EFFECTS OF SLIP ON SHEET-DRIVEN BOUNDARY LAYER FLOW REVISITED

A DISSERTATION

Submitted in partial fulfilment of the requirements for the award of the degree

of

Master of Science in Mathematics

by

MANOJ KUMAR MANDAL

DEPARTMENT OF MATHEMATICS
NATIONAL INSTITUTE OF TECHNOLOGY ROURKELA
ROURKELA, ODISHA – 769 008 (INDIA)

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CANDIDATE'S DECLARATION

I hereby declare that the work which is being presented in the dissertation entitled **EFFECTS OF SLIP ON SHEET-DRIVEN BOUNDARY LAYER FLOW RE-VISITED** in partial fulfillment of the requirement for the award of the Degree of Master of Science and submitted in the Department of Mathematics of the National Institute of Technology Rourkela, Rourkela is an authentic record of my own work carried out during a period from August 2013 to May 2014 under the supervision of Dr. Bikash Sahoo, Assistant Professor, Department of Mathematics of National Institute of Technology Roourkela, Roourkela.

The matter presented in this dissertation has not been submitted by me for the award of any other degree of this or any other institute.

(MANOJ KUMAR MANDAL)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

(BIKASH SAHOO)
Supervisor

Date:

The Viva-Voce Examination of **Mr. Manoj Kumar Mandal**, has been held on ....

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(Manoj Kumar Mandal)

ABSTRACT

The study of laminar boundary layer flow over a stretching sheet has received considerable attention in the past due to its immense applications in the industries. There are adequate papers on Newtonian and non-Newtonian flows past a stretching sheet subject to conventional no-slip boundary conditions. Literature study reveals that more attention is required to see the effects of partial slip on the boundary layer flow over past a stretching sheet. The no-slip boundary condition (the assumption that a liquid adheres to a solid boundary) is one of the central tenets of the Navier-Stokes theory. Mathematically the no-slip condition is given by $v_n = 0$ and $v_t = 0$, where v_n and v_t are the normal and the tangential component of the velocity on the wall. In certain situations, however, the assumption of no-slip does no longer apply and should be replaced by a partial slip boundary condition. Navier [12] proposed a slip boundary condition wherein the amount of relative slip depends linearly on the local shear stress. In this work the steady laminar boundary layer flow over a stretching sheet is studied subject to partial slip boundary condition. The partial slip is controlled by a dimensionless slip factor, which varies between zero (total adhesion) and infinity (full slip). Suitable similarity transformations are used to reduce the resulting non-linear partial differential equation into ordinary differential equation. The resulting highly nonlinear ODE with slip boundary condition is solved by using classical shooting method along with fourth order Runge-Kutta method. It is interesting to find the slip has a prominent effects on the velocity profiles and on the skin friction coefficient. It is observer that with the increase in slip parameter boundary layer thickness decreases.

Contents

1 Introduction

The study of laminar boundary layer flow over a stretching sheet has received considerable attention in the past due to its immense applications in the industries, for example, materials manufactured by extrusion process, the boundary layer along a liquid film in condensation process and heat treated materials traveling between a feed roll and a wind-up roll or on conveyor belt poses the features of a moving continuous surface. In view of these applications, Sakiadis [13] initiated the study of boundary layer flow over a continuous solid surface moving with constant speed. Due to the entrainment of the ambient fluid, this boundary layer is quite different from that in the Blasius flow past a flat plate. Erickson et al [7] extended this problem to the case in which the transverse velocity at the moving surface is non-zero, with heat and mass transfer in the boundary layer being taken into account. These investigations have a bearing on the problem of a polymer sheet extruded continuously from a die. It is often tacitly assumed that the sheet is inextensible, but situations may arise in the polymer industry in which it is necessary to deal with a stretching plastic sheet, as pointed out by McCormack and Crane [11] and Crane [6] respectively. The flow and heat transfer phenomena over stretching surface has promising applications in a number of technological processes including production of polymer films or thin sheets. Following Crane [6], Gupta and Gupta [8] examined the heat and mass transfer using a similarity transformation subject to suction or blowing. The effects of power-law surface temperature and power-law surface heat flux on the heat transfer characteristics of a continuous stretching surface with suction or blowing were investigated by Chen and Char [3]. Again, many metallurgical processes involve the cooling of continuous strips or filaments by drawing them through a quiescent fluid. Mention may be made of drawing, annealing, and tinning of copper wires as processes that produce similar effects. In all the cases, the properties of the final product depend to a great extent on the rate of cooling. By drawing such strips in an electrically conducting fluid, subject to a magnetic field, the cooling rate can be controlled and product of desired characteristics can be obtained. Another interesting application of the hydromagnetics to metallurgy lies in the purification of molten metals from non-metallic inclusions by the application of a magnetic field. In view of these applications, Chakrabarti and Gupta [2] studied the hydromagnetic flow and heat transfer in a fluid over a stretching sheet. The steady two-dimensional stagnation point flow of an incompressible, viscous, electrically conducting fluid over a stretching sheet has been investigated by Mahapatra and Gupta [10]. In fact, realistically stretching of the sheet may not necessarily be linear. This situation was dealt by many authors [4, 5, 9, 14] in their works on boundary layer flow over nonlinear stretching sheet.

