Thesis

Lie group analysis and evolution of weak waves for certain hyperbolic system of partial differential equations

by

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DEPARTMENT OF MATHEMATICS
NATIONAL INSTITUTE OF TECHNOLOGY ROURKELA
ROURKELA

2014

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Lie group analysis and evolution of weak waves for certain hyperbolic system of partial differential equations

A Thesis Submitted

in Partial Fulfilment of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

by

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Under the supervison of

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and

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to the
DEPARTMENT OF MATHEMATICS
NATIONAL INSTITUTE OF TECHNOLOGY ROURKELA
Rourkela, 2014

DECLARATION

It is certified that the work contained in the thesis titled **"Lie group analysis and evolution of weak waves for certain hyperbolic system of partial differential equations"** has done by me under the guidance of Dr. Snehashish Chakraverty, Head & Professor, Department of Mathematics, National Institute of Technology Rourkela and Dr. Raja Sekhar Tungala, Assistant Professor, Department of Mathematics, Indian Institute of Technology Kharagpur for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

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CERTIFICATE

It is certified that the work contained in the thesis titled **"Lie group analysis and evolution of weak waves for certain hyperbolic system of partial differential equations"** by **Bibekananda Bira**, a student in the Department of Mathematics, National Institute of Technology Rourkela for the award of the degree of Doctor of Philosophy has been carried out under our supervision and this work has not been submitted elsewhere for a degree.

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Dedicated

to

my Parents

Acknowledgements

First of all, I wish to express my sincere gratitude and gratefulness to my thesis co-supervisor **Dr. Raja Sekhar Tungala**, Assistant Professor, Department of Mathematics, Indian Institute of Technology Kharagpur for his effective guidance and constant inspirations throughout my research work. His tireless working capacity, devotion towards research work as well as his clarity of presentation have strongly motivated me. Apart from the academic support, his friendly camaraderie even helps me to reach the divine feet of the almighty.

I would also like to thank Prof. Snehashish Chakraverty, Head, Department of Mathematics, National Institute of Technology Rourkela for his support and cordial cooperation for the successful submission of the thesis.

I thank Prof. G.K. Panda, Prof. K.C. Pati, Prof. S. Meher and Prof. S. Murugan for their cooperation and useful suggestions during the course of my work.

I am highly thankful to all the faculty members as well as office bearer of the Department of Mathematics, National Institute of Technology Rourkela for their valuable inspiration and needful help to complete this work.

I would like to thank National Institute of Technology Rourkela for the facilities provided to me during my research work and to Ministry of Minority Affairs, Govt. of India (Maulana Azad National Fellowship for minority students), for providing me financial assistance for the completion of my thesis work.

I would like to acknowledge my deepest sense of gratitude to my mother, elder brother, sisters and brother-in-laws who has helped me morally and have given mental strength to face all the problems throughout my academic career.

I would like to give the special thanks to my wife Mrs. Padmini, for understanding and supporting me to reach the landmark.

Many friends have helped me stay sane through these difficult years. Their support and care helped me overcome setbacks and stay focused on my research work. I greatly value their friendship and I deeply appreciate their belief in me.

Date: **Bibekananda Bira**

Abstract

In the present thesis, we study the applications of Lie group theory to system of quasilinear hyperbolic partial differential equations (PDEs), which are governed by many physical phenomena and having various important physical significance in the real life. Our primary objective in this thesis is to identify the symmetries of system of PDEs in order to obtain certain classes of group invariant solutions. The investigations carried out in this thesis are confined to the applications of Lie group method to the system of quasilinear hyperbolic PDEs arising in magnetogasdynamics, two phase flows and other scientific fields. We organize the whole thesis into 7 chapters, described as follows.

First chapter is introductory and deals with a short background history of Lie group of transformations and symmetries along with some of their important features which are of great importance in the work of proceeding chapters and the motivation behind our interest. In the second chapter, we obtain exact solutions to the quasilinear system of PDEs, describing the one dimensional unsteady simple flow of an isentropic, inviscid and perfectly conducting compressible fluid, subjected to a transverse magnetic field. Lie group of point transformations are used for constructing similarity variables which lead the governing system of PDEs to system of ordinary differential equations (ODEs); in some cases, it is possible to solve these equations exactly. A particular solution to the governing system, which exhibits space-time dependence, is used to study the evolutionary behavior of weak discontinuities.

The next chapter deals with system of PDEs, governing the one dimensional unsteady flow of inviscid and perfectly conducting compressible fluid in the presence of magnetic field. For this, Lie group analysis is used to identify a finite number of generators that leave the given system of PDEs invariant. Out of these generators, two commuting generators are constructed involving some arbitrary constants. With the help of canonical variables associated with these two generators, the assigned system of PDEs is reduced to an autonomous system whose simple solutions provide nontrivial solutions of the original system. Using this exact solution, we discuss the evolutionary behavior of weak discontinuity.

The aim of chapter 4, is to carry out symmetry group analysis to obtain important classes of exact solutions from the given system of quasilinear PDEs. Lie group analysis is employed to derive some exact solutions of one dimensional unsteady flow of an ideal isentropic, inviscid and perfectly conducting compressible fluid, subject to a transverse magnetic field for the magnetogasdynamics system. By using Lie group theory, the full one-parameter infinitesimal transformations group leaving the equations of motion invariant is derived. The symmetry generators are used for constructing similarity variables which leads the system of PDEs to a reduced system of ODEs and obtained the solution for some cases. Further, we discuss the evolutionary behavior of weak discontinuity along one of the solution curve.

Chapter 5 concerns with a quasilinear hyperbolic system of PDEs governing unsteady planar and radially symmetric motion of an inviscid, perfectly conducting and non-ideal gas in which the effects of magnetic field is significant. A particular exact solution, to the governing system, which exhibits space-time dependence, is derived using Lie group symmetry analysis. The evolutionary behavior of weak discontinuity across the solution curve is discussed. Further, the evolution of characteristic shock and the corresponding interaction with the weak discontinuity is studied. The amplitudes of reflected wave, transmitted wave and the jump in shock acceleration influenced by the incident wave after interaction are evaluated. Finally, the influence of van der Waals excluded volume in the behavior of weak discontinuity is completely characterized.

Chapter 6 presents an analytical solution for the drift-flux model of two-phase flows using Lie group analysis. The analysis involves an isentropic no-slip conservation of mass for each phase and the conservation of momentum for the mixture. The present analysis employs a complete Lie algebra of infinitesimal symmetries. Subsequent to these theoretical analysis a symmetry group is established. The symmetry generators are used for constructing similarity variables which reduces the model equations to a system of ODEs. In particular, a general framework is discussed for solving the model equations analytically. As a consequence of this, new classes of exact group-invariant solutions are developed. This provides new insights into the fundamental properties of weak discontinuities and helps to understand situations under which solutions exist.

In the last chapter, we present the brief discussion of the conclusions and results along with future plans.

Keywords: Hyperbolic systems; Magnetogasdynamics; Lie group of transformations; Similarity solutions; Exact solutions; Weak discontinuity; Two-phase flows; Characteristic shock; van der Waals gas; Interactions

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Chapter 1

Introduction

Many physical problems in this universe are modelled by hyperbolic system of PDEs in the form of either conservation laws, or balance laws. In the recent past, study of hyperbolic systems of PDEs has been the subject of great interest both from mathematical and physical point of view due to it's applications in variety of fields such as magnetogasdynamics, astrophysics, engineering physics, multi phase flow models, aerodynamics and plasma physics etc. The most significant behavior of the solution of such system of quasilinear hyperbolic PDEs lies in the fact that a smooth solution breaks down within a finite span of time. The breaking of these smooth solutions gives rise to one of the most interesting nonlinear phenomena that occur in nature, i.e. the appearance of shock waves, which are abrupt jumps in pressure, density and velocity. A shock is indeed, an admissible discontinuity which satisfies Rankine-Hugoniot jump conditions and the entropy condition. The interaction of waves within the context of quasilinear hyperbolic system of PDEs is another interesting feature. Today many engineering and science researchers routinely studying the problems of interaction between weak discontinuity and shock waves to gain a better understanding of such nonlinear phenomena.

The explicit determination of exact solutions to such system of nonlinear PDEs are of great interest and is an important task. Unlike linear systems, where the exact solutions are often easy to derive by treating several techniques, we do not have the luxury of complete exact solutions for such nonlinear PDEs. For such type of problems we rely on some approximate analytical and numerical methods which may be useful to set the scene and provide useful information towards our understanding of the complete physical phenomena involved. To solve such system of nonlinear PDEs, there is no such general theory is available in the literature and it is also very difficult to construct their exact solutions systematically.

In view of this specific interest, it is desirable to undertake a systematic investigation and develop a comprehensive fundamental analysis to solve such models. Unfortunately, finding solutions for such nonlinear PDEs is an arduous task. Lie group analysis, based on symmetry and invariance principles (see, Ames et. al. [3], Bluman and Cole [10], Hydon [39]), is one of the most powerful and systematic method for solving nonlinear differential equations analytically. Symmetries of the differential equations are pivotal to a profound understanding of the physics of the underlying the problems under investigations. Symmetry group analysis of differential equations on the basis of Lie groups unify a wide variety of ad-hoc methods to analyze and solve differential equations exactly. The applications of continuous groups to differential equations make no use of the global aspects of Lie groups. Lie's fundamental theorem shows that such group is completely characterized by their infinitesimal generators. In turn these form a Lie algebra determined by structure constants (see, Kac [46]). An infinitesimal transformation is a limiting form of small continuous transformation group which is applied to solve differential equations. From the Lie's first fundamental theorem of Lie group of transformations, infinitesimal transformation is equivalent to infinitesimal generator.

Lie group of transformations and hence their infinitesimal generators, can be naturally extended or prolonged to act on the space of independent variables, dependent variables and derivatives of the dependent variables up to any finite order. As a consequence, the seemingly intractable nonlinear conditions of group invariance of the given system of differential equations reduced to linear homogeneous equations determining the infinitesimal generators of the group. Since these determining equations form an over determined system of linear homogeneous partial differential equations, one can determine the infinitesimal generators in closed form. For a given system of differential equations, the setting up of the determining equations is entirely routine. To solve the determining equations explicitly, power series method is used.

The mathematical discipline known today as the Lie group analysis, was originated by an outstanding mathematician of 19th century Sophus Lie [51]. According to Lie, it was during the winter of 1873-74 that his theory of groups was born. The creation of theory of continuous groups are related to Lie's prodigious research activity during the four year period from the fall of 1869 to the fall of 1873. It was during this period that continuous groups of transformations and infinitesimal transformations entered his work through geometrical considerations and through related interest in differential equations [5, 38]. The theory of Lie groups and their representations is a vast subject with an extraordinary range of applications [15]. We find them in diverse roles, in many major areas of mathematics and mathematical physics. Despite of its important features, the Lie's approach to differential equations was not exploited for half a century and only the abstract theory of Lie groups grew. In the late 1950, it was developed to an advanced state through the pioneering efforts of Ovsiannikov [68]. Contemporarily the group-theoretic problems were first posed and new applications of group theory were being developed by a number of researchers including Bluman and Anco [8], Cantwell [17], Stephani [94]. Lie introduced the notion of continuous groups, now known as Lie groups, in order to unify and extend various specialized solution methods for ODEs as well as PDEs (see, Ibragimov [40], Ovsiannikkov [67]).

Although today Sophus Lie is rightfully recognized as the creator of the theory of continuous groups, Lie's ideas did not stand in isolation from the rest of mathematics. Weyl brought the early period of the development of the theory of Lie groups to fruition and put Lie's theory itself on firmer footing by clearly enunciating the distinction between Lie's infinitesimal groups (see, Hawkins [37]). The theory of Lie groups was systematically reworked in modern mathematical language in a monograph by Chevalley [20].

Since the time of Lie, many mathematicians and scientists have used this method and extension of this method to solve ODEs and PDEs. Bluman and Cole [9] in the year 1969, developed an extension of Lie's symmetries method called the nonclassical method and used to obtain new exact solutions of heat equation. The reductions of Bossinesq equation and Fitzhugh-Nagumo equation using the nonclassical method can be found in the work of Burzon and Gandarias [16], Clarkson and Kruskal [22], Levi and Winternitz [50], Nucci and Clarkson [61]. In the year 1987, Olver and Rosenau [65, 66], used the nonclassical method to construct some special solutions, i.e. group invariant solutions for the PDEs. However, their frame work has proved to be too general to be practical but they concluded that: the unifying theme behind finding special solutions of PDEs is not as commonly supposed, group theory but rather the more analytic subject of over-determined systems of PDEs (see, Olver [65]). The use of Lie groups of point symmetries in order to construct a mapping which transforms a given differential equation to another differential equation in the sense that any solution of the source differential equation is mapped into a solution of the target differential equation is a well known procedure widely applied in literature (see, Clarkson and Mansfield [23], Dickson [28], Gandarias and Bruzon [35], Mekheimer et al. [54], Moitsheki [55], Migranov and Tomchuk [56], Chaharborj et al.[87]).

Today Lie group theoretical approach to differential equations has been extended to new situations and has become applicable to the majority of equations that frequently occur in applied sciences. Some of the greatest mathematicians and physicists of twentieth century have created the tools of the subject that we all use. This technique has been applied by many researchers to solve different flow phenomena over different geometries. The analysis of the nonlinear PDEs through Lie group analysis is well illustrated in the book of Hermann [38] and Anderson et. al. [4]. The symmetries admitted by given PDEs enable us to look for appropriate canonical variables which transform the original system to an equivalent one whose simple solutions provide nontrivial solutions of the original system (see, Ames and Donato [2], Donato and Ruggeri [30], Oliveri and Speciale [62, 63], Razvan and Ozer [83]). The application of Lie groups to gasdynamics, magnetogasdynamics and Euler equations may be found in the work of Pandey et. al. [69], Radha and Sharma [74], Raja Sekhar and Sharma [80], Sharma and Radha [88].

