Numerics of Singularly Perturbed Differential Equations

A report

submitted by

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Declaration

I hereby certify that the work which is being presented in the report entitled "Numerics of singularly perturbed differential equations" in partial fulfillment of the requirement for the award of the degree of Master of Science, submitted to the Department of Mathematics, National Institute of Technology Rourkela is a review work carried out under the supervision of Dr. Jugal Mohapatra. The matter embodied in this report has not been submitted by me for the award of any other degree.

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This is to certify that the above statement made by the candidate is true to the best of my knowledge.

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Date: May 2014

Acknowledgement

Many lives and destinies are destroyed due to the lack of proper guidance, direction and opportunities. It is in this respect I feel that I am in a much better condition today due to continuous process of motivation and focus provided by my parents and faculty in general. I would like to highlight the role played by individual towards this.
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Last but not the least, I would like to extend my heartfelt thanks to my loving family, all my classmates and friends. They are always supporting me and encouraging me with their best wishes.

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Abstract

The main purpose of this report is to carry out the effect of the various numerical methods for solving singular perturbation problems on non-uniform meshes. When a small parameter 'ε' known as the singular perturbation parameter is multiplied with the higher order terms of the differential equation, then the differential equation becomes singularly perturbed. In this type of problems, there are regions where the solution varies very rapidly known as boundary layers and the region where the solution varies uniformly known as the outer region. Standard finite difference/element methods are applied on the singularly perturbed differential equation on uniform mesh give unsatisfactory result as $\varepsilon \to 0$. Due to presence of boundary layer, standard difference schemes unable to capture the layer behaviour until the mesh parameter and perturbation parameter are of the same size which results vast computational cost. In order to overcome this difficulty, we adapt non-uniform meshes. The Shishkin mesh and the adaptive mesh are two widely used special type of non-uniform meshes for solving singularly perturbed problem. Here, in this report singularly perturbed problems namely convection-diffusion and reaction-diffusion problems are considered and solved by various numerical techniques. The numerical solution of the problems are compared with the exact solution and the results are shown in the shape of tables and graphs to validate the theoretical bounds.

Contents

Chapter 1

Introduction

Singularly perturbed problems(SPP) are wide-spread in nature. These problems can be algebraic equations, ordinary differential equations, partial differential equations or their systems. Singularly perturbed differential equations is one of the area of increasing interest in the applied mathematics and engineering since recent years. Differential equations where the highest order derivative is multiplied by an arbitrarily small parameter "ε" (known as the singular perturbation parameter) are known as singularly perturbed and the solution cannot be approximated by setting the parameter value to zero. The solution of singularly perturbed problem has special character: there are thin layers where the solution varies very rapidly known as boundary layer, while away from the layers the solution behaves uniformly and varies slowly known as outer region . These equation arises in many areas of science and engineering such as fluid dynamics, quantum mechanics, control theory, chemical science etc.

Singularly perturbed differential equation arises in modeling many real life problem. For example modeling of semiconductor device,

$$\left.\begin{aligned} v''(x) = d(x) - p(x) \\ \lambda p''(x) + v'(x)p'(x) + (d(x) - p(x))p(x) = 0 \end{aligned}\right\},$$

where p is the density of positive charge holes, d is the dopant density of negative charges, v is the electrostatic potential and λ is small diffusion coefficient.

Another example is the fluid flow past over a cylindrical system. Considering the system to be conservative i.e. div $u = 0$. Then by the Navier-Stokes equation

$$\frac{\partial u}{\partial t} - \frac{1}{Re} \triangle u + (u.\bigtriangledown)u + \bigtriangledown p = F,$$

where u, p and F are the velocity, pressure and the external force respectively. This equation becomes singularly perturbed with appropriate boundary condition when Re, the Reynolds number is very high.

SPP was first introduced by Prandtl during his talk on fluid motion with small friction in the third international conference of mathematicians in Heidelberg in 1904 in which he demonstrated that fluid flow past over a body can be divide in two regions, a boundary layer and outer region. Prandtl introduced the term boundary layer but Wasow had generalized the term. Many outstanding work has been done by many researchers in past few decades ([1], [3], [8], [10], [12], [16]).

The solution of singularly perturbed problem has boundary layer or narrow region (as shown in figure (1.1, 1.2, 1.3) where, the solution varies very rapidly. In these problems, the standard numerical techniques used on uniform mesh leads to the error in the solution as $\varepsilon \to 0$ and so the error depends upon the value of ε. Since the boundary layer is of the width $O(\varepsilon)$, in order to capture the layer phenomena, one has to take the mesh length of size ϵ, which results vast computational cost . That is why our interest is to device special techniques to solve these problems, in which the error is independent of ε and we call these techniques *ε-uniformly convergent methods* or robust techniques. There are various methods available to solve the singularly perturbed differential equations. Numerical method and asymptotic analysis methods are two mainly used method to solve the singularly perturbed problem . Solving singularly perturbation problems by asymptotic methods. one need to have some idea about the behavior of the solution. So we go for numerical treatments. Numerical techniques do not give the exact solution of the problem but the approximation to the solution. We can solve problems by numerical techniques those faces difficulty with the asymptotic methods. We can examine the behaviour and quantitative information about a particular problem by using numerical techniques.

One may refer Farrrell et al.[6], Miller et al.[9], Shishkin et al.[18] for theoretical and numerical treatment of singularly perturbed differential equations. Many numerical schemes have been devised to solve these type of problems. Roos and Stynes[19] has used midpoint upwind scheme to solve the singularly perturbed boundary value problem. Kadalbajoo[7] used cubic B-spline method to solve the singularly perturbed problem.

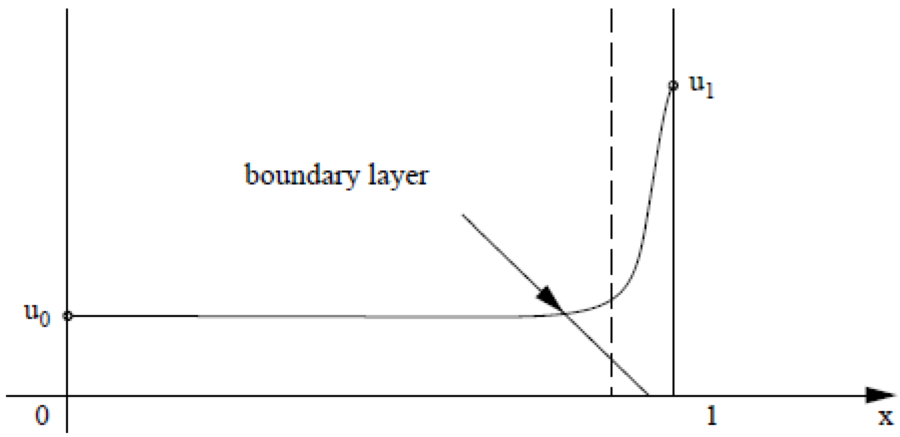


Figure 1.1: Right layer

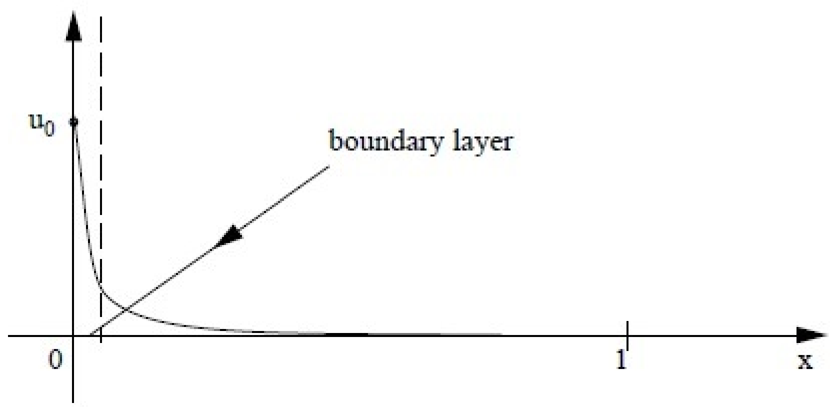


Figure 1.2: Left layer.

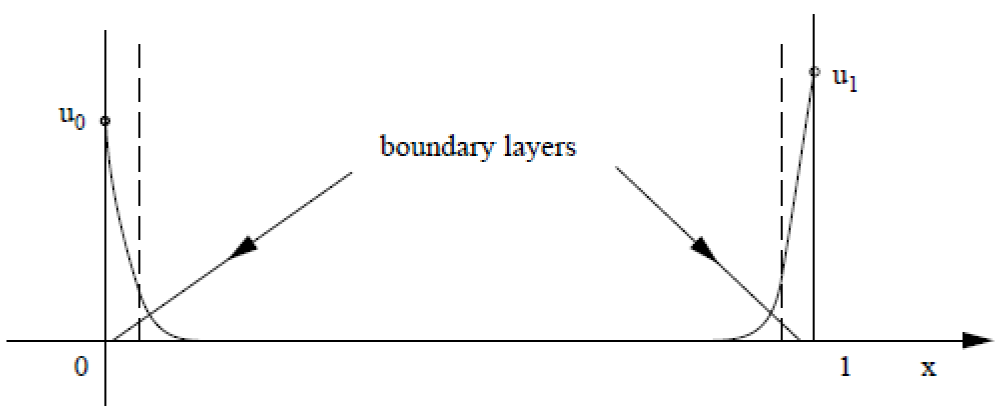


Figure 1.3: Twins layer

1.1 Preliminaries

Perturbation theory is a subject in which we study the effect of small parameter in the mathematical model problems in differential equations. In perturbation theory we deal with two types of problems. They are Regularly perturbed and Singularly perturbed.

