ROOT DIAGRAM OF EXCEPTIONAL LIE ALGEBRA G_2 AND F_4

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CERTIFICATE

This is to certify that the project work embodied in the dissertation **"Root Diagram of Exceptional Lie algebra G_2 and F_4"** which is being submitted by **Manasi Mishra, Roll No.412MA2064**, has been carried out under my supervision.

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Contents

Abstract

In this report, there is a compilation of many basic notations and examples from the theory of the Lie algebras. We describe some definition of General linear Lie algebra , Simple and semisimple Lie algebras , Lie groups and Lie algebra. We have now our disposal all the equipment needed to investigate the general structure of complex semisimple Lie algebras. We explain the strategy which is followed below . By using Root space decomposition , cartan-killing form and cartan matrix we describe the semisimple Lie algebra .We describe root systems and their associated Dynkin diagrams. The Cartan matrix and Dynkin diagram are introduced to suggest the applications of root systems . Finally we show how to construct the simple exceptional Lie algebra of type G_2 and F_4 (rank 2 and 4 respectively) and its total roots.

Chapter-1

1 Introduction

Studying continuous transformation groups in the end of nineteenth century, Sophus Lie discovered new algebraic structures now known as Lie algebras. They were not only interesting on their own right but also played an important role in twentieth century mathematical physics. Furthermore, mathematicians discovered that every Lie algebra could be associated to a continuous group, or a Lie group, which in turn considerably expanded the theory. Today, more than a century after Lie's discovery, we have a vast algebraic theory studying objects like Lie algebras, Lie groups, Root systems,Weyl groups, etc. It is called Lie theory and the intensive current research indicates its importance in modern mathematics.

Lie For the sake of coherence, we start from the very beginning and first briefly discuss the basic structure theory of Lie algebras. This will hopefully enable any reader with basic algebraic background to follow the text. We particularly lay stress on the root space decomposition and root systems since they are the tools needed for the classification of simple Lie algebras.In this thesis we discuss the classification of simple Lie algebras. We introduce representations of semisimple Lie algebras to decompose them into their root spaces. A root system, encoded in its associated Dynkin diagram, bears all the information about its Lie algebra. As the roots of semi simple Lie algebras satisfy several restrictive geometrical properties, we can classify all irreducible root systems by a brief series of combinatorial arguments. After unwinding the equivalences between this cast of objects, the result will finally provide a classification of simple complex Lie algebras.classical Lie algebras A_n, B_n, C_n and D_n construct the exceptional ones and show that the corresponding Dynkin diagrams are connected in each case, i.e. they are all simple. Furthermore, we briely describe of The classication of simple Lie algebras Killing form and Cartan matrix . There are ve exceptional Lie algebras of types G_2, F_4, E_6, E_7 and E_8 of dimensions $14, 56, 78, 133$ and 248 respectively, which we denote by G_2, F_4 etc. The construction of these exceptional Lie algebras is subtle and has been treated by several authors beginning with Cartan. A general but cumbersome construction valid for all simple Lie algebras was given by Harish-Chandra , while for the exceptional Lie algebras has given a uniform but perhaps complicated treatment. In principle the Serre relations allow us to write down structure constants for any simple Lie algebra from the Dynkin diagram, but this is somewhat abstract. In many concrete applications one desires to know the explicit structure constants with respect to a convenient basis.

Chapter-2

2 Basic Structure Theory

2.1 Definitions and basic properties

The definition of lie algebra consist of essentially of two part revealing its structure. A lie algebra L is firstly a vector space , Secondly there is defined on L a particular kind of binary operation i.e. a mapping $L \times L \longleftrightarrow L$, denoted by $[.,.]$. The axiom characterizing the so-called Lie bracket $[.,.]$ are given in definition (2.1.1) below. The dimension of a Lie algebra is by definition dimension of its vector space. It may be finite or infinite. The vector space is taken either over the real numbers R or the complex numbers C. We will use F to denote either$R$ or C.

Definition 2.1.1 A Lie algebra L is a vector space with a binary operation.

$$(x, y) \in L \times L \longrightarrow [x, y] \in L$$

called Lie bracket or commutator, which satisfies

1. For all $x, y \in L$ one has

$$[x, y] = -[x, y] \qquad (antisymmetry)$$

2. The binary operation is linear in each of its entries:

$$[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z] \qquad (bilinearity)$$

For all $x, y \in L$ and all $\alpha, \beta \in F$ 3. For all $x, y, z \in L$ one has

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad (jacobiidentity)$$

A Lie algebra is called real or complex when its vector space is respectively real or complex($F = C$)

Definition 2.1.2 A Lie algebra L is called abelian or commutative if $[x, y] = 0$ for all x and y in L

Definition 2.1.3 A subset K of a Lie algebra L is called a subalgebra of L if for all $x, y \in K$ and all $\alpha, \beta \in F$ one has $\alpha x + \beta y \in K, [x, y] \in K$

Definition 2.1.4 An ideal I of a Lie algebra L is a subalgebra of L with the property.

$$[I, L] \subset I$$

i.e for all $x \in I$ and all $y \in L$ one has

$$[x, y] \in I$$

Note

Every (non-zero) Lie algebra has at least two ideals namely the Lie algebra L itself and the subalgebra 0 consisting of the zero element only.

$$0 \equiv \{0\}$$

Both these ideal are called trivial. All non-trivial ideal are called proper.

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2.2 General linear Lie algebra

Let $L(V)$ be the set of all linear operators on a vectors space V . This vector space V will often be a finite-dimensional one.

We have for all $\alpha, \beta \in F$ and all $\nu \in V$

$$(\alpha A + \beta B)\nu = \alpha(A\nu) + \beta(B\nu)$$

and

$$(AB)\nu = A(B\nu)$$

clearly , $L(V)$ is an associative algebra with unit element 1; defined by $1\nu = \nu$ for all ν in V.

Defining on $L(V)$ a bracket operation by

$$[A, B] := AB - BA$$

turns $L(V)$ into a Lie algebra. This Lie algebra is called general Lie algebra $gl(V)$.

2.3 Derived algebra of a Lie algebra

Consider in L the set $L^{'} = [L, L]$. this is the set of element of the form $[x, y](x, y \in L)$ and possible linear combinations of such elements. It is called the derived algebra of L.

Lemma -: Let

$$L^{'} := [L, L]$$

then this is an ideal in L.

Proof: The derived algebra $L^{'}$ is by definition a subspace of L.

since $L^{'} \subset L$

We have $[L^{'}, L^{'}] \subset [L, L] = L^{'}$ in order to prove that $L^{'}$ is an ideal. However from (1) it follows

$$[L^{'}, L] \subset [L, L] = L^{'} \qquad .$$

Definition 3.1 Let L be a Lie algebra. The sequence $L^0, L^1, ..., L^n, ...$ defined by

$$L^0 := L, L^1 := [L, L], L^2 := [L, L^1], ...$$

$$L^n := [L, L^{n-1}], ...$$

is called the descending central sequence.

Definition 2.3.1 Let L be a Lie algebra. The sequence $L_0, L_1, ..., L_n, ...$ defined by

$$L_0 := L, L_1 := [L_0, L_0], ..., L_n := [L_{n-1}, L_{n-1}], ...$$

is called the derived sequence.

Definition 2.3.2 A Lie algebra L is called nilpotent if there exist an $n \in N$ such that $L^n = 0$.

Remark : If $L^n = 0$ then of course $L^q = 0$ for $q \geq 0$. Let $L \neq 0$ and let p be the smallest integer for which $L^p = 0$. Then We have $[L, L_{p-1}] = 0$ and $L^{p-1} \neq 0$. This means that L^{p-1}

is an abelian ideal in L. Hence each nilpotent Lie algebra $L \neq 0$ contains an ideal unequal to 0.

Definition 2.3.3 A Lie algebra L is called solvable if $L_n = 0$ for some $n \in N$.