This thesis comprises five problems, which are briefly described as follows. The second chapter presents the Lie group analysis for the quasilinear system of partial differential equations (PDEs), describing the one dimensional unsteady simple flow of an isentropic, inviscid and perfectly conducting compressible fluid in the presence of magnetic field. Since the system of equations has at most first-order derivatives, the first prolongation of the generator is used to investigate the infinitesimal transformations under which the governing system of equations remains invariant. For distinguished cases, similarity variables and similarity forms of field variables are constructed which are further used to reduce the system of PDEs to the system of ODEs. The reduced system of ODEs are solved analytically and the solutions of original system of PDEs are obtained. One of the solution is used to discuss the evolutionary behavior of the weak discontinuity.

The main aim of next chapter is to identify a finite number of generators that leave the given system of PDEs invariant by using the analysis mentioned in [73, 80]. Out of these generators, two commuting generators are constructed involving some arbitrary constants. Then we introduce some canonical variables and solve the characteristic conditions associated with the generator to obtain the transformation of variables which transform the given system of PDEs to an autonomous system of PDEs [29, 62, 63]. The autonomous system of PDEs is solved by using compatible condition and the solution of the governing system of PDEs is derived. Further, the evolution of weak discontinuity for a hyperbolic quasilinear system of equations satisfying the Bernoulli's law has been studied quite extensively and it is found that the presence of magnetic field enhances the decay rate of weak discontinuity and reduces the shock formation time as compared to what it would be in absence of magnetic field.

The next chapter is all about the explicit determination of exact solutions to system of PDEs governing one dimensional unsteady flow of an ideal isentropic, inviscid and perfectly conducting compressible fluid, subject to a transverse magnetic field for the magnetogas-dynamics. Besides its own intrinsic interest, these solutions may be used for modeling, designing and testing numerical procedures for solving special initial and/or boundary value problems. Finally, we study the evolutionary behavior of weak discontinuity across one of the solution curve.

The fifth chapter concerns with the evolution of characteristic shock, weak discontinuity and their interaction in a non-ideal gas, where the non-ideal gas obeys the va der Waals equation of state. Here first the governing system of non autonomous PDEs is reduced into an autonomous form via Lie group analysis [31]. The autonomous system of PDEs is solved and the particular exact solution of the original system is obtained, in which one of the solution exhibits linear dependence with the particle velocity on the spatial coordinate [21, 71, 90]. Indeed, Pert studied the usefulness of this form of the solution in modeling the free expansion of polytropic gases and is attained in the limit of large time. Further, this solution is used to discuss the evolutionary behavior of weak discontinuity. The effect of van der Waals excluded volume on the evolutionary behavior of weak discontinuity is studied in detail. It has been noticed that the presence of the van der Waals excluded volume enhances the decaying of an expansion wave, whereas it fastens the decaying rate of the compression wave as compared to what it would be in a corresponding ideal gas ($b = 0$). Further, the results corresponding to interaction theory are used to study the existence of reflected and transmitted wave amplitudes (see, [13, 85]). Finally, the jump in shock acceleration together with the amplitudes of reflected and transmitted waves can be determined in terms of incident wave.

In chapter 6, within the context of two-phase fluid flow problems, we consider drift-flux model. The model has been widely described and investigated in the literature for industrial and computational purposes [33, 41, 100]. Here we investigate the most general Lie group of transformations which leaves the given system of PDEs invariant. The symmetry group transformations obtained are used to reduce the system of PDEs to a system of ODEs. In literature there are two types of reduction methods; one of them is to analyze the relations between the parameters of the symmetry group and the other is to find the optimal system of the Lie algebra of the symmetry group [64]. At this point, we apply both of these methods to see the differences between them. Primarily, we try to find the optimal system of the Lie algebra L_5 with the symmetry group in hand. The infinitesimal generators of the group are used to construct the commutator table. This commutator table shows that the set of infinitesimal generators becomes a closed Lie algebra under the Lie bracket operations and also this Lie algebra is solvable. For this reason, the symmetry group of transformations are used to obtain the similarity-reduced forms of the given system of PDEs. For the similarity-reduced forms of the system, the infinitesimal generators are used to reconstruct the adjoint representation of a Lie group on its Lie algebra. We obtain the reduced forms of the system of PDEs for each subalgebra in the optimal system of L_5. On the other hand, we obtain the reduced forms of the system of PDEs by analyzing the relations between the parameters of the symmetry group to show the difference between the two methods. From the reduced forms and the reduced equations we conclude that the second method is more effective than the first one. Finally, we study the fundamental properties of weak discontinuities and which helps to understand situations under which solutions exist.

Chapter 2

Symmetry group analysis and exact solutions of isentropic magnetogasdynamics

2.1 Introduction

Many flow fields involving wave phenomena are governed by quasilinear hyperblic system of PDEs. For nonlinear systems involving discontinuities such as shocks we do not have the luxury of complete exact solutions, and for analytical work we have to rely on some approximate analytical or numerical methods which may provide useful information to understand the complex physical phenomena completely. Lie group of point transformations [11, 64, 67] is the most powerful method to determine particular solutions to such nonlinear PDEs based upon the study of their invariance. The invariance of the transformations allows to introduce a new similarity variable which reduces the number of independent variables by one. With the help of similarity variables, we can reduce the system of PDEs to a system of ODEs, which in general nonlinear. Applications of this method for unsteady one dimensional problems may be found in [29]. A different approach has been described by Oliveri and Speciale for unsteady equations of perfect gases and ideal magnetogasdynamics equations using substitution principles [62, 63]. Sahin et al. [86] have discussed Lie symmetry group properties and similarity solutions of gravity currents in two-layer flow with shallow-water approximations. Radha et al. [69, 73] discussed symmetry analysis and obtained exact solutions for Euler equations of gasdynamics and magnetogasdynamic equations. Lie group transformations for self-similar shocks in a gas with dust particles have been discussed by Jena [44]. Raja Sekhar and Sharma [77] discussed the evolution of weak discontinuities in classical shallow water equations. Evolution of weak discontinuities in a state characterized by invariant solutions is given by Ames and Donato [2]. Singh et al. [91] discussed self-similar solutions of exponential shock waves in non-ideal magnetogasdynamics. For nonlinear wave propagation in quasilinear hyperbolic systems, the reader is refer to the book by Sharma [89]. Solution to the Riemann problem in magnetogasdynamics and elementary wave interactions in isentropic magnetogasdynamics have been discussed by Raja Sekhar and Sharma [78, 81].

In the present chapter, we consider quasilinear system of PDEs which governs the one dimensional unsteady simple flow of an isentropic, inviscid and perfectly conducting compressible fluid, subject to a transverse magnetic field. Lie group of transformations method is used to obtain exact solutions of nonlinear PDEs. Usage of similarity variable we reduce PDEs to ODEs and discuss the evolution of weak discontinuities in the medium characterized by particular solution of the governing system.

2.2 Lie group analysis

The system of equations which governs the one dimensional unsteady simple flow of an isentropic, inviscid and perfectly conducting compressible fluid, subject to a transverse magnetic field in non-conservative form can be written as follows [78]:

$$\begin{aligned} \rho_t + u\rho_x + \rho u_x &= 0, \\ u_t + \frac{w^2}{\rho}\rho_x + uu_x &= 0, \end{aligned} \tag{2.1}$$

where ρ is the fluid density, u is the velocity, $w = \sqrt{b^2 + c^2}$ is the magneto-acoustic speed with $c = \sqrt{\gamma p/\rho}$ as the local sound speed and $b = \sqrt{\frac{B^2(\rho)}{\mu\rho}}$ the Alfven speed; here prime denotes differentiation with respect to ρ. p and B are known functions defined as $p = k_1\rho^\gamma$

and $B = k_2\rho$ where k_1 and k_2 are positive constants and γ is the adiabatic constant that lies in the range $1 < \gamma \leq 2$ for most gases. The independent variables t and x denote the time and space respectively.

Here we investigate the most general Lie group of transformations which leaves the system (2.1) invariant. Now, we consider Lie group of transformations with independent variables x, t : and dependent variables ρ, u for the problem

$$\begin{aligned} \tilde{t} &= \tilde{t}(t, x, \rho, u; \epsilon), \qquad \tilde{x} = \tilde{x}(t, x, \rho, u; \epsilon), \\ \tilde{\rho} &= \tilde{\rho}(t, x, \rho, u; \epsilon) \qquad \tilde{u} = \tilde{u}(t, x, \rho, u; \epsilon), \end{aligned} \tag{2.2}$$

where ϵ is the group parameter. The infinitesimal generator of the group (2.2) can be expressed in the following vector form

$$V = \phi^{(1)}\frac{\partial}{\partial t} + \phi^{(2)}\frac{\partial}{\partial x} + \psi^{(1)}\frac{\partial}{\partial \rho} + \psi^{(2)}\frac{\partial}{\partial u}$$

in which $\phi^{(1)}$, $\phi^{(2)}$, $\psi^{(1)}$ and $\psi^{(2)}$ are infinitesimal functions of the group variables. Then the corresponding one-parameter Lie group of transformations are given by

$$\begin{aligned} \tilde{t} &= t + \epsilon\phi^{(1)}(t, x, \rho, u; \epsilon) + O(\epsilon^2), \qquad \tilde{x} = x + \epsilon\phi^{(2)}(t, x, \rho, u; \epsilon) + O(\epsilon^2), \\ \tilde{\rho} &= \rho + \epsilon\psi^{(1)}(t, x, \rho, u; \epsilon) + O(\epsilon^2), \qquad \tilde{u} = u + \epsilon\psi^{(2)}(t, x, \rho, u; \epsilon) + O(\epsilon^2). \end{aligned}$$

Since the system of equations has at most first-order derivatives, the first prolongation of the generator should be considered in the form:

$$Pr'V = V + \tau_t^{\rho}\frac{\partial}{\partial \rho_t} + \tau_x^{\rho}\frac{\partial}{\partial \rho_x} + \tau_t^{u}\frac{\partial}{\partial u_t} + \tau_x^{u}\frac{\partial}{\partial u_x}, \tag{2.3}$$

where

$$\begin{aligned} \tau_t^{\rho} &= \psi_t^{(1)} + \psi_\rho^{(1)}\rho_t + \psi_u^{(1)}u_t - \rho_t(\phi_t^{(1)} + \phi_\rho^{(1)}\rho_t + \phi_u^{(1)}u_t) - \rho_x(\phi_t^{(2)} + \phi_\rho^{(2)}\rho_t + \phi_u^{(2)}u_t), \\ \tau_x^{\rho} &= \psi_x^{(1)} + \psi_\rho^{(1)}\rho_x + \psi_u^{(1)}u_x - \rho_t(\phi_x^{(1)} + \phi_\rho^{(1)}\rho_x + \phi_u^{(1)}u_x) - \rho_x(\phi_x^{(2)} + \phi_\rho^{(2)}\rho_x + \phi_u^{(2)}u_x), \\ \tau_t^{u} &= \psi_t^{(2)} + \psi_\rho^{(2)}\rho_t + \psi_u^{(2)}u_t - u_t(\phi_t^{(1)} + \phi_\rho^{(1)}\rho_t + \phi_u^{(1)}u_t) - u_x(\phi_t^{(2)} + \phi_\rho^{(2)}\rho_t + \phi_u^{(2)}u_t), \\ \tau_x^{u} &= \psi_x^{(2)} + \psi_\rho^{(2)}\rho_x + \psi_u^{(2)}u_x - u_t(\phi_x^{(1)} + \phi_\rho^{(1)}\rho_x + \phi_u^{(1)}u_x) - u_x(\phi_x^{(2)} + \phi_\rho^{(2)}\rho_x + \phi_u^{(2)}u_x), \end{aligned}$$

where the infinitesimals $\phi^{(1)}$, $\phi^{(2)}$, $\psi^{(1)}$ and $\psi^{(2)}$ can be obtained using a straight forward procedure outlined in [77, 80, 86] as follows

$$\phi^{(1)} = \alpha_1 t + \alpha_4, \quad \phi^{(2)} = \alpha_1 x + \alpha_2 t + \alpha_3, \quad \psi^{(1)} = 0, \quad \psi^{(2)} = \alpha_2, \tag{2.4}$$

where α_1, α_2, α_3 and α_4 are arbitrary constants. In order to reduce PDEs (2.1) to a system of ODEs, we construct similarity variables and similarity forms of field variables. Using a straight forward analysis, the characteristic equations used to find similarity variables are

$$\frac{dt}{\phi^{(1)}} = \frac{dx}{\phi^{(2)}} = \frac{d\rho}{\psi^{(1)}} = \frac{du}{\psi^{(2)}}. \tag{2.5}$$

Integration of first order differential equations corresponding to pair of equations involving only independent variables of (2.5) leads to a similarity variable, called η, which is given as a constant in the solution. We distinguish two cases:

Case I: $\phi^{(1)} \neq 0$, i.e, $\alpha_1 \neq 0$ or $\alpha_4 \neq 0$.

Case II: $\phi^{(1)} = 0$, i.e, $\alpha_1 = 0$ and $\alpha_4 = 0$.

In the former case, one obtains a non-homogeneous autonomous system of ordinary differential equations if $\alpha_1 = 0$. Therefore, we distinguish the case $\alpha_1 \neq 0$ and $\alpha_1 = 0$. The system of ordinary differential equations yield different type of solutions corresponding to the following cases.

Case I: $\alpha_1 \neq 0$ or $\alpha_4 \neq 0$.

Case Ia: $\alpha_1 = 0$.

Case Ib: $\alpha_1 = 0$ and $\alpha_4 \neq 0$.

Case II: $\alpha_1 = 0$ and $\alpha_4 = 0$.

This case corresponds to $\phi^{(1)} = 0$. We obtain from (2.5) that the similarity variable is t.