The stretching sheet flow problem got a new dimension after Andersson [1] considered the slip flow past the linearly stretching sheet. In fact, the no-slip boundary condition (the assumption that a liquid adheres to a solid boundary) is one of the central tenets of the Navier-Stokes theory. Mathematically the no-slip condition is given by $v_n = 0$ and $v_t = 0$, where v_n and v_t are the normal and the tangential component of the velocity on the wall. In certain situations, however, the assumption of no-slip does no longer apply and should be replaced by a partial slip boundary condition. Navier [12] proposed a slip boundary condition wherein the amount of relative slip depends linearly on the local shear stress. The equations of motion are still valid for these flows, but the boundary conditions have to be changed appropriately. This is a condition which was discovered empirically, and which is satisfied well within the framework of continuum mechanics. Literature survey reveals that some rarefied gases and most of the non-Newtonian fluids exhibit the slip boundary conditions. Although the Navier condition looks simple, analytically it is much more difficult than the no-slip condition and only a few simple exact slip flow solutions have been found.

In this project, we have reconsidered the work by Andersson [1] and solved the resulting nonlinear similarity equation by shooting method along with fourth order Runge-Kutta method.

2 Mathematical formulation

In this case, the sheet is stretched horizontally along the x-axis by pulling it on both sides with equal forces keeping the origin fixed and with speed U, that varies with the distance from the y-axis, i.e. $U = cx$. Here we assume that flow of fluid is steady and incompressible and the body force ($\overrightarrow{F}$) is zero. The viscous fluid is only partially

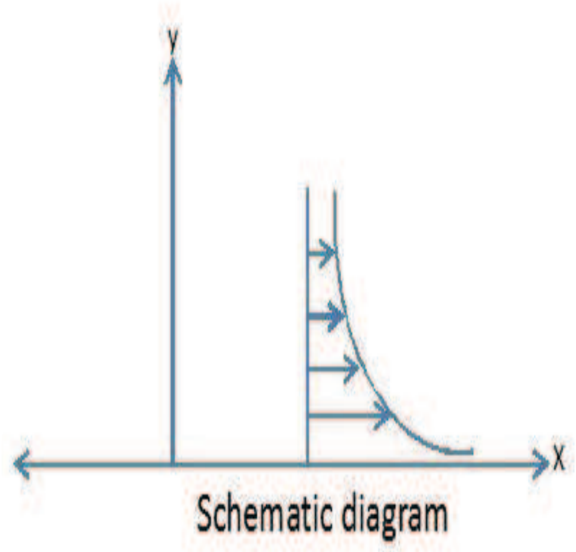

Figure 1: Schematic diagram of the flow domain.

adhering to the stretching sheet and the fluid motion is thus subject to the slip flow condition.