Definition 2.3.4 The maximal solvable ideal of a Lie algebra L is called the radical of L and it is denoted by $R \equiv radL$.

2.4 Simple and Semisimple Lie algebras

Definition 2.4.1 A Lie algebra L is called simple if L is non-abelian and has no proper ideals.

Corollary For a simple Lie algebra L the derived algebra $L^{'} = [L, L]$ is equal to L.

Proof: The derived algebra $L^{'}$ is an ideal. Since L is simple, $L^{'}$ has to be a trivial ideal. One has only two alternatives,either $L^{'} = 0$ or $L^{'} = L$. The first option is rule out since L is non-abelian. Therefore $L^{'} = L$.

Example Consider in $gl(n)$ the subset of matrices with trace equal to zero. This is a Lie algebra called the $sl(n)$ and which is simple.

Definition 2. 4.2 Lie algebra L is called semisimple if $L \neq 0$ and L has no abelian $ideal \neq 0$.

Levi's theorem Let L be a finite-dimensional Lie algebra and $R \equiv radL$ its radical. Then there exists a semisimple subalgebra S of L such that L is the direct sum of its linear subspaces R and S.

$$L = R \oplus S$$

2.5 Idealizer and Centralizer

Definition 2.5.1 Let k be a subalgebra of a Lie algebra L. The idealizer $N_L(k)$ of k in L is defined as

$$N_L(k) := \{x \in L | \forall y \in k : [x, y] \in K\} \qquad .$$

Definition 2.5.2 A subalgebra k is called self-idealizing in L if $N_L(k) = k$. **Definition 2.5.3** A subalgebra of L. The centralizer $C_L(k)$ of k is defined by

$$C_L(k) := \{x \in L | \forall y \in k : [x, y] = 0\} \qquad .$$

Definition 2.5.4 The centralizer$C_L(L)$ of L itself is called the center of L. It is usually denoted by $Z(L) \equiv C_L(L)$ and it is given by

$$Z(L) := \{x \in L | \forall y \in L : [x, y] = 0\}$$

2.6 Derivations of a Lie algebra

Definition 2.6.1 A derivation δ of a Lie algebra L is a linear map

$$\delta : L \longrightarrow L \qquad satisfying$$

$$\delta[x, y] = [\delta x, y] + [x, \delta y]$$

The collection of all derivation of L is denoted by $DerL$

2.7 Structure constants of a Lie algebra

Let L be a finite dimensional Lie algebra . Let n be the dimension of L. $\{e_1, e_2, ..., e_n\}$ is a basis for L. the every element of Lie algebra can be written as

$$x = \Sigma_{i=1}^{n} x^i e_i$$

Let $y \in L$ $y = \Sigma_{k=1}^{n} y^k e_k$

$$[x, y] = [\Sigma_{i=1}^{n} x^i e_i, \Sigma_{k=1}^{n} y^k e_k] = \Sigma_{i,k=1}^{n} x^i y^k [e_i, e_k]$$

So commutator $[x, y]$ of two element $x, y \in L$ is completely determine by the lie bracket $[e_i, e_k]$ of pairs of basis element $[e_i, e_k] \in L$. hence $[e_i, e_k]$ can again expanded w.r.t. the basis $\{e_1, e_2, ..., e_n\}$.

2.8 Special linear Lie algebra

The structure of the special linear Lie algebra $sl(n, C)$.

Definition 2.8.1 $sl(n, C)$ is the set of all $n \times n$ matrics with complete entries having trace zero. The lie bracket of element of $sl(n, C)$ is commutator of there matrices.

$$dim(sl(n, C)) = n^2 - 1$$

Remark

We are interested for matrices $n \geq 2$ putting $n = k + 1$ for $k \geq 1$ with this convention the special linear lie algebra is denoted by $asl(k + 1, C)$ or A_k.

The dimension of A_k is

$$dim(sl(k + 1, C))(k + 1)^2 - 1 = k^2 + 2k + 1 - 1 = k(k + 2)$$

2.9 Lie groups and Lie algebra

We discuss briefly the relationship between Lie algebras and Lie groups. Since we will be dealing with linear Lie groups, i.e. groups the elements of which are linear operators on some vector space.

Our discussion is based on the complex general linear Lie groups $GL(V)$, the groups of bijective linear operators on a complex $n-$ dimensional vector space V. Denoting the Groups elements by capitals A, B etc. We define a matrix representation of these operators by taking a basis in V. Then the matrix (a_{ij}) representing the operators A is defined by

$$Ae_i = \Sigma_{j=1}^{n} e_j a_{ji} \qquad (i = 1, 2, ..., n)$$

In this way we obtain the isomorphism $A \in GL(V) \longrightarrow (a_{ij}) \in GL(n, C)$ where $GL(n, C)$ is the groups of all invertible $n \times n$ matrices. Next we indicate why both groups, $GL(V)$ and $GL(n, C)$, are n^2- dimensional complex Lie groups. Let $M(n, C)$ be the set of all complex $n \times n$ matrices, then map

$$k : (a_{ij}) \in M(n, C) \longrightarrow (a_{11}, a_{12}, ..., a_{nn}) \in C^{n^2}$$

is a bijection. From the fact that the map

$$det : (a_{ij}) \longrightarrow det(a_{ij}) \in C$$

is a continuous function of the matrix elements it follows that $GL(n,C) = \{A \in M(n,C|detA \neq 0\}$ is an open set in C^{n^2}.

This implies that the restriction $k|GL(n,C)$ map the open set $GL(n,C)$ bijectively onto the open set $C^{n^2}|k$, where

$$k : \{(a_{11}, a_{12}, ..., a_{nn}) \in C^{n^2}|det(a_{ij}) = 0$$

In the general theory of Lie groups it is shown that the vector space structure of the Lie algebra of a Lie groups is isothermic with the tangent space at the unit element of the group manifold. For a linear Lie group the tangent space is readily obtained. Consider in $GL(n,C)$ a subset of operators $A(t)$depending smoothly on areal parameter t and such that $A(0) = 1$ where 1 is the identity operators on V. Such a subset is called a curve through the unit element. The tangent vector at $t = 0$ is obtained by making the Taylor expansion of $A(t)$ upto the first order term.

$$A(t) = A(0) + M(t) + O(t^2)$$

with M the derivative of $A(t)$ at $t = 0$;

$$M = A(0)$$

Next we define subgroups of $GL(n,C)$ by considering subset of elements in $GL(n,C)$ that leave a specific non-degenerate bilinear form on the vector space V invariant.

A bilinear form , denotes by $(.,.)$ is a map $(.,.) : V \times V \longrightarrow C$ such that for all $\alpha, \beta \in C$ and all $x, y, z \in V$ one has

$$(\alpha x + \beta y, z) = \alpha(x,z) + \beta(y,z)$$

and

$$(x, \alpha y + \beta z) = \alpha(x,y) + \beta(x,z)$$

A bilinear form is called symmetric if for all $x, y \in V$

$$[x, y] = [y, x]$$

A bilinear form is called skew-symmetric if $\forall x, y \in V$

$$[x, y] = -[y, x]$$

A skew-symmetric form $(.,.)$ is called non-degenerate
if $(x, y) = 0$for all $y \in V \Rightarrow x = 0$
We will now point out that a bilinear form $(.,.)$ on a vector space V can be used to define a subgroup of $GL(n, C)$ consider in $GL(n, C)$ the subset S of elements which leave invariant the form $(.,.)$. An elements $A \in Gl(n, C)$ is said to leave form $(.,.)$ invariant if we have for all $x, y \in V$

$$(Ax, Ay) = (x, y) \qquad .....(1)$$

Let $A, B \in S$ then

$$(ABx, ABy) = (Bx, By) = (x, y)$$

This means that the product $AB \in S$. From the fact that the unit operators 1 belongs to S one obtains as follows that $A \in S$ implies $A^{-1} \in S$.

$$(x, y) = (1x, 1y) = (AA^1x, AA^1y) = (A^{-1}x, A^{-1}y)$$

We conclude that the set S is a subgroup of $GL(n, C)$ The condition of invariance (1) on the groups can be translated to a condition on the Lie-algebra. Taking instead of A a curve $A(t) = expMt$, one has Differentiation of this express with respect to t and taking $t = 0$ gives

$$(Mx, y) + (x, My) = 0$$

Example:- The simplest examples of an r parameter Lie groups is the abelian Lie group R^r . The groups operation is given by vector addition. The identity element is the zero vector and the inverse of a vector x is the vector $-x$.