Definition 1.1.1. *If the mathematical problem involving a small parameter ε converges to the reduced problem (which is obtained by setting the ε to zero) as ε goes to zero is categorized as regularly perturbed problem. Mathematically, if P_ϵ is the solution of any problem and if $P_\varepsilon \to P_0$ as $\varepsilon \to 0$, P_0 is the solution obtained by setting ε to zero.*

Example 1.1.2. $u'' + u = \varepsilon u^2, \quad u(0) = 1, \quad u'(0) = -1.$

Definition 1.1.3. *The perturbed problem is said to be singularly perturbed, when the the solution P_ε behaves differently from P_0 of problem as $\varepsilon \to 0$, where P_0 is the solution obtained by setting ε to zero. In other words, order of the problem is reduced when we set $\varepsilon = 0$.*

Example 1.1.4. $\varepsilon u'' + u' = 2x + 1, \quad u(0) = 1, \quad u(1) = 3.$

Definition 1.1.5. *In the domain of problem the region where the solution varies rapidly is known as boundary layer region or inner region and the region where solution varies uniformly is known as outer region.*

Definition 1.1.6. *The letters 'O' and 'o' are order symbols. The function f is big-oh of function g if $\lim_{x\to x_0} \frac{f(x)}{g(x)} = C$, where C is finite and it is written as $f(x) = O(g(x))$ as $x \to x_0$ or in other words, the function $f(x)$ approaches to a limiting value at the same rate of another function $g(x)$ as $x \to x_0$. And if $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$ then f is little-oh of g as $x \to x_0$ that means $f(x)$ is smaller than $g(x)$ as $x \to x_0$.*

Definition 1.1.7. *A method is said to be uniform convergence if*

$$\underbrace{sup}_{0<\varepsilon\leq 1} \|U_\epsilon - u_\epsilon\|_{\Omega^N} \leq CN^{-p},$$

where u_ϵ is the unique solution of family of mathematical problems P_ε parameterized by a singular parameter ε, where $0 < \varepsilon \leq 1$ and U_ϵ is the approximation of u_ϵ on the mesh $\overline{\Omega}^N$. Here p is called the ε-convergence rate and C is called ε-uniform error constant.

To study the convergence of numerical solutions to the exact solution of SPP's the appropriate norm is maximum norm.

$$\|u\| = \underbrace{\max}_{\overline{\Omega}} |u(x)|.$$

Since max $\|.\|$ easily capture the layer phenomena. Let us define the finite difference operator as follows:
Let U be the solution then the forward, backward and central difference operators D^+, D^-, D^0 and second order central difference operator D^+D^- are define respectively as:

$$D^+U(x_i) = \frac{U(x_{i+1}) - U(x_i)}{x_{i+1} - x_i}$$

$$D^-U(x_i) = \frac{U(x_i) - U(x_{i-1})}{x_i - x_{i-1}}$$

$$D^0U(x_i) = \frac{U(x_{i+1}) - U(x_{i-1})}{x_{i+1} - x_{i-1}}$$

$$D^+D^-U(x_i) = \frac{2(D^+U(x_i) - D^-U(x_i))}{x_{i+1} - x_{i-1}}$$

Objective

Main objective of my report is to analyze theoretical bound, analytical properties and numerical techniques namely upwind scheme, midpoint scheme, hybrid scheme and B-spline method to solve two kind of singularly perturbed problems which are convection diffusion and reaction diffusion problems and also in my report I will adapt the non-uniform meshes instead of uniform meshes namely Shishkin mesh and adaptive grid. Hence, in my report I will try to analyze and develop the numerical techniques to solve convection diffusion and reaction diffusion problem on Shishkin mesh and adaptive grid.

1.2 Non-uniform meshes

In my study, I will adopt various numerical schemes to examine the solution of various models. But we cannot apply the difference scheme (forward difference, backward difference, central difference) arbitrarily. For example, when we use the central difference scheme we get oscillation in the solution. Moreover, since the solution of these problems have boundary layer where the solution varies sharply, we get unsatisfactory results on the uniform mesh as $\varepsilon \to 0$. The numerical methods constructed on uniform-mesh result in arising of serious difficulties ([6], [18]). These difficulties can be overcome by the use of appropriate kind of non-uniform-mesh. The simplest kind of non-uniform mesh to construct ε-uniform method is a piece-wise uniform mesh. The first special kind of non-uniform mesh was introduced by Bakhvalov(1969). Some other special kinds of non-uniform meshes are Shishkin mesh(1982) and adaptive mesh [8].

Shishkin mesh

Shishkin mesh is a special piece-wise uniform mesh. This mesh is different from other meshes in the sense that the choice of transition parameter where the solution changes very rapidly. Let $\overline{\Omega} = [0, 1]$. To establish the Shishkin mesh, one has to known the location and width of boundary layer. Let us assume that a layer of width $O(\varepsilon)$ is at $x = 0$ on the left end of the domain. If the boundary layer $O(\varepsilon)$ is near 0 (left end) then piece-wise uniform mesh Ω_ϵ^N is constructed by dividing $\overline{\Omega}$ into two sub-interval $\Omega = (0, \sigma) \cup (\sigma, 1)$, where

$$\sigma = \min\left\{\frac{1}{2}, \frac{\varepsilon}{\alpha}\ln N\right\}.$$

Shishkin mesh is obtained by dividing both sub-intervals in $N/2$ mesh elements. When $\sigma = 1/2$ the mesh is uniform and when $\sigma = \frac{\varepsilon}{\alpha}\ln N$ then it has boundary layer at $x = 0$ (see the figure). When the boundary points is near to the $x = 1$ then piece-wise uniform mesh Ω^N is constructed by dividing $\overline{\Omega}$ into two sub-interval $\Omega = (0, 1-\sigma) \cup (1-\sigma, 1)$.

Adaptive mesh

To solve the SPP's, we have to use meshes that are fine in layer regions and coarse in the outer region. Shishkin mesh has the demerit of prior knowing of location and width of

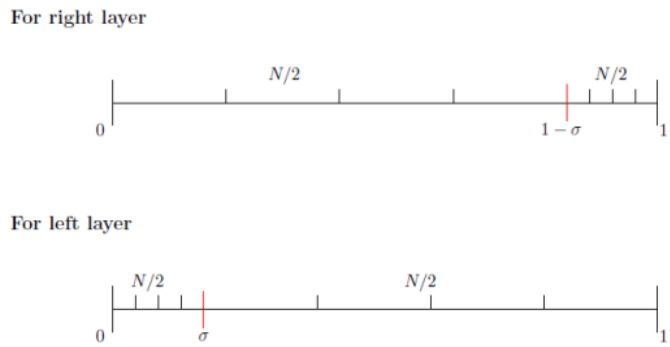


Figure 1.4: Shishkin mesh for N=8.

layer, so we go for adaptive grid. Adaptive mesh is one of the special kind of mesh. To construct such mesh we use the adaptive algorithm. Initially the mesh used is uniform with N sub interval. One approaches to the adaptive mesh construction for SPP's is the use of monitor function. A monitor function $M(x)$ is an arbitrary non-negative function defined on $[0, 1]$. A mesh is said to be equidistribute if

$$\int_{x_{i-1}}^{x_i} M(x)dx = \frac{1}{N}\int_0^1 M(x)dx, \text{ for } i = 1, ..., N.$$

Different monitor functions are used by the researchers. Some of the monitor functions are,

(1) $M(x) = |u'(x)|$

(2) $M(x) = 1 + |u'(x)|^{1/m}$, $m \geq 2$

(3) $M(x) = \sqrt{1 + (u'(x))^2}$ (Known as arc-length Monitor function)

(4) $M(x) = 1 + \alpha|(u'(x))|^p$, $p > 0$

(5) $M(x) = 1 + |(u''(x))|^{1/2}$

Since the exact solution u is unknown, a priori adaptive algorithm used some discrete analogues of certain monitor functions. We will consider arc length monitor function in our report. We have to find (x_i, u_i), with u_i^N computed from the x_i by $Lu_i = f_i, u_0 = A, u_N = B$ such that,

$$\sqrt{|u_i - u_{i-1}|^2 + |x_i - x_{i-1}|^2} = \frac{1}{N}\sqrt{|u_j - u_{j-1}|^2 + |x_j - x_{j-1}|^2}, \text{ for every } i$$

We can write,

$$h_i M_i = \frac{1}{N} \sum_{i=1}^{N} h_j M_j, \text{ for } i = 1, 2, ...N.$$

Where, $M_i = \sqrt{1 + \left(\frac{u_i - u_{i-1}}{x_i - x_{i-1}} \right)^2}$ and $h_i = x_i - x_{i-1}$.

Adaptive mesh generation algorithm

Step 1: Consider the initial mesh $\{0, 1/N, 2/N, ...1\}$ is uniform.
Step 2: For $k = 0, 1, ...$ assuming the mesh $\{x_i^k\}$ is given. Compute the discrete solution $\{u_i^k\}$ satisfying, $L^k u^k = f^k$ on x_i^k with $u_0^k = A$, $u_N^k = B$.
Let $l_i^k = h_i^k \sqrt{1 + \left(\frac{u_i^k - u_{i-1}^k}{x_i^k - x_{i-1}^k} \right)^2}$, where $h_i^k = x_i^k - x_{i-1}^k$ be the arc length of points (x_{i-1}^k, u_{i-1}^k) and (x_i^k, u_i^k). Thus total arc length is, $L^k = \sum_{i=1}^{N} l_i^k$.
Step 3: Let C_0 be the user chosen constant $C_0 > 1$. If, $\frac{\max_i l_i^k}{L^k} \leq \frac{C_0}{N}$ then go to step 5 otherwise continue to step 4.
Step 4: Chose the points $0 = x_0^{k+1} < x_1^{k+1} < ... < x_N^{k+1} = 1$ such that for each i the distance from $(x_{i-1}^{k+1}, u^k(x_{i-1}^{k+1}))$ to $(x_i^{k+1}, u^k(x_i^{k+1}))$, measured along polygon solution curve $u^k(x)$ equals L^k/N. Our new mesh is defined to be x_i^{k+1}.
Step 5: Set $x_0, x_1, ...x_N = x_i^k$ and $u = u^k$ then stop.

