terms are successively smaller. An approximate *"perturbation solution"* is obtained by truncating the series, usually by keeping only the first two terms.

2.1 Regular Perturbation Theory

Very often, a mathematical problem can not be solved exactly or, if the exact solution is available it exhibits such an intricate dependency in the parameters that it is hard to use as such. It may be the case however, that a parameter can be identified, say, ϵ ,such that the solution is available and reasonably simple for $\epsilon = 0$. Then one may wonder how this solution is altered for non zero but small ϵ . Perturbation theory gives a systematic answer to this question.

Example-2.1 : Consider an quadratic equation

$$x^2 - (3 + 2\epsilon)x + 2 + \epsilon = 0 \tag{2.1.1}$$

when $\epsilon = 0$ then (2.1.1) reduce to

$$x^2 - 3x + 2 = 0 \Rightarrow (x - 2)(x - 1) = 0 \tag{2.1.2}$$

whose roots are $x = 1$ and 2. Equation (2.1.1) is called perturbed equation where as equation (2.1.2) is called un-perturbed or reduced equation.

*Step*1 : In determining an approximate solution is to assume the form of the expansion. Let us assume that the roots have expansion in the form

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + ... \tag{2.1.3}$$

Here the first term x_0 is the zeroth-order term, the second term ϵx_1 is the first order term and the third term $\epsilon^2 x_2$ as the second order term.

*Step*2 : Substitute equation (2.1.3) in equation (2.1.1)

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...)^2 - (3 + 2\epsilon)(x_0 + \epsilon x_1 + ...) + 2 + \epsilon = 0 \tag{2.1.4}$$

*Step*3 : Using binomial theorem to expand the first term

$$\begin{aligned}(x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...)^2 &= x_0^2 + 2x_0(\epsilon x_1 + \epsilon^2 x_2 + ...) + (\epsilon x_1 + \epsilon^2 x_2 + ...)^2 \\ &= x_0^2 + 2\epsilon x_0 x_1 + 2\epsilon^2 x_0 x_2 + \epsilon^2 x_1^2 + 2\epsilon^3 x_1 x_2 + \epsilon^4 x_2^2 + ... \\ &= x_0^2 + 2\epsilon x_0 x_1 + \epsilon^2(2x_0 x_2 + x_1^2) + ... \end{aligned} \tag{2.1.5}$$

Similarly,

$$\begin{aligned}(3 + 2\epsilon)(x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...) &= 3x_0 + 3\epsilon x_1 + 3\epsilon^2 x_1 + 2\epsilon x_0 + 2\epsilon^2 x_1 + ... \\ &= 3x_0 + \epsilon(3x_1 + 2x_0) + \epsilon^2(3x_2 + 2x_1) + ... \end{aligned} \tag{2.1.6}$$

Substitute equation (2.1.5)and(2.1.6) in equation (2.1.4)

$$x_0^2 + 2\epsilon x_0 x_1 + \epsilon^2(2x_0 x_2 + x_1^2) - (3x_0 + \epsilon(3x_1 + 2x_0) + \epsilon^2(3x_2 + 2x_1)) + 2 + \epsilon = 0$$

Collect the co-efficient of like powers of ϵ yields,

$$(x_0^2 - 3x_0 + 2) + \epsilon(2x_0x_1 - 3x_1 - 2x_0 + 1) + \epsilon^2(2x_0x_2 + x_1^2 - 3x_2 - 2x_1) + ... = 0 \tag{2.1.7}$$

<u>*Step*4 :</u> Equating the co-efficient of each power of ϵ to *Zero*.

$$x_0^2 - 3x_0 + 2 = 0 \tag{2.1.8}$$

$$2x_0x_1 - 3x_1 - 2x_0 + 1 = 0 \tag{2.1.9}$$

$$2x_0x_2 + x_1^2 - 3x_2 - 2x_1 = 0 \tag{2.1.10}$$

From equation (2.1.8),$x_0 = 1, 2$, when $x_0 = 1$ equation (2.1.9) becomes

$$x_1 + 1 = 0 \Rightarrow x_1 = -1$$

When $x_0 = 1$ and $x_1 = -1$ equation (2.1.10) becomes

$$2x_2 + 1 - 3x_2 + 2 = 0$$

$$\Rightarrow x_2 - 3 = 0 \Rightarrow x_2 = 3$$

When $x_0 = 2$, equation (2.1.9) becomes

$$x_1 - 3 = 0 \Rightarrow x_1 = 3$$

equation (2.1.10) $\Rightarrow x_2 + 3 = 0 \Rightarrow x_2 = -3$

<u>*Step*5 :</u> When $x_0 = 1$, $x_1 = -1$ and $x_2 = 3$

$$Equ^n(3) \Rightarrow x = 1 - \epsilon + 3\epsilon^2 + ... \tag{2.1.11}$$

When $x_0 = 2$, $x_1 = 3$ and $x_2 = -3$

$$Equ^n(3) \Rightarrow x = 2 + 3\epsilon - 3\epsilon^2 \tag{2.1.12}$$

$\therefore$ Hence Equ^n(2.1.11) and (2.1.12) are the approximations for the two roots of (2.1.1).