Chapter-3

3 CLASSIFICATION OF SIMPLE LIE ALGEBRA

3.1 Cartan matrix

Definition: The cartan matrix $(A_{ij})_{i,j=1}^{k}$ of semisimple Lie algebra is defned by means of the dual contraction between Π and Π^v.

$$A_{ij} = \langle \alpha_j, \alpha_i^v \rangle = \frac{2\,(\alpha_j \mid \alpha_i)}{(\alpha_i \mid \alpha_i)} \tag{1}$$

The matrix elements of the cartan matrix are the cartan integers of simple roots. An immediate consequence of (1) is a relation between the ratios of lengths of simple roots and matrix elements of the caran matrix

$$\frac{A_{ij}}{A_{ji}} = \frac{\|\alpha_j\|^2}{\|\alpha_i\|^2}$$

3.2 Cartan Subalgebra

Definition: A subalgebra $K \subset L$ is called a cartan subalgebra of L if h is nilpotent and equal to its own normalizer.

The adjoint representation

For every $x \in L$ a linear operator adx on the vector space L by means of the Lie bracket on L, namely $adx := [x, y]$, for all $y \in L$ the map

$(x, y) \in L \times L \longrightarrow adx(y) \in L$

$$ad[x, y] = [adx, ady]$$

$$ad[x, y](z) = [[x, y], z] = [x, [y, z]] - [y, [x, z]]$$

$$= (adxady)(z) - (adyadx)(z)$$

$$= [adx, ady](z)$$

$$ad : x \in L \longrightarrow \in gl(L)$$

is a representations of L with representation space L. This representation is called the adjoint representations.

$$ade_i(e_j) = [e_i, e_j] = \sum_k c_{ij}^k e_k$$

This yields a matrix representation of ade_i.

3.3 Cartan-killing form

Definition: Let L be a Lie algebra.The map $K : L \times L \longrightarrow F$ given by $K(x, y) = Tr(adxady)$

Properties

1. $K(\alpha x+\beta y, z) = \alpha K(x, z)+\beta K(y, z)$

   $K(x, \alpha y + \beta z) = \alpha K(x, y) + \beta K(x, z)$ $\qquad (bilinearity)$

2. $K(x, y) = K(y, x)$ $\qquad (symmetry)$

3. $K([x, y], z) = K(x, [y, z])$ $\qquad (associativity)$

3.4 Root Space Decomposition

Let H be a maximal toral subalgebra of L. The linear Lie algebra $ad_L H$ is a commuting set of digonalizable linear operators on the vector space L. So L has a basis consisting of the simulteneous eigen vectors of the set operators $\{adh|h \in H\}$. This means that L decomposes into a direct sum of subspace.

This subspace which will be denoated by L_α. The vectors $x \neq 0$ in $L_\alpha \subset L$ aer by definition eigen vectors of adh for all $h \in H$. Denoting eigen value of adh by $\alpha(h)$ one has for all $h \in H$.

$adh(x) = \alpha h(x)$

Clearly the lable α is the function $\alpha : h \in H \to \alpha(h) \in C$ which associates the eigenvalues $\alpha(h)$ of the eigen vector x to the element h.

Definition: Let H be a maximal toral sub algebra of a finite dimensional complex semisimple Lie algebra L. The eigenvalues of the linear operator adh will be denoated by αh and one defines the subspace L_α of L by

$L_\alpha := \{x \in L | \ \forall \ h \in H : adh(x) = \alpha(h)x\}$

Then the Lie algebra L is a vector space direct sum of the subspace L_α :

$L = \oplus_\alpha L_\alpha$

This is called the rootspace decomposition of L w.r.t H.

3.5 Different types of Simple Lie algebra (A_n, B_n, C_n, D_n)

Without loss of generality we shall work over C in this entire subsection. We consider the classical Lie algebras $sl(n, C); so(n, C)$ and $sp(n, C)$ for $n \geq 2$. We want to find their root systems and to show that their Dynkin diagrams.

$A_n - Type(Sl(n+1, C)$

(1) The root space decomposition of $L = Sl(n+1, C)$ is

$$L = H \oplus \bigoplus_{i \neq j} L_{\varepsilon_i - \varepsilon_j}$$

Where $\varepsilon_i(h)$ is the $i-$th entry of h and the root space $L_{\varepsilon_i - \varepsilon_j}$ is spanned by e_{ij}. Thus $\Phi = \{\pm(\varepsilon_i - \varepsilon_j) : 1 \leq i < j \leq n+1\}$.

(2) If $i < j$, then we have $[e_{ij}, e_{ji}] = e_{ii} - e_{jj} = h_{ij}$ and $[h_{ij}, e_{ij}] = 2e_{ij} \neq 0$ and thus $[[L_\alpha, L_{-\alpha}], L_\alpha]$ for each root $\alpha \in \phi$.

(3) The root system Φ has as a base $\{\alpha_i : 1 \leq i \leq n\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$

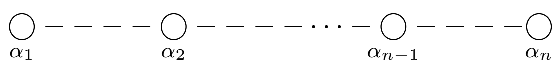
(4) We have already computed the Cartan matrix for this root system. We have simply the following

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2, & i = j \\ -1, & if\, |i-j| = 1; \\ 0, & otherwise \end{cases}$$

We shall also notice that from (2) follows that the standard basis elements for the subalgebra $Sl(\alpha_i)$ can be taken as

$$e_{\alpha_i} = e_{i,i+1}, f_{\alpha_i} = e_{i+1,i}, h_{\alpha_i} = e_{i,i} - e_{i+1,i+1}$$

Calculation shows that .Here L is simple. We say that the root system of $sl(n+1, C)$ has type A_n.
and the Dynkin diagram is :



. This diagram is connected, so L is simple.

B_n**-Type** $(So(2n+1, C))$

$so(2n+1, C)$ is represented by the block matrices of the type

$$x = \begin{pmatrix} 0 & c^t & -b^t \\ b & m & p \\ -c & q & -m^t \end{pmatrix},$$

with $p = -p^t$ and $q = -q^t$. As usual, let H be the set of diagonal matrices in L. We label the matrix entries from 0 to $2n$ and thus every element $h \in H$ can be written in the form $h = \Sigma_{i=1}^{n} a_i(e_{ii} - e_{i+n,i+n})$, where $0, a_1, ...a_n, -a_1, ... - a_n$ are exactly the diagonal entries of h.
(1) We first start by finding the root spaces for H and then using them we find the root space decomposition of L. Now consider the subspace of L spanned by the matrices whose non-zero entries lie only in the positions labeled by b and c. Now using our labeling and looking at the block matrix above we easily see that this subspace has a basis

$b_i = e_{i,0} - e_{0,n+i} and c_i = e_{0,i} - e_{n+i,0}$ for $1 \leq i \leq n$.