Chapter 2

Methods for convection-diffusion problems

Introduction

The convection-diffusion equation is a combination of the convection and diffusion equations and describes physical phenomena where particles, energy or other physical quantities are transferred inside a physical system due to two processes: convection and diffusion. In this chapter, we discuss numerical techniques to solve singularly perturbed convection-diffusion equations. The various numerical schemes viz. the upwind scheme, the midpoint scheme, the hybrid scheme and the B-spline collocation method is applied to obtain approximate solutions and then it is compared with the exact solution.

2.1 Model problem

Consider the following singularly perturbed differential equation

$$\left.\begin{aligned} \varepsilon u'' + p(x)u' + q(x)u = f(x), \;\; x \in \Omega = (0,1) \\ u(0) = A, \;\; u(1) = B \end{aligned}\right\}, \tag{2.1}$$

where $0 < \varepsilon \leq 1$ and $p(x) \geq \alpha > 0$ for all $x \in \overline{\Omega}$, A, B are constants.
when $\varepsilon = 0$, the reduced problem is $p(x)u' + q(x)u = f(x)$ which is first order differential equation . So it can not be made to satisfy the both boundary conditions simultaneously. And hence it has boundary layer at $x = 0$ or $x = 1$ depending upon the sign of $p(x)$.
Let L_ϵ be the differential operator, then $L_\epsilon u = \varepsilon u'' + pu' + qu$. The differential operator L_ϵ satisfy the maximum principle.

Maximum Principle

Lemma 2.1.1. *Assume that $u(0) \geq 0$ and $u(1) \geq 0$. Then $L_\epsilon u \geq 0$ for all $x \in \Omega$ implies that $u(x) \geq 0$ for all $x \in \overline{\Omega}$.*

Proof. See [15]. □

2.2 Properties of the solution

Lemma 2.2.1. *Let u be the solution of model problem (2.1). Then u satisfying following bound:*

$$\|u\| \leq \frac{\|f\|}{\alpha} + \max(u(0), u(1)), \quad forall \;\; x \in \bar{\Omega}.$$

Proof. Let us consider $\Psi^{\pm}(x) = x\frac{\|f\|}{\alpha} + \max\left(u(0), u(1)\right) \pm u(x)$. So we have,
$\Psi^{\pm}(0) = \max\left(u(0), u(1)\right) \pm u(0) \geq 0$ and $\Psi^{\pm}(1) = \dfrac{\|f\|}{\alpha} + \max\left(u(0), u(1)\right) \pm u(1) \geq 0$.

$$L\psi^{\pm} = \varepsilon(\psi^{\pm(x)})'' + p(x)(\psi^{\pm}(x))' + q(x)\psi^{\pm} = p(x)\frac{\|f\|}{\alpha} + p(x)(\max(u(0), u(1)) + x\frac{\|f\|}{\alpha} \pm f(x).$$

Since $p(x) \geq \alpha > 0$ and $\|f\| \geq f(x)$, we have $p(x)\alpha^{-1}\|f\| \pm f(x) \geq 0$. Using this inequality we get $L\psi^{\pm} \geq 0$, for all $x \in \Omega$. Hence by Lemma(2.1.1) $\psi^{\pm} \geq 0$, for all $x \in \Omega$ which is the required result. □

Lemma 2.2.2. *Let u be the solution of model problem (2.1) then for $0 \le i \le 3$ we have*

$$|u^k(x)| \le C(1 + \varepsilon^{-k} \exp(-\alpha x/\varepsilon)).$$

Proof. By mean value theorem, $\exists$ a point z in $(0, \varepsilon)$ such that

$$u'(z) = \frac{u(0) - u(\varepsilon)}{\varepsilon}.$$

Thus

$$|\varepsilon u'(z)| < 2 \|u\| . \tag{2.2}$$

Integrating (2.1) from 0 to z we get

$$\varepsilon u' - \varepsilon u'(0) = \int_0^z (f(t) - p(t)u'(t) - q(t)u(t))dt. \tag{2.3}$$

$$\left| \int_0^z (f(t) - p(t)u'(t) - q(t)u(t))dt \right| \le \|f\| \, |z| + \int_0^z (p(t)u'(t))dt + \|u\| \, z. \tag{2.4}$$

Here, $\int_0^z (p(t)u'(t))dt = p(t)u(t)|_0^z - \int p'(t)u(t)dt$. Taking modulus on both side

$$\left| \int_0^z (p(t)u'(t))dt \right| \le 2 \|p\| \, \|u\| - \|p'\| \, \|u\| \, \|z\| . \tag{2.5}$$

Now combining the inequalities (2.2) and (2.4)

$$\left| \int_0^z (f(t) - p(t)u'(t) - q(t)u(t))dt \right| \le \|f\| + (2 \|p\| + \|p'\| \, \|q\|) \, \|u\| . \tag{2.6}$$

Using (2.6) with (2.2) and (2.3), we get $|u'(0)| \le C\varepsilon^{-1}$. Now using (2.3) with $z = x \in \Omega$ we get $|u'(x)| \le C\varepsilon^{-1}$, for all $x \in \Omega$. Similarly, by using the bound of u and u', we can show that $|u^k(x)| \le C(1 + \varepsilon^{-k} \exp(-\alpha x/\varepsilon))$, $x \in \ \Omega, \ \ 0 \le i \le 3$. $\square$

The solution of model problem (2.1) can be decomposed in $u = v + w$ where v and w are regular and singular components of u respectively. The v component can be written in asymptotic expansion as $v(x) = v_0 + \varepsilon v_1 + \varepsilon^2 v_2$. Now

$$p(x)v_0' + q(x)v_0 = f(x) \quad v_0(0) = u_0(0)$$

$$p(x)v_1' + q(x)v_1 = v_0''(x) \quad v_1(0) = 0$$

$$\varepsilon v_2'' + p(x)v_2' + q(x)v_2 = v_1''(x) \;\; v_2(0) = 0, \;\; v_1(0) = 0$$

Thus v is the solution of

$$Lv(x) = f(x) \text{ with } v(0) = u(0) \text{ and } v(1) = v_0(1) + \varepsilon v_1(1),$$

and singular component w is the solution of

$$Lw(x) = 0 \text{ with } w(0) = 0 \text{ and } w(1) = u(1) - v(1),$$

and we have,

Lemma 2.2.3. *The regular and singular components v(x) and u(x)respectively satisfying the following bond:*

$$|v^k(x)| \leq C(1 + \varepsilon^{-(k-2)} \exp(-\alpha x/\varepsilon)),$$

and

$$|w^k(x)| \leq C\varepsilon^{-k} \exp(-\alpha x/\varepsilon).$$

Proof. From above it is very well known

$$\left|v_0^k\right| \leq C, \;\; 0 \leq k \leq 3 \text{ and} \left|v_1^k\right| \leq C, \quad 0 \leq k \leq 3.$$

From previous lemma, $\left|v_2^k\right| \leq (1 + \varepsilon^{-1} \exp(-\alpha x/\varepsilon)), \;\; 0 \leq k \leq 3$. Thus we have required bound.

Now construct $\psi^{\pm}(x) = |w(0)| \exp(-\alpha x/\varepsilon) \pm w(x)$. Clearly, $\psi^{\pm}(0) \geq 0, \quad \psi^{\pm}(1) \geq 0$ and $L\psi^{\pm}(x) \geq 0$. Hence by maximum principle $\psi^{\pm}(x) \geq 0$ for all $x \in \Omega$ which gives $|w(x)| \leq C \exp(-\alpha x/\varepsilon)$. □

2.3 Numerical methods

In this section, we apply different numerical schemes on various convection-diffusion problems to find the numerical solutions and then they will be compared with the exact solutions. The results are shown in the shape of tables and graphs. Here we discritized the model problem (2.1) on non-uniform mesh as defined in Chapter 1. $\overline{\Omega^N} = \{0 = x_0 < x_1 < x_2 < ... < x_{N-1} < x_N = 1\}$.

Scheme 1: Upwind scheme

Let us consider the upwind scheme to solve the model problem (2.1)

$$\left.\begin{aligned} \varepsilon D^+D^-U(x_i) + p(x_i)D^+U(x_i) + q(x_i)U(x_i) = f(x_i), \;\; x_i \in \Omega^N \\ U(0) = u(0), \;\; U(1) = u(1) \end{aligned}\right\}, \tag{2.7}$$

where Ω^N is non- uniform mesh as defined in chapter 1.

Theorem 2.3.1. *The upwind method defined in(2.7) satisfy the error bound, for all* $N \geq 4$

$$|u(x_i) - u_i| \leq CN^{-1} \ln N, \tag{2.8}$$

where $u(x)$*is the exact solution and* u_i *is the numerical solution obtained by (2.7).*

Proof. Refer [6]. □

Scheme 2: Midpoint upwind scheme

The midpoint upwind scheme to solve model problem (2.1) is defined as follows

$$\left.\begin{aligned} \varepsilon D^+D^-U(x_i) + p(x_{i-1/2})D^+U(x_i) + q(x_{i-1/2})U(x_i) = f(x_{i-1/2}), \; x_i \in \Omega^N \\ \text{where } x_{i-1/2} = (x_i + x_{i-1})/2 \\ U(0) = u(0), U(1) = u(1) \end{aligned}\right\}, \tag{2.9}$$

where Ω^N is piece-wise uniform mesh as defined in Chapter 1.