Now, to verify this approximation are correct, we compare with the exact solution.

$$\begin{aligned} x^2 - (3 + 2\epsilon)x + 2 + \epsilon = 0 & \\ \Rightarrow x &= \frac{1}{2}[3 + 2\epsilon \pm \sqrt{(3 + 2\epsilon)^2 - 4(2 + \epsilon)} \\ \Rightarrow x &= \frac{1}{2}[3 + 2\epsilon \pm \sqrt{1 + 8\epsilon + 4\epsilon^2}] \end{aligned} \tag{2.1.13}$$

Using binomial theorem, we have

$$\begin{aligned}(1+8\epsilon+4\epsilon^2)^{\frac{1}{2}} &= 1+\frac{1}{2}(8\epsilon+4\epsilon^2)+\frac{(\frac{1}{2})(\frac{-1}{2})}{2!}(8\epsilon+4\epsilon^2)^2+...\\ &= 1+4\epsilon+2\epsilon^2-\frac{1}{8}(64\epsilon^2+...)\\ &= 1+4\epsilon+2\epsilon^2-8\epsilon^2+...\\ &= 1+4\epsilon-6\epsilon^2+...\end{aligned}$$

Substitute this value in $Equ^n(13)$, we have

$$\begin{aligned}x &= \frac{1}{2}(3+2\epsilon+1+4\epsilon-6\epsilon^2+...)\\ &= 2+3\epsilon-3\epsilon^2+...\\ x &= \frac{1}{2}(3+2\epsilon-1-4\epsilon+6\epsilon^2+...)\\ &= 1-\epsilon+3\epsilon^2+...\end{aligned}$$

Which are same as equation (2.1.11) and (2.1.12).

2.2 Singular Perturbation Theory

It concern the study of problems featuring a parameter for which the solution of the problem at a limiting value of the parameter are different in character from the limit of the solution of the general problem. For regular perturbation problems, the solution of the general problem converge to the solution of the limit problem as the parameter approaches the limit value.

Example-2.2: Consider,

$$\epsilon x^2+x+1=0 \tag{2.2.1}$$

Since equation (2.2.1) is a quadratic equation, it has two roots. For $\epsilon \longrightarrow 0$ Equation (2.2.1) reduce to

$$x+1=0 \tag{2.2.2}$$

Which is of first order. Thus x is discontinuous at $\epsilon = 0$. Such perturbation are called *singular perturbation problem.*

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + ... \tag{2.2.3}$$

Putting this value in Equation (1)

$$\begin{aligned}
\epsilon\left(x_0 + \epsilon x_1 + ...\right) + x_0 + \epsilon x_1 + ... + 1 = 0 \\
\Rightarrow \epsilon\left(x_0^2 + 2\epsilon x_0 x_1 + ...\right) + x_0 + \epsilon x_1 + ... + 1 = 0 \\
\Rightarrow \epsilon x_0^2 + 2\epsilon^2 x_0 x_1 + ... + x_0 + \epsilon x_1 + ... + 1 = 0 \\
\Rightarrow \epsilon\left(x_0^2 + x_1\right) + x_0 + 1 = 0
\end{aligned}$$

Equating co-efficient of like power of ϵ gives

$$\begin{aligned}
x_0 + 1 = 0 \\
x_1 + x_0^2 = 0
\end{aligned}$$

When $x_0 = -1$, $x_1 = -1$ So one of the root is

$$x = -1 - \epsilon + ... \tag{2.2.4}$$

Thus as expected the above procedure yielded only one root. We investigate the exact solution i.e. ,

$$x = \frac{1}{2\epsilon}\left(-1 \pm \sqrt{1 - 4\epsilon}\right) \tag{2.2.5}$$

Using binomial theorem we have

$$\begin{aligned}
\sqrt{1 - 4\epsilon} &= 1 - 2\epsilon + \frac{(\frac{1}{2})(\frac{-1}{2})}{2!} \times (-4\epsilon)^2 + ... \\
&= 1 - 2\epsilon - 2\epsilon^2 + ... \qquad (2.2.6)
\end{aligned}$$