$$u(x,y) - U(x) = L\frac{\partial u}{\partial y},\ at\ \ y = 0 \tag{2.1}$$

2.1 Governing equations

Hence, the Navier-Stokes equation along x-axis is given be

$$u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right) \tag{2.2}$$

and along y-axis is given by

$$u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right) \tag{2.3}$$

Equation of continuity is given by

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{2.4}$$

and the boundary conditions are

$$u_y = 0, \quad u_x - U(x) = L\frac{\partial u_x}{\partial y} \quad \text{at } y = 0,$$
$$u_x = U_\infty \quad \text{as } y \to \infty \tag{2.5}$$
$$\tag{2.6}$$

2.2 Boundary layer approximation

We now see the familiar strategy in boundary layer theory, which is to scale the cross-stream distance by a much smaller length scale, and adjust that length scale in order to achieve a balance between convection and diffusion. The dimensionless x co-ordinate is defined as $x^* = (x/L)$, while the dimensionless y co-ordinate is defined as $y^* = (y/\delta)$, where the length δ is determined by a balance between convection and diffusion. In momentum boundary layers, it is also necessary to scale the velocity components and the pressure. In the stream wise direction, the natural scale for the velocity is the free-stream velocity U_∞. So we define a scaled velocity in the x direction as $u_x^* = (u_x/U_\infty)$. The scaled velocity in the y direction is determined from the mass conservation condition, when the above equation is expressed in terms of scaled variables $x^* = (x/L)$, $y^* = (y/\delta)$ and $u_x^* = (u_x/U_\infty)$, and multiplied throughout by (L/U_∞), we obtain,

$$\frac{\partial u_x^*}{\partial x^*} + \frac{L}{\delta U_\infty}\frac{\partial u_y}{\partial y^*} = 0 \tag{2.7}$$

The above Eq. (2.7) indicates that the approximate scaled velocity in the y direction is $u_y^* = (u_y/(U_\infty\delta/L))$. Note that the magnitude of the velocity u_y in the cross-stream y direction, $(U_\infty\delta/L)$ is small compared to that in the stream wise direction. This is a feature common to all the boundary layers in incompressible flows.

Next we turn to the x momentum Eq., (2.2). When it is expressed in terms of scaled spatial and velocity coordinates, and divided throughout by $(\rho U_\infty^2/L)$, we obtain,

$$u_x^* \frac{\partial u_x^*}{\partial x^*} + u_y^* \frac{\partial u_y^*}{\partial y^*} = -\frac{1}{\rho U_\infty^2}\frac{\partial p}{\partial x^*} + \frac{\mu}{\rho U_\infty L}\left(\frac{L^2}{\delta^2}\right)\left(\frac{\partial^2 u_x^*}{\partial {y^*}^2} + \frac{\delta^2}{L^2}\frac{\partial^2 u_x^*}{\partial {x^*}^2}\right) \tag{2.8}$$

The above equation indicates that it is appropriate to define the scaled pressure as $p^* = (p/\rho U_\infty^2)$. Also note that the factor $(\mu/\rho U_\infty L)$ on the right side of the Eq. (2.8) is the inverse of the Reynolds number based on the free stream velocity and the length of the plate. In the right side of the Eq. (2.8), we can also neglect the stream wise gradient $(\partial^2 u_x^*/\partial x_*^2)$, since this is multiplied by the factor $(\delta/L)^2$, which is small in the limit $(\delta/L) \ll 1$. With this simplifications, Eq. (2.8) reduces to,

$$u_x^* \frac{\partial u_x^*}{\partial x^*} + u_y^* \frac{\partial u_y^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + Re^{-1}\left(\frac{L^2}{\delta^2}\right)\frac{\partial^2 u_x^*}{\partial {y^*}^2} \tag{2.9}$$

From the above equation, it is clear that a balance is achieved between convection and diffusion only for $(\delta/L) \sim \mathrm{Re}^{-1/2}$ in the limit of the right Reynolds number. This indicates that the boundary layer thickness is $\mathrm{Re}^{-1/2}$ smaller than the length of the plate. Without loss of generality, we substitute $\delta = \mathrm{Re}^{-1/2}L$ in Eq. (2.9), to get the scaled momentum equation in the stream wise direction,

$$u_x^* \frac{\partial u_x^*}{\partial x^*} + u_y^* \frac{\partial u_y^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u_x^*}{\partial {y^*}^2} \tag{2.10}$$