Theorem 2.3.2. *The midpoint method defined in(2.9) satisfy the error bound.*

$$|u(x_i) - u_i| \leq \begin{cases} CN^{-1}(\varepsilon + N^{-1}), & 0 \leq i \leq 3N/4, \\ CN^{-1}(\varepsilon + N^{-4(1-i/N)} \ln N), & 3N/4 \leq i \leq N, \end{cases} \tag{2.10}$$

where u(x) is the exact solution and u_i *is the numerical solution obtained by (2.9).*

Proof. For the proof one may refer [19]. $\square$

Scheme 3: Hybrid scheme

Consider the hybrid scheme to solve the model problem (2.1)

$$\begin{cases} \varepsilon D^+D^-U(x_i) + p(x_i)D^0U(x_i) + q(x_i)U(x_i) = f(x_i), & \text{if } i = 0, 1, \ldots N/2, \\ \varepsilon D^+D^-U(x_i) + p(x_{i-1/2})D^+U(x_i) + q(x_{i-1/2})U(x_i) = f(x_{i-1/2}), & \text{if } i = N/2 + 1 \ldots N, \end{cases} \tag{2.11}$$

with the boundary conditions $U(0) = u(0)$, $U(1) = u(1)$.

Theorem 2.3.3. *The hybrid method defined in(2.11) satisfy the following error bound.*

$$|u(x_i) - u_i| \leq \begin{cases} CN^{-1}(\varepsilon + N^{-1}), & 0 \leq i \leq N/2, \\ CN^{-2}\ln^2 N, & N/2 \leq i \leq N, \end{cases} \tag{2.12}$$

where u(x) is the exact solution and u_i is the numerical solution obtained by (2.11).

Proof. For the detail proof refer [19]. $\square$

Scheme 4: B-Spline method

Now, we consider the cubic B-spline S(x) to find the numerical estimation of model problem (2.1).

$$S(x) = \sum_{i=-1}^{N+1} \alpha_i B_i(x),$$

where cubic B-splines B_i is defined as

$$B_i(x) = \frac{1}{h^3}\begin{cases} (x - x_{i-2})^3, & x_{i-2} \le x \le x_{i-1}, \\ h^3 + 3h^2(x - x_{i-1}) + 3h^2(x - x_{i-1})^2 + (x - x_{i-1})^3, & x_{i-1} \le x \le x_i, \\ h^3 + 3h^2(x_{i+1} - x) + 3h^2(x_{i+1} - x)^2 + (x_{i-1} - x)^3, & x_i \le x \le x_{i+1}, \\ (x_{i+2} - x)^3, & x_{i+1} \le x \le x_{i+2}, \\ 0, & Otherwise. \end{cases} \tag{2.13}$$

Suppose the approximate solution of (2.1) is given as

$$U(x) = \sum_{i=-1}^{N+1} \alpha B_i(x).$$

From equation (2.13), we have

$$B_i(x_j) = \begin{cases} 4 & \text{if } \ j = i, \\ 1 & \text{if } \ j = i-1 \ \text{ or } \ j = i+1, \\ 0 & \text{if } \ j = i+2 \ \text{ or } \ j = i-2, \end{cases} \tag{2.14}$$

and $B_i(x) = 0$ if $x \ge x_{i+2}$ and $x \le x_{i-2}$. Thus we have

$$U(x) = \alpha_{i-1} + 4\alpha_i + \alpha_{i+1}$$

$$U'(x) = 3(\alpha_{i-1} - \alpha_{i+1})/h$$

$$U''(x) = (6\alpha_{i-1} - 12\alpha_i + 6\alpha_{i+1})/h^2$$

Substituting the value of U_i, U_i', U_i'' in the equation (2.1) we have

$$\varepsilon h^2 U''(x_i) + p(x_i)h^2U'(x_i) + q(x_i)h^2U(x_i) = h^2 f(x_i)$$

$$\varepsilon(6\alpha_{i-1} - 12\alpha_i + 6\alpha_{i+1}) + 3p(x_i)h(\alpha_{i-1} - \alpha_{i+1}) + q(x_i)h^2(\alpha_{i-1} + 4\alpha_i + \alpha_{i+1}) = h^2 f(x_i). \tag{2.15}$$

The boundary conditions are

$$\left.\begin{aligned} U(0) = u(0) = A \Rightarrow \alpha_{-1} + 4\alpha_0 + \alpha_1 = A \\ U(1) = u(1) = B \Rightarrow \alpha_{N-1} + 4\alpha_N + \alpha_{N+1} = B \end{aligned}\right\}. \tag{2.16}$$

From the equation (2.15) and (2.16), we obtain $(N + 3) \times (N + 3)$ system with $(N + 3)$ unknowns $\alpha = \{\alpha_{-1}, ..., \alpha_{N+1}\}$. On eliminating α_{-1} from first equation of (2.15) and α_{N+1}

from the last equation of (2.15), we obtain the $(N+1)\times(N+1)$ system with $(N+1)$ unknowns $\alpha = \{\alpha_0, ..., \alpha_N\}$. It is easily seen that matrix for above system of equation strictly diagonally dominant and non-singular. Therefore, we can solve the above linear system uniquely for real unknowns $\alpha_0, ..., \alpha_N$ and then using the boundary conditions (2.16) we obtain α_{-1} and α_{N+1}. Hence, the method of B-spline using a basis of cubic B-splines applied to model problem (2.1) has a unique solution.

Theorem 2.3.4. *The B-spline method defined above for model problem (2.1) satisfy the following error bound for sufficiently large N.*

$$|u(x_i) - u_i| \leq CN^{-2}(\ln N)^2, \tag{2.17}$$

where $u(x)$ is exact solution and u_i is numerical solution obtained by B-spline method.

Proof. One may see [7]. □

2.4 Results and discussion

Example 2.4.1. *Consider the following constant coefficient problem*

$$\varepsilon u'' + 2u' = 0, u(0) = 1,\ u(1) = 0\ x \in\ (0,1).$$

The exact solution is given as $u(x) = \dfrac{\exp(-2x/\varepsilon) - \exp(-2/\varepsilon)}{(1 - \exp(-2/\varepsilon))}$.

Example 2.4.2. *Let us consider the following variable coefficient problem*

$$-\varepsilon u'' + (1 + x(1-x))u' = f(x), u(0) = 0,\ u(1) = 0\ x \in\ (0,1),$$

where f is chosen such that $u(x) = \dfrac{1 - \exp(-(1-x)/\varepsilon)}{1 - \exp(-1/\varepsilon)} - \cos\dfrac{\pi}{2}x$.

Figure 2.1 is the graph which is plotted for Example 2.4.1 using upwind scheme on Shishkin mesh with $\varepsilon = 1e-2$ and $N = 32$. From the Figure 2.1(a) it can be seen that the approximate solution on Shishkin mesh is identical for the most of the range. Figure

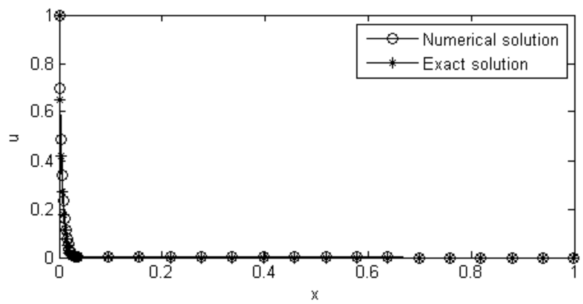
2.1(b) show that the error is more in the layer region and less in the outer region. Figure 2.2 display the graph which is plotted for Example 2.4.1 with the same perturbation parameter and the number of nodes using upwind scheme on adaptive grid. From Figure 2.2(a) we can see that the adaptive grid approximate the solution very well in the layer region. Error distribution using upwind scheme on adaptive grid for Example 2.4.1 is shown in the Figure 2.2(b) . Solution of Example 2.4.1 with $\varepsilon = 1e-2$ and $N = 32$ using B-spline method with corresponding error is plotted in the Figure 2.3. From Figure 2.3(a) and (b) we can observe that the approximate solution is identical in the outer region and the error in the boundary layer is much greater than the error in the outer region. Figures 2.4, 2.5 and 2.6 are the graphs plotted for the solution and error distribution for Example 2.4.2 with $\varepsilon = 1e-2$ and $N = 32$ using midpoint-scheme, hybrid scheme and B-spline method. From the Figures 2.4(b), 2.5(b) and 2.6(b) we can observe that the error distribution in the hybrid scheme is less than that on the midpoint scheme and B-spline method.

Let $u(x)$ be the exact solution and u_i be the numerical solution then the maximum point-wise error is calculated as:

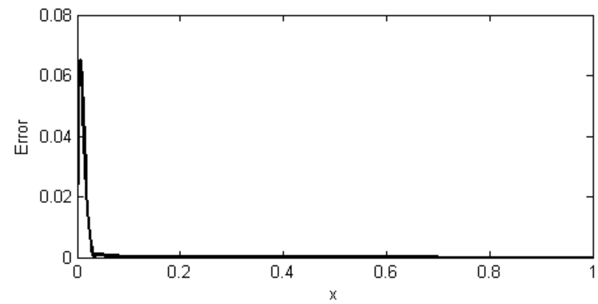
$$E^N = \|u(x_i) - u_i\|,$$

and the rate of convergence $r^N = log_2\left(\frac{E^N}{E^{2N}}\right)$.

