Free Flexural Vibration of Multiple Stepped Beams by Spectral Element Method

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Dedicated to

My Family & Guide...

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May , 2014

Certificate

This is to certify that the work in the thesis entitled *Free Flexural Vibration of Multiple Stepped Beams by Spectral Element Method* by *Jitendra Kumar Meher*, bearing Roll Number 212CE2044, is a record of an original research work carried out by him under my supervision and guidance in partial fulfillment of the requirements for the award of the degree of *Master of Technology* in *Structural Engineering, Department of Civil Engineering*. Neither this thesis nor any part of it has been submitted for any degree or academic award elsewhere.

Manoranjan Barik

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Abstract

The free vibration analyses of multiple-stepped Bernoulli-Euler beam with various boundary conditions have been studied by many researchers using different methods of analysis such as Differential Quadrature Element Method (DQEM), Composite Element Method (CEM), Admonian Decomposition Method (ADM), Differential Quadrature Method (DQM), Local adaptive Differential Quadrature Method (LaDQM), Discrete Singular Convolution (DSC) algorithm etc., besides the conventional analytical methods and finite element methods. In this work the Spectral Element Method (SEM) for analysis of stepped-beams has been used. The second part of the work is concerned with the free flexural vibration of multiple-stepped Timoshenko beam with various boundary conditions using the Spectral Element Method (SEM).

Accurate computation of even the higher modes of vibration frequencies with consideration of least number of degrees of freedom is possible using SEM thus promising very high computational efficiency. Validation of this method is performed with various numerical solutions. A comparison between application of both Euler-Bernoulli and Timoshenko beam theory to the same beam is carried out and various important physical parameters are also investigated.

***Keywords*:** Multiple-stepped Beam , Bernoulli-Euler Beam, Timoshenko Beam, Spectral Element Method (SEM),Analytical Methods,Finite Element Methods .

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Chapter 1

Inroduction

Prediction of dynamic characteristics of the structures in the field of engineering is of utmost importance. Now a days the aim of researchers are towards the achievement of more accurate results.

1.1 Finite Element Method

Among the numerical tools finite element method (FEM) is a competent one for the dynamic analysis of structures. Frequency-independent polynomial shape functions are used in the formulation of conventional FEM models. These can work for dynamic problems with lower frequencies wave modes but solutions become increasingly inaccurate with higher modes, where FEM model needs very large number of elements for better accuracy.

1.2 Dynamic Stiffness Method

It is an exact solution method. Here exact wave solutions to the governing differential equations is obtained to derive exact dynamic shape function leading to formulation of exact dynamic stiffness matrix in the frequency domain. In Dynamic Stiffness Method (DSM), governing differential equations adopted in the formulation of exact dynamic stiffness matrix decide the accuracy level.

For example, the solution provided by DSM based Timoshenko-beam model is more accurate frequency-domain solutions than the Bernoulli-Euler beam based model. DSM still provides better results than conventional FEM as the severity in assumptions done for DSM will be less. The need to make multiple fine meshes to a regular part of structure is finished as only one single element suffices the work that significantly reduces the number of elements and degrees-of-freedom(DOFs) in total. So the computation time is significantly reduced. At the mean time this reduces computer round-off errors or numerical errors leading to improved accurate solution that are of extreme importance for most large size problems.

1.3 Spectral Analysis Method

Among the frequency-domain methods the spectral analysis method (SAM) is one corresponding to the solutions by continuous Fourier transformation. Instead of continuous Fourier Transform, Discrete Fourier Transform (DFT) is widely practiced. This approach involves determining an infinite set of spectral components (or Fourier coefficients) in the frequency domain and performing the inverse Fourier transform to reconstruct the time histories of the solutions. transform.

1.4 Spectral Element Method

Assembly and meshing of finite elements, exactness of the dynamic stiffness matrix with minimum number of DOFs from DSM and superposition of wave modes via DFT theory and FFT algorithm from SAM is found in Spectral element method (SEM).

1.5 Objectives

The primary objectives of this research work are summarized as follows:

1. To study the free vibration of multiple-stepped Bernoulli-Euler beam and Timoshenko beam with combination of different classical boundary conditions using the Spectral Element Method (SEM).

2. To compair the freuencies for stepped-beam obtained by application of both Bernoulli-Euler and Timoshenko beam theory.

3. To compare the natural frequencies found using SEM with those found by other methods.

4. To find higher mode of natural frequencies for stepped beams by SEM as these are rare in literature.

Chapter 2

Literature Review

2.1 Euler-Bernoulli Stepped Beam

There is wide application of stepped beams in many engineering fields such as robot arm, aircraft wing, long span bridges etc. The high-frequency vibrations are of crucial importance to aerospace structures such as aircraft, rotorcraft, satellite and space shutter, jet fighter, rocket, and missile [1]. Hence, accurate estimation of both low and high order frequencies is of utmost importance and demands efficient methodology. Though the existing numerical methods have developed quite fast in the last decades, numerical evaluation of high frequencies is still a daunting task.

Many researchers have worked on free vibration of stepped beams. Klein [2] developed a method combining the advantages of a finite element approach and a Rayleigh-Ritz analysis. Sato [3] combined transfer matrix method with partly used finite element method for the analysis of beams with abrupt changes in cross-section. Exact and numerical solutions to a single stepped beam were derived by Jang and Bert [4], [5]. An analytical method for the vibration analysis of stepped Euler-Bernoulli beam on classical and elastic end supports was proposed by Naguleswaran [6], [7]. It is preferred to have a numerical method for the solution as the analytical or exact methods becomes more difficult with increase in number of steps. Though the finite element method (FEM) is more versatile for the numerical method, there are few alternative methods which are of better advantage over the FEM. Recently Differential Quadrature Element Method (DQEM) [8] has been used for the free vibration analysis of multiple-stepped beams and extensive reviews of literature pertaining to the advantages and disadvantages of various other methods employed for the stepped-beams vibration analysis have been reported. According to Duang and Wang [1] the rate of convergence of DQEM is very high and the DQEM can yield very accurate results with a small number of grid points and is a highly accurate method of analysis. Some other methods reported are the Composite Element Method (CEM) [9], Admonian Decomposition Method(ADM) [10], Differential Quadrature Method (DQM) [11], Local adaptive Differential Quadrature Method (LaDQM) [12] and Discrete Singular Convolution (DSC) algorithm [1], [13].

Levinson (1976) [14] observed that the frequency equation for a stepped beam consisting of only two distinct parts is quite complicated. He suggested using numerical methods for solution to vibration of continuous systems having discontinuous properties. Balasubramanian and Subramanian (1985) [15] compare the frequency values obtained by using 2DPN elements (deflection, slope) and 4DPN elements (deflection, slope, bending moment and shear force) in FEM for uniform, stepped and continuous beams for various boundary conditions to show the superior performance of the 4DPN element. Balasubramanian et al. (1990) [16] introduced complete polynomials up to 15th degree successively so that the end nodal degrees of freedom progressively involve up to and including the seventh derivative (8DPN). A way to employ very high order derivatives as degrees of freedom in beam vibration was demonstrated.

Jang and Bert (1989) [5] considered the higher mode frequencies of a stepped beam with two different cross-sections and investigated the effects of steps on frequency of a beam. They found that the stepping up the beam results in lowering the natural frequencies, thus weakening the structure. Also, stepping down the beam decreases the stiffness of the structure in most cases. Subramanian and Balasubramanian (1987) [17] used circular rod, rod of rectangular section of cross-sections to understand the effect of steps on frequencies of vibration.

According to their findings stepping down at anti-nodal locations can dynamically stiffen the structure and stepping up can dynamically weakens the structure unless the ends are held down. The steps can be judiciously incorporated for dynamic tuning. Laura et al. (1991) [18] studied the fundamental frequency of transverse vibration, referring to through study of beneficial effects of beam by introducing step variations of the cross-sectional area and moment of inertia, predicting its use in lighter structures. The outcome was not satisfactory however it shows beneficial effects of cross-section discontinuities in a very eloquent fashion, when one considers the values of natural frequencies which correspond to the transversely vibrating beam of uniform cross-sectional area and moment of inertia.

Bert and Newberry (1986) [19] applied non-integer-polynomial concept to the finite element technique. They observe it as difficult to find closed form solutions for two-section stepped beams. Bapat and Bapat (1987) [20] used exact general solution for a uniform Euler beam, together with the continuity of displacement and slope and the relationship between the shear force and bending moment at a support. They found good agreement with previously found results. Rao and Mirza (1989) [21] derived exact frequency and normal mode expressions for generally restrained Bemoulli-Euler beams. According to them the translational and rotational spring constants have a significant effect on the first three frequencies and mode shapes of vibration and the higher mode frequencies, comparatively, do not show much variation with the range of spring constants considered by them. Jang and Bert (1989) [4] found the exact solutions and compared the results obtained by the use of the finite element method (FEM) with non-integer polynomial shape function and with a commercial code, MSC/pal. FEM showed quite good agreement with the exact solutions.

Reyes et al. (1987) [22] studied the numerical results using trial functions which contain an unknown parameter when implementing the methodology in a finite element formulation in case of vibrating beams and frames and compared it with the experimental results. No computational economy and/or advantage was acquired at least when using the classical Bernouilli theory of vibrating beams. Lee and Bergman (1994) [23] proposed a method for concise and efficient solution of the free and forced vibration of a class of complex like structures using Greens function. A dynamic flexibility method was used to formulate and solve the free and forced vibration of stepped beams. They found good agreement with others. Popplewell and Chang (1996) [24] proposed a unified treatment for finding the free vibration of a non-uniform beam having material or cross-sectional discontinuities, intermediate spring supports, or non-classical end supports. They mentioned that, this approach can vary accurately approximate the natural frequencies, bending moments and shear forces of these beams.

2.2 Timoshenko Stepped Beam

Many researches are done on multi-stepped Timoshenko beam with different boundary and loading conditions.Bhashyam [25] used finite element modelling for analysis of Timoshenko beam. Akella [26] modified the Timoshenko beam-shaft element to include the effect of disks within its length with formulation of a stepped element which performs better than the linearly tapered element in representing shaft discontinuities. Wang [27] found the effect of elastic foundation on the vibration of stepped beams, noticeable for both frequencies and mode shapes, especially in the lowest mode. Farghaly [28] investigated a beneficial effect of the relative span and relative thickness parameters on the natural frequency of Timoshenko beam by making second span stepped.

Farghaly et al. [29] observed additional gain in natural frequencies for a one-span beam with a stepwise variable cross-section made of unidirectional fiber composite materials of different fiber volume fraction than those made of conventional materials. Popplewell [30] employ polynomial based generalized force mode functions with the method of Galerkin for stepped Timoshenko beam.Here polynomials are found on each side of a discontinuity that satisfy the conditions at the contiguous end which is chosen so that the transverse deflection and its slope or the slope due to bending for a Timoshenko beam are continuous at the location of a discontinuity. Wu [31] used modified CTMM (Combined transfer matrix method) for the analysis of multi-step Timoshenko beam. Dong [32] investigated on stepped laminated composite Timoshenko beam.

In the present study the Spectral Element Method (SEM) is used to study the free vibration of multiple-stepped Timoshenko beam with combination of different classical boundary conditions. The SEM is easy to implement as it is similar to the conventional Finite Element Method (FEM). Further the efficacy of this method can be realized as the number of elements needed is only one more than the number of steps used in the stepped beam. For a stepped beam of only one number of change of cross section, consideration of only two number of elements suffices the analysis, thus highly reducing the number of degrees of freedom in comparison to the other methods of analysis.