Next we analyse the momentum equation in the cross-stream direction (see Eq. (2.3)). This equation is expressed in terms of the scaled spatial co-ordinates, velocities and pressure, to obtain,

$$\frac{\rho U_\infty^2 \delta}{L^2}\left(u_x^* \frac{\partial u_y^*}{\partial x^*} + u_y^* \frac{\partial u_y^*}{\partial y^*}\right) = -\frac{\rho U_\infty^2}{L^2}\frac{\partial p^*}{\partial y^*} + \frac{\mu U_\infty}{\delta L}\left(\frac{\partial^2 u_y^*}{\partial {y^*}^2} + \left(\frac{\delta}{L}\right)^2 \frac{\partial^2 u_y^*}{\partial {x^*}^2}\right) \tag{2.11}$$

By examining all the terms in the above equation, it is easy to see that the largest terms is the pressure gradient in the cross-stream direction. We divide throughout by the pre-factor of this term, and substitute $(\delta/L) = Re^{-1/2}$, to obtain,

$$Re^{-1}\left(u_x^*\frac{\partial u_y^*}{\partial x^*}+u_y^*\frac{\partial u_y^*}{\partial y^*}\right)=-\frac{\partial p^*}{\partial y^*}+Re^{-1}\left(\frac{\partial^2 u_y^*}{\partial {y^*}^2}+Re^{-1}\frac{\partial^2 u_y^*}{\partial {x^*}^2}\right) \tag{2.12}$$

In the limit, Re $\gg$ 1, the above momentum conservation equation reduces to ,

$$\frac{\partial p^*}{\partial y^*}=0 \tag{2.13}$$

Thus, the pressure gradient in the cross-stream direction is zero in the leading approximation, and the pressure at any stream wise location in the boundary layer is same as that in the free-stream at that same stream-wise location. This is a salient feature of the flow in a boundary layers. Thus, the above scaling analysis has provided us with the simplified 'boundary layer equations' (2.7), (2.10) and (2.13), in which we neglect all terms that are O(1) in an expansion in the parameter Re$^{-1/2}$. Expressed in dimensional form, the mass conservation equation is (2.4), while the approximate momentum conservation equations are,

$$\rho\left(u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u_x}{\partial y^2} \tag{2.14}$$

$$\frac{\partial p}{\partial y}=0 \tag{2.15}$$

From the equation (2.15), the pressure is only a function of the leading edge of the plate x, and not a function of cross-stream distance y. Therefore, the pressure at a displacement x from the leading edge is independent of the normal distance form the plate y. However, in the limit $y\rightarrow\infty$, we know that the free stream velocity U_∞ is a constant, and the pressure is a constant independent of x. This implies that the pressure is also independent of x as well, and the term $(\partial p/\partial x)$ in equation (2.14) is equal to zero. With this, equation (2.14) simplifies to,

$$\rho\left(u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}\right)=\mu\frac{\partial^2 u_x}{\partial y^2} \tag{2.16}$$

Taking a similarity variable

$$\eta = \sqrt{\frac{c}{\nu}}y \tag{2.17}$$

and defining the stream function ψ as

$$\psi = \sqrt{c\nu}xf(\eta) \tag{2.18}$$

Where, $f(\eta)$ is dimensionless function of similarity variable η. The stream wise velocity can be expressed as

$$u_x = \frac{\partial\psi}{\partial y} = cxf'(\eta) \tag{2.19}$$

And the cross-stream velocity is given by

$$u_y = -\frac{\partial\psi}{\partial x} = -\sqrt{c\nu}f(\eta) \tag{2.20}$$

Eq. (2.16) also contains derivatives of the stream wise velocity, which can be expressed in terms of the similarity variable η as,

$$\frac{\partial u_x}{\partial x} = cf^{'}(\eta) \tag{2.21}$$

$$\frac{\partial u_x}{\partial y} = \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial y} = cxf^{''}(\eta)\sqrt{\frac{c}{\nu}} \tag{2.22}$$

$$\frac{\partial^2 u_x}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial y}\right) = \frac{c^2x}{\nu}f^{'''}(\eta) \tag{2.23}$$