The maximum point-wise error and the rate of convergence of the Example 2.4.1 on upwind scheme on the Shishkin mesh and on the adaptive grid is shown in Tables 2.1 and 2.2. From the Table it can be seen that upwind scheme on adaptive grid give better result than on the Shishkin mesh. The rate of convergence on adaptive grid is lies in the range $(0.7, 1)$ while on the Shishkin mesh convergence rate lies in $(0.4, 1)$ for Example 2.4.1. The Table 2.3 show the maximum point-wise error and the rate of convergence of the Example 2.4.1 on B-spline method on Shishkin mesh. It is clear from the Tables 2.4 and 2.5 that rate of convergence of the midpoint upwind scheme is less than the hybrid scheme. The rate of convergence of the hybrid scheme is lies in the range $(1, 2)$ and the result of hybrid scheme is better than the midpoint upwind scheme. The maximum point-wise error and the rate of convergence for the Example 2.4.2 calculated on the B-spline method on Shishkin mesh is shown in Table 2.6.

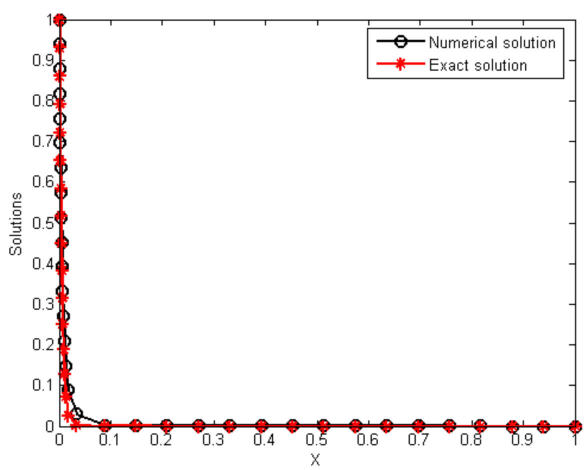


(a) Solutions.

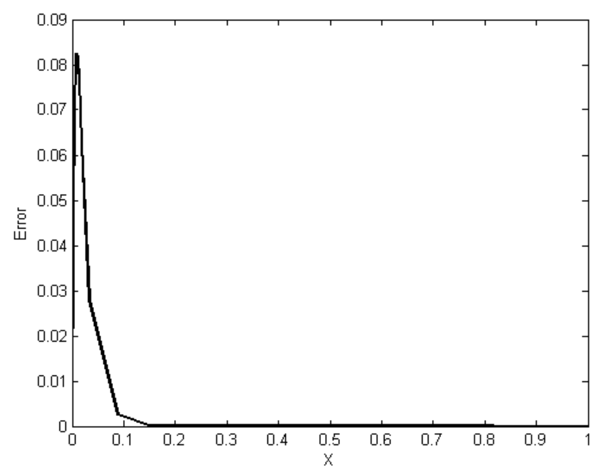


(b) Error.

Figure 2.1: *Numerical solution with the exact solution and the corresponding error of Example (2.4.1)for* $\varepsilon = 10^{-2}$ *and* $N = 32$ *on upwind scheme on Shishkin mesh.*

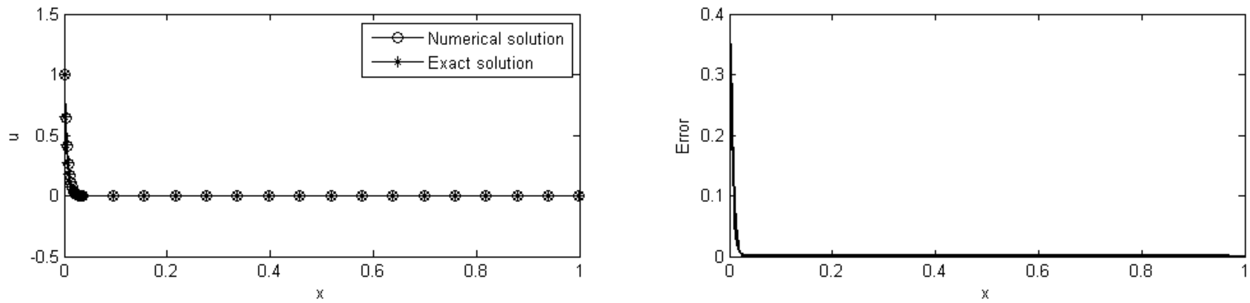


(a) Solutions.



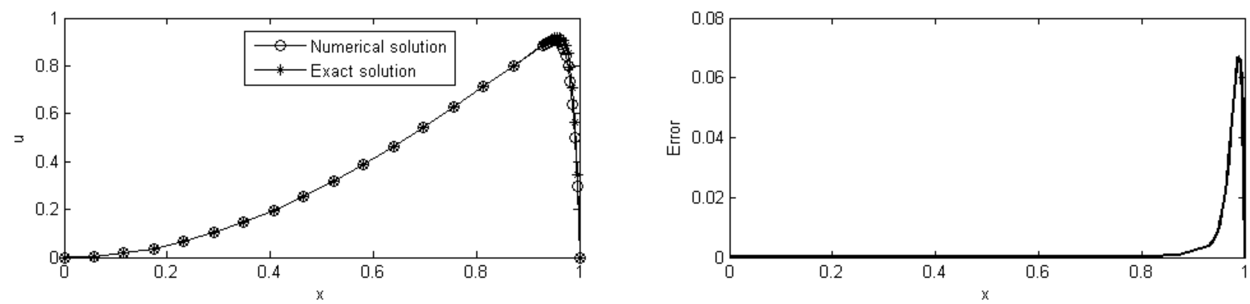
(b) Error.

Figure 2.2: *Numerical solution with the exact solution and the corresponding error of Example (2.4.1)for* $\varepsilon = 10^{-2}$ *and* $N = 32$ *on upwind scheme on adaptive grid.*



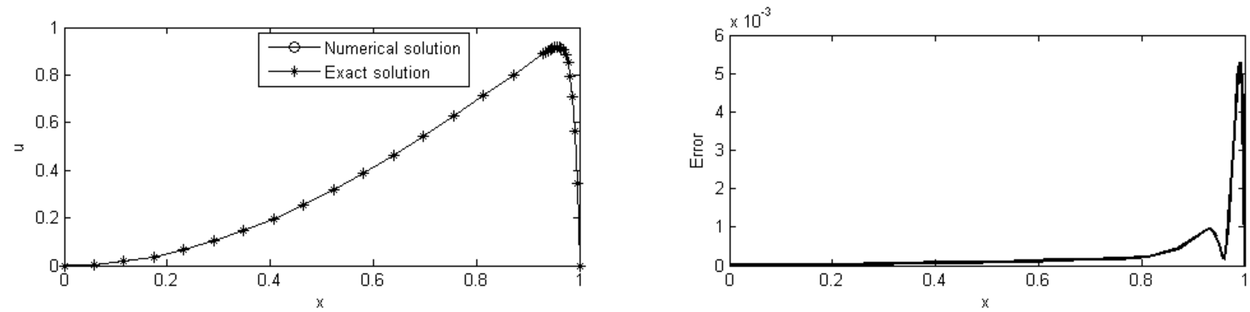
(a) Solutions. *(b) Error.*

Figure 2.3: *Numerical solution with the exact solution and the corresponding error of Example (2.4.1)for $\varepsilon = 10^{-2}$ and $N = 32$ on on B-spline scheme.*



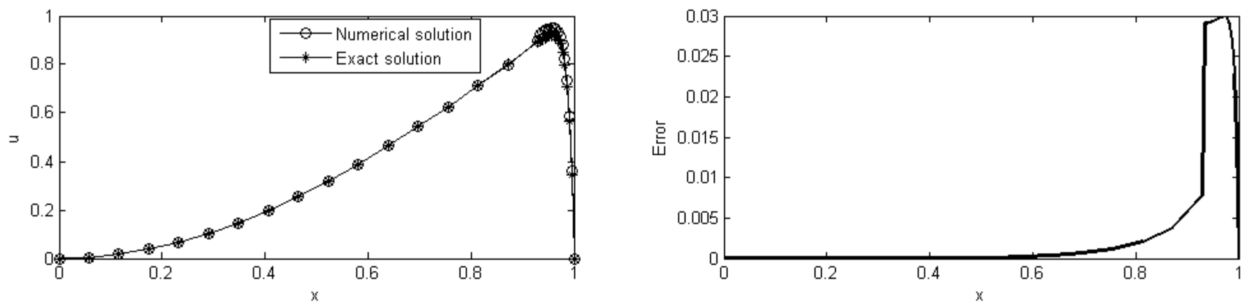
(a) Solutions. *(b) Error.*

Figure 2.4: *Numerical solution with the exact solution and the corresponding error of Example (2.4.2)for $\varepsilon = 10^{-2}$ and $N = 32$ on midpoint scheme.*



(a) Solutions. *(b) Error.*

Figure 2.5: *Numerical solution with the exact solution and the corresponding error of Example (2.4.2)for $\varepsilon = 10^{-2}$ and $N = 32$ on hybrid scheme.*



(a) Solutions. *(b) Error.*

Figure 2.6: *Numerical solution with the exact solution and the corresponding error of Example (2.4.2)for $\varepsilon = 10^{-2}$ and $N = 32$ on B-spline scheme.*

Table 2.1: *Maximum point-wise errors E^N and the rate of convergence r^N for Example (2.4.1)on upwind scheme on Shishkin mesh.*