Chapter 3

Stepped Bernoulli- Euler Beam Theory

3.1 Theoretical Formulation

The formulation of the spectral element model for a stepped beam is fairly similar to the formulation of the conventional finite element method. However the major difference is that in general the spectral element formulation begins with the transformation of the governing partial differential equations of motion from the time domain to the frequency domain by using the Discrete Fourier Transform (DFT). Here the time variable disappears and the frequency becomes a parameter to transform the original time-domain partial differential equations into the frequency-domain ordinary differential equations which are then solved exactly and the exact wave solutions are used to derive frequency-dependent dynamic shape functions. The exact dynamic stiffness matrix called the spectral element matrix is finally formulated by using the dynamic shape functions [33]. The spectral element matrices are then assembled to form the global dynamic stiffness matrix and the boundary conditions are applied similar to the conventional finite element method thus producing the reduced dynamic stiffness matrix. The determinant of the reduced dynamic stiffness matrix which is a function of the natural frequencies when equated to zero and solved gives rise to the required natural frequencies.

3.1.1 Spectral Element Matrix for Stepped Beam

The spectral element matrix for a stepped beam can be derived by considering one Bernoulli-Euler beam element of the stepped beam which is of uniform cross section whose free flexural vibration is represented by

$$EIw'''' + \rho A\ddot{w} = 0 \tag{3.1}$$

where $w(x,t)$=transverse displacement, E=Young's Modulus, A=area of cross-section, I=moment of inertia, ρ=mass density.

$$M_t(x,t) = EIw''(x,t) \tag{3.2}$$

$$Q_t(x,t) = -EIw'''(x,t) \tag{3.3}$$

where $M_t(x,t)$=bending moment, $Q_t(x,t)$=internal transverse shear force. Let the solution to Eq.(4.1) in spectral form be

$$w(x,t) = \frac{1}{N}\sum_{n=0}^{N-1} W_n(x;\omega_n)e^{i\omega_n t} \tag{3.4}$$

Substituting Eq.(4.4) into Eq.(4.1) gives an Eigenvalue problem for a specific discrete frequency such as $\omega = \omega_n$

$$EIW'''' - \omega^2\rho AW = 0 \tag{3.5}$$

Let the general solution to Eq.(4.5) be

$$W(x) = ae^{-ik(\omega)x} \tag{3.6}$$

Substituting Eq.(4.6) into Eq.(4.5) yields a dispersion relation

$$k^4 - k_F^4 = 0 \tag{3.7}$$

where k_F = wave number for pure bending (flexural) wave-mode defined by

$$k_F = \sqrt{\omega}\left(\frac{\rho A}{EI}\right)^{\frac{1}{4}} \tag{3.8}$$

Eq.(4.7) gives two real roots and two imaginary roots as

$$k_1 = -k_2 = k_F\ , \quad k_3 = -k_4 = ik_F \tag{3.9}$$

For the finite Euler-Bernoulli beam element of length L, the general solution of Eq.(4.5) can be obtained in the form of

$$W(x;\omega) = a_1 e^{-ik_F x} + a_2 e^{-k_F x} + a_3 e^{ik_F x} + a_4 e^{k_F x} = \mathbf{e}(x;\omega)\mathbf{a} \tag{3.10}$$

where

$$\mathbf{e}(x;\omega) = \begin{bmatrix} e^{-ik_F x} & e^{-k_F x} & e^{ik_F x} & e^{k_F x} \end{bmatrix} \tag{3.11}$$

and

$$\mathbf{a} = \{a_1 \ \ a_2 \ \ a_3 \ \ a_4\} \tag{3.12}$$

The spectral nodal displacements and slopes of the beam element are related to displacement field by

$$\mathbf{d} = \begin{Bmatrix} W_1 \\ \Theta_1 \\ W_2 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} W(0) \\ W'(0) \\ W(L) \\ W'(L) \end{Bmatrix} \tag{3.13}$$

Substituting Eq.(4.10) in to right hand side of Eq.(4.13) gives

$$\mathbf{d} = \begin{bmatrix} e(0;\omega) \\ e'(0;\omega) \\ e(L;\omega) \\ e'(L;\omega) \end{bmatrix} \mathbf{a} = \mathbf{H}_B(\omega)\mathbf{a} \tag{3.14}$$

where

$$\mathbf{H}_B(\omega) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -ik_F & -k_F & ik_F & k_F \\ e^{-ik_F} & e^{-k_F} & e^{ik_F} & e^{k_F} \\ -ik_F e^{-ik_F} & -k_F e^{-k_F} & ik_F e^{ik_F} & k_F e^{k_F} \end{bmatrix} \tag{3.15}$$

From Eq.(3.14) we have

$$\mathbf{a} = \mathbf{H}_B(\omega)^{-1}\mathbf{d} \tag{3.16}$$

Substituting the value of **a** from Eq.(3.16) in to Eq.(4.10),the displacement field within the beam element is represented as

$$W(x) = \mathbf{e}(x;\omega)\mathbf{H}_B^{-1}\mathbf{d} \tag{3.17}$$

From the Eq.(4.2) and Eq.(4.3) the spectral components of the bending moment and transverse shear force can be related to W(x) as

$$M(x) = EIW''(x) \tag{3.18}$$

$$Q(x) = -EIW'''(x) \tag{3.19}$$

The spectral nodal bending moments and transverse shear forces defined for the beam element correspond to the moments and the forces as given below (Fig. 1).

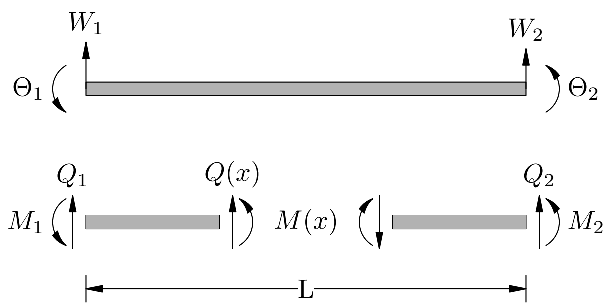

Figure 3.1: Bernoulli-Euler spectral beam element with nodal forces and displacements

$$\mathbf{f}_c = \begin{Bmatrix} Q_1 \\ M_1 \\ Q_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} -Q(0) \\ -M(0) \\ +Q(L) \\ +M(L) \end{Bmatrix} \tag{3.20}$$

Substituting Eq.(4.17) into Eq.(4.18) and Eq.(4.19) and its results into right-hand side of Eq.(4.20) we have

$$\mathbf{f}_c = \begin{Bmatrix} EIW'''(0) \\ -EIW''(0) \\ -EIW'''(L) \\ EIW''(L) \end{Bmatrix} = EI \begin{Bmatrix} e'''(0;\omega) \\ -e''(0;\omega) \\ -e'''(L;\omega) \\ e''(L;\omega) \end{Bmatrix} \mathbf{H}_B^{-1}\mathbf{d} = \mathbf{S}_B(\omega)\mathbf{d} \tag{3.21}$$

where $\mathbf{S}_B(\omega)$ is spectral element (dynamic stiffness) matrix for the beam element given by

$$\mathbf{S}_B(\omega) = EI \begin{Bmatrix} e'''(0;\omega) \\ -e''(0;\omega) \\ -e'''(L;\omega) \\ e''(L;\omega) \end{Bmatrix} \mathbf{H}_B^{-1} \tag{3.22}$$

Chapter 4

Stepped Timoshenko Beam Theory

4.1 Theoretical Formulation

The formulation of the spectral element model for a stepped beam is fairly similar to the formulation of the conventional finite element method. However the major difference is that in general the spectral element formulation begins with the transformation of the governing partial differential equations of motion from the time domain to the frequency domain by using the Discrete Fourier Transform (DFT). Here the time variable disappears and the frequency becomes a parameter to transform the original time-domain partial differential equations into the frequency-domain ordinary differential equations which are then solved exactly and the exact wave solutions are used to derive frequency-dependent dynamic shape functions. The exact dynamic stiffness matrix called the spectral element matrix is finally formulated by using the dynamic shape functions [33]. The spectral element matrices are then assembled to form the global dynamic stiffness matrix and the boundary conditions are applied similar to the conventional finite element method thus producing the reduced dynamic stiffness matrix. The determinant of the reduced dynamic stiffness matrix which is a function of the natural frequencies when equated to zero and solved gives rise to the required natural frequencies.

4.1.1 Spectral Element Matrix for Stepped Beam

The spectral element matrix for a stepped beam can be derived by considering one Timoshenko beam element of the stepped beam which is of uniform cross section whose free vibration is represented by

$$\kappa GA(w^{''} - \theta^{'}) - \rho A\ddot{w} = 0 \tag{4.1}$$

$$EI(\theta^{''} + \kappa GA(w^{'} - \theta) - \rho I\ddot{\theta} = 0 \tag{4.2}$$

where $w(x,t)$=transverse displacement, $\theta(x,t)$= slope due to bending, E=Young's modulus, G=shear modulus , κ=shear correction factor,which depends upon shape of the cross-section, A=area of cross-section and I=moment of inertia about the neutral axis.

$$M_t(x,t) = EI\theta^{'}(x,t) \tag{4.3}$$

$$Q_t(x,t) = \kappa GA[w^{'}(x,t) - \theta(x,t)] \tag{4.4}$$

where $M_t(x,t)$=internal bending moment, $Q_t(x,t)$=transverse shear force. Let the solution to Eq.(4.1) & Eq.(4.2) in spectral form be

$$w(x,t) = \frac{1}{N}\sum_{n=0}^{N-1} W_n(x;\omega_n)e^{i\omega_n t} \tag{4.5}$$

$$\theta(x,t) = \frac{1}{N}\sum_{n=0}^{N-1} \theta_n(x;\omega_n)e^{i\omega_n t} \tag{4.6}$$

Substituting Eq.(4.5 & 4.6) into Eq.(4.1 & 4.2) gives an Eigenvalue problem

$$\kappa GA(W^{''} - \theta^{'}) + \rho A\omega^2 W = 0 \tag{4.7}$$

$$EI\theta^{''} - \kappa GA(W^{'} - \theta^{'}) + \rho I\omega^2\theta = 0 \tag{4.8}$$

Let the general solution to Eq.(4.7 & 4.8) be

$$W(x) = ae^{-ik(\omega)x} \tag{4.9}$$

$$\theta(x) = \beta ae^{-ik(\omega)x} \tag{4.10}$$

Substituting Eq.(4.9 & 4.10) into Eq.(4.7 & 4.8) yields an eigenvalue problem as

$$\begin{bmatrix} \kappa GAk'' - \rho A\omega^2 & -ik\kappa GA \\ ik\kappa GA & EIk^2 + \kappa GA - \rho A\omega^2 \end{bmatrix} \begin{Bmatrix} 1 \\ \beta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{4.11}$$

Equation 4.11 gives a dispersion relation as

$$k^4 - \eta k_F^4 k^2 - k_F^4(1 - \eta_1 k_G^4) \tag{4.12}$$

where

$$k_F = \sqrt{\omega}\left(\frac{\rho A}{EI}\right) \ , \ \ k_G = \sqrt{\omega}\left(\frac{\rho A}{\kappa EI}\right) \tag{4.13}$$

and

$$\eta = \eta_1 + \eta_2 \ , \ \ \eta_1 = \frac{\rho I}{\rho A} \ , \ \ \eta_2 = \frac{EI}{\kappa GA} \tag{4.14}$$

Solving Eq.4.12 gives four roots as

$$\begin{aligned} k_1 = -k_2 = \frac{1}{\sqrt{2}} k_F \sqrt{\eta k_F^2 + \sqrt{\eta^2 k_F^4 + 4(1 - \eta_1 k_G^4)}} = k_t \\ k_3 = -k_4 = \frac{1}{\sqrt{2}} k_F \sqrt{\eta k_F^2 - \sqrt{\eta^2 k_F^4 + 4(1 - \eta_1 k_G^4)}} = k_e \end{aligned} \tag{4.15}$$