Corresponding to the cases distinguished above, the dependent variables can be found by integrating one of the two system of characteristic equations

$$\frac{dt}{\phi^{(1)}} = \frac{d\rho}{\psi^{(1)}} = \frac{du}{\psi^{(2)}},$$
$$\frac{dx}{\phi^{(2)}} = \frac{d\rho}{\psi^{(1)}} = \frac{du}{\psi^{(2)}}.$$

After solving any of the system of equations, the solution contains integration constants which are functions of η; these are new dependent variables, called $R(\eta)$ and $U(\eta)$. In any case, substitution of new variables into (2.1) leads to a system of ODEs with independent variable η.

For **Case Ia:**

$$\eta = \frac{x + \frac{\alpha_3}{\alpha_1} - \frac{\alpha_2\alpha_4}{\alpha_1^2}}{(t + \frac{\alpha_4}{\alpha_1})^2} - \frac{\alpha_2}{\alpha_1}\ln(t + \frac{\alpha_4}{\alpha_1}), \quad U(\eta) = u(x,t) - \frac{\alpha_2}{\alpha_1}\ln(t + \frac{\alpha_4}{\alpha_1}), \quad R(\eta) = \rho(x,t). \tag{2.6}$$

Using the new dependent variables in (2.1), we obtain a system of ODEs, namely

$$(U - \eta - \frac{\alpha_2}{\alpha_1})R' + RU' = 0,$$
$$(U - \eta - \frac{\alpha_2}{\alpha_1})U' + (k_3 + \gamma k_1 R^{\gamma-1})R' + \frac{\alpha_2}{\alpha_1} = 0,$$

where $'$ denotes differentiation with respect to η and $k_3 = \frac{k_2}{\mu}$ is a constant. The above system of ODEs can be solved numerically.

For **Case Ib:**

The similarity variable, $\eta = x - \frac{\alpha_2}{2\alpha_4}t^2 - \frac{\alpha_3}{\alpha_4}t$ and the new dependent variables are $U(\eta) = u(x,t) - \frac{\alpha_2}{\alpha_4}t$ and $R(\eta) = \rho(x,t)$. Usage of these new dependent variables into (2.1) leads to the following system of ODEs with independent variable η, namely,

$$(U - \eta - \frac{\alpha_3}{\alpha_4})R' + RU' = 0,$$
$$(U - \eta - \frac{\alpha_3}{\alpha_4})U' + (k_3 + \gamma k_1 R^{\gamma-1})R' + \frac{\alpha_2}{\alpha_4} = 0.$$

Solving the above system of ODEs we obtain the solution of (2.1), as follows:

$$u(x,t) = \frac{\alpha_3}{\alpha_4} + \frac{\alpha_2}{\alpha_4}t + \frac{k_4}{\rho(x,t)},$$
$$\frac{k_4}{\rho^2(x,t)} + \frac{\alpha_3 k_4}{\alpha_4\rho(x,t)} + k_3\rho(x,t) + \frac{\gamma k_1}{(\gamma-1)}\rho^{(\gamma-1)}(x,t) + \frac{\alpha_2}{\alpha_4}x - \frac{\alpha_2}{2\alpha_4}t^2 - \frac{\alpha_3}{\alpha_4}t = k_5,$$

where k_4 and k_5 are arbitrary integration constants.

For **Case II:**

In this case the similarity variables is $\eta = t$ and the new dependent variables are $U(\eta) = u(x,t) - \frac{\alpha_2 x}{(\alpha_2 t + \alpha_3)}$ and $R(\eta) = \rho(x,t)$. Using the variables in the system of PDEs (2.1) which reduces to a system of ODEs given below:

$$R' + \frac{\alpha_2 R}{(\alpha_2\eta + \alpha_3)} = 0,$$
$$U' + \frac{\alpha_2 R}{(\alpha_2\eta + \alpha_3)} = 0. \tag{2.7}$$

Solving the system of ODEs (2.7), we obtain the following solution

$$\rho = \frac{k_6}{(\alpha_2 t + \alpha_3)}, \qquad u = \frac{(\alpha_2 x + k_7)}{(\alpha_2 t + \alpha_3)}, \tag{2.8}$$

where k_6 and k_7 are arbitrary integration constants. It may be remarked that the state such as this, where the particle velocity exhibits linear dependence on the spatial coordinate, has been discussed by Clarke [21], Pert [71] and Sharma et al. [90]; Pert has shown that such a form of velocity distribution is useful in modeling the free expansion of polytropic fluids, and is attained in the large time limit.

2.3 Evolution of weak discontinuities

The governing system of equations can be written in the matrix form as

$$W_t + AW_x = 0 \tag{2.9}$$

where $W = (\rho, u)^T$ is a column vector with superscript T denoting transposition, while A is a 2×2 matrix with elements $A_{11} = A_{22} = u$, $A_{12} = \rho$, $A_{21} = \frac{w^2}{\rho}$. The matrix A has the eigenvalues

$$\lambda^{(1)} = u - w, \qquad \lambda^{(2)} = u + w$$

where $w = \sqrt{b^2 + c^2}$ with the corresponding left and right eigenvectors

$$\begin{aligned} l^{(1)} &= (-w, \rho), & r^{(1)} = (\rho, -w)^T, \\ l^{(2)} &= (w, \rho), & r^{(2)} = (\rho, w)^T. \end{aligned}$$

The evolution of weak discontinuity for a hyperbolic quasilinear system of equations satisfying the Bernoulli's law has been studied quite extensively in the literature [12, 77]. The transport equation for the weak discontinuities across the second characteristic of a hyperbolic system of equations (2.9) is given by

$$l^{(2)}(\frac{d\Lambda}{dt} + (W_x + \Lambda)(\nabla\lambda^{(2)})\Lambda) + ((\nabla l^{(2)})\Lambda)^T \frac{dW}{dt} + (l^{(2)}\Lambda)((\nabla\lambda^{(2)})W_x + \lambda^{(2)}_x) = 0 \tag{2.10}$$

where the coefficient matrix possesses two distinct eigenvalues $\lambda^{(1)}$, $\lambda^{(2)}$ together with four linearly independent left and right eigenvectors, $\Lambda = \beta r^{(2)}$ and $\nabla = (\frac{\partial}{\partial\rho}, \frac{\partial}{\partial u})$. For the system under consideration, Λ denotes the jump in W_x across the weak discontinuity wave with amplitude β, propagating along the curve determined by $\frac{dx}{dt} = \lambda^{(2)}$ originating from the point (x_0, t_0). Now from equation (2.9) we obtain the following Bernoulli type of equation for the amplitude β

$$\frac{d\beta}{dt} + l_1(x,t)\beta^2 + l_2(x,t)\beta = 0, \qquad \frac{dx}{dt} = u + w$$

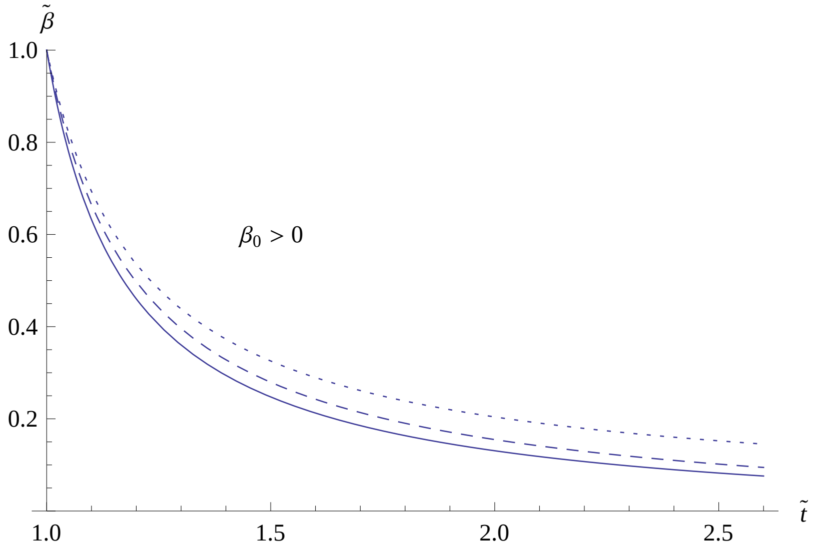

Figure 2.1: Behavior of $\tilde{\beta}$ with $\tilde{t}$ for $\beta_0 > 0$ here $k_2 = 0$ (dotted line), $k_2 = 0.5$ (dashed line) and $k_2 = 1$ (solid line).

where

$$l_1(x,t) = \frac{\frac{k_3 k_6}{(\alpha_2 t+\alpha_3)} + \frac{2k_1\gamma(3-\gamma)k_6^{\gamma-1}}{(\alpha_2 t+\alpha_3)^{\gamma-1}}}{2\sqrt{\frac{k_3 k_6}{(\alpha_2 t+\alpha_3)} + \frac{k_1\gamma k_6^{\gamma-1}}{(\alpha_2 t+\alpha_3)^{\gamma-1}}}}$$

$$l_2(x,t) = \frac{\left[5k_3 + \frac{k_1\gamma(\gamma-2)k_6^{\gamma-2}}{(\alpha_2 t+\alpha_3)^{\gamma-1}} - \frac{(\alpha_2 x+k_7)}{(\alpha_2 t+\alpha_3)}\sqrt{\frac{k_3 k_6}{(\alpha_2 t+\alpha_3)} + \frac{k_1\gamma k_6^{\gamma-1}}{(\alpha_2 t+\alpha_3)^{\gamma-1}}}\right]\frac{\alpha_2}{(\alpha_2 t+\alpha_3)}}{2(k_3 + \frac{k_1\gamma(\gamma-2)k_6^{\gamma-2}}{(\alpha_2 t+\alpha_3)^{\gamma-1}})}$$

with the initial conditions $\beta = \beta_0$ and $x = x_0$ at $t = t_0$. The solution of (2.11) can be written in quadrature form as $\beta(t) = \frac{\beta_0 I(t)}{1+\beta_0 J(t)}$ where $I(t) = exp(\int_1^t -l_1(x(s),s)ds)$ and $J(t) = \int_1^t l_2(x(t'),t')exp(\int_1^t -l_1(x(s),s)ds)dt'$. For the functions l_1 and l_2, given as above, we find that both the integrals $I(t)$ and $J(t)$ are finite and continuous on $[1,\infty)$. Indeed, $I(t) \to 0$ as $t \to \infty$, where as $J(\infty) < \infty$, implying thereby that when $\beta_0 > 0$, which corresponds to an expansion wave, the wave decays and dies out eventually, the corresponding situation is illustrated by the curve in Figure 2.1. The effect of magnetic field, which enters through the parameter k_2, and β_0 on the amplitude $\tilde{\beta}$ are shown in Figures 2.1-2.3, where $\tilde{\beta}$, $\tilde{t}$, and $\tilde{x}$ are dimensionless variables. We noticed that the presence of magnetic field makes the amplitude of expansion wave decreases and decays fastly. However, when $\beta_0 < 0$, which corresponds to a compressive wave, the wave terminates into a shock after a finite time. In fact, there exists a positive quantity $\beta_c > 0$, such that when $|\beta_0| > \beta_c$, $\beta(t)$ increases from

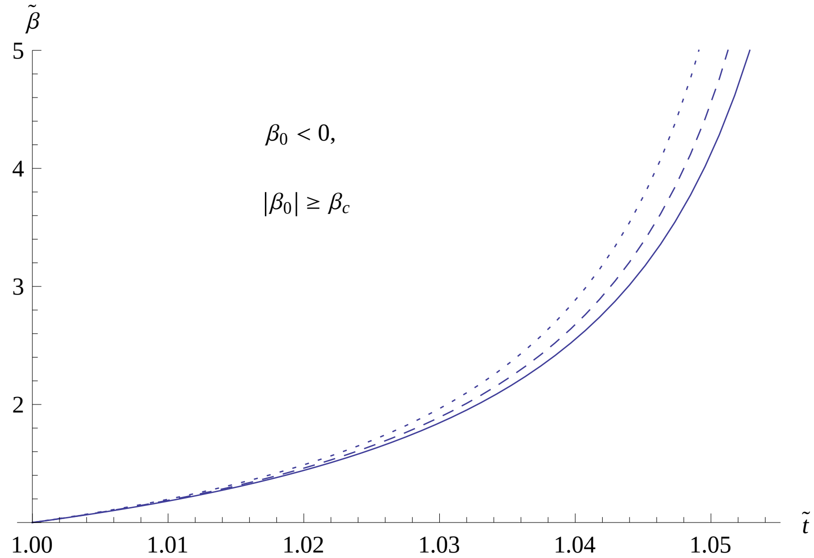

Figure 2.2: Behavior of $\tilde{\beta}$ with $\tilde{t}$ for $\beta_0 < 0$ and $|\beta_0| \geq \beta_c$ here $k_2 = 0$ (dotted line), $k_2 = 0.5$ (dashed line) and $k_2 = 1$ (solid line).

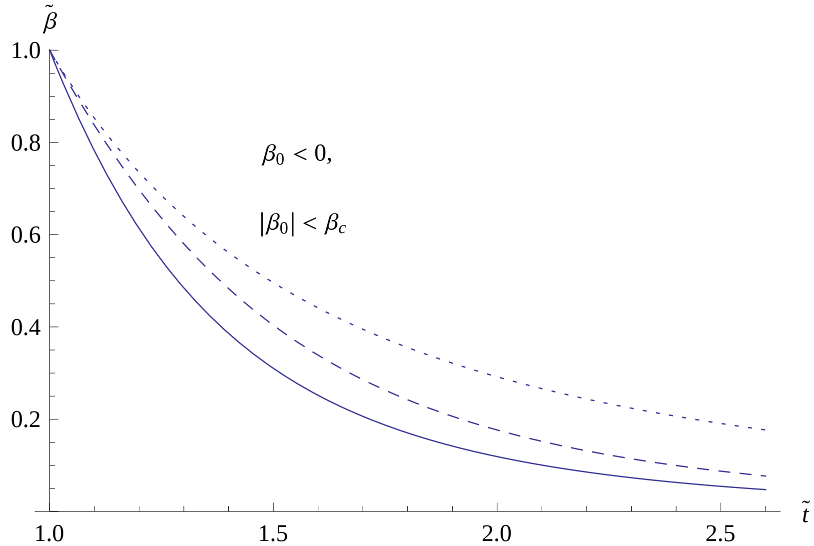

Figure 2.3: Behavior of $\tilde{\beta}$ with $\tilde{t}$ for $\beta_0 < 0$ and $|\beta_0| < \beta_c$ here $k_2 = 0$ (dotted line), $k_2 = 0.5$ (dashed line) and $k_2 = 1$ (solid line).