Substituting (6) in (5)

$$x = \frac{-1 + 1 - 2\epsilon - 2\epsilon^2 + ...}{2\epsilon} = -1 - \epsilon + ... \tag{2.2.7}$$

$$x = \frac{-1 - 1 + 2\epsilon + 2\epsilon^2 + ...}{2\epsilon} = \frac{-1}{\epsilon} + 1 + \epsilon + ... \tag{2.2.8}$$

Therefore, both of the roots go in powers of ϵ but one starts with ϵ^{-1}. Hence it is not surprising that the assumed expansion in (2.2.3) is failed to produce the root (2.2.8). consequently one can not determine the second root by a perturbation technique unless its form is known. In those cases, we recognize that, if the order of the equation is not to be reduced, the other tends to ∞ as $\epsilon \longrightarrow 0$ and hence, assume that the leading term has the form

$$x = \frac{y}{\epsilon^{v}} \tag{2.2.9}$$

Where v must be greater than zero and needs to be determined in the course of analysis. Substitute (2.2.9) in (2.2.1)

$$\epsilon^{1-2v}y^2 + \epsilon^{v}y + 1 + ... = 0$$

Since $v > 0$, th second term is much bigger than 1 . Hence the dominant part of (2.2.9) is

$$\epsilon^{1-2v}y^2 + \epsilon^{v}y = 0 \tag{2.2.10}$$

which demands that power of ϵ be the same.

$$1 - 2v = -v \qquad \Rightarrow v = 1$$

For $v = 1 \qquad \Rightarrow y = o$ or -1.

The first value $y = 0$, correspond to the first root $x = -1 - \epsilon$. For $y = -1$, it corresponds to second root. Thus it follows from (2.2.9)

$$x = \frac{-1}{\epsilon} + ...$$

To determine more terms in the expansion of second root, we try

$$x = \frac{-1}{\epsilon} + x_0 + ... \tag{2.2.11}$$

Substitute it in equation (2.2.1)

$$\Rightarrow \epsilon\left(\frac{-1}{\epsilon} + x_0 + ...\right)^2 - \frac{-1}{\epsilon} + x_0 + ... + 1 = 0$$
$$\Rightarrow \epsilon\left(\frac{-1^2}{\epsilon} + \frac{2x_0}{\epsilon} + x_0^2 + ...\right) - \frac{-1}{\epsilon} + x_0 + 1 + ... = 0$$
$$\Rightarrow -2x_0 + x_0 + 1 + \bigcirc(\epsilon) = 0$$

$\Rightarrow x_0 = 1$and equation (2.2.11) becomes

$$x = -\frac{1}{\epsilon} + 1 + ...$$

Alternatively, once v has been determined. We view (2.2.9) as a transformation from x to y. Then putting $x = \frac{y}{\epsilon}$ in (2.2.1) yields,

$$y^2 + y + \epsilon = 0 \tag{2.2.12}$$

Which can be solved to determine both the roots because ϵ does not multiply the highest order.

2.3 Perturbation Theory For Differential Equation

Example-2.3 : Consider,

$$\frac{d^2y}{d\tau^2} = -\epsilon\frac{dy}{d\tau} - 1, \quad y(0) = o, \quad \frac{dy}{d\tau}(0) = 1 \tag{2.3.1}$$

Let us assume the expansion

$$y(\tau) = y_0(\tau) + \epsilon y_1(\tau) + \epsilon^2 y_2(\tau) + \bigcirc(\epsilon^3) \tag{2.3.2}$$

Substitute Equation (2.3.2) in (2.3.1)

$$\frac{d^2y}{d\tau^2} + \epsilon\frac{dy}{d\tau} + 1 = 0$$

$$\frac{d^2}{d\tau^2}\left(y_0(\tau) + \epsilon y_1(\tau) + \epsilon^2 y_2(\tau) + \bigcirc(\epsilon^3)\right) + \epsilon\frac{d}{d\tau}\left(y_0(\tau) + \epsilon y_1(\tau) + \epsilon^2 y_2(\tau) + \bigcirc(\epsilon^3)\right) + 1 = 0$$

$$\Rightarrow \frac{d^2y_0}{d\tau^2} + 1 + \epsilon\left(\frac{d^2y_1}{d\tau^2} + \frac{dy_0}{d\tau}\right) + \epsilon^2\left(\frac{d^2y_2}{d\tau^2} + \frac{dy_1}{d\tau}\right) + \bigcirc(\epsilon^3) = 0$$

Equating the co-efficient of ϵ , it becomes

$$\begin{aligned}
&\Rightarrow \frac{d^2y_0}{d\tau^2} + 1 = 0, \quad y_0(0) = 0, \quad \frac{dy_0}{d\tau}(0) = 1 \\
&\Rightarrow \frac{d^2y_1}{d\tau^2} + \frac{dy_0}{d\tau} = 0, \quad y_1(0) = 0, \quad \frac{dy_1}{d\tau}(0) = 0 \\
&\Rightarrow \frac{d^2y_2}{d\tau^2} + \frac{dy_1}{d\tau} = 0, \quad y_1(0) = 0, \quad \frac{dy_1}{d\tau}(0) = 0
\end{aligned} \tag{2.3.3}$$