From the first line of Eq. 4.11 we can obtain the wavemode ratio as

$$\beta_p(\omega) = \frac{1}{ik_p}(k_p^2 - k_G^2) = -ir_p(\omega) \ (p = 1, 2, 3, 4) \tag{4.16}$$

where

$$r_p(\omega) = \frac{1}{k_p}(k_p^2 - k_G^4) \tag{4.17}$$

By using the four wavenumbers given by Eq. 3.15 the general solution of Eq.(4.7 & 4.8) can be obtained as

$$\begin{aligned} W(x) = a_1 e^{-ik_t x} + a_2 e^{k_t x} + a_3 e^{-ik_e x} + a_4 e^{ik_e x} = \mathbf{e}(x;\omega)\mathbf{a} \\ \theta(x) = a_1 e^{-ik_t x} + a_2 e^{k_t x} + a_3 e^{-ik_e x} + a_4 e^{ik_e x} = \mathbf{e}(x;\omega)\mathbf{a} \end{aligned} \tag{4.18}$$

where

$$\mathbf{a} = \begin{Bmatrix} a_1 & a_2 & a_3 & a_4 \end{Bmatrix}^T \tag{4.19}$$

and

$$\mathbf{e}_w(x;\omega) = \begin{bmatrix} e^{-ik_tx} & e^{ik_tx} & e^{-ik_ex} & e^{ik_ex} \end{bmatrix}$$
$$\mathbf{e}_\theta(x;\omega) = \mathbf{e}_w(x;\omega)\mathbf{B}(\omega) \tag{4.20}$$
$$\mathbf{B}(\omega) = diag[\beta_p(\omega)]$$

The spectral nodal displacements and slopes of the beam element of length L are related to displacement field by

$$\mathbf{d} = \begin{Bmatrix} W_1 \\ \Theta_1 \\ W_2 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} W(0) \\ \theta^(0) \\ W(L) \\ \theta(L) \end{Bmatrix} \tag{4.21}$$

Substituting Eq.(4.18) in to right hand side of Eq.(4.21) gives

$$\mathbf{d} = \begin{bmatrix} e_\omega(0;\omega) \\ e_\theta(0;\omega) \\ e_\omega(L;\omega) \\ e_\theta(L;\omega) \end{bmatrix} \mathbf{a} = \mathbf{H}_T(\omega)\mathbf{a} \tag{4.22}$$

where

$$\mathbf{H}_T(\omega) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -ir_t & ir_t & -ir_e & ir_e \\ e_t & e_t^{-1} & e_e & e_e^{-1} \\ -ir_te_t & ir_te_t^{-1} & -ir_ee_e & ir_ee_e^{-1} \end{bmatrix} \tag{4.23}$$

with the use of following definations :

$$e_t = e^{-ik_tL} \ , \ e_e = e^{-ik_eL}$$
$$r_t = \frac{1}{k_t}(k_t^2 - k_G^4) \ , \ r_e = \frac{1}{k_e}(k_e^2 - k_G^4) \tag{4.24}$$

From Eq.4.22 we have

$$\mathbf{a} = \mathbf{H}_T^{-1}(\omega)\mathbf{d} \tag{4.25}$$

Substituting the value of **a** from Eq.(4.25) in to Eq.(4.18),the general solution can be expressed as

$$W(x) = \mathbf{e}_w(x;\omega)\mathbf{H}_T^{-1}\mathbf{d}$$
$$\theta(x) = \mathbf{e}_\theta(x;\omega)\mathbf{H}_T^{-1}\mathbf{d} \tag{4.26}$$

From the Eq.(4.3) and Eq.(4.4) the spectral components of the bending moment and transverse shear force can be related to $W(x)$ and $\theta(x)$ as

$$Q = \kappa GA(W^{'} - \theta) \quad , \; M = EI\theta^{'} \tag{4.27}$$

The spectral nodal bending moments and transverse shear forces defined for the beam element correspond to the moments and the forces as given below (Fig. 1).

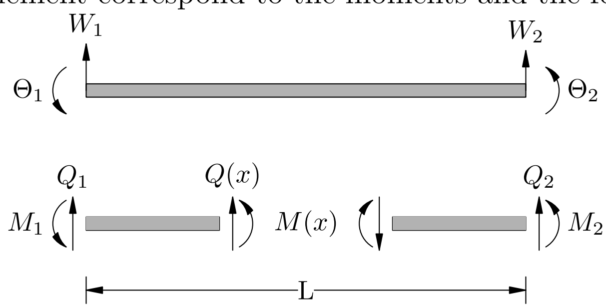

Figure 4.1: Timoshenko spectral beam element with nodal forces and displacements

$$\mathbf{f}_c(\omega) = \begin{Bmatrix} Q_1 \\ M_1 \\ Q_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} -Q(0) \\ -M(0) \\ +Q(L) \\ +M(L) \end{Bmatrix} \tag{4.28}$$

Substituting Eq.(4.26) into Eq.(4.27) and and its results into right-hand side of Eq.(4.28) we have

$$\mathbf{f}_c = \begin{Bmatrix} EIW^{'''}(0) \\ -EIW^{''}(0) \\ -EIW^{'''}(L) \\ EIW^{''}(L) \end{Bmatrix} = \begin{Bmatrix} -\kappa GA\left\{e_\omega^{'}(0;\omega) - e_\theta(0;\omega)\right\} \\ -EIe_\theta^{'}(0;\omega) \\ \kappa GA\left\{e_\omega^{'}(L;\omega) - e_\theta(L;\omega)\right\} \\ EIe_\theta^{'}(L;\omega) \end{Bmatrix} \mathbf{H}_T^{-1}\mathbf{d} = \mathbf{S}_T(\omega)\mathbf{d} \tag{4.29}$$

where $\mathbf{S}_T(\omega)$ is spectral element (dynamic stiffness) matrix for the beam element given by

$$\mathbf{S}_T(\omega) = \begin{Bmatrix} -\kappa GA\left\{e_\omega^{'}(0;\omega) - e_\theta(0;\omega)\right\} \\ -EIe_\theta^{'}(0;\omega) \\ \kappa GA\left\{e_\omega^{'}(L;\omega) - e_\theta(L;\omega)\right\} \\ EIe_\theta^{'}(L;\omega) \end{Bmatrix} \mathbf{H}_T^{-1} \tag{4.30}$$

Chapter 5

Globalisation of Dynamic Stiffness Matrix and Solution Procedure

5.1 Globalisation of Dynamic Stiffness Matrix

After spectral element matrix for each element is computed, they are assembled into the spectral global matrix following the similar convention of the classical finite element method where the inter-element continuity conditions are automatically maintained. The classical boundary conditions are applied by eliminating the corresponding rows and columns of those restrained degrees of freedom thus forming the reduced spectral global matrix $\mathbf{S}_g(\omega)$ and the eigenvalue problem is formed as

$$\mathbf{S}_g(\omega)\mathbf{d}_g = \mathbf{0} \tag{5.1}$$

where $\mathbf{d}_g$ is the global spectral nodal degrees of freedoms vector.

5.2 Solution Procedure

The eigenfrequencies $\omega_i (i = 1, 2, ..., \infty)$ are determined by equating the determinant of $\mathbf{S}_g(\omega)$ to zero at $\omega = \omega_i$, i.e.,

$$\left|\mathbf{S}_g(\omega_i)\right| = 0 \tag{5.2}$$

The value of ω is found by following an iterative procedure with a very high degree of accuracy which makes the value of $\left|\mathbf{S}_g(\omega_i)\right|$ close to zero. For practical purposes the tolerance value of 10^{-6} is sufficient, however our solution procedure is able to sustain up to a tolerance limit of 10^{-11} uninterruptedly. The solution procedure followed is detailed below.

In this method we begin with a starting value of frequency ω_1 and suitable values of an increment δ and a tolerance ϵ are considered so that no frequency may be skipped. The value '$\omega_1 + \delta$' is assigned to ω_2 and the values of $\left|\mathbf{S}_g(\omega_1)\right|$ and $\left|\mathbf{S}_g(\omega_2)\right|$ are checked for whether they are of the opposite sign or not. When they are of the same sign each value is incremented by δ otherwise $\left|\mathbf{S}_g(\omega_0)\right|$ is determined where $\omega_0 = (\omega_1 + \omega_2)/2$. In the first case if $\left|\mathbf{S}_g(\omega_1)\right| > 0$ & $\left|\mathbf{S}_g(\omega_2)\right| < 0$, then $\omega_1 = \omega_0$ if $\left|\mathbf{S}_g(\omega_0)\right| > 0$ or $\omega_2 = \omega_0$ if $\left|\mathbf{S}_g(\omega_0)\right| \leq 0$. In the second case if $\left|\mathbf{S}_g(\omega_1)\right| < 0$ & $\left|\mathbf{S}_g(\omega_2)\right| > 0$, then $\omega_2 = \omega_0$ if $\left|\mathbf{S}_g(\omega_0)\right| > 0$ or $\omega_1 = \omega_0$ if $\left|\mathbf{S}_g(\omega_0)\right| \leq 0$. This iterative procedure is repeated as long as $(\omega_2 - \omega_1) > \delta$ and the corresponding ω_0 value becomes the desired natural frequency. For next mode of natural frequency we begin with a new ω_1 value which is set at slightly greater value than that of previously found natural frequency (for example 1.0005 times of ω_0)and the process is repeated till the next mode of natural frequency is obtained as per requirement.

Chapter 6

Results & Discussion for Stepped Euler-Bernoulli Beam

6.1 Free-Free Beam with single step change

A free-free (F-F) beam with single step change in cross-section schematically shown in Fig. 6.1 is considered. The geometrical dimensions and material properties are [34] $L_1 = 254mm,\ L_2 = 140mm,\ b = 25.4mm,\ h_1 = 19.05mm,\ h_2 = 5.49mm,\ E = 71.7GPa$ and $\rho = 2830Kg/m^3$. The first three numbers of natural frequencies are computed by considering two numbers of spectral elements and the results are compared with those obtained by other methods in Table 6.1 . There is an excellent agreement of the SEM results with those of DQEM and FEM methods.

Table 6.1: Nondimensional natural frequencies of the single-stepped free-free beam

Mode No.	SEM (2)	DQEM [12]	FEM [12]	CEM [9]	Experimental [34]
1	292.44379	292.44	292.44	291.9	291
2	1181.31992	1181.30	1181.30	1176.2	1165
3	1804	1804.10	1804.10	1795	1771

6.2 Single-stepped beam with a circular and a rectangular cross-section having sliding-pinned boundary condition

A single-stepped beam having a circular and a rectangular cross-section as shown in the Fig.6.1 with left support sliding and right support pinned is considered.The diameter of the circular section is taken as 0.125m. Keeping the area of both the section equal, the height to width ratio of the rectangular section is fixed at 0.7. The first 15 non-dimensional natural frequencies of the stepped-beam are computed.The material and geometric properties used are $L1 = 1m$, $L2 = 1m$, diameter of circular cross-section $d = 0.125m$, $\rho A = 10Kg/m$, $EI_1 = 10000Nm^2$ where ρ and A are the density and cross-sectional area respectively of both the beam section and EI_1 is the flexural rigidity of the beam of circular cross-section. The stepped-beam was analyzed for free vibration and the non-dimensional natural frequencies $\left(\dfrac{\omega L^2}{\sqrt{EI_1/\rho A}}\right)$ obtained were compared between the SEM (2 elements) and FEM (10, 50, 100 and 400 elements) results in Table 6.2. The numbers in parenthesis represent number of elements .The FEM results converge to the SEM ones when number of finite elements are increased.