β_0 and terminates into a shock after finite time, i.e., there exist a finite time t_c given by the solution of $J(t_c) = \frac{1}{|\beta_0|}$ such that $|\beta_c| \to \infty$ as $t \to t_c$; this means that when the amplitude of the incident discontinuity exceeds the critical value in magnitude, the wave culminates into a shock in a finite time; the corresponding situation is shown by the curve in Figure 2.2 with $\beta_0 < 0$ and $|\beta_0| > \beta_c$. It is also observed that the presence of magnetic field would make the solution existing for a longer time in the sense that it further delays the shock formation. However, for $|\beta_0| < \beta_c$, $\beta(t)$ initially decreases from β_0 and reaches to minimum at finite time; the corresponding situation is illustrated in Figure 2.3.

2.4 Conclusions

Lie group analysis is used to obtain some exact solutions of quasilinear PDEs that describe one-dimensional unsteady simple flow of an isentropic, inviscid and perfectly conducting compressible fluid, subject to a transverse magnetic field. We obtained special exact solutions to the governing system of PDEs. It is worth remarking that these solutions play a major role in designing, analyzing and testing of numerical methods for solving special initial and/or boundary-value problems. The evolution of weak discontinuities in a state characterized by exact solution is studied. It is shown that a weak discontinuity wave culminates into a shock after finite time, only if the initial discontinuity associate with it exceeds a critical time i.e. $|\beta_0| > \beta_c$ (see, Figure 2.2). However, when $|\beta_0| < \beta_c$ and $\beta_0 < 0$ or $\beta_0 > 0$, in both the cases the wave decays eventually (see, Figures 2.1 and 2.3). It is noticed that the presence of magnetic field enhances the decay rate of weak discontinuity and reduces the shock formation time as compared to what they would be in the absence of magnetic field. It is also observed that the presence of magnetic field would make the solution existing for a long time in the sense that it further delays the shock formation.

Chapter 3

Exact solutions to magnetogasdynamics using Lie point symmetries

3.1 Introduction

Lie group of transformations has been extensively applied to the linear and nonlinear differential equations in the mathematical physics, engineering, applied mathematics, gasdynamics and mechanics to deal with symmetry reductions, similarity solutions and conservation laws. The method of Lie symmetry groups is the most important approach to obtain analytical solutions of nonlinear PDEs. The basic tool in the study is the use of the corresponding infinitesimal representations of Lie algebras. By an expanded Lie group of transformations of partial differential equations we mean a continuous group of transformations acting on the expanded space of variables which includes the equation parameters in addition to independent and dependent variables. One of the most powerful method to determine particular solutions to PDEs is based upon the study of their invariance with respect to one parameter Lie group of point transformations (see, [10, 11, 44, 64, 67, 83, 86]). Indeed, with the help of symmetry generators of these equations, one can construct similarity variables which can reduce these equations to ordinary differential equations (ODEs); in some cases, it is possible to solve these ODEs exactly [79]. Besides these similarity solutions, the symmetries admitted by given PDEs enable us to look for appropriate canonical variables which transform the original system to an equivalent one whose simple solutions provide nontrivial solutions of the original system (see, [29, 62, 73]). Using this procedure, Ames and Donato [2] obtained solutions for the problem of elastic-plastic deformation generated by a torque and analyzed the evolution of a weak discontinuity in a state characterized by invariant solutions. A self similar method is used to analyze numerically the one-dimensional, unsteady flow of a strong cylindrical shock wave driven by a piston moving with time according to an exponential law in a plasma of constant density by Singh et al. [91]. Donato and Ruggeri [29] used this procedure to study similarity solutions for the system of a monoatomic gas, within the context of the theory of extended thermodynamics, assuming spherical symmetry. Self similar solution of a shock wave propagation in a mixture of a non-ideal gas and small solid particles has been studied in [60]. Wave features and group analysis for axisymmetric flow of shallow water equations have been studied by Raja Sekhar and Bira [76].

In this chapter, we use Lie group analysis approach to characterize a class of solutions of the basic equations governing the one dimensional unsteady flow of inviscid and perfectly conducting compressible fluid, subjected to a transverse magnetic field. Since the system involves two independent variables, we need two commuting Lie vector fields, which are constructed by taking a linear combination of the infinitesimal operators of the Lie point symmetries admitted by the system at hand. Finally, we discuss the behavior of weak discontinuity by using invariant solution of the governing system.

3.2 Symmetry analysis

We consider the PDEs, governing the one dimensional unsteady flow of inviscid and perfectly conducting compressible fluid, subjected to a transverse magnetic field can be written in non-conservative form as follows [81]:

$$\begin{aligned} \rho_t + u\rho_x + \rho u_x &= 0, \\ u_t + \frac{b^2}{\rho}\rho_x + u u_x + \frac{1}{\rho}p_x &= 0, \\ p_t + \rho c^2 u_x + u p_x &= 0. \end{aligned} \tag{3.1}$$

where ρ is the fluid density, p is the pressure, u is the velocity and $b = \sqrt{B^2/\mu\rho}$ the Alfven speed here $B = k_2\rho$ with μ being the magnetic permeability and k_2 is a positive real constant, $c = \sqrt{\gamma p/\rho}$ is the local sound speed and $\gamma = c_p/c_\nu$ is ratio of specific heat capacities. The independent variables x and t denote space and time respectively. Using the straight forward analysis mentioned in [80, 86], it is found that the system (3.1) gives the invariance group of transformations as follows:

$$\begin{aligned} \zeta^{(1)} &= a_0 + a_1 t, \qquad \zeta^{(2)} = a_2 + a_3 x, \qquad \sigma^{(1)} = 2(a_3 - a_1)\rho, \\ \sigma^{(2)} &= (a_3 - a_1)u, \qquad \sigma^{(3)} = 4(a_3 - a_1)p. \end{aligned} \tag{3.2}$$

These transformations provides the following four Lie point generators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = t\frac{\partial}{\partial t} - 2\rho\frac{\partial}{\partial \rho} - u\frac{\partial}{\partial u} - 4p\frac{\partial}{\partial p},$$
$$X_3 = x\frac{\partial}{\partial x} + 2\rho\frac{\partial}{\partial \rho} + u\frac{\partial}{\partial u} + 4p\frac{\partial}{\partial p}, \qquad X_4 = \frac{\partial}{\partial x}.$$

The knowledge of the Lie point symmetries admitted by a system of PDEs may be employed to characterize classes of invariant solutions. But one may look for the introduction of suitable transformations allowing one to map the given system of PDEs to an equivalent form for which the classes of exact solutions may be found. In order to construct two generators V_1, V_2 such that $[V_1, V_2] = 0$, let

$$\begin{aligned} V_1 &= \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4 \\ &= (\alpha_1 + \alpha_2 t)\frac{\partial}{\partial t} + (\alpha_3 x + \alpha_4)\frac{\partial}{\partial x} + 2(\alpha_3 - \alpha_2)\rho\frac{\partial}{\partial \rho} + (\alpha_3 - \alpha_2)u\frac{\partial}{\partial u} + 4(\alpha_3 - \alpha_2)p\frac{\partial}{\partial p} \end{aligned}$$

Similarly

$$\begin{aligned} V_2 &= \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 \\ &= (\beta_1 + \beta_2 t)\frac{\partial}{\partial t} + (\beta_3 x + \beta_4)\frac{\partial}{\partial x} + 2(\beta_3 - \beta_2)\rho\frac{\partial}{\partial \rho} + (\beta_3 - \beta_2)u\frac{\partial}{\partial u} + 4(\beta_3 - \beta_2)p\frac{\partial}{\partial p} \end{aligned}$$

where $\alpha_1\beta_2 - \alpha_2\beta_1 = 0$ and α_i and β_i $(i = 1, 2, 3, 4)$ are arbitrary constants. Since the system is invariant under the group generated by V_1, we introduce a set of canonical variables $\bar{\tau}$, $\bar{\xi}$,

$\bar{R}$, $\bar{U}$ and $\bar{P}$ such that $V_1\bar{\tau} = 1$, $V_1\bar{\xi} = 0$, $V_1\bar{R} = 0$, $V_1\bar{U} = 0$ and $V_1\bar{P} = 0$, which implies that

$$(\alpha_1 + \alpha_2 t)\frac{\partial\bar{\tau}}{\partial t} + (\alpha_3 x + \alpha_4)\frac{\partial\bar{\tau}}{\partial x} + 2(\alpha_3 - \alpha_2)\rho\frac{\partial\bar{\tau}}{\partial\rho} + (\alpha_3 - \alpha_2)u\frac{\partial\bar{\tau}}{\partial u} + 4(\alpha_3 - \alpha_2)p\frac{\partial\bar{\tau}}{\partial p} = 1. \quad (3.3)$$

As V_1 is translation with respect to $\bar{\tau}$, the characteristic conditions associated with (3.3) are

$$\frac{dt}{(\alpha_1 + \alpha_2 t)} = \frac{dx}{(\alpha_3 x + \alpha_4)} = \frac{d\rho}{2(\alpha_3 - \alpha_2)\rho} = \frac{du}{(\alpha_3 - \alpha_2)u} = \frac{dp}{4(\alpha_3 - \alpha_2)p} = \frac{d\bar{\tau}}{1},$$

where $\alpha_2 \neq \alpha_3$; without loss of generality one can assume that $\alpha_4 = 0$, due to freedom in translation. So the characteristic conditions will be

$$\frac{dt}{(\alpha_1 + \alpha_2 t)} = \frac{dx}{\alpha_3 x} = \frac{d\rho}{2(\alpha_3 - \alpha_2)\rho} = \frac{du}{(\alpha_3 - \alpha_2)u} = \frac{dp}{4(\alpha_3 - \alpha_2)p} = \frac{d\bar{\tau}}{1}.$$

Solving the above characteristic equations one can obtain,

$$\bar{\tau} = \frac{1}{\alpha_2}\ln(\alpha_1 + \alpha_2 t), \qquad \bar{\xi} = (\alpha_1 + \alpha_2 t)\ x^{-\frac{\alpha_2}{\alpha_3}}, \qquad \bar{R} = \rho\ x^{-\frac{2(\alpha_3 - \alpha_2)}{\alpha_3}}, \quad (3.4)$$

$$\bar{U} = u\ x^{-\frac{(\alpha_3 - \alpha_2)}{\alpha_3}}, \qquad \bar{P} = p\ x^{-\frac{4(\alpha_3 - \alpha_2)}{\alpha_3}}$$

where α_1, α_2 and α_3 are nonzero constants. In terms of these new variables, V_2 becomes

$$\bar{V}_2 = V_2\bar{\tau}\frac{\partial}{\partial\bar{\tau}} + V_2\bar{\xi}\frac{\partial}{\partial\bar{\xi}} + V_2\bar{R}\frac{\partial}{\partial\bar{R}} + V_2\bar{U}\frac{\partial}{\partial\bar{U}} + V_2\bar{P}\frac{\partial}{\partial\bar{P}},$$

i.e.

$$\bar{V}_2 = \frac{\beta_2}{\alpha_2}\frac{\partial}{\partial\bar{\tau}} + \frac{(\alpha_3\beta_2 - \alpha_2\beta_3)}{\alpha_3}\bar{\xi}\frac{\partial}{\partial\bar{\xi}} - \frac{2(\alpha_3\beta_2 - \alpha_2\beta_3)}{\alpha_3}\bar{R}\frac{\partial}{\partial\bar{R}} - \frac{(\alpha_3\beta_2 - \alpha_2\beta_3)}{\alpha_3}\bar{U}\frac{\partial}{\partial\bar{U}} - \frac{4(\alpha_3\beta_2 - \alpha_2\beta_3)}{\alpha_3}\bar{P}\frac{\partial}{\partial\bar{P}}.$$

Now, we introduce canonical variables τ, ξ, R, U and P such that $\bar{V}_2\tau = 0$, $\bar{V}_2\xi = 1$, $\bar{V}_2 R = 0$, $\bar{V}_2 U = 0$ and $\bar{V}_2 P = 0$. Thus, we get

$$\frac{\beta_2}{\alpha_2}\frac{\partial\xi}{\partial\bar{\tau}} + \frac{(\alpha_3\beta_2 - \alpha_2\beta_3)}{\alpha_3}\bar{\xi}\frac{\partial\xi}{\partial\bar{\xi}} - \frac{2(\alpha_3\beta_2 - \alpha_2\beta_3)}{\alpha_3}\bar{R}\frac{\partial\xi}{\partial\bar{R}} - \frac{(\alpha_3\beta_2 - \alpha_2\beta_3)}{\alpha_3}\bar{U}\frac{\partial\xi}{\partial\bar{U}} - \frac{4(\alpha_3\beta_2 - \alpha_2\beta_3)}{\alpha_3}\bar{P}\frac{\partial\xi}{\partial\bar{P}} = 1. \quad (3.5)$$