ε	Number of intervals N						
	16	32	64	128	256	512	1024
1	1.2729e-2 0.9434	6.6191e-3 0.9735	3.3708e-3 0.9865	1.7012e-3 0.9932	8.5460e-4 0.9965	4.2831e-4 0.9982	2.1441e-4
$1e-2$	8.9324e-2 0.4578	6.5034e-2 0.5992	4.2930e-2 0.7130	2.6189e-2 0.7687	1.5372e-2 0.8073	8.7840e-3 0.8349	4.9243e-3
$1e-4$	8.9044e-2 0.4549	6.4959e-2 0.5981	4.2914e-2 0.7125	2.6189e-2 0.7687	1.5371e-2 0.8072	8.7838e-3 0.8349	4.9243e-3
$1e-6$	8.9041e-2 0.4549	6.4959e-2 0.5981	4.2914e-2 0.7125	2.6185e-2 0.7687	1.5371e-2 0.8072	8.7838e-3 0.8349	4.9243e-3
$1e-8$	8.9041e-2 0.4549	6.4959e-2 0.5981	4.2914e-2 0.7125	2.6185e-2 0.7687	1.5371e-2 0.8072	8.7838e-3 0.8349	4.9243e-3
$1e-10$	8.9041e-2 0.4549	6.4959e-2 0.5981	4.2914e-2 0.7125	2.6185e-2 0.7687	1.5371e-2 0.8072	8.7838e-3 0.8349	4.9243e-3

Table 2.2: *Maximum point-wise errors E^N and the rate of convergence r^N for Example (2.4.1)on upwind scheme on adaptive grid.*

ε	Number of intervals N						
	16	32	64	128	256	512	1024
1	1.2694e-2 0.9397	6.6175e-3 0.9690	3.3806e-3 0.9838	1.7093e-3 0.9919	8.5943e-4 0.9959	4.3092e-4 0.9979	2.1576e-4
$1e-2$	1.3692e-1 0.7324	8.2410e-2 0.7926	4.7575e-2 0.8556	2.6291e-2 0.8986	1.4102e-2 0.9369	7.3660e-3 0.9622	3.7806e-3
$1.e-4$	1.5209e-1 0.6808	9.4878e-2 0.7546	5.6235e-2 0.7878	3.2573e-2 0.8162	1.8499e-2 0.8389	1.0342e-2 0.8584	5.7038e-3
$1e-6$	1.5401e-1 0.6801	9.6126e-2 0.7516	5.7093e-2 0.7829	3.3182e-2 0.8093	1.8935e-2 0.8303	1.0649e-2 0.8472	5.9194e-3
$1e-8$	1.5416e-1 0.6799	9.6228e-2 0.7512	5.7168e-2 0.7824	3.3237e-2 0.8087	1.8975e-2 0.8294	1.0678e-2 0.8461	5.9405e-3
$1e-10$	1.5416e-1 0.6799	9.6230e-2 0.7512	5.7170e-2 0.7824	3.3239e-2 0.8086	1.8977e-2 0.8293	1.0680e-2 0.8459	5.9419e-3

Table 2.3: *Maximum point-wise errors E^N and the rate of convergence r^N for Example (2.4.1) on B-spline scheme.*

ε	Number of intervals N						
	16	32	64	128	256	512	1024
1	1.1509e-1 0.9402	5.9980e-2 0.9703	3.0614e-2 0.9852	1.5465e-2 0.9912	7.7725e-3 0.9963	3.8962e-3 0.9981	1.9506e-3
$1e-2$	4.9777-1 0.5031	3.5121e-1 0.6179	2.2884e-1 0.7019	1.4068e-1 0.7613	8.2995e-2 0.8030	4.7568e-2 0.8324	2.6713e-2
$1e-4$	4.9752e-1 0.5026	3.5117e-1 0.6179	2.2883e-1 0.7019	1.4068e-1 0.7613	8.2995e-2 0.8030	4.7568e-2 0.8324	2.6713e-2
$1e-6$	4.9752e-1 0.5026	3.5117e-1 0.6179	2.2883e-1 0.7019	1.4068e-1 0.7613	8.2995e-2 0.8030	4.7568e-2 0.8324	2.6713e-2
$1e-8$	4.9752e-1 0.5026	3.5117e-1 0.6179	2.2883e-1 0.7019	1.4068e-1 0.7613	8.2995e-2 0.8030	4.7568e-2 0.8324	2.6713e-2
$1e-10$	4.9752e-1 0.5026	3.5117e-1 0.6179	2.2883e-1 0.7019	1.4068e-1 0.7613	8.2995e-2 0.8030	4.7568e-2 0.8324	2.6713e-2

Table 2.4: *Maximum point-wise errors* E^N *and the rate of convergence* r^N *for Example (2.4.2)on midpoint scheme.*

ε	Number of intervals N						
	16	32	64	128	256	512	1024
1	1.3053e-2 0.9643	6.6901e-3 0.9836	3.3831e-3 0.9851	1.7018e-3 0.9958	8.5336e-4 0.9979	4.2730e-4 0.9989	2.1381e-4
$1e-2$	1.0259e-1 0.6157	6.6944e-2 0.6457	4.2790e-2 0.7253	2.5882e-2 0.7722	1.5155e-2 0.8089	8.6530e-3 0.8353	4.8497e-3
$1e-4$	1.0621e-1 0.5981	6.8717e-2 0.6282	4.3575e-2 0.7277	2.6313e-2 0.7733	1.5395e-2 0.8089	8.7877e-3 0.8353	4.9246e-3
$1e-6$	1.0626e-1 0.6284	6.8739e-2 0.6573	4.3584e-2 0.7279	2.6318e-2 0.7734	1.5397e-2 0.8088	8.7892e-3 0.8355	4.9254e-3
$1e-8$	1.0626e-1 0.6284	6.8739e-2 0.6573	4.3584e-2 0.7279	2.6318e-2 0.7734	1.5397e-2 0.8088	8.7892e-3 0.8355	4.9254e-3
$1e-10$	1.0626e-1 0.6284	6.8739e-2 0.6573	4.3584e-2 0.7279	2.6318e-2 0.7734	1.5397e-2 0.8088	8.7892e-3 0.8355	4.9254e-3

Table 2.5: *Maximum point-wise errors* E^N *and the rate of convergence* r^N *for Example (2.4.2)on hybrid scheme.*

ε	Number of intervals N						
	16	32	64	128		256 512	1024
1	6.2314e-3 1.0417	3.0270e-3 1.0209	1.4917e-3 1.0099	7.4074e-4 1.0049	3.6909e-4 1.0025	1.8422e-4 1.0012	9.2032e-5
$1e-2$	1.2763e-2 1.2757	5.2712e-3 1.4782	1.8920e-3 1.5783	6.3357e-4 1.6878	1.9666e-4 1.7926	5.6763e-5 1.9372	1.4822e-5
$1e-4$	1.2598e-2 1.2483	5.3028e-3 1.4369	1.9586e-3 1.5322	6.7716e-4 1.5983	2.2364e-4 1.6503	7.1190e-5 1.6917	2.2038e-5
$1e-6$	1.2596e-2 1.2481	5.3028e-3 1.4366	1.9590e-3 1.5319	6.7747e-4 1.5977	2.2384e-4 1.6503	7.1311e-5 1.6897	2.2106e-5
$1e-8$	1.2596e-2 1.2481	5.3028e-3 1.4366	1.9590e-3 1.5319	6.7747e-4 1.5977	2.2385e-4 1.6503	7.1311e-5 1.6897	2.2106e-5
$1e-10$	1.2596e-2 1.2481	5.3028e-3 1.4366	1.9590e-3 1.5319	6.7747e-4 1.5977	2.2385e-4 1.6503	7.1311e-5 1.6897	2.2106e-5

Table 2.6: *Maximum point-wise errors E^N and the rate of convergence r^N for Example (2.4.2) on B-spline scheme.*

ε	Number of intervals N						
	16	32	64	128	256	512	1024
1	2.0584e-4 2.0000	5.1458e-5 2.0000	1.2864e-5 2.0000	3.2161e-6 1.9999	8.0406e-7 2.0000	2.0101e-7 1.9999	5.0254e-8
$1e-2$	7.5422e-2 1.3334	2.9930e-2 1.9356	7.8239e-3 1.06635	4.7361e-3 1.2120	3.4381e-3 1.5717	2.8385e-3 1.3621	1.0421e-3
$1e-4$	7.1510e-2 1.0146	3.5396e-2 0.9028	1.8931e-2 0.7616	1.1166e-2 0.9027	5.9726e-3 1.0388	2.9070e-3 1.0812	1.3739e-3
$1e-6$	7.1030e-2 1.0523	3.4250e-2 1.0342	1.6720e-2 1.0097	8.3042e-3 0.9777	4.2167e-3 0.9034	2.2544e-3 1.1354	1.0262e-3
$1e-8$	7.1025e-2 1.0527	3.4238e-2 1.0363	1.6694e-2 1.0170	8.2490e-3 1.0061	4.1069e-3 1.0014	2.0514e-3 0.9967	1.0282e-3
$1e-10$	7.1025e-2 1.0527	3.4238e-2 1.0363	1.6693e-2 1.0170	8.2484e-3 1.0065	4.1057e-3 1.0026	2.0491e-3 1.0013	1.0236e-3

Chapter 3

Methods for reaction-diffusion problems

Introduction

Reaction-diffusion systems are naturally occurred in chemistry. However, the system can also describe dynamical processes of non-chemical nature. Examples are found in biology, geology and physics and ecology. Reaction-diffusion systems are mathematical models which explain how the concentration of one or more substances distributed in space changes under the influence of two processes: local chemical reactions in which the substances are transformed into each other and diffusion which causes the substances to spread out over a surface in space.
In this chapter, we discuss the properties and behaviour of singularly perturbed reaction-diffusion problem using some numerical techniques.