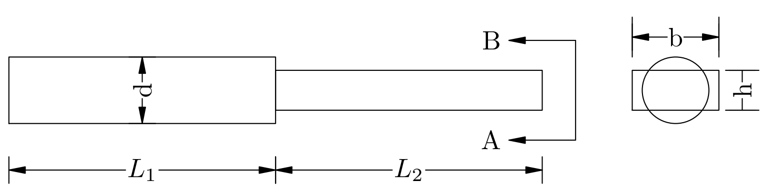

Figure 6.1: A typical stepped-beam with circular and rectangular cross sections

Table 6.2: Nondimensional natural frequencies of the single-stepped beam

Mode No.	SEM (2)	FEM (10)	FEM (50)	FEM (100)	FEM (400)
1	2.38943	2.38943	2.38943	2.38943	2.38932
2	20.19200	20.19270	20.19200	20.19200	20.19198
3	57.51455	57.52969	57.51457	57.51455	57.51454
4	111.01278	111.12345	111.01296	111.01279	111.01278
5	185.47285	185.96520	185.47368	185.47290	185.47285
6	274.92855	276.52207	274.93130	274.92872	274.92855
7	386.20370	390.37986	386.21126	386.20417	386.20370
8	511.99479	521.58200	512.01241	511.99590	511.99480
9	659.66144	677.80844	659.69911	659.66381	659.66145
10	822.24681	869.72192	822.31931	822.25137	822.24682
11	1005.82327	1065.99306	1005.95653	1005.83168	1005.82330
12	1205.69502	1325.71872	1205.92214	1205.70938	1205.69508
13	1424.69221	1611.40766	1425.06947	1424.71610	1424.69230
14	1662.32295	1941.64345	1662.91417	1662.36048	1662.32310
15	1916.29790	2332.20936	1917.21113	1916.35600	1916.29812

6.3 Simply Supported Beam with three step changes

A simply supported (SS-SS) beam with three step changes in cross-section, schematically shown in Fig.4 is analysed. The geometrical dimensions and material properties are [10] $L_1 = L_2 = L_3 = L_4 = 5m,\ h_1 = h_3 = 0.1m,\ h_2 = 0.2m,\ E = 34GPa,\ \rho = 2830kg/m^3$. The first ten numbers of natural frequencies are computed by considering four numbers of spectral elements (one for each cross-section) and the results are compared with those obtained by other methods in Table 6.3. There is an excellent agreement of the present SEM results with those of DQEM and FEM.

Table 6.3: Natural Frequencies (Hz) of three-stepped SS-SS beam

Mode No.	SEM (2)	DQEM [8]	CEM [9]	FEM [8]
1	0.43369	0.43369	0.433	0.43369
2	1.80276	1.80276	1.799	1.80276
3	4.41470	4.41470	4.411	4.41470
4	9.54133	9.54133	9.522	9.54133
5	13.26609	13.26609	13.246	13.26609
6	19.35885	19.35885	19.301	19.35885
7	25.76032	25.76032	25.721	25.76032
8	35.00419	35.00420	34.959	35.00420
9	43.21882	43.21882	43.174	43.21882
10	55.66242	55.66242	55.473	55.66242

6.4 Circular stepped beam with change of diameter ratio

A circular stepped beam is considered for different diameter ratios (d_2/d_1) where d_1 and d_2 are the diameters of first and second part of the beam respectively. The geometrical dimensions and material properties are $d_1 = 0.125m,\ L_1 = L_2 = 1m,\ EI_1 = 10000N - m2$ and $\rho A_1 = 10kg/m$. The non-dimensional fundamental mode frequencies $\left(\dfrac{\omega L^2}{\sqrt{EI_1/\rho A}}\right)$ for various boundary conditions are computed by considering two numbers of spectral elements and the results are compared with those of exact solution obtained by Jang and Bert [5]. There is an excellent agreement of the results.

Table 6.4: Nondimensional frequencies for various diameter ratios d_1/d_2 (1)

	BOUNDRY CONDITIONS							
	PI-PI		CL-CL		CL-PI		CL-FR	
d_1/d_2	SEM	[5]	SEM	[5]	SEM	[5]	SEM	[5]
0.1	0.23758	0.2376	8.91225	8.9213	6.15221	6.1522	1.40555	1.4056
0.2	0.92177	0.9218	13.27014	13.2701	11.33539	11.3354	2.78514	2.7851
0.3	1.97154	1.9715	13.28106	13.2811	12.10922	12.1092	4.01104	4.0110
0.4	3.26801	3.2680	13.66506	13.6651	11.93550	11.9355	4.82492	4.8249
0.5	4.67691	4.6769	14.66967	14.6697	11.99690	11.9969	5.06998	5.0700
0.6	6.07055	6.0706	16.17564	16.1756	12.41533	12.4153	4.90326	4.9033
0.7	7.34312	7.3431	17.90924	17.9092	13.10913	13.1091	4.56353	4.5635
0.8	8.42016	8.4202	19.61332	19.6133	13.92864	13.9286	4.18861	4.1886
0.9	9.26347	9.2635	21.12320	21.1232	14.73212	14.7321	3.83381	3.8338
1.0	9.86960	9.8696	22.37329	22.3733	15.41821	15.4182	3.51602	3.5160
1.5	10.40655	10.4066	25.98529	25.9853	16.27296	16.2730	2.43013	2.4301
2.0	9.35382	9.3538	29.33933	29.3393	14.69946	14.6995	1.83966	1.8397
2.5	8.17002	8.1700	34.16265	34.1626	12.80360	12.8036	1.47712	1.4771
3.0	7.14851	7.1485	40.03545	40.0354	11.15494	11.1549	1.23315	1.2332
3.5	6.31198	6.3120	46.44274	46.4427	9.81149	9.8115	1.05807	1.0581
4.0	5.63099	5.6310	53.07683	53.0768	8.72556	8.7256	0.92640	0.9264
4.5	5.07238	5.0724	59.75883	59.7588	7.84065	7.8406	0.82382	0.8238
5.0	4.60886	4.6089	66.35072	66.3507	7.11044	7.1104	0.74166	0.7417
5.5	4.21953	4.2195	72.67600	72.6760	6.49991	6.4999	0.67438	0.6744
6.0	3.88872	3.8887	78.40642	78.4064	5.98307	5.9831	0.61829	0.6183
6.5	3.60461	3.6046	82.95125	82.9512	5.54053	5.5405	0.57080	0.5708
7.0	3.35825	3.3582	85.82174	85.8217	5.15775	5.1577	0.53008	0.5301
7.5	3.14275	3.1428	87.33512	87.3351	4.82362	4.8236	0.49478	0.4948
8.0	2.95278	2.9529	88.12355	88.1235	4.52957	4.5296	0.46389	0.4639
8.5	2.78413	2.7841	88.56540	88.5654	4.26889	4.2689	0.43662	0.4366
9.0	2.63346	2.6335	88.83299	88.8330	4.03629	4.0363	0.41238	0.4124
9.5	2.49806	2.4981	89.00559	89.0056	3.82750	3.8275	0.39069	0.3907
10.0	2.37576	2.3758	89.12251	89.1225	3.63907	3.6391	0.37117	0.3718

Table 6.5: Nondimensional frequencies for various diameter ratios d_1/d_2 (2)

	BOUNDRY CONDITIONS							
	FR-FR		SL-SL		SL-PI		CL-SL	
d_1/d_2	SEM	[5]	SEM	[5]	SEM	[5]	SEM	[5]
0.1	1.48439	1.4844	2.24354	2.2435	0.06910	0.0691	2.23499	2.2350
0.2	3.35631	3.3563	4.48753	4.4875	0.27331	0.2733	4.39604	4.3960
0.3	5.73282	5.7328	6.55550	6.5555	0.59780	0.5978	6.11348	6.1135
0.4	8.49433	8.4943	8.05671	8.0567	1.00405	1.0040	6.84770	6.8477
0.5	11.42574	11.4257	8.77935	8.7794	1.42795	1.4280	6.77284	6.7728
0.6	14.30734	14.3073	8.99071	8.9907	1.80244	1.8024	6.42466	6.4247
0.7	16.95171	16.9517	9.05716	9.0572	2.08839	2.0884	6.07856	6.0786
0.8	19.21922	19.2192	9.18426	9.1843	2.28249	2.2825	5.82156	5.8216
0.9	21.03073	21.0307	9.45005	9.4500	2.40184	2.4018	5.66522	5.6652
1.0	22.37329	22.3733	9.86960	9.8696	2.46740	2.4674	5.59332	5.5933
1.5	24.15954	24.1595	13.55254	13.5525	2.43567	2.4357	5.69168	5.6917
2.0	22.85148	22.8515	17.55870	17.5587	2.23424	2.2342	5.46995	5.4699
2.5	21.23582	21.2358	20.14177	20.1418	2.01838	2.0184	4.90946	4.9095
3.0	19.86112	19.8611	21.41713	21.4171	1.81997	1.8200	4.31444	4.3144
3.5	18.77867	18.7787	21.99647	21.9965	1.64596	1.6460	3.79695	3.7970
4.0	17.94074	17.9407	22.26008	22.2601	1.49584	1.4958	3.37001	3.3700
4.5	17.29062	17.2906	22.38182	22.3818	1.36686	1.3669	3.02052	3.0205
5.0	16.78156	16.7816	22.43766	22.4377	1.25586	1.2559	2.73251	2.7325
5.5	16.37827	16.3783	22.46177	22.4618	1.15990	1.1599	2.49250	2.4925
6.0	16.05489	16.0549	22.47021	22.4702	1.07645	1.0765	2.29008	2.2901
6.5	15.79249	15.7925	22.47082	22.4708	1.00345	1.0035	2.11740	2.1174
7.0	15.57719	15.5772	22.46760	22.4676	0.93918	0.9392	1.96853	1.9685
7.5	15.39869	15.3987	22.46261	22.4626	0.88227	0.8823	1.83897	1.8390
8.0	15.24927	15.2493	22.45694	22.4569	0.83157	0.8316	1.72525	1.7252
8.5	15.12308	15.1231	22.45116	22.4512	0.78617	0.7862	1.62466	1.6247
9.0	15.01564	15.0156	22.44556	22.4456	0.74531	0.7453	1.53508	1.5351
9.5	14.92348	14.9235	22.44029	22.4403	0.70836	0.7084	1.45481	1.4548
10.0	14.84388	14.8439	22.43540	22.4354	0.67481	0.6748	1.38248	1.3825

Table 6.6: Nondimensional frequencies for various diameter ratios d_1/d_2 (3)

	BOUNDRY CONDITIONS							
	FR-SL		FR-PI		SL-FR	SL-CL	PI-FR	PI-CL
d_1/d_2	SEM	[5]	SEM	[5]	SEM	SEM	SEM	SEM
0.1	0.13300	0.1330	0.36110	0.3611	1.41008	0.13825	1.44366	0.36391
0.2	0.47972	0.4797	1.38260	1.3826	2.82699	0.54650	3.08264	1.42209
0.3	0.94914	0.9491	2.90759	2.9076	4.18458	1.18773	4.99729	3.06833
0.4	1.48568	1.4857	4.74762	4.7476	5.30869	1.96378	7.12053	5.12144
0.5	2.06966	2.0697	6.73611	6.7361	6.00997	2.73497	9.27974	7.34973
0.6	2.69685	2.6968	8.74628	8.7463	6.26719	3.40623	11.26845	9.52212
0.7	3.36648	3.3665	10.68413	10.6841	6.22724	3.97590	12.91732	11.46199
0.8	4.07641	4.0764	12.47694	12.4769	6.04719	4.49689	14.14278	13.08090
0.9	4.82151	4.8215	14.06801	14.0680	5.82137	5.02411	14.95371	14.37940
1.0	5.59332	5.5933	15.41821	15.4182	5.59332	5.59332	15.41821	15.41821
1.5	9.39408	9.3941	18.61899	18.6190	4.70795	9.27908	15.07657	19.28423
2.0	12.01995	12.0199	18.55947	18.5595	4.13931	13.54569	13.47222	23.99381
2.5	13.27172	13.2717	17.80133	17.8013	3.71420	17.11924	11.86906	29.83874
3.0	13.78832	13.7883	17.06655	17.0666	3.36606	19.43707	10.48480	36.14780
3.5	13.99959	13.9996	16.47795	16.4779	3.07006	20.71897	9.33140	42.43105
4.0	14.08709	14.0871	16.02593	16.0259	2.81461	21.39864	8.37593	48.27994
4.5	14.12245	14.1225	15.68010	15.6801	2.59256	21.76872	7.58065	53.19277
5.0	14.13493	14.1349	15.41318	15.4132	2.39861	21.98018	6.91301	56.67694
5.5	14.13714	14.1371	15.20451	15.2045	2.22844	22.10713	6.34705	58.74088
6.0	14.13477	14.1348	15.03911	15.0391	2.07846	22.18681	5.86260	59.86985
6.5	14.13049	14.1305	14.90623	14.9062	1.94566	22.23879	5.44408	60.50007
7.0	14.12557	14.1256	14.79814	14.7981	1.82753	22.27385	5.07943	60.87186
7.5	14.12062	14.1206	14.70917	14.7092	1.72196	22.29818	4.75921	61.10376
8.0	14.11591	14.1159	14.63516	14.6352	1.62719	22.31550	4.47600	61.25545
8.5	14.11156	14.1116	14.57300	14.5730	1.54175	22.32810	4.22389	61.35863
9.0	14.10760	14.1076	14.52032	14.5203	1.46440	22.33744	3.99813	61.43113
9.5	14.10403	14.1040	14.47531	14.4753	1.39410	22.34448	3.79488	61.48347
10.0	14.10082	14.1008	14.43658	14.4366	1.32997	22.34987	3.61097	61.52213