We do the following calculation:

$$[h, b_i] = [\Sigma_{i=1}^{n} a_i(e_{ii} - e_{i+n,i+n}), e_{i,0} - e_{0,n+i}]$$

$$= \Sigma_{i=1}^{n} a_i([e_{ii}, e_{i,0}] - [e_{ii}, e_{0,n+i}] + [e_{n+i,n+i}, e_{i,0}])$$

$$= \Sigma_{i=1}^{n} a_i(e_{ii} - e_{i+n,i+n}) = a_i b_i$$

where we use the following relations

$[e_{ii}, e_{i,0}] = e_{i0}, [e_{ii}, e_{0,n+i}] = 0, [e_{n+i,n+i}, e_{i,0}] = 0, [e_{n+i,n+i}, e_{0,n+i}] = -e_{n+i,n+i}$.

Similarly, we get $[h, c_i] = -a_i c_i$. Further, we extend to a basis of L by the matrices:

$$m_{ij} = e_{ij} - e_{n+j,n+i} \qquad for 1 \leq i \neq j \leq n,$$

$$p_{ij} = e_{i,n+j} - e_{j,n+i} \qquad for 1 \leq i < j \leq n,$$

$$q_{ij} = p_{ij}^t = e_{n+j,i} - e_{n+i,j} \qquad for 1 \leq i < j \leq n.$$

We now calculate the following relations:

$$[h, m_{ij}] = (a_i - a_j) m_{ij},$$

$$[h, p_{ij}] = (a_i + a_j) p_{ij},$$

$$[h, q_{ij}] = -(a_i + a_j) q_{ji}.$$

We can now list the roots. For$1 \leq i \leq n$, let $\varepsilon_i \in H^*$ be the map sending h to a_i, its entry position i.

(2) It suffices to show that $[h_\alpha, x_\alpha] \neq 0$, where $h_\alpha = [x_\alpha, x_{-\alpha}]$. We do this in three steps. First, for$\alpha = \varepsilon_i$, we have $h_i = [b_i, c_i] = e_{ii} - e_{n+i,n+i}$ and by (1) we have $[h_i, b_i] = b_i$. Second, for $\alpha = \varepsilon_i - \varepsilon_j$ and$i < j$, we have$h_{ij} = [m_{ij}, m_{ji}] = (e_{ii} - e_{n+i,n+i}) - (e_{jj} - e_{n+j,n+j})$ and again by (1) we obtain $[h_{ij}, m_{ij}] = 2m_{ij}$. Finally, for $\alpha = \varepsilon_i + \varepsilon_j$ and $i < j$, we have $k_{ij} = [p_{ij}, q_{ji}] = (e_{ii} - e_{n+i,n+i}) + (e_{jj} - e_{n+j,n+j})$ whence $[k_{ij}, p_{ij}] = 2p_{ij}$.

(3) The base for our root system is given by $\Delta = \}\alpha_i : 1 \leq i < n\} \cup \}\beta_n$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$

and $\beta_n = \varepsilon_n$. For $1 \leq i < n$ we see that $\varepsilon_i = \alpha_i + \alpha_{i+1} + ... + \alpha_{n-1} + \beta_n$
and for $1 \leq i < j \leq n$,
$\varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + ... + \alpha_{j-1}$,
$\varepsilon_i + \varepsilon_j = \alpha_i + \alpha_{i+1} + ... + \alpha_{j-1} + 2(\alpha_i + \alpha_{i+1} + ... + \alpha_{n-1} + \beta_n)$.
Now using our table of roots we see that if $\gamma \in \Phi$ then either γ or γ appears above as a non-negative linear combination of elements of Δ. Since $dimH$ is the same as the number of elements of Δ, precisely n ,we conclude that Δ is a base for Φ.

(4) For $i < j$ take $e_{\alpha_i} = m_{i,i+1}$ and by (2) follows $h_{\alpha_i} = h_{i,i+1}$. Taking $e_{\beta_n} = \beta_n$ we see that $h_{\beta_n} = 2(e_{nn} - e_{2n,2n})$. For $1 \leq i, j \leq n$, we calculate that

$$[h_{\alpha_j}, e_{\alpha_i}] = \begin{cases} 2e_{\alpha_i}, & i = j \\ -e_{\alpha_i}, & if\ |i-j| = 1; \\ 0, & otherwise \end{cases}$$

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2, & i = j \\ -1, & if\ |i-j| = 1; \\ 0, & otherwise \end{cases}$$

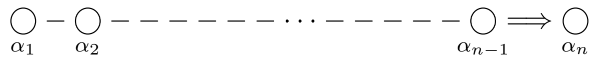
Similarly, by calculating $[h_{\beta_n}; e_{\alpha_i}]$ and $[h_{\alpha_i}; e_{\beta_n}]$, we find that

$$\langle \alpha_i, \beta_n \rangle = \begin{cases} -2, & i = n-1 \\ 0, & otherwise \end{cases}$$

$$\langle \beta_n, \alpha_i \rangle = \begin{cases} -1, & i = n-1 \\ 0, & otherwise \end{cases}$$

.

This shows that the Dynkin diagram of ϕ is :



(2)

and since it is connected, ϕ is irreducible and so L is simple. The root system of $so(2n+1, C)$ is said to have type B_n.

C_n**-Type** $(sp(2n, C))$

The elements of this algebra as block matrices as follows:

$$\begin{pmatrix} m & p \\ q & m^t \end{pmatrix}$$

where $p = p_t$ and $q = q_t$. The first observation to make is that for $n = 1$ we have $sp(2, C) \cong sl(2, C)$, since we have 2×2 matrices with entries numbers and not block matrices. Thus, without loss of generality we will assume that $n > 2$. As above, H is the set of diagonal matrices in L. We also use the same labeling of the matrix entries so $h = \Sigma_{i=1}^{n} a_i(e_{ii} - e_{i+n,i+n})$.

(1) Take the following basis for the root space of L:

$$m_{ij} = e_{ij} - e_{n+j,n+j} \qquad for 1 \leq i \neq j \leq n,$$

$$p_{ij} = e_{i,n+j} - e_{j,n+j} \qquad for 1 \leq i < j \leq n, p_{ii} = e_{i,n+i} \quad for 1 \leq i \leq n$$

$$q_{ij} = p_{ij}^t = e_{n+j,i} - e_{n+i,j} \qquad for 1 \leq i < j \leq n \qquad .$$

Calculations show that:

$$[h, m_{ij}] = (a_i - a_j)m_{ij},$$

$$[h; p_{ij}] = (a_i + a_j)p_{ij},$$

$$[h; q_{ij}] = -(a_i + a_j)q_{ji}.$$

Clearly, for $i = j$ the eigenvalues for p_{ij} and q_{ji} are $2a_i$ and $-2a_i$ respectively.

(2) Now we must check that $[h, x_\alpha] \neq 0$ with $h = [x_\alpha; x_{-\alpha}]$ holds for each root . It has been done for $\alpha = \varepsilon_i - \varepsilon_j$ for $so(2n+1, C)$. If $\alpha = \varepsilon_i + \varepsilon_j$, then $x_\alpha = p_{ij}$ and $x_{-\alpha} = q_{ji}$ and $h = (\varepsilon_{ii} - \varepsilon_{l+i,l+i}) + (\varepsilon_{jj} - e_{n+j,n+j})$ for $i = j$. We then have $[h, x_\alpha] = 2x_\alpha$ in both cases.

(3) Choose $\alpha_i = \varepsilon_i - \varepsilon_{i+1} for 1 \leq i \leq n-1$ as before, and $\beta_n = 2\varepsilon_n$. Our claim now is that $\{\alpha_1, ..., \alpha_{n-1}, \alpha_n\}$ is a base for the root system ϕ of $sp(2n, C)$. For $1 \leq i < j \leq n$ we have $\varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + ... + \alpha_{j-1}$, $\varepsilon_i + \varepsilon_j = \alpha_i + \alpha_{i+1} + ... + \alpha_{j-1} + 2(\alpha_j + \alpha_{j+1} + ... + \alpha_{n-1} + \beta_n)$, $2\varepsilon_i = \alpha_i + \alpha_{i+1} + ... + \alpha_{n-1} + \beta_n$,

Thus using the same arguments as above we conclude that $(\alpha_1, ... + \alpha_{n-1}, \beta_n)$ is the base of ϕ.