The characteristic conditions associated with (3.5) yield the following transformation of variables

$$\tau = \bar{\tau} - \frac{\beta_2}{\alpha_2}\xi, \quad \xi = \frac{\alpha_3}{(\alpha_3\beta_2 - \alpha_2\beta_3)}\log(\bar{\xi}), \quad R = \bar{R}(\bar{\xi})^2, \quad U = \bar{U}(\bar{\xi}), \quad P = \bar{P}(\bar{\xi})^4, \quad (3.6)$$

where R, U and P arbitrary function of τ and ξ. In view of (3.4) the variables (3.6) become

$$\tau = -\frac{\beta_3}{(\alpha_3\beta_2 - \alpha_2\beta_3)}\log[(\alpha_1 + \alpha_2 t)\ x^{-\frac{\beta_2}{\beta_3}}], \quad \xi = \frac{\alpha_3}{(\alpha_3\beta_2 - \alpha_2\beta_3)}\log[(\alpha_1 + \alpha_2 t)\ x^{-\frac{\alpha_2}{\alpha_3}}],$$

$$\rho = R(\alpha_1 + \alpha_2 t)^{-2}\ x^2, \qquad u = U(\alpha_1 + \alpha_2 t)^{-1}\ x, \qquad p = P(\alpha_1 + \alpha_2 t)^{-4}\ x^4. \quad (3.7)$$

Using (3.7) in (3.1), we get

$$\frac{(\beta_2 U-\alpha_2\beta_3)}{(\alpha_3\beta_2-\alpha_2\beta_3)}\frac{\partial R}{\partial \tau}+\frac{(\alpha_3\alpha_2-\alpha_2 U)}{(\alpha_3\beta_2-\alpha_2\beta_3)}\frac{\partial P}{\partial \xi}+\frac{\beta_2 R}{(\alpha_3\beta_2-\alpha_2\beta_3)}\frac{\partial U}{\partial \tau} -\frac{\alpha_2 R}{(\alpha_3\beta_2-\alpha_2\beta_3)}\frac{\partial U}{\partial \xi}+(3U-2\alpha_2)=0, \tag{3.8}$$

$$\frac{(\beta_2 U-\alpha_2\beta_3)}{(\alpha_3\beta_2-\alpha_2\beta_3)}\frac{\partial U}{\partial \tau}+\frac{(\alpha_3\alpha_2-\alpha_2 U)}{(\alpha_3\beta_2-\alpha_2\beta_3)}\frac{\partial U}{\partial \xi}+\frac{\beta_2 k}{(\alpha_3\beta_2-\alpha_2\beta_3)}\frac{\partial R}{\partial \tau}-\frac{\alpha_2 k}{(\alpha_3\beta_2-\alpha_2\beta_3)}\frac{\partial R}{\partial \xi} +\frac{\beta_2}{R(\alpha_3\beta_2-\alpha_2\beta_3)}\frac{\partial P}{\partial \tau}-\frac{\alpha_2}{R(\alpha_3\beta_2-\alpha_2\beta_3)}\frac{\partial P}{\partial \xi}+U(U-\alpha_2)+2kR+\frac{4P}{R}=0, \tag{3.9}$$

$$\frac{(\beta_2 U-\alpha_2\beta_3)}{(\alpha_3\beta_2-\alpha_2\beta_3)}\frac{\partial P}{\partial \tau}+\frac{(\alpha_3\alpha_2-\alpha_2 U)}{(\alpha_3\beta_2-\alpha_2\beta_3)}\frac{\partial P}{\partial \xi}+\frac{\gamma\beta_2 P}{(\alpha_3\beta_2-\alpha_2\beta_3)}\frac{\partial U}{\partial \tau} -\frac{\gamma\alpha_2 P}{(\alpha_3\beta_2-\alpha_2\beta_3)}\frac{\partial U}{\partial \xi}+((\gamma+4)U-4\alpha_2)=0, \tag{3.10}$$

where $k=k_2^2/\mu$. The above equations can be solved completely when $U=$ constant. For this, we consider the following cases:

Case-I

Let $U=$ constant $\neq \alpha_3$. Then equations (3.8) and (3.10) has the closed form solutions as

$$\begin{aligned} R(\xi,\tau) &= R_1(\eta)\ \exp(-\frac{(3U-2\alpha_2)(\alpha_3\beta_2-\alpha_2\beta_3)}{\alpha_2(\alpha_3-U)}\xi), \\ P(\xi,\tau) &= P_1(\eta)\ \exp(-\frac{((\gamma+4)U-4\alpha_2)(\alpha_3\beta_2-\alpha_2\beta_3)}{\alpha_2(\alpha_3-U)}\xi) \end{aligned} \tag{3.11}$$

where

$$\eta=\tau-\frac{(\beta_2U-\alpha_2\beta_3)}{\alpha_2(\alpha_3-U)}\xi=\frac{1}{(\alpha_3-U)}\log((\alpha_1+\alpha_2 t)^{\frac{-U}{\alpha_2}}x).$$

Using (3.11) in (3.9), we get the compatibility condition on $R_1(\eta)$, U and $P_1(\eta)$ as

$$U=\alpha_2 \qquad or \qquad U=\frac{2\alpha_2}{\gamma+1} \tag{3.12}$$

and

$$\begin{aligned} \frac{dR_1(\eta)}{d\eta}+[2(\alpha_3-\alpha_2)+U]R_1(\eta)=0, \\ \frac{dP_1(\eta)}{d\eta}+(\gamma U+4\alpha_2-4)P_1(\eta)=0. \end{aligned} \tag{3.13}$$

Thus, in view of (3.7), (3.11) and (3.12), the solution of the system (3.1) can be expressed as follows.

Case-Ia: When $U = \alpha_2 \neq \alpha_3$, the solution of the system (3.1) is as follows

$$\rho = c_1(\alpha_1 + \alpha_2 t)^{-1}, \qquad u = \alpha_2 x(\alpha_1 + \alpha_2 t)^{-1}, \qquad p = c_2(\alpha_1 + \alpha_2 t)^{-\gamma}, \tag{3.14}$$

where c_1 and c_2 are arbitrary constants and $\eta = \frac{1}{(\alpha_3 - \alpha_2)} \log((\alpha_1 + \alpha_2 t)^{-1} \ x)$.

Case-Ib: When $U = \frac{2\alpha_2}{(\gamma+1)} \neq \alpha_3$, the solution of the system (3.1) is

$$\rho = c_3(\alpha_1 + \alpha_2 t)^{-\dfrac{2}{\gamma+1}}, \qquad u = \frac{2\alpha_2 x}{(\alpha_1 + \alpha_2 t)}, \qquad p = c_4(\alpha_1 + \alpha_2 t)^{-\dfrac{2\gamma}{\gamma+1}} \tag{3.15}$$

where c_3 and c_4 are arbitrary constants and $\eta = \log[(\alpha_1 + \alpha_2 t)^{-\dfrac{2}{\alpha_3(\gamma+1) - 2\alpha_2}} x^{\dfrac{(\gamma+1)}{(\alpha_3(\gamma+1) - 2\alpha_2)}}]$.

Case-II

When $U = \alpha_3$.

The equations (3.8) and (3.10) implies that

$$\begin{aligned} R(\xi, \tau) &= R_1(\xi) + (2\alpha_2 - 3\alpha_3)\tau, \\ P(\xi, \tau) &= P_1(\xi) + (4(\alpha_2 - \alpha_3) - \alpha_3\gamma)\tau, \end{aligned} \tag{3.16}$$

where τ and ξ are same as defined in (3.8) and $R_1(\xi)$ and $P_1(\xi)$ are arbitrary functions of ξ. Moreover, on using (3.16) into (3.9), we get the compatibility condition for $R_1(\xi)$, U and $P_1(\xi)$ as $U = \alpha_3 = \alpha_2$,

$$R_1^{'}(\xi) - 2(\beta_2 - \beta_3)R_1 = -\beta_2 - 2\alpha_3(\beta_2 - \beta_3)\tau, \tag{3.17}$$

$$P_1^{'}(\xi) - 4(\beta_2 - \beta_3)P_1 = -\beta_2\gamma - 4\alpha_3(\beta_2 - \beta_3)\tau.$$

Thus, in view of the equations (3.7), (3.16) and (3.17), the solution of the system (3.1) can be written as

$$\begin{aligned} \rho &= (\alpha_1 + \alpha_3 t)^{-2} x^2 \ (\frac{\beta_2}{2(\beta_2 - \beta_3)} + c_5 \exp(2(\beta_2 - \beta_3)\xi)), \\ u &= \frac{\alpha_3 x}{(\alpha_1 + \alpha_3 t)}, \\ p &= (\alpha_1 + \alpha_3 t)^{-4} x^4 \ (\frac{\beta_2\gamma}{4(\beta_2 - \beta_3)} + c_6 \exp(4(\beta_2 - \beta_3)\xi) \end{aligned} \tag{3.18}$$

where c_5 and c_6 are arbitrary constants and $\xi = \frac{\alpha_3}{(\alpha_3\beta_2 - \alpha_2\beta_3)} \log((\alpha_1 + \alpha_2 t) \ x^{-\dfrac{\alpha_2}{\alpha_3}})$.

3.3 Behavior of weak discontinuities

The governing system of equations can be written in the matrix form

$$H_t + AH_x = 0, \tag{3.19}$$

where $H = (\rho, u, p)^T$ is a column vector with superscript T denoting transposition, while A is a matrix with elements $A_{11} = A_{22} = A_{33} = u$, $A_{12} = \rho$, $A_{21} = \frac{b^2}{\rho}$, $A_{13} = A_{31} = 0$, $A_{23} = \frac{1}{\rho}$, $A_{32} = \rho c^2$. The matrix A has the eigenvalues

$$\lambda^{(1)} = u - w, \qquad \lambda^{(2)} = u, \qquad \lambda^{(3)} = u + w,$$

where $w = \sqrt{c^2 + b^2}$ with the corresponding left and right eigenvectors

$$\begin{aligned} l^{(1)} &= \left(k, -w, \frac{1}{\rho}\right), \qquad & r^{(1)} &= \left(\rho, -w, \rho c^2\right)^T, \\ l^{(2)} &= \left(-c^2, 0, 1\right), \qquad & r^{(2)} &= \left(1, 0, -b^2\right)^T, \\ l^{(3)} &= \left(k, w, \frac{1}{\rho}\right), \qquad & r^{(3)} &= \left(\rho, w, \rho c^2\right)^T. \end{aligned} \tag{3.20}$$

The evolution of weak discontinuity for a hyperbolic quasilinear system of equations satisfying the Bernoulli's law has been studied quite extensively in the literature [12, 89]. The transport equation for the weak discontinuities across the third characteristic of a hyperbolic system of equations is given by

$$l^{(3)}\left(\frac{d\Lambda}{dt} + (H_x + \Lambda)(\nabla\lambda^{(3)})\Lambda\right) + ((\nabla l^{(3)})\Lambda)^T\frac{dH}{dt} + (l^{(3)}\Lambda)((\nabla\lambda^{(3)})H_x + \lambda^{(3)}_x) = 0 \tag{3.21}$$

where the coefficient matrix possesses three distinct eigenvalues $\lambda^{(1)}$, $\lambda^{(2)}$ and $\lambda^{(3)}$ together with six linearly independent left and right eigenvectors, where $\Lambda = \theta r^{(3)}$ and $\nabla = (\frac{\partial}{\partial\rho}, \frac{\partial}{\partial u}, \frac{\partial}{\partial p})$. For the system under consideration, Λ denotes the jump in H_x across the weak discontinuity wave with amplitude θ, propagating along the curve determined by $\frac{dx}{dt} = \lambda^{(3)}$ originating from the point (x_0, t_0). Now from equation (3.21) we obtain the following Bernoulli type of equation for the amplitude θ

$$\frac{d\theta}{dt} + \Phi_1(x,t)\theta^2 + \Phi_2(x,t)\theta = 0, \qquad \frac{dx}{dt} = u + w \tag{3.22}$$

where

$$\Phi_1(x,t) = \frac{\frac{3k_2^2c_1}{\mu(\alpha_2t+\alpha_1)} + \frac{\gamma(\gamma+1)c_2}{c_1(\alpha_2t+\alpha_1)^{\gamma-1}}}{2\sqrt{\frac{k_2^2c_1}{(\alpha_2t+\alpha_1)} + \frac{\gamma c_2}{c_1(\alpha_2t+\alpha_1)^{\gamma-1}}}},$$

$$\Phi_2(x,t) = \frac{5k_2^2c_1\alpha_2}{2(\alpha_2t+\alpha_1)} - \frac{\left(\frac{k_2^2c_1^2\alpha_2}{\mu(\alpha_2t+\alpha_1)^2} + \frac{\gamma c_2\alpha_2}{c_1(\alpha_2t+\alpha_1)^{\gamma-1}}\right)\left(2\sqrt{\left(\frac{k_2^2c_1^2}{\mu(\alpha_2t+\alpha_1)^2} + \frac{\gamma(\gamma-1)c_2}{c_1(\alpha_2t+\alpha_1)^{\gamma-1}}\right)} + \frac{\alpha_2^2x}{(\alpha_2t+\alpha_1)}\right)}{\frac{4k_2^2c_1}{\mu(\alpha_2t+\alpha_1)}\left(\frac{k_2^2c_1^2}{\mu(\alpha_2t+\alpha_1)^2} + \frac{\gamma c_2}{c_1(\alpha_2t+\alpha_1)^{\gamma-1}}\right)^{\frac{3}{2}}}.$$

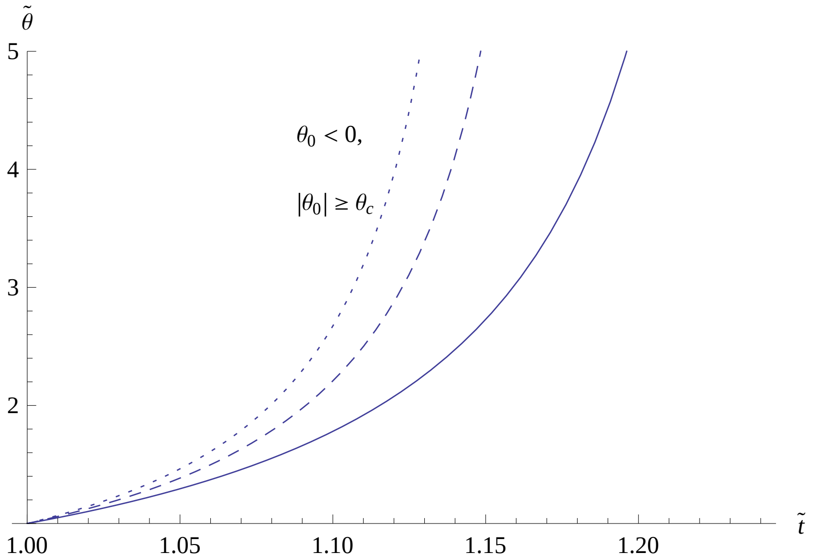

Figure 3.1: The behavior of $\tilde{\theta}$ with $\tilde{t}$ for $\theta_0 < 0$ and $|\theta_0| \geq \theta_c$ here $k_2 = 0$ (dotted line), $k_2 = 0.5$ (dashed line) and $k_2 = 1$ (solid line).