Eqs. (2.19) to (2.23) are inserted into the Eq. (2.16) to obtain

$$f^{'''} + ff^{''} = f^{'2}, \tag{2.24}$$

after some simplification. This equation has to be solved, subject to the appropriate boundary conditions, which are as follows. At the surface of the sheet, the slip condition requires that the velocity components $u_x - U(x) = L\frac{\partial u_x}{\partial y}$, which reduces to,

$$\frac{df}{d\eta} = 1 + \gamma\frac{d^2f}{d\eta^2}, \quad \eta = 0 \tag{2.25}$$

Using Eq. (2.20) for the cross-stream velocity u_y, along with condition (2.25) at the surface, the condition $u_y = 0$ at $y = 0$ reduces to,

$$f = 0 \quad \eta = 0 \tag{2.26}$$

Finally, we require that the velocity u_x is equal to the free stream velocity U_∞ in the limit $y \to \infty$. Using Eq. (2.19) for u_x, we obtain,

$$\frac{df}{d\eta} = 0, \quad \eta \to \infty \tag{2.27}$$

2.3 Skin friction coefficient

The boundary layer normally generates a drag on the sheet as a result of the viscous stresses, which are developed at the wall. This drag is normally referred to as skin friction. Skin friction occurs from the interaction amid the fluid and the skin of the body, and is directly associated to the wetted surface, the area of the facade of the body that is in contact with the fluid.

$$C_f = \tau_{xy}|_{y=0} \tag{2.28}$$

$$\tau_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \tag{2.29}$$

Where

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta} . \frac{\partial \eta}{\partial y} = cxf''(\eta) \sqrt{\frac{c}{\nu}} \tag{2.30}$$

$$\frac{\partial u}{\partial y}|_{\eta=0} = cxf''(0) \sqrt{\frac{c}{\nu}} \tag{2.31}$$

$$\frac{\partial v}{\partial x}|_{\eta=0} = \frac{\partial v}{\partial \eta} . \frac{\partial \eta}{\partial x} = 0 \tag{2.32}$$

Using (2.31) and (2.32) in (2.28), we have

$$C_f = \tau_{xy}|_{\eta=0} = \mu \left(cxf''(0) \sqrt{\frac{c}{\nu}} \right) \tag{2.33}$$

3 Numerical tools used

Solving a non-linear differential equation analytically is very complicated and very limited number of methods are available to solve it. Moreover, there are huge number of problems which do not have a analytical solutions. So here we adopt numerical techniques to solve the stretching sheet problem with slip boundary conditions.

3.1 Shooting method

Consider the two point boundary value problem

$$u'' = f(x, u, u'), x \in (a, b) \tag{3.1}$$

with following three boundary conditions.

(1) Dirichlet boundary condition: In this problem we have

$$u(a) = A, u(b) = B \tag{3.2}$$

In order to apply initial value method, we guess the value for $u'(a) = s$ (say).

(2)Neumann boundary condition: In this problem we have

$$u'(a) = A \tag{3.3}$$

In order to apply initial value method, we guess the value for $u(a) = s$ (say).

(3) Robin or Mixed boundary condition: In this problem we have

$$a_0 u(a) - a_1 u'(a) = A \tag{3.4}$$

or

$$b_0 u(b) + b_1 u'(b) = B \tag{3.5}$$

In order to apply the initial value method, we guess the value $u(a) = s$ then from Eq.(37), we get

$$u'(a) = \frac{a_0 s + A}{a_1} \tag{3.6}$$

In this method we convert the boundary value problem in to system of first order initial value problem and use the Runge-Kutta method to solve. Now we compare the solution at $x = b$. If it does not satisfy, then we guess another value for $u'(a)$ or $u(a)$ and again solve the initial value problem and find u at $x = b$. There solution are used to obtain the better guess of $u(a)$ or $u'(a)$. This technique of solving the boundary value problems by using the methods for solving the initial value problems is called shooting Method.