By solving the above equation we will get

$$y_0(\tau) = \tau - \frac{\tau^2}{2} \tag{2.3.4}$$

$$y_1(\tau) = \frac{-\tau^2}{2} + \frac{\tau^3}{6} \tag{2.3.5}$$

$$y_2(\tau) = \frac{\tau^3}{6} - \frac{\tau^4}{24} \tag{2.3.6}$$

Putting these values in equation (2.3.2), we have the solution

$$y(\tau) = \tau - \frac{\tau^2}{2} + \epsilon\left(\frac{-\tau^2}{2} + \frac{\tau^3}{6}\right) + \epsilon^2\left(\frac{\tau^3}{6} - \frac{\tau^4}{24}\right) + \bigcirc(\epsilon^3)$$

CHAPTER 3

3 Homotopy Perturbation Method

In recent years, the Homotopy Perturbation Method has been successfully applied to solve many types of differential equation. It was proposed by ”Ji-Huan He” in 1999 . Dr. He used HPM to solve

1. Lighthill equation
2. Duffing equation
3. Non-linear wave equation
4. Schrodinger equation

In the *homotopy perturbation technique* we will first propose a new perturbation technique coupled with the homotopy technique. In topology two continuous function from one topological space to another is called ”homo-topic”. Formally a homotopy between two continuous function f and g from a topological space X to a topological space Y is defined to be a continuous function

$$H : X \times [0,1] \longrightarrow Y$$

such that

$$H(x,0) = f(x) \quad and \quad H(x,1) = g(x) \quad , \forall x \in X$$

The homotopy perturbation method does not depend upon a small parameter in the equation. By the homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$ which is considered as a small parameter.

3.1 Basic idea of HPM

Let us consider the non-linear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{3.1.1}$$

with boundary condition

$$B(u, \frac{\partial u}{\partial n}) \quad , \quad r \in \Gamma \tag{3.1.2}$$

Where A is a general differential operator , B is a boundary operator. Γ is the boundary of domain Ω. $f(r)$ is a known analytic function. Now, the operator A can be divided into two parts L and N, where L is linear and N is non-linear. Equation (3.1.1) can be written as follows

$$L(u) + N(u) - F(r) = 0 \tag{3.1.3}$$

By the homotopy technique, we construct a homotopy

$$v(r,p) : \Omega \times [0,1] \longrightarrow R,$$

Which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0,1], \quad r \in \Omega \tag{3.1.4}$$

or

$$H(v,p) = L(v) - L(u_0 + pL(u_0 + p[N(v) - f(r)] = 0$$

Where, u_0 is an initial approximation of equation (3.1.1), which satisfies the boundary condition. From equation (3.1.4)

$$H(v,o) = L(v) - L(u_0) = 0 \tag{3.1.5}$$

$$H(v,1) = A(v) - f(r) = 0 \tag{3.1.6}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called *deformation* and $L(v)-L(u_0)$ and $A(v)-f(r)$ are called *homotopic*.

In this paper, we will first use the embedding parameter p as a *small parameter* and assume that the solution of equ^n(3.1.4) can be written as a power series of p.

$$v = v_0 + pv_1 + p^2v_2 + ... \tag{3.1.7}$$

setting $p = 1$, results the approximate solution of equ^n(3.1.1)

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ... \tag{3.1.8}$$

The series (3.1.8) is convergent for most cases, however the convergent rate depends upon the non-linear operator $A(v)$.

Example 3.2: We will consider the Lighthill equation

$$(x+\epsilon y)\frac{dy}{dx}+y=0, \quad y(1)=1 \tag{3.2.1}$$

By the method, we can construct a homotopy which satisfies

$$(1-p)\left[\epsilon Y\frac{dY}{dx}-\epsilon y_0\frac{dy_0}{dx}\right]+p\left[(x+\epsilon y)\frac{dY}{dx}+Y\right]=0, \quad p\in[0,1] \tag{3.2.2}$$