6.5 Beams with three step changes in cross-section

Beams with three step changes in cross-section and various boundary conditions for Type I beams with (i=i) considered by Wang and Wang [8] are analysed using the SEM. The geometrical dimensions and material properties are $L_R = 1m,\ \rho A_R = 1kg/m^3,\ EI = 1N - m^2,\ \mu_1 = 1.0,\ \mu_2 = 0.8,\ \mu_3 = 0.65,\ \mu_4 = 0.25,\ L_1 = 0.25L_R,\ L_2 = 0.3L_R,\ L_3 = 0.25L_R,\ L_4 = 0.2L_R$. The non- dimensional frequency parameters are $\alpha_i = L_R$ Fig. 5

Table 6.7: The non- dimensional frequency parameters for Type-I rectangular beam ($I_i = b_i h^3/12$)with $\mu_i = \phi_i$ (1)

Mode	BOUNDRY CONDITIONS							
	SS-SS		SS-CL		SS-FR		SS-SL	
No.	SEM	[5]	SEM	[5]	SEM	[5]	SEM	[5]
1	3.09682	3.09682	3.63486	3.63486	4.31315	4.31315	1.48529	1.48529
2	6.18383	6.18383	6.83813	6.83813	7.33089	7.33089	4.77480	4.77480
3	9.34252	9.34252	10.01235	10.01235	10.24003	10.24003	7.96461	7.96461
4	12.60534	12.60534	13.24531	13.24531	13.29720	13.29720	11.04210	11.04210
5	15.76680		15.76680		15.76680		14.04575	
6	15.81630		16.52995		16.52117		15.76680	
7	18.87773		19.74543		19.72102		17.19378	
8	21.90109		22.83102		22.80939		20.44233	
9	23.65020		23.65020		23.65020		23.65020	
10	25.04958		25.83140		25.82136		23.67958	

6.6 Cantilever twelve-stepped beam

A cantilever beam with twelve step changes in cross-section, schematically shown in Fig. 6.2, is considered. The material properties of the beam are $E = 60.6\ GPa$, $\rho = 2664Kg/m^3$. The geometrical dimensions shown in the figure are all in mm. Rest of the dimensions are; $b = 3.175mm,\ h_1 = 12.7mm,\ h_2 = 25.4mm.$

Table 6.8: The non- dimensional frequency parameters for Type-I rectangular beam ($I_i = b_i h^3/12$)with $\mu_i = \phi_i$ (2)

Mode	BOUNDRY CONDITIONS							
	CL-CL		CL-SS		CL-FR		CL-SL	
No.	SEM	[5]	SEM	[5]	SEM	[5]	SEM	[5]
1	4.54053	4.54053	3.97252	3.97252	2.28469	2.28469	2.57248	2.57248
2	7.66031	7.66031	6.99941	6.99941	5.13316	5.13316	5.63072	5.63072
3	10.80888	10.80888	10.15323	10.15323	8.08297	8.08297	8.76359	8.76359
4	14.06436	14.06436	13.41665	13.41665	10.97825	10.97825	11.78485	11.78485
5	15.76680		15.76680		14.09371		14.81653	
6	17.34903		16.59827		15.76680		15.76680	
7	20.53023		19.63297		17.33378		18.00677	
8	23.57073		22.66272		20.50527		21.25416	
9	23.65020		23.65020		23.55163		23.65020	
10	26.17735		25.85839		23.65020		24.45693	

Table 6.9: The non- dimensional frequency parameters for Type-I rectangular beam ($I_i = b_i h^3/12$)with $\mu_i = \phi_i$ (3)

Mode	BOUNDRY CONDITIONS							
	FR-FR		FR-CL		FR-SS		FR-SL	
No.	SEM	[5]	SEM	[5]	SEM	[5]	SEM	[5]
1	5.05064	5.05064	1.45296	1.45296	3.77438	3.77438	2.07263	2.07263
2	8.03610	8.03610	4.35361	4.35361	6.93279	6.93279	5.53861	5.53861
3	10.96791	10.96791	7.60522	7.60522	10.13313	10.13313	8.72535	8.72535
4	14.09853	14.09853	10.79606	10.79606	13.41982	13.41982	11.78046	11.78046
5	15.76680		14.06924		15.76680		14.82220	
6	17.33938		15.76680		16.60412		15.76680	
7	20.50769		17.35466		19.63608		18.01179	
8	23.55208		20.53263		22.66359		21.25604	
9	23.65020		23.57117		23.65020		23.65020	
10	26.17735		23.65020		25.85826		24.45709	

Both Type 1 (Flap-wise bending) beam ($I_i = h_i b^3/12$) and Type 2 (Chord-wise bending) beam ($I_i = bh_i^3/12$) are analyzed. A comparison of natural frequencies found by SEM and other methods are presented in Table 6.10 & 6.11 . The numbers in parenthesis and bracket represent number of elements and the reference number respectively. The SEM results compare well with FEM (900 elements) and DSC results. Natural frequencies of various modes found by SEM are mentioned by several boundary conditions (FR-Free, CL-Clamped, PI-Pinned and SL-Sliding) are mentioned.

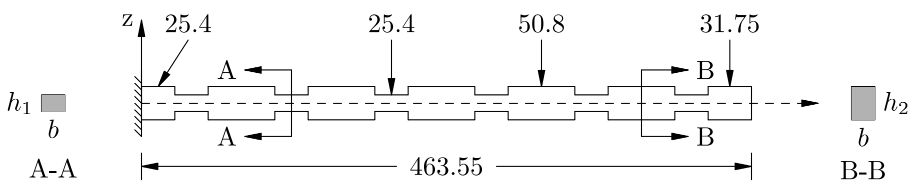

Figure 6.2: Clamped stepped-beam with twelve stepped change in cross-section

6.7 Single-stepped beam with a circular and a rectangular cross-section

The problem in Example 1 is considered for several boundary conditions (FR-Free, CL-Clamped, PI-Pinned and SL-Sliding) with different step ratios.The stepped-beam was analyzed for free vibration and the non-dimensional natural frequencies $\left(\dfrac{\omega L^2}{\sqrt{EI_1/\rho A}}\right)$obtained are presented in Table 6.14. It is observed that for a particular boundary condition, with increase in step ratio the natural frequencies also increases as the structure becomes more stiff. For a fixed step ratio the higher mode natural frequencies become nearly equal if the clamped support is replaced with the free one or vice versa.

Table 6.10: Natural frequencies of twelve-stepped clamped beam in Hz for Flap-wise Bending mode

Mode No.	SEM (2)	FEM (900)	DSC [1]	DQEM [1]	CEM [9]	Ritz [35]	CMA [35]	FEM [1]	EXP [35]
1	10.74507	10.78182	10.745	10.746	10.758	10.752	10.816	10.745	10.63
2	67.47321	67.47365	67.470	67.473	67.553	67.429	67.463	67.473	66.75
3	189.55922	189.55908	189.546	189.559				189.559	
4	373.46128	373.46168	373.426	373.461				373.460	
5	622.27380	622.27401	622.198	622.274				622.271	
10	2867.62872	2867.62869	2867.061	2867.629				2867.583	
80	207033.05279	201636.97395	2.018E+5	2.193E+5				2.016E+5	
120	472058.68468	456338.31916	3.009E+5	8.966E+5				4.562E+5	
140	648581.50503	622475.42444	5.531E+5	2.450E+6				6.222E+5	
200	1323716.20612	1271489.87715							

Table 6.11: Natural frequencies of twelve-stepped clamped beam in Hz for Chord-wise Bending mode

Mode No.	SEM (2)	FEM (900)	DSC [1]	DQEM [1]	CEM [9]	Ritz [35]	CMA [35]	FEM [1]	EXP [35]
1	54.49652	53.30129	54.496	54.495	54.699	54.795	54.985	54.499	49.38
2	344.80793	344.78381	344.793	344.808				344.807	
3	977.81252	977.82623	977.740	977.812				977.809	
4	1951.40933	1951.41769	1951.199	1951.409				1951.398	
5	3301.63914	3301.64345	3301.141	3301.639				3301.606	
10	17464.10020	17464.09993	17460.834	17464.100				17463.810	
120	2819538.69460	2819612.41278	2.538E+6	4.569E+6				2.819E+6	
140	3840285.33057	3840471.84604	3.852E+6	1.300E+7				3.838E+6	
200	8037765.48518	7885224.73548							

Table 6.12: Natural frequency (Hz) of 12 stepped beam with Flap wise (Type I) bending mode for different boundary conditions by SEM

	Mode Number					
BC	1	2	3	4	5	6
SS-SS	30.33066	121.50868	274.14638	489.35463	768.38650	1096.85753
SS-CL	47.88263	155.84275	326.91131	562.70885	865.77188	1215.20944
SS-SL	7.58536	68.34593	190.29945	374.42017	622.34332	936.63296
SS-FR	47.00734	152.33466	318.42167	546.82404	840.56000	1179.71556
SL-SS	7.57989	68.28672	190.08724	373.84259	620.85088	931.37324
SL-CL	17.33849	94.06037	233.39781	436.59905	705.55500	1048.26917
SL-SL	30.35447	121.62721	274.50280	490.27771	770.94102	1136.356
SL-FR	17.06272	92.13502	227.71211	424.68969	685.23496	1015.23593
CL-SS	47.57420	154.91703	325.19860	560.21325	862.10536	1222.58119
CL-CL	69.85864	194.08228	383.82375	640.84719	972.37651	1337.57776
CL-SL	17.20854	93.56830	232.36983	435.24088	704.67591	1052.90263
CL-FR	10.74507	67.47321	189.55922	373.46128	622.27380	942.35133
FR-SS	47.23720	152.95705	319.34302	547.53062	839.81794	1184.49502
FR-CL	10.87556	68.22281	191.38826	376.29277	625.41163	945.08048
FR-SL	17.19180	92.63076	228.79062	426.31378	687.32392	1021.36866
FR-FR	67.93730	186.85817	366.35623	607.67800	916.81307	1264.24096

Table 6.13: Natural frequency (Hz) of 12 stepped beam with chord wise (Type II) bending mode for different boundary conditions by SEM

	Mode Number					
BC	1	2	3	4	5	6
SS-SS	153.58118	616.69449	1396.17734	2496.33906	3859.09412	4930.95355
SS-CL	247.36409	812.52175	1724.14679	3001.43775	4559.55400	6754.77451
SS-SL	38.63037	348.88449	975.89760	1932.19548	3220.99213	4672.16686
SS-FR	238.09269	773.98277	1626.70779	2811.96773	4307.14790	5361.80343
SL-SS	38.35893	346.11003	966.35242	1906.72651	3156.57778	4516.71696
SL-CL	89.23032	488.00942	1223.58728	2316.76980	3767.92032	6649.31887
SL-SL	154.72654	622.12294	1412.01136	2536.80644	3962.27867	7905.37733
SL-FR	86.35536	467.26006	1159.79141	2176.99142	3528.45047	5182.77175
CL-SS	242.65662	796.87005	1689.37625	2931.11694	4397.17392	7280.80219
CL-CL	364.69000	1030.21859	2078.06938	3541.19406	6476.86031	7571.29326
CL-SL	87.88448	482.83080	1212.73063	2302.82271	3760.45543	7102.96592
CL-FR	54.49652	344.80793	977.81252	1951.40933	3301.63914	5105.41746
FR-SS	239.20688	776.54087	1628.04218	2801.91432	4240.59215	5685.56035
FR-CL	55.84962	353.04653	999.81690	1991.85638	3367.48744	5474.28624
FR-SL	87.70718	473.02081	1174.53901	2206.41607	3586.47993	5577.46452
FR-FR	344.14366	950.04271	1875.87170	3147.51455	4900.07519	5814.99309