3.2 Runge-Kutta method

The fourth order Runge-Kutta method to solve the single initial value problem

$$\frac{dy}{dx} = f(x, y);\ f(x_0) = y_0$$

is given by

$$y_{j+1} = y_j + h\frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where,

$$\begin{aligned}
K_1 &= f(x_j,\ y_j), \\
K_2 &= f\left(x_j + \frac{h}{2},\ y_j + \frac{k_1}{2}\right), \\
K_3 &= f\left(x_j + \frac{h}{2},\ y_j + \frac{k_2}{2}\right), \\
K_4 &= f(x_j + h,\ y_j + k_3)
\end{aligned}$$

and h is the step length.

Similarly, the fourth ordered classical Runge-Kutta method for the system of equations

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = \mathbf{f}(x, y_1, y_2, ...y_n)\ ;\ \mathbf{Y}(\mathbf{X}_0) = \mathbf{Y}_0$$

may be written as

$$\mathbf{Y}_{j+1} = \mathbf{Y}_j + h\frac{1}{6}(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + \mathbf{K}_4)$$

where,

$$\mathbf{K}_1 = \begin{pmatrix} k_{11} \\ k_{21} \\ \vdots \\ k_{n1} \end{pmatrix}, \mathbf{K}_2 = \begin{pmatrix} k_{12} \\ k_{22} \\ \vdots \\ k_{n2} \end{pmatrix}, \mathbf{K}_3 = \begin{pmatrix} k_{13} \\ k_{23} \\ \vdots \\ k_{n3} \end{pmatrix}, \mathbf{K}_4 = \begin{pmatrix} k_{14} \\ k_{24} \\ \vdots \\ k_{n4} \end{pmatrix}$$

and,

$$
\begin{aligned}
k_{i1} &= f_i\left(x_j,\ y_{1j},\ y_{2j},\ \ldots,\ y_{nj}\right) \\
k_{i2} &= f_i\left(x_j + \frac{h}{2},\ y_{1j} + \frac{k_{11}}{2},\ y_{2j} + \frac{k_{21}}{2},\ \ldots,\ y_{nj} + \frac{k_{n1}}{2}\right) \\
k_{i3} &= f_i\left(x_j + \frac{h}{2},\ y_{1j} + \frac{k_{12}}{2},\ y_{2j} + \frac{k_{22}}{2},\ \ldots,\ y_{nj} + \frac{k_{n2}}{2}\right) \\
k_{i4} &= f_i\left(x_j + h,\ y_{1j} + k_{13},\ y_{2j} + k_{23},\ \ldots,\ y_{nj} + k_{n3}\right)
\end{aligned}
$$

Where, i = 1(1)n

Now let us discuss briefly the shooting method for our problem. Introducing the new parameters y_1, y_2 and y_3, we have

$$
\begin{aligned}
y_1 &= f \\
y_2 &= f' \\
y_3 &= f''
\end{aligned}
\tag{3.7}
$$

Then Eq. (2.24) can be written as,

$$
\begin{aligned}
\frac{dy_1}{d\eta} &= f' = y_2 = f_1(\eta, y_1, y_2, y_3) \\
\frac{dy_2}{d\eta} &= f'' = y_3 = f_2(\eta, y_1, y_2, y_3) \\
\frac{dy_3}{d\eta} &= f''' = f'^2 - ff'' = y_2^2 - y_1 y_3 = f_3(\eta, y_1, y_2, y_3) \qquad (3.8)
\end{aligned}
$$

(3.9)

The boundary conditions in terms of the new variables are

$$
y_1(0) = 0, \quad y_2(0) = 1 + \gamma y_3(0), \quad y_2(\eta_\infty) = 0 \tag{3.10}
$$