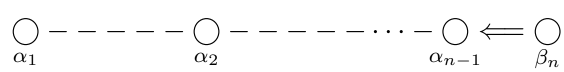
(4) In the end we need to calculate the Cartan integers. The numbers $\langle \alpha_i, \alpha_j \rangle$ are al- ready known. Taking $(e_{\beta_n} = p_{nn}$ we find that $h_{\beta_n} = e_{n,n} - e_{2n,2n}$ and so.

$$\langle \alpha_i, \beta_n \rangle = \begin{cases} -1, & i = n-1 \\ 0, & otherwise \end{cases}$$

$$\langle \beta_n, \alpha_j \rangle = \begin{cases} -2, & i = n-1 \\ 0, & otherwise \end{cases}$$

The Dynkin diagram of this root system.

C_n



which is connected, so L is simple. The root systems of $sp(2n, C)$ is said to have type C_n.

D_n**-Type** $(so(2n, C))$

All the elements of this classical algebra as block matrices:

where $p = -p^t$ and $q = -q^t$.

We observe that for$n = 1$ our Lie algebra is one dimensional so by definition is neither simple nor semisimple. In particular, the classical Lie algebra $so(2; C)$ is neither simple or semisimple. Again H is the set of diagonal matrices in L and we do the same labeling as in the former case. Thus we can use the calculations above by simply ignoring the row and column of matrices labeled by 0.

(1) We now simply copy the second half of the calculations for $so(2n+1;C)$.

(2) The calculations done above immediately yield that $[[L_\alpha, L_{-\alpha}], L_\alpha] \neq 0$ for each root α.

(3) We now claim that the base for our root system is $\Delta = \{\alpha_i : 1 \leq i < n\} \cup \{\beta_n\}$, where $\alpha = \varepsilon_i - \varepsilon_{i+1}$ and $\beta = \varepsilon_{n-1} - \varepsilon_n$. For $1 \leq i < j \leq n$, we have the following:

$\varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + ... + \alpha_{j-1}$,

$\varepsilon_i + \varepsilon_j = \alpha_i + \alpha_{i+1} + ... + \alpha_{n-2} + (\alpha_j + \alpha_{j+1} + ... + \alpha_{n-1} + \beta_n)$.

Then if $\gamma \in \phi$ then either γ or $-\gamma$ is a non-negative Z-linear combination of elements of Δ. Therefore, Δ is a base for our root system.

(4) Now we calculate the Cartan integers. The work already done for $so(2n+1, C)$ gives us the Cartan numbers $\langle \alpha_i, \alpha_j \rangle$ for $i, j < n$. To calculate the remaining ones we take $e_{\beta_n} = p_{n-1,n}$ and use (2) from $so(2n+1;C)$. Thus we obtain that $h_{\beta_n} = (e_{n-1,n-1} - e_{2n-1,2n-1}) + (e_{n,n} - e_{2n,2n})$. Hence

$$\langle \alpha_j, \beta_n \rangle = \begin{cases} -1, & j = n-2 \\ 0, & otherwise \end{cases}$$

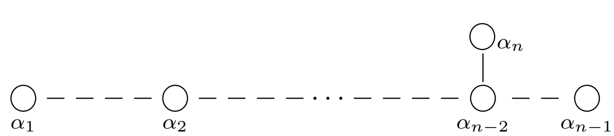
$$\langle \beta_n, \alpha_j \rangle = \begin{cases} -2, & j = n-2 \\ 0, & otherwise \end{cases}$$

.

If $n = 2$, then the base has only two orthogonal roots α_1 and β_2, so in this case , ϕ is reducible and hence $so(4, C)$ is not simple. If $n > 3$.

If $n = 2$, then the base has only two orthogonal roots α_1 and β_2, so in this case , ϕ is reducible and hence $so(4, C)$ is not simple. If $n > 3$, then our calculations show that the Dynkin diagram of ϕ is .

D_n

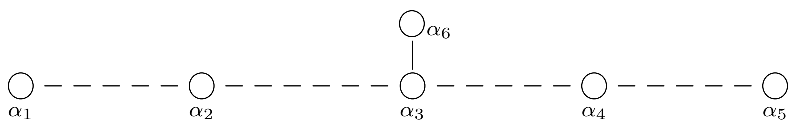


(4)

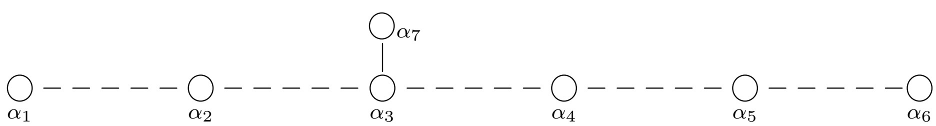
As this diagram is connected, the Lie algebra is simple. When $n \geq 3$, the Dynkin diagram is the same as A_3, the root system of $sl(4, C)$, so we have that $so(6, C) \cong sl(4, C)$.For $n > 4$, the root system of $so(2n, C)$ is said to have type D_n. So far we have that only $so(2, C)$ and $so(4, C)$ are not simple. Therefore, it now remains to show that $sp(2n, C)$ is simple . Besides of these simple Lie algebra some others Lie algebra are also simple called Exceptional Lie algebra i.e. E_6, E_7, E_8, G_2, F_4.

The Dynkin diagram of above Exceptional Lie algebra

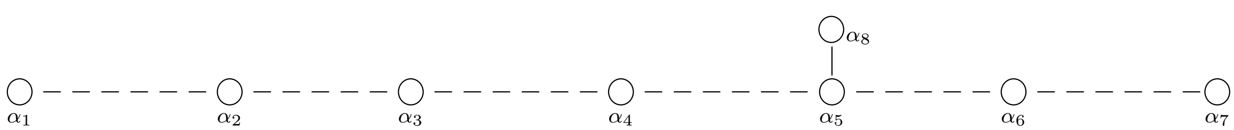
E_6



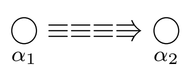
E_7



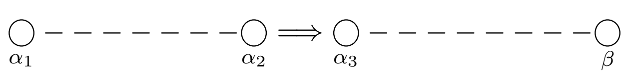
E_8



G_2



F_4



Chapter-4

4 Root System and Dynkin diagram

4.1 Root System

An abstract root system is a finite set of elements $R \subset E \setminus 0$, where E is a real vector space with a positive definite inner product, such that the following properties hold:

(1) R generates E as a vector space.

(2) For any two roots α, β, the number

$$n_{(\alpha\beta)} := 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}$$

is an integer.

(3) Let $s_\alpha : E \longrightarrow E$ be defined by

$$s_\alpha(\lambda) = \lambda - 2\frac{(\alpha, \lambda)}{(\alpha, \alpha)}\alpha$$

Then for any roots $\alpha, \beta, s_\alpha(\beta) \in R$. The number $r = dimE$ is called the rank of R. If, in addition, R satisfies the following property

(4) If $\alpha, c\alpha$ are both roots, then $c = 1$. then R is called a reduced root system.

4.2 Root Chain

In this section Let α and $\beta \neq \pm\alpha$ be roots.Repeated application of the linear operator ad e_α on e_β yields sequence of non-zero vectors $e_\beta, e_{\beta+\alpha}, e_\alpha, ..., e_{\beta+q\alpha}$ of course deleting the zero vectors. Likewise repeated application of the linear operators ad $f_\alpha (f_\alpha \in L_{-\alpha})$ on e_β yields also a sequence of non-zero vectors $e_\beta, e_{\beta-\alpha}, e_{\beta-2\alpha}, ..., e_{\beta-p\alpha}$. In this way we obtain the following sequence of roots.

$$\beta - p\alpha, ..., \beta - 2\alpha, \beta - \alpha, \beta + \alpha, \beta + 2\alpha, ..., \beta + q\alpha$$

.