The solution of (3.22) can be written in quadrature form as $\theta(t) = \frac{\theta_0 P(t)}{1+\theta_0 Q(t)}$ where $P(t) = exp(\int_{t_0}^{t} -\Phi_1(x(s), s)ds)$ and $Q(t) = \int_{t_0}^{t} \Phi_2(x(t'), t')exp(\int_{t_0}^{t'} -\Phi_1(x(s), s)ds)dt'$, for the functions Φ_1 and Φ_2, given as above, we find that both the integrals $P(t)$ and $Q(t)$ are finite and continuous on $[t_0, \infty)$. Indeed, $P(t) \to 0$ as $t \to \infty$, where as $Q(t)$ is finite as $t \to \infty$, implying thereby that when $\theta_0 < 0$, which corresponds to a compressive wave, the wave terminates into a shock after a finite time. In fact, there exists a positive quantity $\theta_c > 0$ such that when $|\theta_0| \geq \theta_c$, $\theta(t)$ increases from θ_0 and terminates into a shock after finite time, i.e. there exist a finite time t_c given by the solution of $Q(t_c) = \frac{1}{|\theta_0|}$ such that $|\theta_c| \to \infty$ as $t \to t_c$; this means that when the amplitude of the incident discontinuity exceeds the critical value in magnitude, the wave culminates into a shock in a finite time; the corresponding situation is illustrated in Figure 3.1. The effect of magnetic field, through the parameter k_2 and θ_0 on $\tilde{\theta}$ are shown in Figures 3.1-3.3, where $\tilde{\theta}$ and $\tilde{t}$ are dimensionless variables. It is observed that the presence of magnetic field would make the solution existing for a longer time in the sense that it further delays the shock formation.

However, for $|\theta_0| < \theta_c$, $\theta(t)$ initially decreases from θ_0 and reaches to minimum at finite

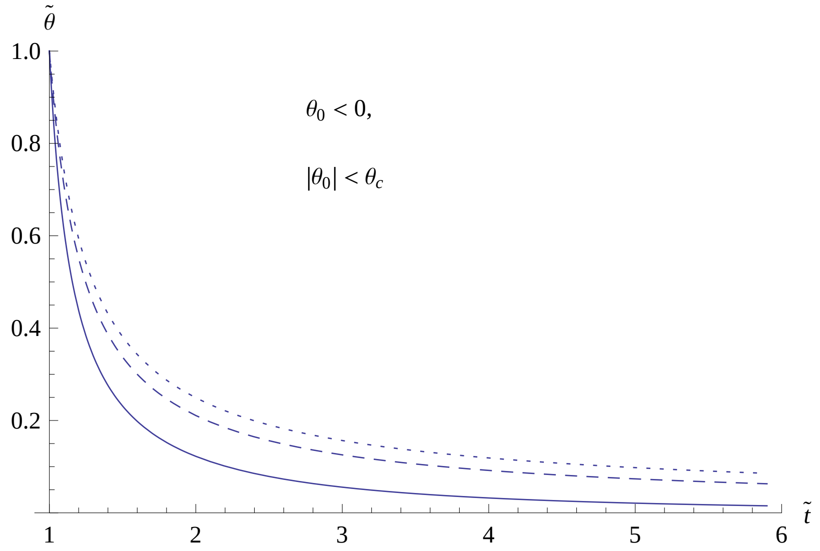

Figure 3.2: The behavior of $\tilde{\theta}$ with $\tilde{t}$ for $\theta_0 < 0$ and $|\theta_0| < \theta_c$ here $k_2 = 0$ (dotted line), $k_2 = 0.5$ (dashed line) and $k_2 = 1$ (solid line).

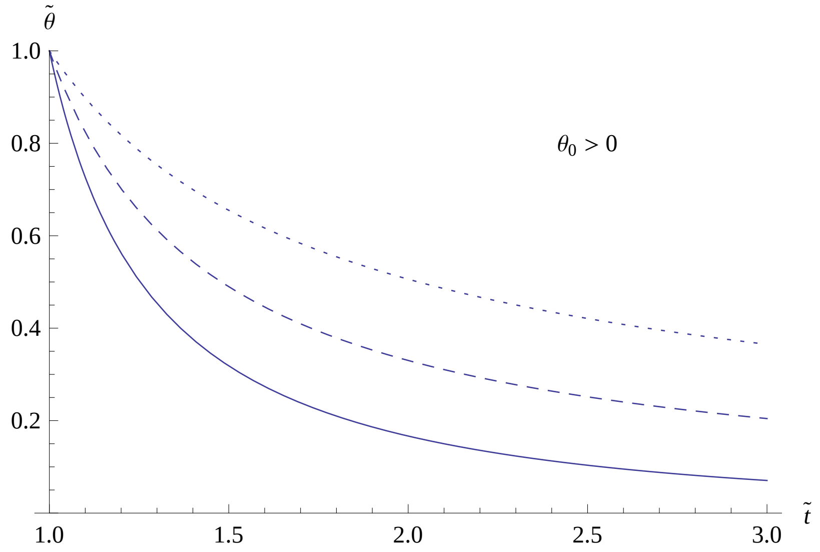

Figure 3.3: The behavior of $\tilde{\theta}$ with $\tilde{t}$ for $\theta_0 > 0$ here $k_2 = 0$ (dotted line), $k_2 = 0$ (dotted line), $k_2 = 0.5$ (dashed line) and $k_2 = 1$ (solid line).

time; the corresponding situation is illustrated in Figure 3.2. When $\theta_0 > 0$, which corresponds to an expansion wave, the wave decays and dies out eventually; the corresponding situation is shown in Figure 3.3. However, an increase in the parameter k_2, when $\theta_0 > 0$, the amplitude of expansion wave decreases and decays.

3.4 Conclusions

For the governing system of magnetogasdynamics equations we obtained some exact solutions, using Lie group of point transformations. These solutions enables one to understand the physical phenomenon completely and has applications in designing, analyzing and testing the numerical methods for solving special initial and/or boundary value problems. With the exact solutions in hand, we have extensively discussed the behavior of weak discontinuities across the solution curve. The behavior of weak discontinuity is well illustrated by the Figures 3.1-3.3. Figure 3.1 shows that, for $|\theta_0| \geq \theta_c$ and $\theta_0 < 0$, the wave culminates into a shock after finite time. It is noticed that the presence of magnetic field enhances the decay rate of weak discontinuity and reduces the shock formation time as compared to what it would be in absence of magnetic field. However, Figures 3.2-3.3 shows that for $|\theta_0| < \theta_c$ and $\theta_0 < 0$ or $\theta_0 > 0$, in both the cases the wave decays eventually. It is also observed that the presence of magnetic field makes the amplitude of expansion wave decreases and decays faster.

Chapter 4

Lie group analysis and propagation of weak discontinuity in one-dimensional ideal isentropic magnetogasdynamics

4.1 Introduction

Lie group symmetry method, originally developed by Sophius Lie in the latter half of the 19th century, is highly algorithmic. The main advantage of this method is that it can be successfully applied to obtain some similarity solutions of the non-linear PDEs. Analysis of PDEs through the use of Lie groups has a rich history (see, [10]-[11]). The primary objective of the Lie symmetry analysis advocated by Lie, is to find one or several-parameter local continuous transformations leaving the equations invariant and then exploit them to obtain the so-called invariant or similarity solutions etc. Lie group of point transformations is the most powerful method to solve nonlinear systems involving discontinuities such as shocks which may provide useful information to understand the complex physical phenomena completely. The explicit determination of exact solutions to system of PDEs of physical relevance is of great interest; besides its own intrinsic interest, these solutions (especially, when they contain arbitrary functions) may be used for modeling asymptotic limits of more complicated solutions, or for testing numerical procedures, or for solving special initial and/or boundary value problems. This technique has been applied by many researchers to solve different flow phenomena over different geometries. Reduction to autonomous form by group analysis and exact solutions of axisymmetric MHD equations have been studied by Donato and Oliveri [29]. Lie group analysis is used to obtain exact and similarity solutions of Euler equations(see [7], [73], [80]). Jena [44] studied Lie group transformations for self-similar shocks in a gas with dust particles. Lie group analysis and Riemann problem for a 2×2 system of balance laws have been discussed by Conforto et.al. [26]. Kovalev [48] applied Lie group technique for studying nonlinear multi-scale systems. Lie algebra of point symmetries and invariant solutions of the integro-partial differential Vlasov-Maxwell system in Lagrangian variables is analyzed by Rezvan and Ozer [82]. Sahin et.al. [86] investigated the self-similarity solutions of the one-layer shallow-water equations representing gravity currents using Lie group analysis. Two-dimensional generalization of the Burgers equation, using Lie group analysis, has been discussed by Ivanova et.al. [42]. Lie group analysis and basic similarity reductions are performed for MHD aligned creeping flow and heat transfer in a second-grade fluid by neglecting the inertial terms by Afify [1]. Ebaid and Khaled [34] obtained new types of exact solutions in terms of Jacobi-elliptic functions and Weierstrass-elliptic functions for Schrodinger equation. Propagation of weak discontinuities in binary mixtures of ideal gases has been discussed by Barbera and Giambo [6]. Radha and Sharma [74] studied the interaction of a weak discontinuity with elementary waves of Riemann problem. Evolution of weak discontinuities in a non-ideal radiating gas has been studied by Singh et.al. [92].

In the present chapter, we consider the one dimensional unsteady flow of an ideal isentropic, inviscid and perfectly conducting compressible fluid, subject to a transverse magnetic field for the magnetogasdynamic system and we obtain certain exact solutions to the governing system of PDEs. With the exact solution in hand, we study the behavior of evolution of weak discontinuity.

4.2 Symmetry group analysis

We consider the PDEs, governing the one dimensional unsteady flow of an ideal isentropic, inviscid and perfectly conducting compressible fluid, subjected to a transverse magnetic field as follows [52]:

$$\begin{aligned} \rho_t + u\rho_x + \rho u_x &= 0, \\ u_t + uu_x + \frac{\gamma}{\rho} p_x + \frac{B}{\rho} B_x &= 0, \\ B_t + uB_x + Bu_x &= 0, \end{aligned} \tag{4.1}$$

where ρ is the fluid density, u is the velocity, p is the pressure with $p = k_1\rho^\gamma$, k_1 is a positive constant, B is the magnetic field strength and $\gamma = c_p/c_\nu$ is the ratio of specific heat capacities. The independent variables x and t denote space and time respectively. Here we consider a one parameter Lie group of infinitesimal transformations

$$\begin{aligned} \tilde{t} &= t + \epsilon\varphi_1(t, x, \rho, u, B; \epsilon) + O(\epsilon^2), \qquad \tilde{x} = x + \epsilon\varphi_2(t, x, \rho, u, B; \epsilon) + O(\epsilon^2), \\ \tilde{\rho} &= \rho + \epsilon\mu_1(t, x, \rho, u, B; \epsilon) + O(\epsilon^2), \qquad \tilde{u} = u + \epsilon\mu_2(t, x, \rho, u, B; \epsilon) + O(\epsilon^2), \\ \tilde{B} &= B + \epsilon\mu_3(t, x, \rho, u, B; \epsilon) + O(\epsilon^2), \end{aligned} \tag{4.2}$$

where the generators φ_1, φ_2, μ_1, μ_2 and μ_3 are functions of t, x, ρ, u and B, which are to be determined in such a way that the PDEs (4.1) are invariant with respect to the transformations (4.2); the group parameter ϵ is so small such that it's square and higher powers may be neglected. Using straight forward analysis mentioned in [80], it is found that the system (4.1) gives the invariance group of transformations as follows:

$$\begin{aligned} \varphi_1 &= a_1 + a_2 t, \qquad \varphi_2 = a_3 + a_4 t + a_5 x, \qquad \mu_1 = \frac{2(a_5 - a_2)}{(\gamma - 1)}\rho, \\ \mu_2 &= a_4 + (a_5 - a_2)u, \qquad \mu_3 = \frac{\gamma(a_5 - a_2)}{(\gamma - 1)}B. \end{aligned} \tag{4.3}$$

The characteristic equations associated with the above transformations to find similarity variables are:

$$\frac{dt}{\varphi_1} = \frac{dx}{\varphi_2} = \frac{d\rho}{\mu_1} = \frac{du}{\mu_2} = \frac{dB}{\mu_3}. \tag{4.4}$$

i.e.,

$$\frac{dt}{a_1 + a_2 t} = \frac{dx}{a_3 + a_4 t + a_5 x} = \frac{d\rho}{\frac{2(a_5 - a_2)}{(\gamma - 1)}\rho} = \frac{du}{a_4 + (a_5 - a_2)u} = \frac{dB}{\frac{\gamma(a_5 - a_2)}{(\gamma - 1)}B}. \tag{4.5}$$

Solving the above characteristic equations, one can obtain similarity variables called ξ, $R(\xi)$, $U(\xi)$ and $B(\xi)$ which are given as constants in the solution. The dependent variables can be found by integrating the above system of characteristic equations considering the following cases:

Case A: $a_1 \neq 0, \quad a_2 \neq 0, \quad a_3 \neq 0, \quad a_4 \neq 0$ and $a_5 \neq 0$.