Table 6.14: Nondimensional natural frequencies of the single-stepped beam (1)

Step Ratio	BC	Mode Number 1	2	3	4	5	6	7	8	9
0.5	CL-CL	18.71086	52.89600	100.85228	170.77356	249.96147	355.11273	466.33404	605.29767	750.54925
	FR-FR	18.45438	52.88655	100.91530	170.77049	249.95467	355.11224	466.33458	605.29773	750.54921
	CL-PI	12.85196	43.18450	86.68466	152.29281	228.18389	327.34562	437.32505	568.06362	568.06362
	FR-PI	12.15812	43.20308	86.79010	152.29845	228.17470	327.34458	437.32568	568.06373	714.26269
	PI-PI	8.16312	33.89914	74.05050	135.04588	206.40595	302.83299	405.89646	536.60414	673.11882
	SL-SL	8.52123	32.83787	76.10028	131.87949	210.65254	297.71719	411.50584	531.01403	678.08271
	SL-PI	2.28479	18.22050	52.88034	101.42435	169.41213	252.27519	351.74903	470.70220	600.17097
	CL-FR	3.43674	18.64132	52.80926	100.84101	170.77752	249.96155	355.11238	466.33406	605.29769
	CL-SL	4.94866	25.81214	63.09798	116.76026	189.31900	273.95821	382.60586	497.74692	642.35580
	FR-SL	4.13706	25.53651	63.22930	116.80355	189.31013	273.95379	382.60623	497.74729	642.35580
	SL-FR	5.37201	24.83578	64.21666	115.77311	189.89515	274.10708	381.37018	500.21132	638.51147
	SL-CL	4.69036	24.79545	64.30214	115.77040	189.89026	274.10749	381.37040	500.21129	638.51146
	PI-FR	13.41706	42.11985	87.81082	151.46459	228.44393	328.02318	435.51502	571.22883	709.74425
	PI-CL	12.97465	42.22695	87.84165	151.45618	228.44277	328.02365	435.51504	571.22881	709.74425
	FR-CL	2.57534	17.91859	53.01229	100.92385	170.76666	249.95458	355.11258	466.33457	605.29772
	PI-SL	1.86777	19.32145	51.37876	102.92139	168.38765	252.34729	353.06385	467.66614	605.11424

Table 6.15: Nondimensional natural frequencies of the single-stepped beam (2)

Step Ratio	BC	Mode Number 1	2	3	4	5	6	7	8	9
0.6	CL-CL	19.74036	55.16444	106.49119	178.51589	263.31994	372.03657	490.19468	635.47030	787.40668
	FR-FR	19.60101	55.16173	106.52503	178.51679	263.31629	372.03642	490.19497	635.47032	787.40666
	CL-PI	13.55235	45.02499	91.42061	159.55253	239.75326	343.89316	458.44917	597.84708	747.73503
	FR-PI	13.05846	45.05717	91.49024	159.55303	239.74738	343.89285	458.44958	597.84712	747.73501
	PI-PI	8.65941	35.33036	78.22291	141.09204	217.55708	317.09137	426.89219	563.07226	706.49564
	SL-SL	8.85637	34.73279	79.41750	139.15562	220.32709	313.45813	431.35120	557.89356	712.22145
	SL-PI	2.34419	19.25855	55.41563	106.45341	178.20126	264.13489	370.56366	492.39604	632.47063
	CL-FR	3.46404	19.63080	55.10098	106.47795	178.52191	263.32033	372.03624	490.19467	635.47032
	CL-SL	5.10826	27.10099	66.02969	123.12267	197.89566	288.86088	400.17934	524.30588	672.76477
	FR-SL	4.50993	26.92199	66.12808	123.14790	197.88833	288.85850	400.17975	524.30607	672.76475
	SL-FR	5.44718	26.31911	67.07401	121.94140	199.10305	287.81603	400.86195	524.14825	672.19524
	SL-CL	4.91935	26.26868	67.14511	121.94186	199.09858	287.81619	400.86217	524.14823	672.19523
	PI-FR	14.03737	44.10827	92.56718	158.33056	240.89945	343.00839	458.89739	598.04129	746.77466
	PI-CL	13.67913	44.18755	92.59637	158.32396	240.89803	343.00879	458.89744	598.04127	746.77465
	FR-CL	2.81357	19.09166	55.26716	106.53552	178.51091	263.31589	372.03675	490.19498	635.47030
	PI-SL	2.02520	20.15617	54.05472	108.10470	176.47488	265.69024	369.44180	492.82124	632.98980

Table 6.16: Nondimensional natural frequencies of the single-stepped beam (3)

Step		Mode Number								
Ratio	BC	1	2	3	4	5	6	7	8	9
0.7	CL-CL	20.61727	57.19060	111.31539	185.24625	274.99671	386.38089	511.50790	660.53779	820.93280
	FR-FR	20.55233	57.19003	111.33105	185.24644	274.99503	386.38085	511.50804	660.53780	820.93279
	CL-PI	14.16276	46.60784	95.59813	165.64692	250.15397	357.60175	477.68789	622.41051	778.30527
	FR-PI	13.83574	46.63932	95.64106	165.64552	250.15051	357.60175	477.68812	622.41052	778.30526
	PI-PI	9.07245	36.61682	81.77803	146.38078	227.25148	329.24850	445.54748	585.15693	736.73656
	SL-SL	9.16488	36.33251	82.35767	145.41573	228.68019	327.29259	448.07721	582.02546	740.47759
	SL-PI	2.38943	20.19200	57.51455	111.01278	185.47285	274.92855	386.20370	511.99479	659.66144
	CL-FR	3.48383	20.45413	57.15200	111.30278	185.25083	274.99724	386.38062	511.50788	660.53781
	CL-SL	5.25598	28.16526	68.70001	128.43249	205.60283	301.49255	415.63970	547.27562	698.83424
	FR-SL	4.84776	28.05721	68.76680	128.44619	205.59770	301.49136	415.64002	547.27571	698.83422
	SL-FR	5.50235	27.60580	69.51114	127.41209	206.80256	300.18898	416.97691	545.98803	699.98713
	SL-CL	5.12694	27.55786	69.56455	127.41430	206.79896	300.18896	416.97710	545.98803	699.98712
	PI-FR	14.52960	45.92347	96.52722	164.52579	251.41159	356.27040	479.00992	621.17934	779.36072
	PI-CL	14.26550	45.97660	96.55136	164.52123	251.41023	356.27069	479.00998	621.17933	779.36072
	FR-CL	3.03087	20.08124	57.27305	111.34088	185.24201	274.99449	386.38112	511.50806	660.53778
	PI-SL	2.16560	20.84488	56.46941	112.39351	183.82993	276.74504	384.31511	513.84371	657.97131

Table 6.17: Nondimensional natural frequencies of the single-stepped beam (4)

Step		Mode Number								
Ratio	BC	1	2	3	4	5	6	7	8	9
0.8	CL-CL	21.37596	59.04598	115.47823	191.32285	285.18883	399.15489	530.33741	682.54548	850.93612
	FR-FR	21.35409	59.04591	115.48348	191.32287	285.18827	399.15489	530.33746	682.54548	850.93612
	CL-PI	14.70035	48.01664	99.29898	170.97972	259.47923	369.45325	495.13345	643.43624	806.29105
	FR-PI	14.51570	48.03913	99.32164	170.97820	259.47750	369.45333	495.13356	643.43624	806.29105
	PI-PI	9.42210	37.79930	84.83976	151.17194	235.68922	340.11292	461.97720	604.61475	763.71244
	SL-SL	9.45334	37.70249	85.03922	150.83512	236.19679	339.40290	462.91929	603.41327	765.19797
	SL-PI	2.42505	21.03826	59.31059	115.16616	191.67450	284.82541	399.50351	530.02967	682.78446
	CL-FR	3.49882	21.14933	59.03257	115.46850	191.32573	285.18931	399.15472	530.33739	682.54549
	CL-SL	5.39426	29.06515	71.16252	132.94510	212.68755	312.29054	429.76457	567.05970	722.42061
	FR-SL	5.15758	29.00959	71.20063	132.95140	212.68462	312.29004	429.76475	567.05974	722.42060
	SL-FR	5.54454	28.73274	71.66809	132.27502	213.51896	311.31471	430.87063	565.84393	723.72839
	SL-CL	5.31890	28.69746	71.70157	132.27746	213.51659	311.31460	430.87077	565.84394	723.72838
	PI-FR	14.92926	47.59870	99.89059	170.22637	260.38384	368.40963	496.29617	642.17118	807.63483
	PI-CL	14.76573	47.62843	99.90694	170.22381	260.38282	368.40980	496.29622	642.17117	807.63482
	FR-CL	3.23149	20.93003	59.10574	115.49036	191.32014	285.18779	399.15506	530.33748	682.54547
	PI-SL	2.29239	21.43265	58.66162	116.05854	190.55349	286.15672	397.98360	531.71325	680.96512

Table 6.18: Nondimensional natural frequencies of the single-stepped beam (5)

Step		Mode Number								
Ratio	BC	1	2	3	4	5	6	7	8	9
0.9	CL-CL	22.04103	60.77116	119.10367	196.93499	294.11433	410.88708	546.90875	702.63920	877.48675
	FR-FR	22.03851	60.77115	119.10427	196.93499	294.11426	410.88708	546.90875	702.63920	877.48675
	CL-PI	15.17786	49.30082	102.59153	175.79213	267.82546	380.07501	510.84238	662.15454	831.64712
	FR-PI	15.11666	49.30958	102.59858	175.79145	267.82495	380.07505	510.84241	662.15453	831.64712
	PI-PI	9.72217	38.90150	87.50413	155.60330	243.06934	350.10530	476.41793	622.40742	787.55004
	SL-SL	9.72578	38.89027	87.52737	155.56382	243.12928	350.02071	476.53136	622.26104	787.73348
	SL-PI	2.45383	21.81033	60.88918	118.95901	197.11531	293.90064	411.13176	483.61357	546.63581
	CL-FR	3.51058	21.74357	60.78243	119.09879	196.93606	294.11454	410.88702	546.90874	702.63921
	CL-SL	5.52465	29.84140	73.44935	136.85107	219.26410	321.65565	442.87030	584.24383	744.27618
	FR-SL	5.44431	29.82465	73.46202	136.85276	219.26314	321.65553	442.87036	584.24384	744.27618
	SL-FR	5.57785	29.72762	73.62742	136.60947	219.56962	321.28699	443.30150	583.75097	744.83002
	SL-CL	5.49878	29.71322	73.63954	136.61071	219.56873	321.28691	443.30155	583.75098	744.83002
	PI-FR	15.25988	49.15503	102.80173	175.51842	268.16257	379.67497	511.30448	661.63103	832.23097
	PI-CL	15.20111	49.16462	102.80805	175.51761	268.16214	379.67502	511.30450	661.63103	832.23097
	FR-CL	3.41838	21.66806	60.80792	119.10624	196.93408	294.11404	410.88714	546.90877	702.63920
	PI-SL	2.40800	21.94769	60.66058	119.27842	196.70567	294.39978	410.54400	547.31116	702.17615

Table 6.19: Nondimensional natural frequencies of the single-stepped beam (6)