Applying the properties of the irreducible representations of $Sl(2, C)$ on this sequence we derive in the following Lemma some importance properties of the root system Δ.

Lemma Let L be complex semi simple Lie algebra Let α and $\beta \neq \pm\alpha$ be roots and M be the set of integers $\{t\}$ such that $\beta + t\alpha$ is a root i.e.

$$M := \{t \in Z \mid \beta + t\alpha \in \Delta\}$$

.

a then M is closed interval $[-p, q] \cap Z$.

with p and q non-negative integers. Moreover one has .

$$p - q = \langle \beta, h_\alpha \rangle$$

.

In particular $\langle \beta, h_\alpha \rangle$ is an integer.

The sequence roots

$$\beta - p\alpha, ..., \beta - 2\alpha, \beta - \alpha, \beta + \alpha, \beta + 2\alpha, ..., \beta + q\alpha$$

.

is called the $\alpha-$chain through β.

b As a special result we mention that $\beta - \langle \beta, h_\alpha \rangle \alpha$ is a root, i.e. $\beta - \langle \beta, h_\alpha \rangle \alpha \in \Delta$ ed root system.

4.3 Automorphisms and Weyl group :

Definition: Let $R_1 \subset E_1, R_2 \subset E_2$ be two root systems. An isomorphism $\phi : R_1 \longrightarrow R_2$ is a vector space isomorphism $\phi : E_1 \longrightarrow E_2$which also gives a bijection $R_1 simeq R_2$ and such that$n_{\phi(\alpha)\phi(\beta)} = n_{(\alpha\beta)}$ for any $\alpha, \beta \in R_1$.

Definition The Weyl group W of a root system R is the subgroup of $GL(E)$ generated by reflections $s_\alpha, \alpha \in R$.

4.4 Pairs of roots and rank two root systems

Our main goal is to give a full classification of all possible reduced root systems, which in turn will be used to get a classification of all semisimple Lie algebras. The first step is considering the rank two case. Throughout this section, R is a reduced root system. The first observation is that conditions (2), (3) impose very strong restrictions on relative position of two roots.

4.5 Positive roots and simple roots

Given a root system R we can always choose a set of positive roots. This is a subset R^+ of R such that For each root $\alpha \in R$ exactly one of the roots α, α is contained in R^+. For any two distinct $\alpha, \beta \in R^+$ such that $\alpha + \beta$ is a root, $\alpha + \beta \in R^+$. If a set of positive roots R^+ is chosen, elements of $-R^+$ are called negative roots.

An element of R^+ is called a simple root if it cannot be written as the sum of two elements of R^+. The set Δ of simple roots is a basis of E with the property that every vector in R is a linear combination of elements of Δ with all coefficients non-negative, or all coefficients non-positive. For each choice of positive roots, the corresponding set of simple roots is the unique set of roots such that the positive roots are exactly those that can be expressed as a combination of them with non-negative coefficients, and such that these combinations are unique.

4.6 Dynkin diagrams and classification of root systems

The main goal of this section will be to give a complete solution of this problem, i.e. give a classification of all root systems.

The first step is to note that there is an obvious construction which allows one to construct larger root systems from smaller ones. Namely, if $R_1 \subset E_1$ and $R_2 \subset E_2$ are two root systems, then we can define $R = R_1 \sqcup R_2 \subset E_1 \oplus E_2$, with the inner product on $E_1 \oplus E_2$ defined so that$E_1 \perp E_2$. It is easy to see that so defined R is again a root system.

Definition: A root system R is called reducible if it can be written in the form $R = R_1 \sqcup R_2$, with $R_1 \perp R_2$. Otherwise, R is called irreducible.

Definition: The Cartan matrix A of a set of simple roots $\Pi \subset R$ is the $r \times r$ matrix with entries

$$a_{ij} = n_{\alpha_j \alpha_i} = \langle \alpha_i^\wedge, \alpha_j \rangle = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

The following properties of Cartan matrix immediately follow from definitions and from known properties of simple roots.

(1) For any $i, a_{ii} = 2$.

(2) For any $i \neq j, a_{ij}$ is a non-positive integer: $a_{ij} \in Z, a_{ij} \leq 0$.

(3) For any $i \neq j, a_{ij}a_{ji} = 4cos^2\phi$, where ϕ is angle between α_i, α_j . If $\phi \neq \frac{\pi}{2}$, then

$$\frac{|\alpha_i|^2}{|\alpha_j|^2} = \frac{a_{ji}}{a_{ji}}$$

.

Definition: Let II be a set of simple roots of a root sytem R. The Dynkin diagram of II is the graph constructed as follows:

• For each simple root α_i, we construct a vertex v_i of the Dynkin diagram (traditionally, vertices are drawn as small circles rather than as dots)

• For each pair of simple roots $\alpha_i \neq \alpha_j$, we connect the corresponding vertices by n edges, where n depends on the angle ϕ between α_i, α_j :

For $\phi = \frac{\pi}{2}, n = 0$ (vertices are not connnected)

For $\phi = \frac{2\pi}{3}, n = 1$ (case of A_2 system)

For $\phi = \frac{3\pi}{4}, n = 2$ (case of B_2 system)

For $\phi = \frac{5\pi}{6}, n = 3$ (case of G_2 system)

• Finally, for every pair of distinct simple roots $\alpha_i \neq \alpha_j$, if $|\alpha_i| \neq |\alpha_j|$ and they are not orthogonal, we orient the corresponding (multiple) edge by putting on it an arrow pointing towards the shorter root.

Chapter-5

5 Exceptional Lie Algebras

.

In this chapter we describe some new types of root systems which are associated with so called exceptional Lie algebras. In each case we use the following set up. Let E be a subspace of R^m and ε_i be the vector with 1 in $i-th$ position and 0 elsewhere. Similarly to chapter 4 we take as many simple roots as possible from the set $\{\alpha_1, ..., \alpha_{m-1}\}$, where $\alpha_1 = \varepsilon_i - \varepsilon_{i+1}$. For these elements we have .

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2, & i = j \\ -1, & if\, |i-j| = 1; \\ 0, & otherwise \end{cases}$$

.

5.1 Type G_2

Let$E = \{v = \Sigma_{i=1}^{3} c_i \varepsilon_i \in R^3 : \Sigma c_i = 0\}$ let $I = \{m_1\varepsilon_1 + m_2\varepsilon_2 + m_3\varepsilon_3 \in R^3 : m_1, m_2, m_3 \in Z\}$, and let $R = \{\alpha \in I \cap E : (\alpha, \alpha) = 2 or (\alpha, \alpha) = 6\}$. This choice is motivated by the fact that the ratio of the length of a long root to the length of short root in this case is $\sqrt{3}$. By direct calculation we have that this root system is given by the set: $R = \{\pm(\varepsilon_i - \varepsilon_j) j i \neq j\} \cup \{\pm(2\varepsilon_1 \varepsilon_j \varepsilon_k) \mid \{i, j, k\} = \{1, 2, 3\}$. This gives 12 roots in total as expected. To find a base, we need to find two roots in R of different lengths making an angle of $\frac{5\pi}{6}$. 6 . One such choice is $\alpha = \varepsilon_1 - \varepsilon_2$ and $\beta = \varepsilon_2 + \varepsilon_3 - 2\varepsilon_1$

$$G_2 \qquad \underset{\alpha_1}{\bigcirc} \equiv\equiv\equiv\Rrightarrow \underset{\alpha_2}{\bigcirc} \qquad (1)$$

Details calculation of G_2 A cartan matrix for finite-dimensional semisimple Lie algebra satisfying:

(1)

$$det A \neq 0$$

(2)

$$A_{ii} = 2 for i = 1, 2, ..., k$$

(3)

$$A_{ij} = 0 \Rightarrow A_{ji} = 0$$

(4)

$$A_{ij} \in \{0, -1, -2, -3\} \qquad i \neq j$$

(5)

$$A_{ij} = -2 \Rightarrow A_{ji} = -1$$

(6)

$$A_{ij} = -3 \Rightarrow A_{ji} = -1$$

Dynkin diagram

$$G_2 \qquad \underset{\alpha_1}{\bigcirc} \equiv\!\equiv\!\equiv\!\Rrightarrow \underset{\alpha_2}{\bigcirc} \qquad (1)$$

$\phi_{(\alpha_1 \alpha_2)}$ be the angle between (α_j and α_i) i.e. 150° Cartan Matrix

$$x = \begin{pmatrix} \frac{2(\alpha_1,\alpha_1)}{(\alpha_1,\alpha_1)} & \frac{2(\alpha_2,\alpha_1)}{(\alpha_1,\alpha_1)} \\ \frac{2(\alpha_1,\alpha_2)}{(\alpha_2,\alpha_2)} & \frac{2(\alpha_2,\alpha_2)}{(\alpha_2,\alpha_2)} \end{pmatrix},$$

by solving this , we get

$$x = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix},$$

$(i, j)th$ entry a_{ij} of cartan matrix of G_2 is given by

$$a_{ij} = \frac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)}$$

All roots of G_2

(i)α_1 **chain through** $\alpha_1 + \alpha_2 \quad \alpha_2$,

$$\alpha_2 - \frac{2(\alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)}\alpha_1 = \alpha_2 + \alpha_1$$

(ii)α_1 **chain through** $\alpha_1 + \alpha_2$

$$\alpha_1 + \alpha_2 - \frac{2(\alpha_1 + \alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)} = \alpha_1 + \alpha_2 - \alpha_1 = \alpha_2$$

(iii)α_2 **chain through** α_1

$$\alpha_1 - \frac{2(\alpha_1 + \alpha_2)\alpha_2}{(\alpha_2, \alpha_2)} = \alpha_1 + 3\alpha_2$$

(iv)α_1 **chain through** $\alpha_1 + 3\alpha_2$

$$\alpha_1 + 3\alpha_2 - \frac{2(\alpha_1 + 3\alpha_2)\alpha_2}{(\alpha_2, \alpha_2)} = \alpha_1$$

(v)α_2 **chain through** $\alpha_1 + \alpha_2$

$$\alpha_1 + \alpha_2 - \frac{2(\alpha_1 + \alpha_2)\alpha_2}{(\alpha_2, \alpha_2)} = \alpha_1 + 2\alpha_2$$

(vi)α_1 **chain through** $\alpha_1 + 2\alpha_2$

$$\alpha_1 + 2\alpha_2 - \frac{2(\alpha_1 + 2\alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)} = \alpha_1 + 2\alpha_2$$

(vii)α_2 **chain through** $\alpha_1 + 2\alpha_2$

$$\alpha_1 + 2\alpha_2 - \frac{2(\alpha_1 + 2\alpha_2, \alpha_2)}{(\alpha_2, \alpha_2)} = \alpha_1 + \alpha_2$$

(viii)α_1 **chain through** $\alpha_1 + 3\alpha_2$

$$\alpha_1 + 3\alpha_2 - \frac{2(\alpha_1 + 3\alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)} = 2\alpha_1 + 3\alpha_2$$

(ix) α_1 **chain through** $2\alpha_1 + 3\alpha_2$

$$2\alpha_1 + 3\alpha_2 - \frac{2(2\alpha_1 + 3\alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)} = \alpha_1 + 3\alpha_2$$

(iix) α_2 **chain through** $2\alpha_1 + 3\alpha_2$

$$2\alpha_1 + 3\alpha_2 - \frac{2(2\alpha_1 + 3\alpha_2, \alpha_2)}{(\alpha_2, \alpha_2)} = 2\alpha_1 + 3\alpha_2$$

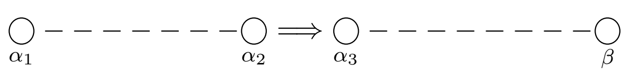
The root system is

$$\Delta = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_2 + \alpha_1), \pm(\alpha_1 + 2\alpha_2), \pm(\alpha_1 + 3\alpha_2), \pm(2\alpha_1 + 3\alpha_2)\}$$

5.2 F_4-Type

This type of root systems can be constructed by simply extending the root system of B_3. We start with given roots $\varepsilon_1 - \varepsilon_1, \varepsilon_2 - \varepsilon_3$ and ε_3, and for a root $\beta \in R_4$ so that $\Delta = \{\varepsilon_1 - \varepsilon_1, \varepsilon_2 - \varepsilon_3, \varepsilon_3 \beta\}$ is a base for F_4. A straightforward computation shows that the length of the root $\alpha_2 is\sqrt{2}$, while the length of α_3 is 1. Furthermore, another simple calculation gives us $\langle \alpha_2, \alpha_3 = -2$ and $\langle \alpha_3, \alpha_2 = -1$ which implies that there must be two edges between α_2 and α_3 . Thus the Dynkin diagram is the following:

F_4



Δ must span R^4 since we want it to be a base of our root system. From linear algebra clearly follows that the first three roots in Δ are linearly independent and we now only need an appropriate β. Easily we observe that the only possibilities are $\beta = \frac{-1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \pm \varepsilon_4$. We now set $R = \{\pm\varepsilon_i : 1 \le i \le 4\}[\{\pm\varepsilon_i \pm \varepsilon_j : 1 \le i \ne j \le 4\}[\{\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}$.

$$\beta_1 = \varepsilon_1 - \varepsilon_2$$

$$\beta_2 = \varepsilon_2 - \varepsilon_3$$

$$\beta_3 = \varepsilon_3$$

$$\beta = \frac{-1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3) + \varepsilon_4$$

.

really defines a base for R. Let $\Sigma_{i=1}^4 c_i\beta_i = 0$ then from $\Sigma_{i=1}^4 c_i\beta_i = (c_1 - \frac{1}{2}c_4) + (-c_1 - \frac{1}{2}c_4 + c_2)\varepsilon_2 + (-c_2 + c_3 - \frac{1}{2}c_4)\varepsilon_3 + \frac{1}{2}c_4\varepsilon_4$.and the fact that $\varepsilon_i^{'}$ are the standard basis in R^4. Thus $\beta_i^{'}$ clearly form another basis of R^4 and hence 5.1 holds. Using similar calculations we verify the axiom 5.9 and we have done.

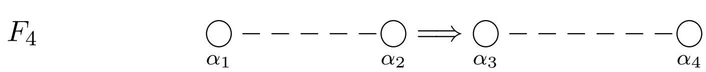
We also easily count that R consists of 48 elements, so we need to find 24 positive roots. Indeed, each ε_i is a positive root and they are 4 in total. If $1 \leq i < j \leq 3$ then both $\varepsilon_i - \varepsilon_j$ and $\varepsilon_i + \varepsilon_j$ are positive as well. Their number is 6. Furthermore,$\varepsilon_4 - \varepsilon_i$ for $1 \leq i \leq 3$ are another 6 positive roots. The rest positive roots are 3 of the form $\beta_4 + \varepsilon_j$, 3 of the form $\beta_4 + \varepsilon_j + \varepsilon_k, \beta_4$ itself and $\beta_4 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3$.