We can obtain a solution of (3.2.2) in the form

$$Y(x)=Y_0(x)+pY_1(x)+p^2Y_2(x)+... \tag{3.2.3}$$

Where $Y_i(x); i=0,1,2,...$ are functions yet to be determined. By considering only first two terms of the above equation substitute equation (3.2.3) into equation (3.2.2)

$$(1-p)\left[\epsilon(Y_0+pY_1)\left(\frac{dY_0}{dx}+\frac{dY_1}{dx}\right)-\epsilon y_0\frac{dy_0}{dx}\right] \\ +p\left[(x+\epsilon Y_0+\epsilon pY_1)\left(\frac{dY_0}{dx}+p\frac{dY_1}{dx}\right)+(Y_0+pY_1)\right]=0$$

$$\Rightarrow (1-p)\left[\epsilon Y_0\left(\frac{dY_0}{dx}+\frac{dY_1}{dx}\right)+\epsilon pY_1\left(\frac{dY_0}{dx}+\frac{dY_1}{dx}\right)-\epsilon y_0\frac{dy_0}{dx}\right] \\ +p\left[(x+\epsilon Y_0+\epsilon pY_1)\left(\frac{dY_0}{dx}+p\frac{dY_1}{dx}\right)+(Y_0+pY_1)\right]=0$$

$$\Rightarrow \epsilon pY_1\frac{dY_1}{dx}+(1-p)\left[\epsilon Y_0\frac{dY_0}{dx}-\epsilon y_0\frac{dy_0}{dx}\right] \\ +p\left[(x+\epsilon Y_0)\frac{dY_0}{dx}+Y_0\right]+\epsilon p^2Y_1\left(\frac{dY_0}{dx}+p\frac{dY_1}{dx}\right)+p^2Y_1=0$$

Now, we get

$$\epsilon Y_0\frac{dY_0}{dx}-\epsilon y_0\frac{dy_0}{dx}=0 \tag{3.2.4}$$

$$\epsilon Y_1\frac{dY_1}{dx}+\left[(x+\epsilon Y_0)\frac{dY_0}{dx}+Y_0\right]=0 \tag{5}$$

The initial approximation $Y_0(x)$ or $y_0(x)$ can be freely chosen. Here I set

$$Y_0(x)=y_0(x)=-\frac{x}{\epsilon}, \quad Y_0(1)=-\frac{1}{\epsilon} \tag{3.2.6}$$

So that, the residual of equation (3.2.1) at $x = 0$ vanishes. Then substitute equation (3.2.6) into equation (3.2.5),

$$\begin{aligned} \epsilon Y_1 \frac{dY_1}{dx} + \left[(x - \epsilon \frac{x}{\epsilon}) \frac{dY_0}{dx} - \frac{x}{\epsilon} \right] &= 0 \\ \Rightarrow \epsilon Y_1 \frac{dY_1}{dx} - \frac{x}{\epsilon} &= 0 \\ \Rightarrow \epsilon Y_1 \frac{dY_1}{dx} &= \frac{x}{\epsilon} \\ \Rightarrow \epsilon^2 Y_1 dY_1 &= x dx \end{aligned}$$

Integrating both sides, we get

$$\begin{aligned} \Rightarrow \epsilon^2 \frac{Y_1^2}{2} &= \frac{x^2}{2} + c \\ \Rightarrow \epsilon^2 Y_1^2 &= x^2 + 2c \\ \Rightarrow Y_1 &= \frac{\sqrt{x^2 + 2c}}{\epsilon} \\ \Rightarrow \epsilon Y_1 &= \sqrt{x^2 + 2c} \end{aligned} \tag{7}$$

Putting the initial condition $Y_1(1) = 1 - Y_0 = 1 + \frac{1}{\epsilon}$,

$$\begin{aligned} \Rightarrow \epsilon \left(1 + \frac{1}{\epsilon} \right) &= \sqrt{1 + 2c} \\ \Rightarrow 1 + \epsilon &= \sqrt{1 + 2c} \\ \Rightarrow 1 + \epsilon^2 + 2\epsilon &= 1 + 2c \\ \Rightarrow c &= \frac{\epsilon^2 + 2\epsilon}{2} \end{aligned}$$

Now, putting this value in equation (3.2.7) we get

$$Y_1 = \frac{1}{\epsilon} \sqrt{x^2 + 2\epsilon + \epsilon^2}$$

Substitute this value in equ^n(3.2.3) ,

$$\Rightarrow Y(x) = Y_0(x) + Y_1(x) = \frac{1}{\epsilon} \left(-x + \sqrt{x^2 + 2\epsilon + \epsilon^2} \right) \tag{8}$$

Which is the exact solution of equ^n(3.2.1).