Step Ratio	BC	Mode Number 1	2	3	4	5	6	7	8	9
1.0	CL-CL	22.63066	62.39111	122.29155	202.18564	301.98526	421.84294	561.54400	721.37519	900.96735
	FR-FR	22.62908	62.39111	122.29193	202.18564	301.98522	421.84294	561.54401	721.37519	900.96735
	CL-PI	15.60503	50.49129	105.53370	180.23072	275.29694	389.84864	524.92857	679.34034	854.42533
	FR-PI	15.65221	50.48369	105.52859	180.23136	275.29728	389.84859	524.92855	679.34034	854.42533
	PI-PI	9.98260	39.93838	159.75184	249.57438	359.44034	489.16701	639.00385	808.62421	998.44232
	SL-SL	9.98485	39.93139	159.72726	249.61171	359.38763	489.23772	638.91253	808.73872	998.30203
	SL-PI	2.47756	22.51834	62.30684	122.42850	202.00623	302.20789	421.57611	561.85620	721.01675
	CL-FR	3.52004	22.25688	62.42597	122.29322	202.18493	301.98503	421.84299	561.54402	721.37519
	CL-SL	5.64824	30.52251	75.58228	140.29207	225.39185	329.92163	455.08001	599.41846	764.64050
	FR-SL	5.71154	30.53422	75.57254	140.29104	225.39257	329.92169	455.07996	599.41846	764.64050
	SL-FR	5.60481	30.61183	75.44018	140.48589	225.14686	330.21755	454.73336	599.81539	764.19362
	SL-CL	5.66891	30.62501	75.43011	140.48458	225.14762	330.21764	454.73332	599.81539	764.19363
	PI-FR	15.53772	50.60732	105.36595	180.45012	275.02644	390.16998	524.55676	679.76231	853.95372
	PI-CL	15.58642	50.60024	105.36038	180.45069	275.02684	390.16995	524.55673	679.76232	853.95372
	FR-CL	3.59373	22.31724	62.40559	122.28727	202.18652	301.98543	421.84289	561.54399	721.37520
	PI-SL	2.51420	22.40850	62.48974	122.17275	202.33456	301.80734	422.04844	561.31256	721.63111

Table 6.20: Nondimensional natural frequencies of the single-stepped beam (7)

Step		Mode Number								
Ratio	BC	1	2	3	4	5	6	7	8	9
2.0	CL-CL	26.33438	74.94788	141.91599	241.49850	352.39796	501.32691	658.50441	853.33035	1061.08569
	FR-FR	25.86262	74.92556	142.03271	241.50566	352.38541	501.32577	658.50539	853.33048	1061.08563
	CL-PI	18.25610	59.76135	123.62687	214.37287	321.61133	463.91635	613.79441	806.82376	1001.65612
	FR-PI	18.94347	59.58855	123.58308	214.38596	321.61271	463.91565	613.79439	806.82380	1001.65612
	PI-PI	11.44855	48.05417	104.16441	191.06531	290.86433	427.69737	572.94652	756.71758	951.39395
	SL-SL	12.10204	46.14643	107.76730	185.67781	297.76708	419.90297	580.71194	750.09829	955.68593
	SL-PI	2.59303	27.33682	72.50954	145.08782	238.12501	355.34492	499.52934	658.55827	855.74201
	CL-FR	3.56329	25.07734	75.11886	142.04256	241.49530	352.38553	501.32626	658.50535	853.33046
	CL-SL	6.62862	34.84322	90.96589	163.24876	267.94888	387.27807	537.37751	707.45685	899.20207
	FR-SL	7.71288	34.88823	90.83509	163.25525	267.95594	387.27731	537.37722	707.45690	899.20208
	SL-FR	5.72964	35.92200	164.39305	267.91808	385.76602	540.95022	701.73479	906.96907	1113.71776
	SL-CL	7.05307	36.39833	164.31436	267.93105	385.77435	540.94985	701.73408	906.96904	1113.71781
	PI-FR	16.95132	61.16215	122.31888	215.01466	322.25192	461.37669	618.41021	800.04220	1010.38578
	PI-CL	18.10301	61.15412	122.13795	215.00085	322.26774	461.37899	618.40917	800.04195	1010.38584
	FR-CL	4.95106	26.25117	74.80893	141.90223	241.50904	352.39782	501.32642	658.50445	853.33037
	PI-SL	3.25952	25.65023	74.63713	143.32215	238.64863	495.26599	665.87906	845.41217	1068.89870

Table 6.21: Nondimensional natural frequencies of the single-stepped beam (8)

Step		Mode Number								
Ratio	BC	1	2	3	4	5	6	7	8	9
3.0	CL-CL	28.41655	83.42371	153.62161	264.59411	386.90395	543.62553	728.37203	922.29456	1172.88103
	FR-FR	27.18683	83.29788	153.93444	264.63192	386.87308	543.61938	728.37380	922.29533	1172.88099
	CL-PI	19.54049	66.60476	132.97496	237.46650	348.93087	508.34568	673.29950	875.89370	1106.68954
	FR-PI	20.52034	66.32609	132.93829	237.48401	348.93064	508.34496	673.29957	875.89372	1106.68953
	PI-PI	12.07935	53.69189	112.39021	210.07469	318.39108	464.70622	633.03301	817.95975	1052.55745
	SL-SL	13.69846	49.38667	119.39299	201.86560	325.23210	462.19853	628.74024	830.16643	1033.28806
	SL-PI	2.63502	29.97477	79.95502	156.62907	264.23515	383.60995	551.54949	716.39854	935.96469
	CL-FR	3.57794	26.23957	83.56572	153.92854	264.61990	386.87434	543.61979	728.37373	922.29532
	CL-SL	7.32114	37.51619	99.83308	179.30768	290.21931	428.89751	580.31702	782.22654	975.79123
	FR-SL	9.04394	37.46597	99.66134	179.32751	290.22610	428.89618	580.31688	782.22660	975.79123
	SL-FR	5.77254	38.33957	99.68296	177.17080	296.04173	419.69581	591.16288	771.57890	983.10249
	SL-CL	8.11877	39.39251	99.37880	176.95714	296.04430	419.72046	591.16536	771.57703	983.10207
	PI-FR	17.48748	67.38550	133.93679	233.42865	356.50201	498.19286	684.56842	866.64634	1111.04561
	PI-CL	19.69857	67.56013	133.55465	233.36097	356.53099	498.20352	684.56743	866.64515	1111.04556
	FR-CL	5.91181	28.32273	83.21237	153.62299	264.60633	386.90268	543.62513	728.37210	922.29457
	PI-SL	3.70492	27.95964	80.95526	257.30450	395.42729	536.37237	731.85870	924.20496	1163.94268

Chapter 7

Results & Discussion for Stepped Timoshenko Beam

7.1 Single stepped Timoshenko beam

A single stepped beam as shown in Fig. 7.1, is considered for the analysis.The material and geometrical properties consider from [36] are $\eta = 0.0036$, $\gamma_L = 0.25$, $\gamma_b = 0.8$ and $\gamma_h = 0.6$. Natural frequencies coefficient $\left(\dfrac{\omega L^2}{\sqrt{EI_1/\rho A}}\right)$ obtained by SEM (2 elements) shows excellent agreement with those found by [36].

Table 7.1: Nondimensional natural frequencies of the single-stepped Timoshenko beam

Mode No.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
Present SEM	4.9435	23.2090	49.8055	89.1255	143.8718
Gutierrez [36]	4.943	23.209	49.805	89.126	143.872

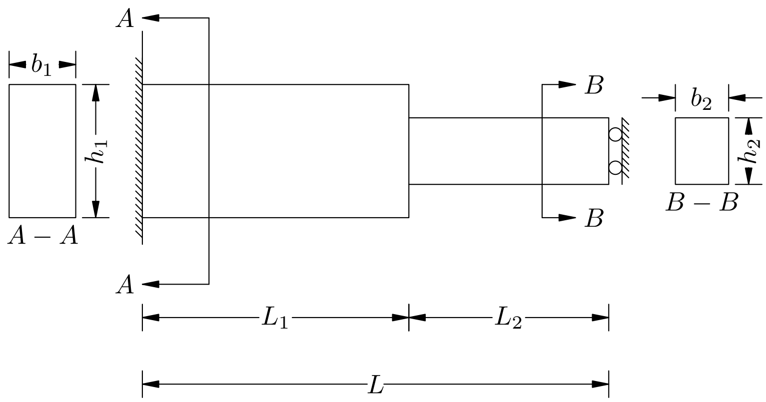

Figure 7.1: Single stepped Timoshenko Beam

7.2 Percentage error in natural frequencies by Euler-Bernoulli beam theory when applied to a single stepped Timoshenko beam

A single stepped cantilevered beam clamped at left end and free at other end, on Fig. 7.1 is considered. The material and geometrical properties in Example 1 are used for analysis using both Euler-Bernoulli beam theory and Timoshenko beam theory. The difference in natural frequencies coefficient along with percentage error in Euler-Bernoulli beam theory application are mentioned in the following table.

7.3 Single stepped cantilevered beam

A single stepped cantilevered beam Fig. 7.2 with parameters [37] & [38] $h_1/h_o = 0.8$, $L_1/L = 2/3$, $\kappa = 5/6$ and $\nu = 0.3$. The non dimensional natural frequencies $\Omega_i = \left(\dfrac{\omega_i L^2}{\sqrt{EI_1/\rho A}}\right)$ obtained compared between the SEM (2 elements) and previously found results show excellent agreement .

Table 7.2: Comparison of Non-dimensional natural frequencies with Timoshenko & Euler-Bernoulli Beam theories for the same beam

Mode No.	Timoshenko	Euler-Bernoulli	Error %
Ω_1	3.2107	3.2148	0.1289
Ω_2	17.5554	17.7031	0.8412
Ω_3	42.3157	43.0725	1.7884
Ω_4	77.7554	80.0712	2.9784
Ω_5	128.9967	135.13752	4.7604
Ω_6	191.5018	205.5301	7.3254
Ω_7	257.0197	283.60479	10.3436
Ω_8	328.0458	368.7040	12.3941
Ω_9	414.2966	474.20011	14.4591
Ω_{10}	511.1571	601.72708	17.7186
Ω_{11}	607.6292	742.29711	22.1628

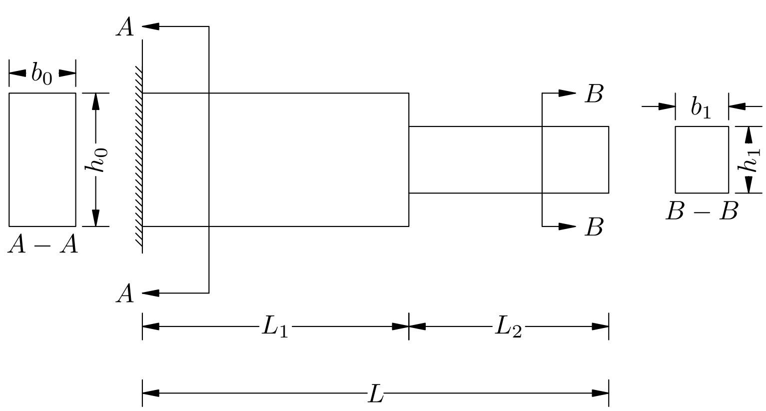

Figure 7.2: Clamped Timoshenko Stepped Beam

Table 7.3: Comparision of nondimensional natural frequencies by different methods

r_0	Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0.0133	Present SEM	3.8243	21.3559	55.0510	107.5298	173.6753
	Tong (1995) [38]	3.8219	21.3540	55.0408	107.4993	173.6322
	Rossi (1990) [37]	3.82	21.35	55.04	107.50	173.62
0.0267	Present SEM	3.8047	20.7275	51.6754	96.3656	148.9066
	Tong (1995) [38]	3.8034	20.7283	51.6851	96.3918	148.9651
	Rossi (1990) [37]	3.80	20.72	51.68	96.39	148.97
0.0400	Present SEM	3.7730	19.8047	47.3531	84.1407	125.0650
	Tong (1995) [38]	3.7716	19.8036	47.3540	84.1399	125.0681
	Rossi (1990) [37]	3.77	19.80	47.35	84.14	125.06