3.1 Model problem

Consider the following two point boundary value problem:

$$\left.\begin{aligned} \varepsilon u'' + p(x)u = f(x) \ \ , x \in \Omega = (0,1) \\ u(0) = A, \ \ u(1) = B \end{aligned}\right\} , \tag{3.1}$$

where $0 < \varepsilon \leq 1$ and $p(x) \geq \alpha > 0$ for all $x \in \overline{\Omega}$, A, B are constants.
When $\varepsilon = 0$ the reduced problem is $p(x)u_0 = f(x)$ which is not a differential equation and so it can not be made to satisfy the boundary condition at $\{0, 1\}$. And hence it has boundary layer at these points. Let L_ϵ be the differential operator, then

$$L_\epsilon u = \varepsilon u'' + +pu. \tag{3.2}$$

3.2 Properties of the solution

Lemma 3.2.1. *Let u be the solution of (3.1). Then for all k, $0 \leq k \geq 3$, $\left\|u^{(k)}\right\| \leq C(1 + \varepsilon^{-k/2})$.*

Proof. Refer [9]. □

Lemma 3.2.2. *The solution of model problem 3.1 can be decomposed in $u = v + w$ where $v = v_0 + \varepsilon v_1$ and $w = w_l + w_r$ are regular and singular components of u respectively. Furthermore,*

$$|v^0(x)|_k \leq C \ \ , |v^1(x)|_k \leq C\varepsilon^{-(k-2)} \ \ \textit{for all} \ \ k \ ,0 \leq k \leq 3,$$

and

$$|w_l^{(k)}(x)| \leq C\varepsilon^{-k/2} \exp(-x\sqrt{\alpha/\varepsilon})$$
$$|w_r^{(k)}(x)| \leq C\varepsilon^{-k/2} \exp(-(1-x)\sqrt{\alpha/\varepsilon}).$$

Proof. One may refer [9] for the proof. □

3.3 Numerical methods

In this section we consider the reaction-diffusion problem. We compare the numerical solution of different numerical techniques with the exact solution. Here we discritized the model problem (3.1) on Shishkin mesh. The Shishkin mesh is constructed by dividing Ω^N in three sub-interval $(0,1) = (0,\sigma) \cup (\sigma, 1-\sigma) \cup (1-\sigma, 1)$, where σ is chosen such that,

$$\sigma = \min\left\{\frac{1}{4}, \frac{\varepsilon}{\alpha}\ln N\right\}.$$

Now let us take the point $N/4$ in the domain $(0,\sigma)$ and $(1-\sigma,1)$ and $N/2$ points in the $(\sigma, 1-\sigma)$.

Scheme 1: Upwind Scheme

Let us apply the upwind scheme to solve the model problem (3.1)

$$\left.\begin{aligned} \varepsilon D^+D^-U(x_i) + p(x_i)U(x_i) = f(x_i),\ \ x_i \in \Omega^N \\ U(0) = u(0),\ \ U(1) = u(1) \end{aligned}\right\}. \tag{3.3}$$

Theorem 3.3.1. *The upwind method defined in(3.3) for model problem(3.1) satisfy the error bound.*

$$|u(x_i) - u_i| \leq CN^{-1}\ln N. \tag{3.4}$$

Proof. The proof is given in [9]. $\square$

Scheme 2: Midpoint upwind Scheme

Let us consider midpoint upwind scheme to solve the model problem (3.1)

$$\left.\begin{aligned} \varepsilon D^{+}D^{-}U(x_i) + p(x_{i-1/2})U(x_i) = f(x_{i-1/2}),\ \ x_i \in \Omega \\ \text{where } x_{i-1/2} = (x_i + x_{i-1})/2 \\ U(0) = u(0),\ \ U(1) = u(1) \end{aligned}\right\}. \tag{3.5}$$

3.4 Results and discussion

Example 3.4.1. *Let us consider the constant coefficient reaction diffusion problem*

$$-\varepsilon u'' + u = -\cos^2 \pi x - 2\varepsilon \cos 2\pi x, \quad u(0) = 0, \quad u(1) = 0,$$

where the exact solution is given by

$$u(x) = \frac{\exp\left(-(1-x)/\sqrt{\varepsilon}\right) + \exp\left(-x/\sqrt{\varepsilon}\right)}{1 + \exp\left(-1/\sqrt{\varepsilon}\right)} - \cos^2 \pi x.$$

Example 3.4.2. *Let us consider the variable coefficient reaction diffusion problem*

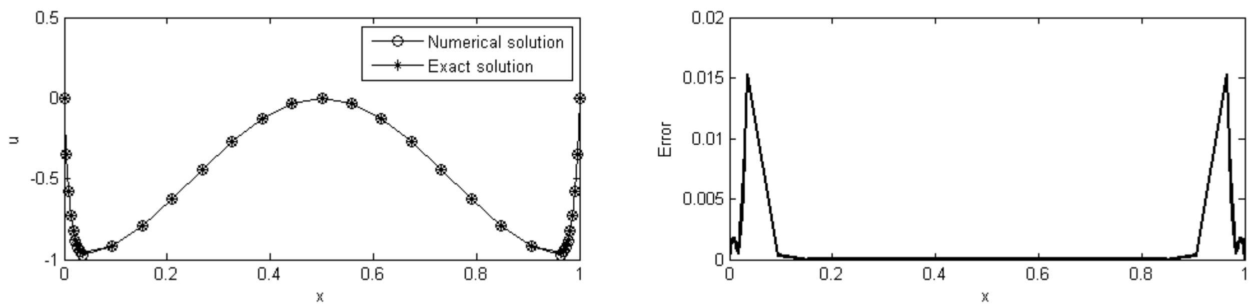
$$-\varepsilon u'' + 1 + x(1-x)u = f(x), \ \ u(0) = 0, \quad u(1) = 0,$$

where the $f(x)$ is chosen in such a way that the exact solution is given by

$$u(x) = 1 + (x-1)\exp\left(-x/\sqrt{\varepsilon}\right) + x\exp\left(-(1-x)/\sqrt{\varepsilon}\right).$$

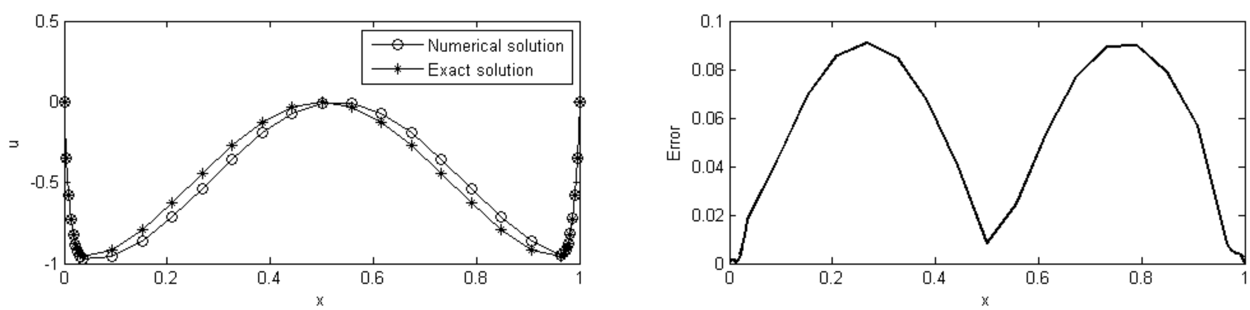
The numerical solution and the exact solution of Example 3.4.1 with corresponding error on upwind scheme and midpoint upwind scheme for $\varepsilon = 1e-4$ and $N = 32$ is shown in Figures 3.1 and 3.2. From Figure 3.1(a), we can observe that the approximate solution matches very well. Figure 3.1(b) and 3.2(b) show that the distribution of error is high in the layer region. Figure 3.3 and 3.4 show the graph of the upwind scheme and midpoint upwind scheme for Example 3.4.2 with the exact solution and corresponding error respectively.

The maximum point-wise error and rate of convergence is calculated as defined in Chapter 2. The maximum point-wise error and the rate of convergence for the Example 3.4.1 on upwind scheme on the Shishkin mesh is shown in Table 3.1. Table 3.2 show the maximum point-wise error and the rate of convergence of the Example 3.4.1 on midpoint upwind method. The maximum point-wise error and the rate of convergence of the Example 3.4.2 calculated on the upwind scheme and midpoint upwind scheme is shown in the Tables 3.3 and 3.4.



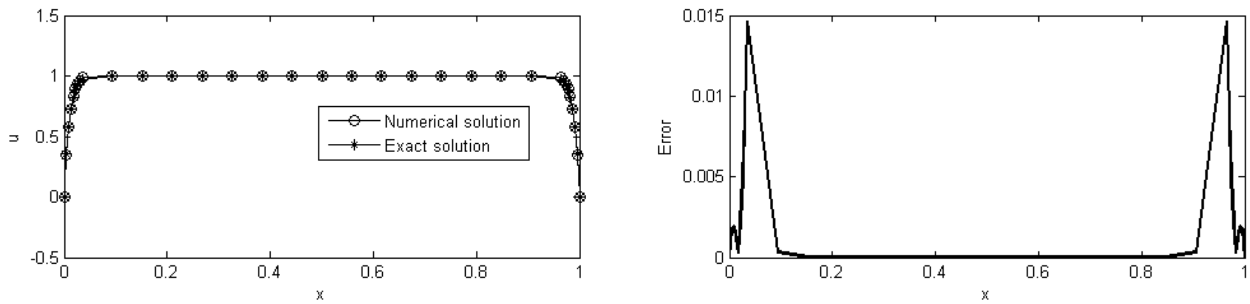
(a) Solutions. *(b) Error.*

Figure 3.1: *Numerical solution with the exact solution and the corresponding error of Example (3.4.1)for* $\varepsilon = 10^{-4}$ *and* $N = 32$ *on upwind scheme.*