Details Classification of F_4

Cartan Matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix},$$

The Dynkin diagram of F_4



All roots of F_4

$\|\alpha_1\| = \|\alpha_3\| = \|\alpha_4\| = 1$ and

$\|\alpha_2\| = \sqrt{2}$

(1)α_1 chain through α_2

$$\alpha_2 - \frac{2(\alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)}\alpha_1 = \alpha_2 + \alpha_1$$

(2)α_2 **chain through** α_3

$$\alpha_3 - \frac{2(\alpha_3, \alpha_2)}{(\alpha_2, \alpha_2)}\alpha_2 = \alpha_3 + 2\alpha_2$$

(3)α_2 **chain through** α_4

$$\alpha_2 - \frac{2(\alpha_2, \alpha_3)}{(\alpha_3, \alpha_3)}\alpha_3 = \alpha_2 + \alpha_3$$

(4)α_1 **chain through** $\alpha_2 + \alpha_3$

$$\alpha_2 + \alpha_3 - \frac{2(\alpha_2 + \alpha_3, \alpha_1)}{(\alpha_1, \alpha_1)}\alpha_1 = \alpha_1 + \alpha_2 + \alpha_3$$

(5)α_2 **chain through** $\alpha_1 + \alpha_2 + \alpha_3$

$$\alpha_1 + \alpha_2 + \alpha_3 - \frac{2(\alpha_1 + \alpha_2 + \alpha_3, \alpha_2)}{(\alpha_2, \alpha_2)}\alpha_2 = \alpha_1 + 2\alpha_2 + \alpha_3$$

In this case we take

$\|\alpha_1\| = \sqrt{2}$

(6)$\alpha_1 + \alpha_2$ **chain through** $\alpha_1 + \alpha_2 + \alpha_3$

$$\alpha_1 + \alpha_2 + \alpha_3 - \frac{2(\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2)}{(\alpha_1 + \alpha_2, \alpha_1 + \alpha_2)}(\alpha_1 + \alpha_2) = 2\alpha_1 + 2\alpha_2 + \alpha_3$$

(7)α_1 **chain through** $2\alpha_1 + 2\alpha_2 + \alpha_3$

$$2\alpha_1 + 2\alpha_2 + \alpha_3 - \frac{2(2\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1)}{(\alpha_1, \alpha_1)}\alpha_1 = 2\alpha_2 + \alpha_3$$

(8)α_2 **chain through** $2\alpha_1 + 2\alpha_2 + \alpha_3$

$$2\alpha_1 + 2\alpha_2 + \alpha_3 - \frac{2(2\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_2)}{(\alpha_2, \alpha_2)}\alpha_2 = 2\alpha_1 + \alpha_3$$

(9)α_4 **chain through** $2\alpha_2 + \alpha_3$

$$2\alpha_2 + \alpha_3 - \frac{2(2\alpha_2 + \alpha_3, \alpha_4)}{(\alpha_4, \alpha_4)}\alpha_4 = 2\alpha_2 + \alpha_3 + \alpha_4$$

(10)α_4 **chain through** $\alpha_2 + \alpha_3$

$$\alpha_2 + \alpha_3 - \frac{2(\alpha_1 + \alpha_2 + \alpha_3, \alpha_4)}{(\alpha_4, \alpha_4)}\alpha_4 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

(11)α_4 **chain through** $2\alpha_1 + 2\alpha_2 + \alpha_3$

$$2\alpha_1 + 2\alpha_2 + \alpha_3 - \frac{2(2\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_4)}{(\alpha_4, \alpha_4)}\alpha_4 = 2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$$

(12)α_4 **chain through** $2\alpha_1 + 2\alpha_2 + 3\alpha_3$

$$2\alpha_1 + 2\alpha_2 + 3\alpha_3 - \frac{2(2\alpha_1 + 2\alpha_2 + 3\alpha_3, \alpha_4)}{(\alpha_4, \alpha_4)}\alpha_4 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4$$

(13)α_1 **chain through** $2\alpha_2 + 2\alpha_3 + \alpha_4$

$$2\alpha_1 + 2\alpha_2 + \alpha_3 - \frac{2(2\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_4)}{(\alpha_4, \alpha_4)}\alpha_4 = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$$

(14)α_2 **chain through** $2\alpha_1 + \alpha_3$

$$2\alpha_1 + \alpha_3 - \frac{2(2\alpha_1 + \alpha_3, \alpha_2)}{(\alpha_2, \alpha_2)}\alpha_2 = 2\alpha_1 + 4\alpha_2 + \alpha_3$$

(15)α_2 **chain through** $2\alpha_1 + 2\alpha_2 + \alpha_3$

$$2\alpha_1 + 2\alpha_2 + \alpha_3 - \frac{2(2\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_2)}{(\alpha_2, \alpha_2)}\alpha_2 = 2\alpha_1 + \alpha_3$$

(16)α_4 **chain through** $2\alpha_1 + \alpha_3$

$$2\alpha_1 + \alpha_3 - \frac{2(2\alpha_1 + \alpha_3, \alpha_4)}{(\alpha_4, \alpha_4)}\alpha_4 = 2\alpha_1 + \alpha_3 + \alpha_4$$

(17)α_4 **chain through** $2\alpha_1 + 4\alpha_2 + \alpha_3$

$$2\alpha_1 + 4\alpha_2 + \alpha_3 - \frac{2(2\alpha_1 + 4\alpha_2 + \alpha_3, \alpha_4)}{(\alpha_4, \alpha_4)}\alpha_4 = 2\alpha_1 + 4\alpha_2 + \alpha_3 + \alpha_4$$

In this case we take

$\|\alpha_1\| = \sqrt{2}$ **(18)**$\alpha_2 + \alpha_3$ **chain through** $2\alpha_1 + \alpha_3$

$$2\alpha_1 + \alpha_3 - \frac{2(2\alpha_1 + \alpha_3, \alpha_2 + \alpha_3)}{(\alpha_2 + \alpha_3, \alpha_2 + \alpha_3)}(\alpha_2 + \alpha_3) = 2\alpha_1 + 4\alpha_2 + 5\alpha_3$$

(19)α_4 **chain through** $2\alpha_1 + 4\alpha_2 + 5\alpha_3$

$$2\alpha_1 + 4\alpha_2 + 5\alpha_3 - \frac{2(2\alpha_1 + 4\alpha_2 + 5\alpha_3, \alpha_4)}{(\alpha_4, \alpha_4)}\alpha_4 = 2\alpha_1 + 4\alpha_2 + 5\alpha_3 + 5\alpha_4$$

(20)α_3 **chain through** $2\alpha_1 + 2\alpha_2 + \alpha_3$

$$2\alpha_1 + 2\alpha_2 + \alpha_3 - \frac{2(2\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_3)}{(\alpha_3, \alpha_3)}\alpha_3 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3$$

In this process we get 48 roots.

The elements of cartan subalgebra are also root of F_4 i.e. h_1, h_2, h_3, h_4.

The total root system of F_4 is $\Delta = \{h_1, h_2, h_3, h_4, \pm\alpha_1, \pm\alpha_2, \alpha_3, \pm\alpha_4, \pm\alpha_2{+}\alpha_1, \pm\alpha_3{+}2\alpha_2, \pm\alpha_2 + \alpha_3, \pm\alpha_1 + \alpha_2 + \alpha_3, \pm\alpha_1 + 2\alpha_2 + \alpha_3, \pm 2\alpha_1 + 2\alpha_2 + \alpha_3, \pm 2\alpha_2 + \alpha_3, \pm 2\alpha_1 + \alpha_3, \pm 2\alpha_2 + \alpha_3 + \alpha_4, \pm\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,$
$\pm 2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4, \pm 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4, \pm\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4, \pm 2\alpha_1 + 4\alpha_2 + \alpha_3, \pm 2\alpha_1 + \alpha_3, \pm 2\alpha_1 + \alpha_3 + \alpha_4, \pm 2\alpha_1 + 4\alpha_2 + \alpha_3 + \alpha_4, \pm 2\alpha_1 + 4\alpha_2 + 5\alpha_3, \pm 2\alpha_1 + 4\alpha_2 + 5\alpha_3 + 5\alpha_4, \pm 2\alpha_1 + 2\alpha_2 + 3\alpha_3\}$.

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