$$
\begin{aligned}
k_{11} &= f_1(\eta_j, y_{1j}, y_{2j}, y_{3j}) \\
k_{21} &= f_2(\eta_j, y_{1j}, y_{2j}, y_{3j}) \\
k_{31} &= f_3(\eta_j, y_{1j}, y_{2j}, y_{3j})
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
k_{12} &= f_1(\eta_j + \frac{h}{2}, y_{1j} + \frac{k_{11}}{2}, y_{2j} + \frac{k_{21}}{2}, y_{3j} + \frac{k_{31}}{2}) \\
k_{22} &= f_2(\eta_j + \frac{h}{2}, y_{1j} + \frac{k_{11}}{2}, y_{2j} + \frac{k_{21}}{2}, y_{3j} + \frac{k_{31}}{2}) \\
k_{32} &= f_3(\eta_j + \frac{h}{2}, y_{1j} + \frac{k_{11}}{2}, y_{2j} + \frac{k_{21}}{2}, y_{3j} + \frac{k_{31}}{2})
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
k_{13} &= f_1(\eta_j + \frac{h}{2}, y_{1j} + \frac{k_{12}}{2}, y_{2j} + \frac{k_{22}}{2}, y_{3j} + \frac{k_{32}}{2}) \\
k_{23} &= f_2(\eta_j + \frac{h}{2}, y_{1j} + \frac{k_{12}}{2}, y_{2j} + \frac{k_{22}}{2}, y_{3j} + \frac{k_{32}}{2}) \\
k_{33} &= f_3(\eta_j + \frac{h}{2}, y_{1j} + \frac{k_{12}}{2}, y_{2j} + \frac{k_{22}}{2}, y_{3j} + \frac{k_{32}}{2})
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
k_{14} &= f_1(\eta_j + h, y_{1j} + k_{13}, y_{2j} + k_{23}, y_{3j} + k_{33}) \\
k_{24} &= f_2(\eta_j + h, y_{1j} + k_{13}, y_{2j} + k_{23}, y_{3j} + k_{33}) \\
k_{34} &= f_3(\eta_j + h, y_{1j} + k_{13}, y_{2j} + k_{23}, y_{3j} + k_{33})
\end{aligned}
\tag{3.14}
$$

4 Results and discussion

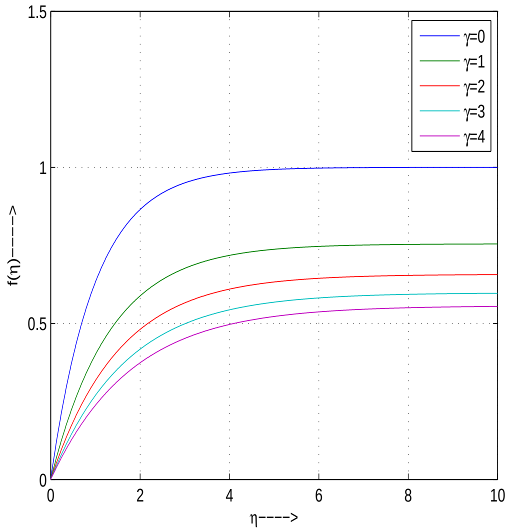

Figure 2: Variation of f with γ.

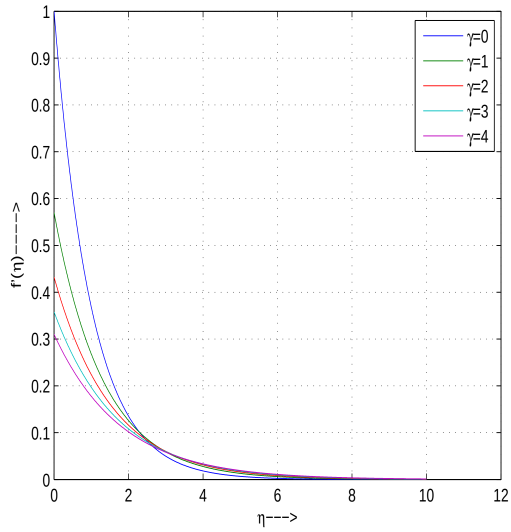

Figure 3: Variation of f' with γ.