CHAPTER 4

4 Application Of Homotopy Perturbation Method

4.1 Derivation of Blasius Equation

For a two-dimensional flow, steady state, incompressible flow with zero pressure gradient over a flat plate, governing equation are simplified to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.1.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{4.1.2}$$

subjected to boundary conditions

$$y = o \quad , \quad u = 0$$
$$y = \infty \quad , \quad u = U_\infty \quad , \quad \frac{\partial u}{\partial y} = 0 \tag{4.1.3}$$

Take

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{\delta}, \quad u^* = \frac{u}{U_\infty}, \quad v^* = \frac{Lv}{\delta U_\infty}, \quad p^* = \frac{p}{\rho U_\infty^2}$$

take the stream function ψ defined by

$$\psi = \sqrt{\nu x U_\infty} f(\eta) \tag{4.1.4}$$

f is a dimensionless function of the similarity variable η .

$$\eta = \frac{y}{\sqrt{\nu x / U_\infty}} \tag{4.1.5}$$

Now,

$$\begin{aligned} u = \frac{\partial \psi}{\partial y} &= \frac{\partial \psi}{\partial \eta}.\frac{\partial \eta}{\partial y} \\ &= \sqrt{\nu x U_\infty} f'(\eta)\frac{1}{\sqrt{\nu x / U_\infty}} \\ &= U_\infty \frac{df}{d\eta} \end{aligned} \tag{4.1.6}$$

similarly,

$$v = -\frac{\partial\psi}{\partial x} = -\left[\frac{\partial}{\partial x}\sqrt{\nu x U_\infty} f(\eta) + \sqrt{\nu x U_\infty}\frac{\partial}{\partial x} f(\eta)\right]$$
$$= -\left[f(\eta)\frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}} + \sqrt{\nu x U_\infty}\frac{df}{d\eta}(-\frac{1}{2})\frac{y x^{-\frac{3}{2}}}{\sqrt{\nu/U_\infty}}\right]$$
$$= -\left[\frac{1}{2}f(\eta)\sqrt{\frac{\nu U_\infty}{x}} - \frac{1}{2}\frac{U_\infty y}{x}\frac{df(\eta)}{d\eta}\right]$$
$$= \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\left[\eta\frac{df}{d\eta} - f\right] \quad (4.1.7)$$

Now,

$$\frac{\partial u}{\partial x} = U_\infty \frac{d^2 f}{d\eta^2}\frac{y}{\sqrt{\nu/U_\infty}}(\frac{1}{2})x^{-\frac{3}{2}}$$
$$= -\frac{U_\infty}{2x}\eta \quad \frac{d^2 f}{d\eta^2} \quad (4.1.8)$$

$$\frac{\partial u}{\partial y} = U_\infty \frac{d^2 f}{d\eta^2}.\frac{1}{\sqrt{\nu x/U_\infty}}$$
$$= \frac{U_\infty}{\sqrt{\nu x/U_\infty}}.\frac{d^2 f}{d\eta^2} \quad (4.1.9)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{U_\infty}{\sqrt{\nu x/U_\infty}}.\frac{d^2 f}{d\eta^2}\right)$$
$$= \frac{U_\infty}{\sqrt{\nu x/U_\infty}}\left(\frac{d^3 f}{d\eta^3}.\frac{1}{\sqrt{\nu x/U_\infty}}\right)$$
$$= \frac{U_{\infty^2}}{\nu x}\frac{d^3 f}{d\eta^3} \quad (4.1.10)$$

Putting this value in equation (4.1.2), we get

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow U_\infty\frac{df}{d\eta}\left[-\frac{U_\infty}{2x}\eta \quad \frac{d^2 f}{d\eta^2}\right] + \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\left[\eta\frac{df}{d\eta} - f\right].\frac{U_\infty}{\sqrt{\nu x/U_\infty}}.\frac{d^2 f}{d\eta^2} = \nu\frac{U_{\infty^2}}{\nu x}\frac{d^3 f}{d\eta^3}$$
$$\Rightarrow -\frac{U_\infty^2}{2x}\eta\frac{df}{d\eta}.\frac{d^2 f}{d\eta^2} + \frac{1}{2}\frac{U_\infty^2}{x}\left[\eta\frac{df}{d\eta} - f\right]\frac{d^2 f}{d\eta^2} = \frac{U_\infty^2}{x}.\frac{d^3 f}{d\eta^3}$$
$$\Rightarrow -\frac{\eta}{2}.\frac{df}{d\eta}.\frac{d^2 f}{d\eta^2} + \frac{\eta}{2}.\frac{df}{d\eta}.\frac{d^2 f}{d\eta^2} - \frac{1}{2}f.\frac{d^2 f}{d\eta^2} = \frac{d^3 f}{d\eta^3}$$

$$\Rightarrow \frac{d^3 f}{d\eta^3} + \frac{1}{2} f . \frac{d^2 f}{d\eta^2} = 0 \tag{4.1.11}$$

With boundary condition,

$$\begin{aligned} \eta = 0 \quad &, \quad f = \frac{df}{d\eta} = 0 \\ \eta \longrightarrow \infty \quad &, \quad \frac{df}{d\eta} = 1 \end{aligned} \tag{4.1.12}$$

4.2 Solution of Blasius Equation By Homotopy Perturbation Method

So, to get a solution of equation (4.1.11) by the homotopy technique, we construct a homotopy

$$v(r,p) : \Omega \times [0,1] \longrightarrow R,$$

Which satisfies,

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0,1], \quad r \in \Omega$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{4.2.1}$$

Where, u_0 is an initial approximation of equation (4.2.1), which satisfies the boundary condition.