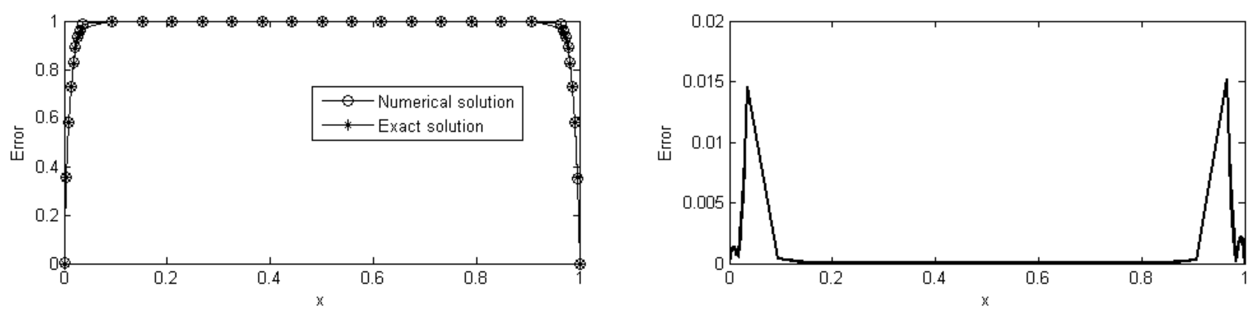
(a) Solutions. *(b) Error.*

Figure 3.2: *Numerical solution with the exact solution and the corresponding error of Example (3.4.1)for* $\varepsilon = 10^{-4}$ *and* $N = 32$ *on midpoint scheme.*



(a) Solutions. *(b) Error.*

Figure 3.3: *Numerical solution with the exact solution and the corresponding error of Example (3.4.2)for* $\varepsilon = 10^{-4}$ *and* $N = 32$ *on upwind scheme.*



(a) Solutions. *(b) Error.*

Figure 3.4: *Numerical solution with the exact solution and the corresponding error of Example (3.4.2)for* $\varepsilon = 10^{-4}$ *and* $N = 32$ *on midpoint scheme.*

Table 3.1: *Maximum point-wise errors E^N and the rate of convergence r^N for Example (3.4.1) on upwind scheme.*

ε	Number of intervals N						
	16	32	64	128	256	512	1024
1	1.1946e-2 2.0000	2.9699e-3 2.0000	7.4144e-4 2.0000	1.8530e-4 2.0000	4.6320e-5 2.0000	1.1580e-5 2.0000	2.8949e-6
$1e-2$	4.9090e-3 1.9338	1.2848e-3 1.9894	3.2355e-4 1.9943	8.1207e-5 1.9993	2.0311e-5 1.9998	5.0784e-6 1.9999	1.2697e-6
$1e-4$	4.2790e-2 1.4864	1.5272e-2 1.9067	4.0730e-3 2.0008	7.4851e-4 2.0162	1.1390e-4 1.6487	3.6325e-5 2.0087	1.1232e-5
$1e-6$	6.0286e-2 1.0458	2.9201e-2 1.0921	1.3697e-2 1.4490	6.0243e-3 1.3662	2.3368e-3 1.6967	7.2086e-4 1.6933	1.5856e-4
$1e-8$	6.2276e-2 1.0045	3.1040e-2 1.0091	1.5422e-2 1.0074	7.6134e-3 1.0369	3.7106e-3 1.0739	1.7627e-3 1.0126	9.1529e-4
$1e-10$	6.2278e-2 1.0046	3.1029e-2 1.0093	1.5405e-2 1.0071	7.6124e-3 0.9699	3.8862e-3 1.1402	1.7632e-3 1.0129	9.1576e-4

Table 3.2: *Maximum point-wise errors E^N and the rate of convergence r^N for Example (3.4.1)on midpoint upwind scheme.*

ε	Number of intervals N						
	16	32	64	128	256	512	1024
1	1.0188e-1 1.0289	4.9930e-2 1.0128	2.4744e-2 1.0059	1.2321e-2 1.0029	6.1480e-3 1.0014	3.0710e-3 1.00075	1.5347e-3
$1e-2$	1.0056e-01 1.0157	4.9737e-02 1.0091	2.4712e-02 1.0049	1.2314e-02 1.0024	6.1466e-3 1.0013	3.0706e-3 1.0056	1.5347e-3
$1e-4$	1.8353e-1 1.0084	9.1230e-2 1.0199	4.4989e-2 1.0216	2.2160e-2 1.0223	1.0910e-2 1.0226	5.3703e-3 1.0239	2.6426e-3
$1e-6$	1.9066e-1 0.9753	9.6975e-2 0.9957	4.8629e-2 1.0007	2.4302e-2 0.9875	1.2257e-2 1.0001	6.0593e-3 1.0020	3.0254e-3
$1e-8$	1.9127e-1 0.9723	9.7489e-2 0.9933	4.8971e-2 1.0109	2.4511e-2 0.9993	1.2269e-2 1.0015	6.1282e-3 1.0118	3.0637e-3
$1e-10$	1.9123e-1 0.9721	9.7540e-2 0.9936	4.8905e-2 1.0103	2.4532e-2 1.2556	1.0274e-2 0.7462	6.1250e-3 1.0117	3.0675e-3

Table 3.3: *Maximum point-wise errors E^N and the rate of convergence r^N for Example (3.4.2)on upwind scheme.*

ε	Number of intervals N						
	16	32	64	128	256	512	1024
1	2.0278e-4 1.9993	5.0719e-5 1.9998	1.2681e-5 1.9999	3.1704e-6 2.0431	7.9260e-7 2.0000	1.9815e-7 1.9999	4.9538e-8
$1e-2$	6.8072e-3 1.8887	1.8382e-3 1.9892	4.6300e-4 1.9972	1.1597e-4 1.9966	2.9006e-5 1.9998	7.2524e-6 1.9999	1.8132e-6
$1e-4$	4.1353e-2 1.4984	1.4636e-2 1.99144	3.8826e-3 2.0008	7.0948e-4 2.0333	1.1675e-4 1.6502	3.7195e-5 2.0087	1.1496e-5
$1e-6$	6.0074e-2 1.0477	2.9060e-2 1.0940	1.3613e-2 1.1865	5.9810e-3 1.0375	2.3180e-3 1.6972	7.1481e-4 1.6079	1.5725e-4
$1e-8$	6.2254e-2 1.0047	3.1025e-2 1.0094	1.5412e-2 1.01856	7.6075e-3 1.0371	3.7072e-3 1.0740	1.7609e-3 1.0226	9.1495e-4
$1e-10$	6.2275e-2 1.0049	3.1027e-2 1.0091	1.5404e-2 1.0171	7.6018e-3 0.9681	3.8859e-3 1.1402	1.7630e-3 1.0229	9.1464e-4

Table 3.4: *Maximum point-wise errors E^N and the rate of convergence r^N for Example (3.4.2)on midpoint upwind scheme.*

ε	Number of intervals N						
	16	32	64	128	256	512	1024
1	5.2437e-4 1.0463	2.5390e-4 1.0272	1.2458e-4 1.0151	6.1641e-5 1.0076	3.0659e-5 1.0038	1.5289e-5 1.0019	7.6344e-6
$1e-2$	1.0644e-2 1.2612	4.4405e-3 1.1814	1.9579e-3 1.1135	9.0485e-4 1.0948	4.3364e-4 1.0312	2.1218e-4 1.0161	1.0491e-4
$1e-4$	4.2492e-2 1.4864	1.5165e-2 1.8994	4.0651e-3 2.0003	7.6131e-4 1.9609	1.9556e-4 1.9360	8.3336e-5 2.0127	3.7783e-5
$1e-6$	6.0240e-2 1.0467	2.9161e-2 1.0928	1.3672e-2 1.1852	6.0123e-3 1.3658	2.3329e-3 1.9655	7.2030e-4 2.1816	1.5877e-4
$1e-8$	6.2271e-2 1.0046	3.1036e-2 1.0092	1.5419e-2 1.0185	7.6112e-3 1.0369	3.7093e-3 1.0739	1.7620e-3 1.0324	9.1515e-4
$1e-10$	6.2277e-2 1.0045	3.1029e-2 1.0091	1.5404e-2 1.0176	7.6121e-3 0.9699	3.8861e-3 1.1402	1.7631e-3 1.0334	9.1571e-4

Chapter 4

Conclusion and Future work

4.1 Conclusion

In this report, we have discussed the analytical properties, theoretical bounds and numerical techniques for solving singularly perturbed differential equations. In Chapter 1, we have discussed some basic terminologies, definitions and formulation of non-uniform meshes like Shishkin mesh and adaptive grids. For Shishkin mesh, one should have prior knowledge of location and width of the boundary layers. Since these information about the location and width of boundary layer are not available easily, one can opt adaptive grid. The variety of numerical methods viz upwind scheme, midpoint upwind scheme, hybrid scheme, B-spline methods are applied to convection diffusion problems in Chapter 2. We have observed that these methods are ε- uniform convergent methods. The upwind scheme on adaptive grid produces the better result than that on the Shishkin mesh. The hybrid scheme give very good result on Shishkin mesh than the midpoint upwind scheme. B-spline method is second order ε- uniformly convergent. In Chapter 3, we have applied the upwind scheme, midpoint upwind scheme for reaction-diffusion problems. We have observed that these methods are second order ε- uniform convergent methods when applied on Shishkin mesh. Hence, we conclude that standard numerical methods produce ε- uniform convergent solution on non-uniform meshes rather than on uniform mesh.

4.2 Future work

The work of the can be extended in various directions. Some of the future works to be carried out are listed below.

1. The numerical method considered in Chapter 2 are applied to convection-diffusion problem on Shishkin mesh. One can also extend those ideas also on adaptive grids for various SPP.
2. The idea of B-spline method, hybrid scheme can be extended for the system of singularly perturbed problems.
3. The idea of adaptive grid can be extended for system of first order and second order singularly perturbed differential equations.

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