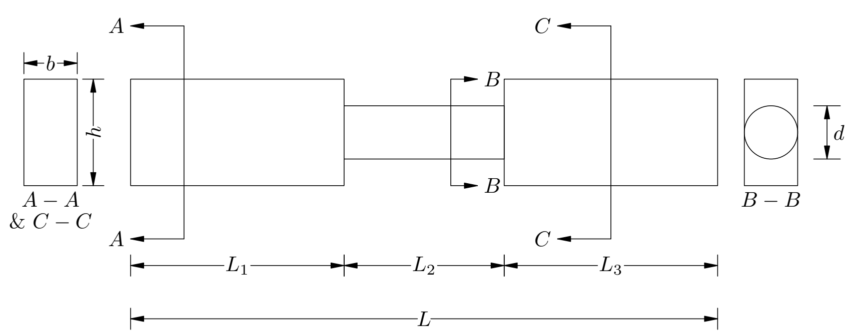

Figure 7.3: Two-stepped Timoshenko Stepped Beam

7.4 Cantilever two-stepped beam

A cantilever beam with two stepped changes in cross-section, schematically shown in Fig. 7.3, is considered. The parameters are total length $L = 1.8$, $\kappa = 5/6$ and $\nu = 0.3$. The cross-sectional areas of both rectangular and circular cross-section are same . The non dimensional natural frequencies $\Omega_i = \left(\frac{\omega_i L^2}{\sqrt{EI_1/\rho A}}\right)$ obtained with various percentage of length of intermediate circular beam $\left(P = \frac{L_2}{L} * 100\right)$. The increase in the value of P form 0 to 90 decreases the non-dimensional natural frequencies.For the lower mode the decrease is less and for higher modes its increases.

Table 7.4: Nondimensional natural frequencies for change in percentage of length

P	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0	3.4633	20.0066	50.5242	88.1030	129.8129	173.7170
10	3.3800	18.3705	50.3999	83.8436	129.0917	168.9506
20	3.2962	17.3046	49.2591	82.8543	124.3630	169.1636
30	3.2079	16.6559	46.8685	82.9772	119.9583	166.1849
40	3.1127	16.2945	44.1042	81.8293	119.1907	160.2150
50	3.0097	16.1055	41.9238	78.4673	118.8456	158.4439
60	2.8991	15.9795	40.6661	74.5483	115.4143	157.9293
70	2.7821	15.8155	40.1572	71.9811	110.4238	153.3781
80	2.6605	15.5343	39.9178	71.1479	107.5236	147.9956
90	2.5365	15.0958	39.4041	70.9209	107.2518	146.7321

7.5 Two stepped cantilever beam with change of circular cross-section location

The same beam in Example 4 is analysed for different boundary conditions considering the length percentage 20% with changing the location of circular cross-section.Plots of non-dimensional natural frequencies with change in location of circular cross-section i.e. the distance of first change in cross-section from left end are shown.

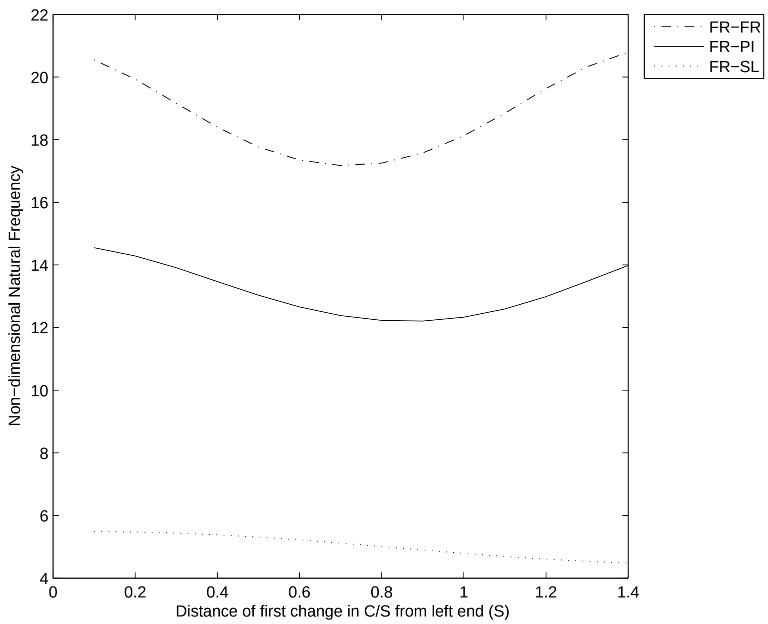

Figure 7.4: Non-dimensional natural frequencies for first mode (1)

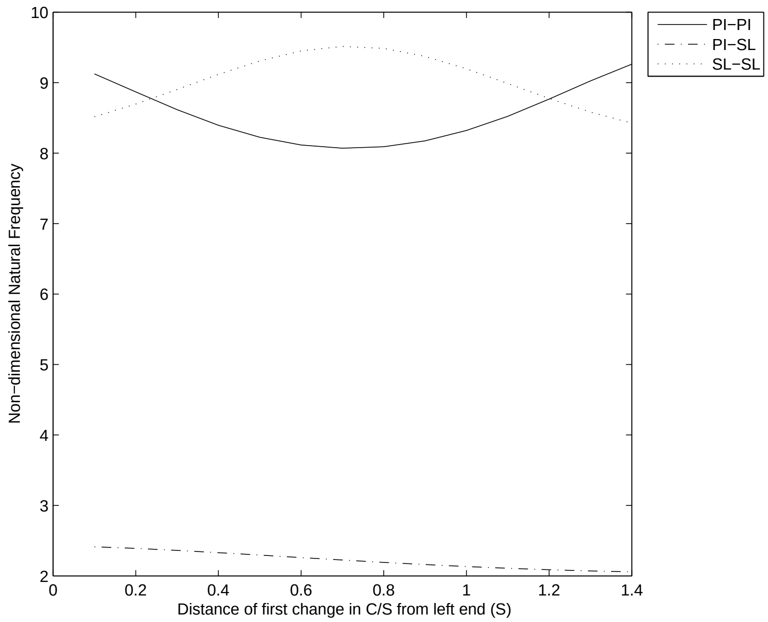

Figure 7.5: Non-dimensional natural frequencies for first mode (2)

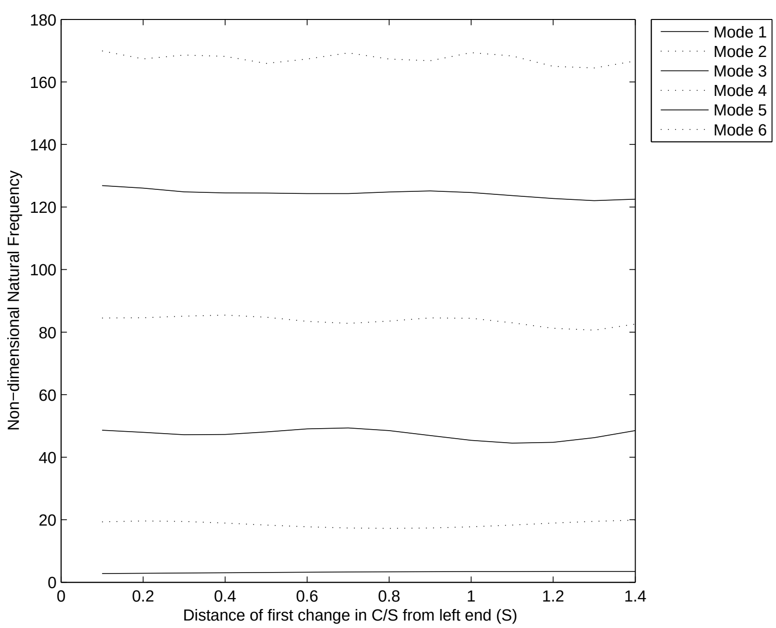

Figure 7.6: Non-dimensional natural frequencies for Cantilevered

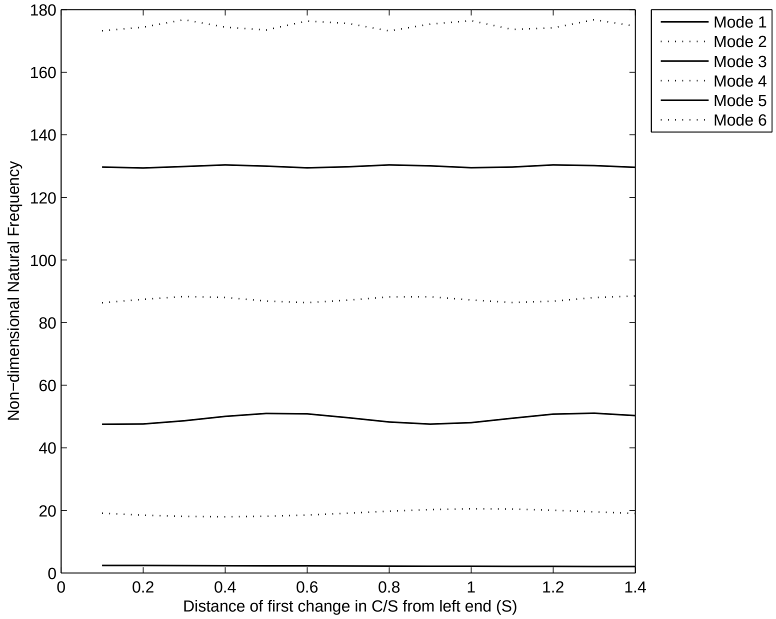

Figure 7.7: Non-dimensional natural frequencies for Pinned-Sliding

Chapter 8

Conclusion

In the whole work the total number of element used in the analysis using SEM is always two. For the same result using other methods the elements number are considerably large. Minimum number of elements and least degree-of-freedom is used in SEM. Thus SEM improves the solution accuracy and efficiency to a considerable extent.

The final conclusion from this research work are summarized as follows:

1. There is excellent agreement between SEM and many other methods along with the experimental results as well . Therefore SEM is a valid method for free vibration analysis of stepped beams.

2. Convergence of FEM is studied comparing with that found using two elements of SEM. For the first few fundamental frequencies the FEM with lesser number of elements shows closer results to SEM. But for higher frequencies FEM needs very high number of elements for good agreement with SEM.

3. The natural frequencies found by DQEM is vary close to SEM.

4. Circular single stepped beam considered using SEM is compared with those found by Jang and Bert [5] which exhibit exact match. The results for the same along with some other boundary conditions are presented in Table

6.4-6.6. With increasing diameter ratio the natural frequencies for FR-FR, SL-SL ,FR-SL , FR-PI , SL-CL and PI-CL boundary conditions increases. In the other hand for PI-FR, SL-FR, CL-SL, SL-PI ,CL-FR and CL-PI it increases for first few step ratio and then decreases.

5. Type I beam with three step changes in cross-section used in example 6.5 by Wang and Wang [8] shows exact match with SEM. Higher mode natural frequencies obtained are presented.

6. Cantilever twelve-stepped beam analysed for both Flap-wise and Chord-wise bending for higher modes up to 200 shows good agreement with those of FEM using 900 elements. First six natural frequencies for different boundary conditions are mentioned.

7. In case of single-stepped beam with a circular and a rectangular cross-section (example 6.7), the nondimensional natural frequencies increases with increase in step ratio.

8. Table 7.2 of (example 7.2) shows the percentage error in natural frequencies by Euler-Bernoulli beam theory when applied to a single stepped Timoshenko beam. The percentage of error increases with increase in mode number.

9. In (example 7.4) the increase in the percentage of length of intermediate circular beam from 0 to 90 decreases the non-dimensional natural frequencies.For the lower mode the decrease is less and for higher modes it increases.

10. With the change of location of circular cross-section in example 7.5 no advantage is obtained. There is vary minimal change in the natural frequencies.

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