Now, from equation (4.1.11)

$$(1-p)\left(\frac{\partial^3 F}{\partial \eta^3} - \frac{\partial^3 f_0}{\partial \eta^3}\right) + p\left(\frac{\partial^3 F}{\partial \eta^3} + \frac{F}{2} + \frac{\partial^2 F}{\partial \eta^2}\right) = 0$$

or,

$$\left(\frac{\partial^3 F}{\partial \eta^3} - \frac{\partial^3 f_0}{\partial \eta^3}\right) + p\left(\frac{\partial^3 f_0}{\partial \eta^3} + \frac{F}{2} + \frac{\partial^2 F}{\partial \eta^2}\right) = 0 \tag{4.2.2}$$

Suppose that the solution of the equation (4.2.2) to be in the following form

$$F = F_0 + pF_1 + p^2 F_2 + ... \tag{4.2.3}$$

Substituting $equ^n(4.2.3)$ in (4.2.2) we get,

$$\frac{\partial^3 F_0}{\partial \eta^3} + p\frac{\partial^3 F_1}{\partial \eta^3} + p^2\frac{\partial^3 F_2}{\partial \eta^3} - \frac{\partial^3 f_0}{\partial \eta^3} + p\frac{\partial^3 f_0}{\partial \eta^3}$$
$$+ p\left[\frac{F_0}{2}\left(\frac{\partial^2 F_0}{\partial \eta^2} + p\frac{\partial^2 F_1}{\partial \eta^2}\right) + p\frac{F_1}{2}\left(\frac{\partial^2 F_0}{\partial \eta^2} + p\frac{\partial^2 F_1}{\partial \eta^2}\right) + ...\right] = 0$$

Re-arranging the co-efficient of the terms with identical powers of p, we have

$$\begin{aligned}
p^0 &: \frac{\partial^3 F_0}{\partial \eta^3} - \frac{\partial^3 f_0}{\partial \eta^3} = 0 \\
p^1 &: \frac{\partial^3 F_1}{\partial \eta^3} + \frac{\partial^3 f_0}{\partial \eta^3} + \frac{F_0}{2}\frac{\partial^2 F_0}{\partial \eta^2} = 0 \\
p^2 &: \frac{\partial^3 F_2}{\partial \eta^3} + \frac{F_1}{2}\frac{\partial^2 F_0}{\partial \eta^2} + \frac{F_0}{2}\frac{\partial^2 F_1}{\partial \eta^2} = 0 \\
p^3 &: \frac{\partial^3 F_3}{\partial \eta^3} + \frac{F_1}{2}\frac{\partial^2 F_1}{\partial \eta^2} + \frac{F_2}{2}\frac{\partial^2 F_0}{\partial \eta^2} + \frac{F_0}{2}\frac{\partial^2 F_2}{\partial \eta^2} = 0 \qquad (4.2.4) \\
. &: . \\
. &: . \\
. &: .
\end{aligned}$$

First we take $F_0 = f_0$. We start iteration by defining f_0 as a Taylor series of order two near $\eta = 0$, so that it could be accurate near $\eta = 0$.

$$F_0 = f_0 = \frac{f''(0)}{2}\eta^2 + f'(0)\eta + f(0)$$

Let us take $f''(0) = 0.332057$, [5] and from the given boundary condition $f = 0$ and $f' = 0$. So,

$$\begin{aligned}
f_0 &= \frac{0.332057}{2}\eta^2 \\
&= 0.1660285\eta^2
\end{aligned}$$

Now, taking this value to solve F_1 from (4.2.4)

$$\begin{aligned}
\frac{\partial^3 F_1}{\partial \eta^3} + \frac{\partial^3 f_0}{\partial \eta^3} + \frac{F_0}{2}\frac{\partial^2 F_0}{\partial \eta^2} &= 0 \\
\frac{\partial^3 F_1}{\partial \eta^3} &= -\frac{F_0}{2}\frac{\partial^2 F_0}{\partial \eta^2} \\
&= -\frac{0.1660285}{2}\eta^2 \quad \frac{\partial^2}{\partial \eta^2}(0.1660285)\eta^2 \\
\frac{\partial^3 F_1}{\partial \eta^3} &= -(0.1660285)^2.\eta^2 \\
F_1 &= -(0.1660285)^2.\frac{\eta^5}{3.4.5} \\
\Rightarrow F_1 = f_1 &= -0.00045942\eta^5
\end{aligned}$$