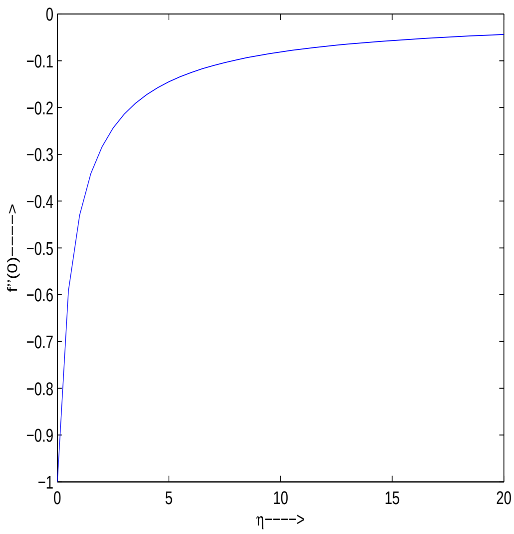

Figure 4: Variation of $f''(0)$ with γ.

The numerical solution of Eq. (2.24) subject to slip boundary conditions (2.25)-(2.27) is obtained by shooting method along with fourth order Runge-Kutta method. The $C++$ code was run on a core $i5$ personal computer. The step size $h = 0.01$. Dimensionless vortical velocity profiles $f(\eta)$ are presented in Fig.2 for some different values of the slip

Table 1: Variation of $f(\eta)$ with γ

η	γ=0	γ=1	γ=2	γ=3	γ=4
0	0	0	0	0	0
0.01	0.00995	0.005677	0.004306	0.003568	0.00309
0.1	0.095163	0.054886	0.041815	0.034734	0.030144
0.2	0.181269	0.105782	0.08097	0.067452	0.058656
0.5	0.393469	0.237322	0.184112	0.154641	0.135257
1.01	0.635781	0.402703	0.318885	0.271266	0.239423
2	0.864665	0.588077	0.48076	0.41737	0.373852
3.1	0.954952	0.682172	0.571628	0.504549	0.457583
4	0.981686	0.718022	0.609884	0.543535	0.496644
5	0.993264	0.737555	0.632728	0.568144	0.522284
6	0.997524	0.746737	0.644567	0.581674	0.53698
10	0.99996	0.754488	0.656387	0.596694	0.554594

Table 2: Variation of $f'(\eta)$ with γ

η	γ=0	γ=1	γ=2	γ=3	γ=4
0	1	0.56984	0.432041	0.357836	0.309908
0.01	0.99005	0.565555	0.429211	0.355702	0.308187
0.1	0.904837	0.528408	0.404556	0.337058	0.293127
0.2	0.818731	0.489988	0.37882	0.317487	0.277254
1.01	0.364219	0.26585	0.222438	0.195566	0.176623
2	0.135336	0.125915	0.116039	0.108168	0.101789
3	0.0497875	0.0591889	0.0601375	0.0594719	0.0583362
4	0.0183162	0.0278233	0.0311666	0.0326984	0.0334339
5	0.0067385	0.0130794	0.0161526	0.0179783	0.0191626
6	0.0024794	0.0061488	0.0083716	0.0098852	0.0109839
10	0.000046	0.0003014	0.000605	0.0009048	0.0011891

Table 3: Varistion of $f''(0)$ with γ

γ	$f''(0)$
0	-1
1	-0.43016
2	-0.28398
3	-0.214055
4	-0.172523

factor γ. It is seen that γ has a substantial effect on the solution. $f(\eta)$ decreases with an increase in γ. Fig.3 depicts near the surface of the sheet, the horizontal component of velocity $f'(\eta)$ decreases with an increase in slip and increases away from the sheet.

This results in a cross-over in the velocity profile. The amount of slip $1 - f'(0)$ increase monotonically with γ. Also from Fig.4 and Table.3 it is seen that $f'(0)$, which is a measure of the skin friction coefficient decreases in magnitude with increasing slip. Our result agrees with the finding of Andersson [1]. As was expected, the boundary layer thickness is found to be decreasing with an increase in slip.

5 Conclusions

In this dissertation we have reconsidered the steady, laminar flow of a viscous fluid past a stretching sheet with slip boundary conditions. Boundary layer approximations are used to reduce the elliptic Navier-Stokes equations in to parabolic equation. Subsequently, similarity transformations are used to reduce the parabolic momentum equation to a non-linear ordinary differential equation, which was solved numerically by shooting method. It is interesting to see that the skin friction coefficient decreases in magnitude with an increase in slip. The boundary layer thickness decreases with an increase in slip. The flow behaves as if inviscid for high values of slip.

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