In this case, the similarity variable and new dependent variables are obtained as follows

$$\xi = L(a_4 t + (a_5 - a_2)x + K)(a_1 + a_2 t)^{\frac{-a_5}{a_2}}, \qquad \rho = (a_1 + a_2 t)^{\frac{2(a_5 - a_2)}{a_2(\gamma - 1)}} R,$$
$$u = \frac{(a_1 + a_2 t)^{\frac{(a_5 - a_2)}{a_2}}}{(a_5 - a_2)} U - \frac{a_4}{(a_5 - a_2)}, \qquad B = (a_1 + a_2 t)^{\frac{\gamma(a_5 - a_2)}{a_2(\gamma - 1)}} P, \tag{4.6}$$

where $L = a_2^{\frac{a_5}{a_2}}$ and $K = \dfrac{a_4 a_1 - a_2 a_3 + a_3 a_5}{a_5}$. Then (4.1) can be reduced into a system of ODEs using the above new dependent variables as follows

$$\begin{aligned}
(U - a_5\xi)\frac{\partial R}{\partial \xi} + R\frac{\partial U}{\partial \xi} + \frac{2(a_5 - a_2)}{a_2(\gamma - 1)} R = 0,& \\
(U - a_5\xi)\frac{\partial U}{\partial \xi} + k_1 (a_5 - a_2)^2 \gamma R^{\gamma - 2}\frac{\partial R}{\partial \xi} + \frac{(a_5 - a_2)^2 P}{R}\frac{\partial P}{\partial \xi} + \frac{(a_5 - a_2)}{L} U = 0,& \\
(U - a_5\xi)\frac{\partial P}{\partial \xi} + P\frac{\partial U}{\partial \xi} + \frac{\gamma(a_5 - a_2)}{a_2(\gamma - 1)} P = 0.&
\end{aligned} \tag{4.7}$$

For $a_5 = 2a_2$ and $k_1 = \frac{2}{L C_1^{\gamma} \gamma} - \frac{C_2}{C_1^{\gamma}(\gamma - 1)}$, we obtain the solution of the above system of ODEs as follows

$$R = C_1(1 - \xi)^{\frac{1}{(\gamma - 1)}}, \qquad U = a_5, \qquad P = C_2(1 - \xi)^{\frac{\gamma}{2(\gamma - 1)}}. \tag{4.8}$$

where C_1 and C_2 are arbitrary constants. Combining (4.6) and (4.8), we obtain the solution of (4.1) as

$$\begin{aligned}
\rho &= C_1(1 - L(a_4 t + (a_5 - a_2)x + K)(a_1 + a_2 t)^{-2})^{\frac{1}{(\gamma - 1)}} (a_1 + a_2 t)^{\frac{2}{\gamma - 1}}, \\
u &= \frac{a_5 (a_1 + a_2 t)^{\frac{(a_5 - a_2)}{a_2}}}{(a_5 - a_2)} - \frac{a_4}{(a_5 - a_2)}, \\
B &= C_2(1 - L(a_4 t + (a_5 - a_2)x + K)(a_1 + a_2 t)^{-2})^{\frac{\gamma}{2(\gamma - 1)}} (a_1 + a_2 t)^{\frac{\gamma}{\gamma - 1}}.
\end{aligned} \tag{4.9}$$

Case B: $a_1 = a_4 = 0$ and $a_2 = a_5$.

The similarity variable and the new dependent variables, are obtained as below;

$$\xi = (a_3 + a_2 x)t^{-1}, \qquad \rho = R, \qquad u = U, \qquad B = P. \tag{4.10}$$

By substituting these new dependent variables in (4.1), we are led to the following new system of ODEs:

$$\begin{aligned}(a_2U - \xi)\frac{\partial R}{\partial \xi} + a_2 R\frac{\partial U}{\partial \xi} &= 0,\\ (a_2U - \xi)\frac{\partial U}{\partial \xi} + k_1 a_2 \gamma R^{\gamma-2}\frac{\partial R}{\partial \xi} + \frac{a_2 P}{R}\frac{\partial P}{\partial \xi} &= 0,\\ (a_2U - \xi)\frac{\partial P}{\partial \xi} + a_2 P\frac{\partial U}{\partial \xi} &= 0.\end{aligned} \tag{4.11}$$

For $\gamma = 2$, $a_2 = a_5 = 2/3$, $k_1 = 1/16C_3$ and $C_4 = \sqrt{C_3}$, we obtain the solution of the system (4.11) as follows

$$R = C_3 \xi^{-\dfrac{a_2}{a_2 - 1}}, \qquad U = \xi, \qquad P = C_4 \xi^{-\dfrac{a_2}{a_2 - 1}}, \tag{4.12}$$

where C_3 and C_4 are arbitrary integration constants. Combining (4.10) and (4.12), we obtain

$$\rho = \frac{C_3(a_3 + 2x/3)^2}{t^2}, \qquad u = \frac{(a_3 + 2x/3)}{t}, \qquad B = \frac{C_4(a_3 + 2x/3)^2}{2\sqrt{2}t^2}. \tag{4.13}$$

Case C: $a_1 = a_3 = a_4 = 0$.

Which yields the similarity and new dependent variables as

$$\xi = xt^{\dfrac{-a_5}{a_2}}, \quad \rho = Rt^{\dfrac{2(a_5 - a_2)}{a_2(\gamma - 1)}}, \quad u = Ut^{\dfrac{(a_5 - a_2)}{a_2}}, \quad B = Pt^{\dfrac{\gamma(a_5 - a_2)}{a_2(\gamma - 1)}}. \tag{4.14}$$

Usage of these new dependent variables in (4.1), we obtain the following system of ODEs

$$\begin{aligned}(U - \frac{a_5}{a_2}\xi)\frac{\partial R}{\partial \xi} + R\frac{\partial U}{\partial \xi} + \frac{2(a_5 - a_2)}{a_2(\gamma - 1)}R &= 0,\\ (U - \frac{a_5}{a_2}\xi)\frac{\partial U}{\partial \xi} + k_1\gamma R^{\gamma-2}\frac{\partial R}{\partial \xi} + \frac{P}{R}\frac{\partial P}{\partial \xi} + \frac{(a_5 - a_2)}{a_2}U &= 0,\\ (U - \frac{a_5}{a_2}\xi)\frac{\partial P}{\partial \xi} + P\frac{\partial U}{\partial \xi} + \frac{\gamma(a_5 - a_2)}{a_2(\gamma - 1)}P &= 0.\end{aligned} \tag{4.15}$$

The solution of the above system of ODEs for $\gamma = 2$, $k_1 = 1/2$ and $C_5 = -C_6$ given as follows

$$R = C_5 \xi^{\dfrac{(2a_5 - a_2)}{(a_5 - a_2)}}, \qquad U = \xi, \qquad P = C_6 \xi^{\dfrac{(2a_5 - a_2)}{(a_5 - a_2)}}, \tag{4.16}$$

where C_5 and C_6 are arbitrary constants. Now (4.14) and (4.16) gives the solution of (4.1) as

$$\rho = C_5 x^{\dfrac{(2a_5 - a_2)}{(a_5 - a_2)}} t^{\dfrac{(3a_5 - 2a_2)}{(a_2 - a_5)}}, \quad u = xt^{-1}, \quad B = C_6 x^{\dfrac{(2a_5 - a_2)}{(a_5 - a_2)}} t^{\dfrac{(3a_5 - 2a_2)}{(a_2 - a_5)}}. \tag{4.17}$$

Case D: $a_1 = a_2 = a_5 = 0$.

For this case, we obtain the similarity variable and new dependent variables as

$$\xi = t, \qquad \rho = R, \qquad u = U + \frac{a_4 x}{(a_3 + a_4 t)}, \qquad B = P. \tag{4.18}$$

Using the above new dependent variables in (4.1), we obtain the system of ODEs as follows:

$$\begin{aligned} \frac{\partial R}{\partial \xi} + \frac{a_4}{(a_3 + a_4\xi)} R = 0, \\ \frac{\partial U}{\partial \xi} + \frac{a_4}{(a_3 + a_4\xi)} U = 0, \\ \frac{\partial P}{\partial \xi} + \frac{a_4}{(a_3 + a_4\xi)} P = 0. \end{aligned} \tag{4.19}$$

Solving the system (4.19) we obtain

$$R = \frac{C_7}{(a_3 + a_4\xi)}, \qquad U = \frac{C_8}{(a_3 + a_4\xi)}, \qquad P = \frac{C_9}{(a_3 + a_4\xi)}, \tag{4.20}$$

where C_7, C_8 and C_9 are arbitrary constants. In view of the equations (4.18) and (4.20), the solution of the system (4.1) can be expressed as follows

$$\rho = \frac{C_7}{(a_3 + a_4 t)}, \qquad u = \frac{a_4 x + C_8}{(a_3 + a_4 t)}, \qquad B = \frac{C_9}{(a_3 + a_4 t)}. \tag{4.21}$$

Case E: $a_3 = a_4 = a_5 = 0$.

This case yields the similarity and dependent variables which are

$$\xi = x, \quad \rho = (a_1 + a_2 t)^{\frac{-2}{(\gamma - 1)}} R, \quad u = (a_1 + a_2 t)^{-1} U, \quad B = (a_1 + a_2 t)^{\frac{-\gamma}{(\gamma - 1)}} P. \tag{4.22}$$

Substituting these variables in (4.1), we obtain

$$\begin{aligned} U\frac{\partial R}{\partial \xi} + R\frac{\partial U}{\partial \xi} - \frac{2a_2}{(\gamma - 1)} R = 0, \\ U\frac{\partial U}{\partial \xi} + k_1 \gamma R^{\gamma - 2}\frac{\partial R}{\partial \xi} + \frac{P}{R}\frac{\partial P}{\partial \xi} - a_2 U = 0, \\ U\frac{\partial P}{\partial \xi} + P\frac{\partial U}{\partial \xi} - \frac{\gamma a_2}{(\gamma - 1)} P = 0. \end{aligned} \tag{4.23}$$

The solution of the above ODEs can be obtained for $\gamma = 2$, $k_1 = 1/2$ and $C_{10} = -C_{11}$ as

$$R = C_{10}\xi, \quad U = a_2\xi, \quad P = C_{11}\xi \tag{4.24}$$

where C_{10} and C_{11} are arbitrary constants. Combining (4.22) and (4.24), we obtain the solution for (4.1) which is given as below:

$$\rho = C_{10} x (a_1 + a_2 t)^{-2}, \quad U = a_2 x (a_1 + a_2 t)^{-1}, \quad B = C_{11} x (a_1 + a_2 t)^{-2}. \tag{4.25}$$

4.3 Propagation of weak discontinuity

The matrix form of the governing hyperbolic system is

$$W_t + HW_x = 0, \tag{4.26}$$

where $W = (\rho, u, B)^T$ is a column vector with superscript T denoting transposition, while H is a matrix with elements $H_{11} = H_{22} = H_{33} = u$, $H_{12} = \rho$, $H_{21} = \frac{c^2}{\rho}$, $H_{13} = H_{31} = 0$, $H_{23} = \frac{B}{\rho}$, $H_{32} = B$, where $c = \sqrt{\gamma p/\rho}$. The matrix H has the eigenvalues

$$\lambda_1 = u - w, \qquad \lambda_2 = u, \qquad \lambda_3 = u + w$$

where $w = \sqrt{c^2 + b^2}$ and $b^2 = \frac{B^2}{\rho}$ with the corresponding left and right eigenvectors

$$\begin{aligned} l_1 &= \left(c^2, -w\rho, B\right), & r_1 &= (\rho, -w, B)^T, \\ l_2 &= (-B, 0, \rho), & r_2 &= \left(-B, 0, c^2\right)^T, \\ l_3 &= \left(c^2, w\rho, B\right), & r_3 &= (\rho, w, B)^T. \end{aligned} \tag{4.27}$$

The evolution of weak discontinuity for a hyperbolic quasilinear system of equations satisfying the Bernoullii's law has been studied quite extensively in the literature (see, [89]). The transport equation for the weak discontinuity across the third characteristic of a hyperbolic system of equations is given by [75]:

$$l_3 \left(\frac{d\Lambda}{dt} + (W_x + \Lambda)(\nabla \lambda_3)\Lambda \right) + ((\nabla l_3)\Lambda)^T \frac{dW}{dt} + (l_3 \Lambda)((\nabla \lambda_3) W_x + (\lambda_3)_x) = 0, \tag{4.28}$$

where Λ, which denotes the jump in W_x across the weak discontinuity, is collinear to the right eigenvector r_3, i.e., $\Lambda = \beta(t) r_3$ with $\beta(t)$ as the amplitude of the weak discontinuity wave and $\nabla = \left(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial u}, \frac{\partial}{\partial B}\right)$. Substitution of (4.21) and (4.27) along with Λ in (4.28) gives the following Bernoulli type of equation for the amplitude $\beta(t)$

$$\frac{d\beta}{dt} + \Psi_1(x,t)\beta^2 + \Psi_2(x,t)\beta = 0, \tag{4.29}$$

and

$$\frac{dx}{dt} = u + w$$

where

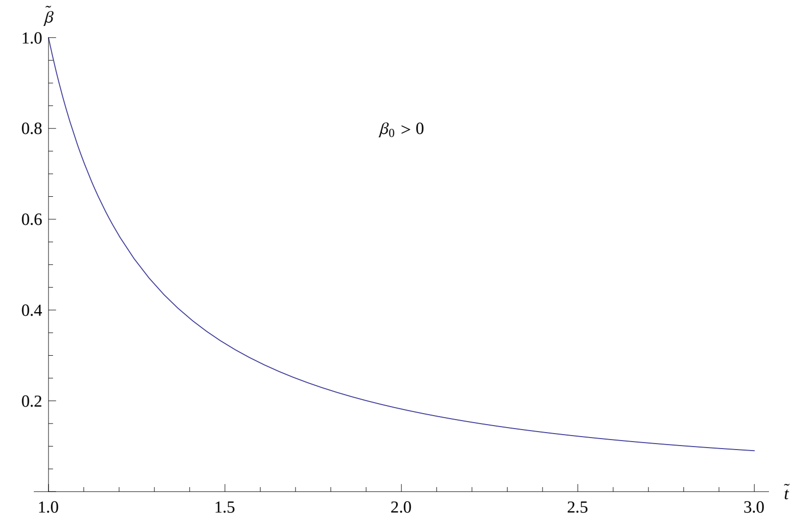

Figure 4.1: The behavior of β with t for $\beta_0 > 0$.