Similarly from (4.2.4) we can easily calculate the value of f_2, f_3,... as

$$f_2 = 0.00000249\eta^8$$
$$f_3 = -0.00000001\eta^{11} \tag{4.2.5}$$

For the assumption p=1, we get

$$f(\eta) = 0.1660285\eta^2 - 0.00045942\eta^5 + 0.00000249\eta^8 - 0.00000001\eta^{11} \tag{4.2.6}$$

Results:

	$f(\eta)$	
η	H.P.M	Blasius
0	0	0
0.5	0.0415	0.0415
1	0.16550	0.1656
1.5	0.3701	0.3701
2	0.6500	0.6500
2.5	0.9962	0.9963
3	1.3964	1.3968
3.5	1.8350	1.8377
4.0	2.2897	2.3057

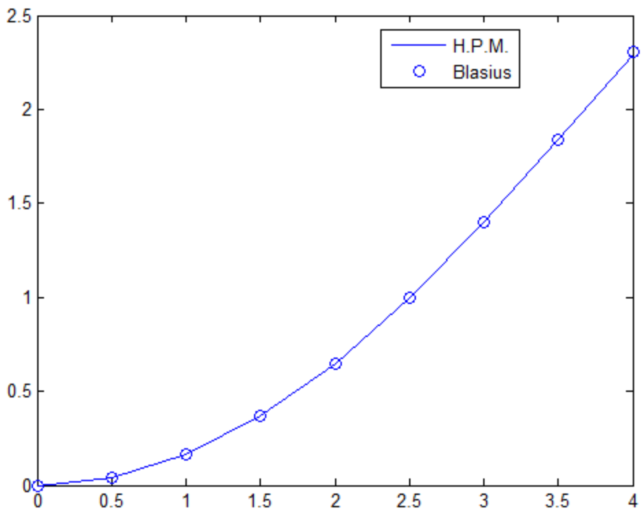

Figure 1: The comparison of answers obtained by H.P.M and Blasius's results for $f(\eta)$.

	$f'(\eta)$	
η	H.P.M	Blasius
0	0	0
0.5	0.1658	0.1659
1	0.3298	0.3298
1.5	0.4867	0.4868
2	0.6297	0.6298
2.5	0.7511	0.7513
3	0.8445	0.8430
3.5	0.9027	0.9130
4.0	0.9028	0.9555

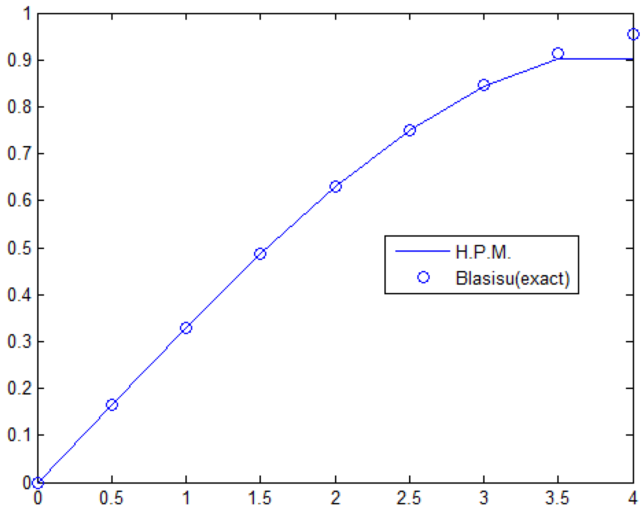

Figure 2: The comparison of answers obtained by H.P.M and Blasius's results for $f'(\eta)$.

5 Conclusion

In this research project paper, we have studied a well known Blasius boundary layer equation. We have applied homotopy perturbation method to solve this non-linear differential equation. From fig. 1 we conclude that the obtained results for $f(\eta)$ have excellent accuracy with the Blasius solution of Howarth [2]. Similarly in fig. 2 we also have approximate accuracy for $f'(\eta)$. The proposed method does not require small parameters in the equations, so the limitation of the traditional perturbation technique can be eliminated. The initial approximation can be freely selected with possible unknown constants. The approximation obtained by this method are valid not only for small parameter, but also for every large parameters. So, the homotopy perturbation method can applied to various non-liner differential equation. In this project paper, I came to know about perturbation method and homotopy perturbation method to solve various non-linear differential equation. I also learned the latex software to write mathematical code. In my future work I will employed all this methods so that I can solve any non-linear problems easily.

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