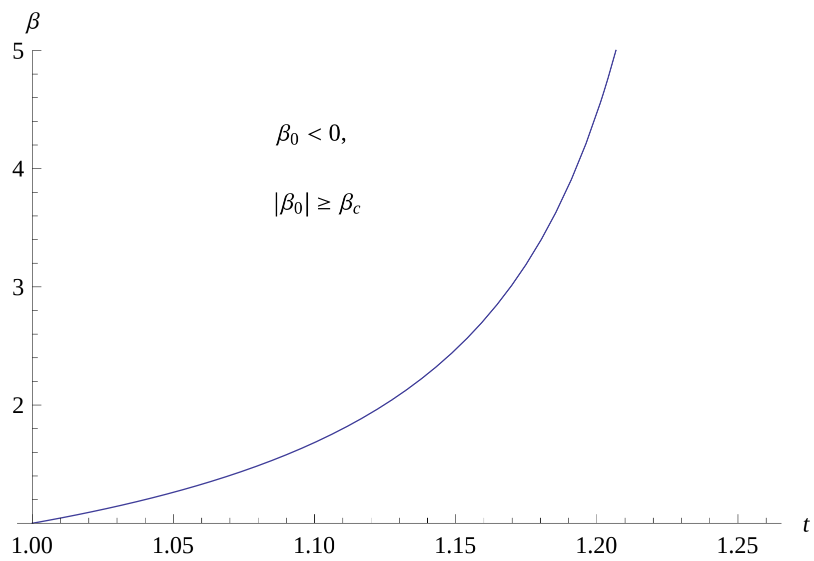

Figure 4.2: The behavior of β with t for $\beta_0 < 0$ and $|\beta_0| \geq \beta_c$.

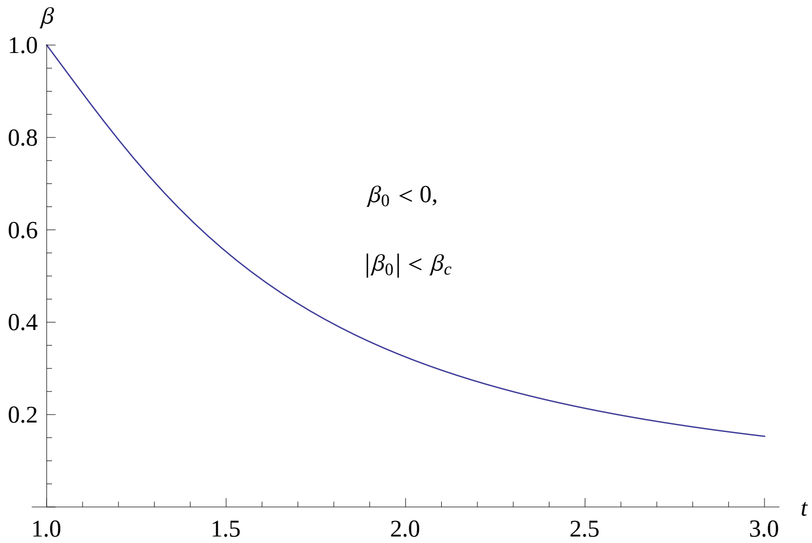

Figure 4.3: The behavior of β with t for $\beta_0 < 0$ and $|\beta_0| < \beta_c$.

$$\Psi_1(x,t) = \frac{\gamma(\gamma+1) + 3t^{\gamma-2}}{2t^{\frac{\gamma-1}{2}}\sqrt{\gamma + t^{\gamma-2}}},$$

$$\Psi_2(x,t) = \frac{2a_4}{(a_3+a_4t)} - \frac{a_4\sqrt{C_7}(a_4x+C_8)\left(k_1\gamma(\gamma+1)C_7^\gamma + 3C_9^2(a_3+a_4t)^{\gamma-2}\right)}{4(a_3+a_4t)^{\frac{5-\gamma}{2}}\left(k_1\gamma(\gamma+1)C_7^\gamma + C_9^2(a_3+a_4t)^{\gamma-2}\right)^{\frac{3}{2}}} - \frac{a_4((a_3+a_4t)^{\gamma-2} + k_1\gamma(\gamma-1))C_7^\gamma}{2(a_3+a_4t)(k_1\gamma C_7^\gamma + C_9^2(a_3+a_4t)^{\gamma-2})}.$$

On integration, (4.29) yields the wave amplitude β as

$$\beta(t) = \frac{\beta_0 S(t)}{1+\beta_0 Q(t)} \tag{4.30}$$

where

$$S(t) = exp(\int_{t_0}^{t} -\Psi_1(x(s),s)ds)$$

and

$$Q(t) = \int_{t_0}^{t} \Psi_2(x(t'),t')exp(\int_{t_0}^{t'} -\Psi_1(x(s),s)ds)dt'.$$

It may be noticed that, the function $S(t)$ is non zero, finite and continuous on $[1,t)$, and it approaches zero as $t \to \infty$ with $Q(\infty) < \infty$. Thus, it follows that for $\beta_0 > 0$, (i.e., an expansion wave), $\beta \to 0$ as $t \to \infty$, implying there by that the wave decays and dies out eventually; the corresponding situation is illustrated by the curve in Figure 4.1. However, for $\beta_0 < 0$ (i.e., a compression wave), there are two possibilities:

(i) Let $|\beta_0| \leq \beta_c$, where $\beta_c = 1/(Q(\infty))$. Then β is finite, non-zero and continuous over

$[1, \infty)$ and $\beta \to 0$ as $t \to \infty$, since $\lim_{t\to\infty} S(t) = 0$. Thus, there exist a critical value β_c of the initial discontinuity such that if $|\beta_0| \leq \beta_c$, then the wave decays; the corresponding situation is shown by the curve in Figure 4.3.

(ii) Let $|\beta_0| > \beta_c$. Then there exist a finite time $t_c > 1$, given by $Q(t_c) = 1/|\beta_0|$ such that β is finite, non-zero, and continuous on $[1, t_c)$ and $|\beta_0| \to \infty$ as $t \to t_c$. This signifies the appearance of a shock wave at an instant t_c; indeed, a compression wave culminates into a shock in a finite time only when the initial discontinuity associated with the wave exceeds a critical value. The corresponding situation is illustrated by the curve in Figure 4.2.

4.4 Conclusions

In this chapter, we derived some exact solutions for the one dimensional unsteady flow of an ideal isentropic, inviscid and perfectly conducting compressible fluid, subject to a transverse magnetic field for the magnetogasdynamics system. Inspite of their own intrinsic interest, these solutions play a major role in modelling, designing, testing numerical procedures, or solving special initial and/or boundary value problems. With the exact solutions in hand, we have extensively discussed the behavior of weak discontinuities across the solution curve. The behavior of weak discontinuity is well illustrated by the Figures 4.1-4.3. Figure 4.1 and Figure 4.3 shows that for $\beta_0 > 0$ or $|\beta_0| \leq \beta_c$ and $\beta_0 < 0$, in both the cases the wave decays and dies out eventually. However, Figure 4.2 shows that, for $|\beta_0| > \beta_c$ and $\beta_0 < 0$, the wave culminates into a shock after finite time.

Chapter 5

Collision of characteristic shock with weak discontinuity in non-ideal magnetogasdynamics

5.1 Introduction

Many physical phenomena in the field of astrophysics and hypersonic aerodynamics are modelled by first order quasilinear hyperbolic system of PDEs. The most significant behavior of the solution of such system of PDEs is that it's solution commonly encountered the elementary waves such as shock waves and weak discontinuity waves. For the safety assessments and prediction of disaster due to explosion point of view, one should have the clear understanding of the behavior of the solution of such system of PDEs. To study such physical phenomenon completely we have to solve the governing system of PDEs. But we don't have the luxury to solve the system of PDEs exactly. So to obtain the exact solutions we rely on Lie group analysis method which is one of the systematic and most powerful technique to solve such nonlinear system of PDEs. Again the shock waves phenomena which is associated with such solution have been a field of continuing research interest over the years. Unfortunately, in the past years most studies of the shock waves phenomena have been limited to ideal gas flows, however, real-gas effects become significant and they need to be considered for many of these flow studies. Real-gas effects can have a noticeable impact on flow features, such as the shock formation, shock stand off distance in a blunt body flow. Because of their importance, real-gas effects have recently been the focus of several studies. A detail study towards gaining a better understanding of the wave interaction problem within the context of hyperbolic systems has been carried out by Jeffrey [43]. Further the extension of this work to elasticity and magneto-fluid dynamics is found in the work of Morro [57, 58].

The evolutionary behaviour of the characteristic shock and its interaction with a weak discontinuity; together with the properties of incident, reflected and transmitted waves via Lie group analysis has been studied in [45, 70]. In [13], the authors examined the effect of an incident wave to create a discontinuity in the acceleration of the shock, the amplitudes of the reflected and transmitted waves with special attention being given to the cases of a weak shock and of a characteristic shock, whereas in [14] the author considered the Euler's variational equations and studied the structure of the characteristic shocks with particular attention to the generalized Born-Infeld Lagrangian describing the electron with spin. The work in [85], accounts the time of shock formation for the fastest transmitted wave when it has overtaken and interacted with a shock wave with an application to the case of a polytropic gas. The interaction of a weak discontinuity with shock wave in an axi-symmetric dusty gas flow is found in [24, 97]. Radha et. al [75] discussed the interaction of shock waves with discontinuities. Planar and nonplanar shock waves in relaxing gas have been studied in [88].

The structure of this chapter is organized as follows: Section 2 describes Lie symmetry analysis for the governing system of PDEs with van der Waals gas equation of state, to reduce it into an autonomous system of PDEs and obtain an exact solution. In Section 3, evolution of characteristic shock is studied using the Rankine-Hugoniot conditions. Evolution of C^1 discontinuity across the solution curve takes place in Section 4. Section 5 deals with interaction of the weak discontinuity with the characteristic shock and derivation of amplitudes of the reflected and transmitted waves and the jump in shock acceleration influenced by the incident wave amplitude after interaction. A brief discussion of the results and conclusion are presented in Section 6.

5.2 Basic equations and their symmetry analysis

Before we introduce the set of equations those are being considered in this chapter, it is worth briefing the magnetohydrodynamic equations in a vector form (see [49]). Following usual notations which are explained below the governing equations (PDEs) for the continuous motion of a non-ideal fluid in the absence of viscosity and thermal conduction, can be written as

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u}) = \mathbf{0}, \tag{5.1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = -\frac{1}{\rho}\nabla(P + \frac{\mathbf{H}^2}{8\pi}) + \frac{1}{4\pi\rho}(\mathbf{H} \cdot \nabla)\mathbf{H}, \tag{5.2}$$

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p + \rho a^2 \nabla \cdot \mathbf{u} = 0, \tag{5.3}$$

$$\frac{\partial \mathbf{H}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{H} = (\mathbf{H} \cdot \nabla)\mathbf{H} - \mathbf{H}(\nabla \cdot \mathbf{u}), \tag{5.4}$$

$$\nabla \cdot \mathbf{H} = 0. \tag{5.5}$$

Here equation (5.1) is the conservation of mass and (5.2) represents conservation of momentum corresponding to the hydrodynamic flow. The energy equation in terms of pressure is given by (5.3). The equation (5.4) represents the momentum balance corresponding to magnetic forces. Finally the magnetic field is divergence force as indicated in equation (5.5), where ρ is fluid density, p the pressure, a the speed of sound, $\mathbf{u} = (u_1, u_2, u_3)$ the velocity vector and $\mathbf{H} = (H_1, H_2, H_3)$ the magnetic field vector. For a non-ideal fluid, $a = \sqrt{\frac{\gamma p}{\rho(1-b\rho)}}$, where γ is specific heat ratio with $1 < \gamma < 2$ and b is the van der Waals excluded volume. The above governing equations are more generic in nature and depending on the geometry under consideration these get simplified. For the current investigation, we consider a unidirectional velocity along a fixed axis and the direction of the corresponding magnetic field is chosen in perpendicular direction to have an induced flow. Accordingly, we choose

$$\mathbf{u} = u(\xi, t)e_i\delta_{1i}, \quad \mathbf{H} = H(\xi, t)e_j\delta_{2j}, \qquad i, \quad j = 1, 2, 3 \tag{5.6}$$

where δ denote the Kronecker delta, t denote the time, ξ denote the spatial co-ordinate and $\mathbf{e_i}$ denote the unit vector in i^{th} direction. It may be noted that ξ and $\mathbf{e_i}$ depend on the co-ordinate system. For example in case of (x, y, z) Cartesian co-ordinates, we have $\xi = x$, $\mathbf{e_1} = \mathbf{i}$, $\mathbf{e_2} = \mathbf{j}$, $\mathbf{e_3} = \mathbf{k}$ and in case of cylindrical co-ordinates $\xi = r$, $\mathbf{e_1} = \mathbf{e_r}$, $\mathbf{e_2} = \mathbf{e_\theta}$,

$\mathbf{e_3} = \mathbf{e_z}$. Accordingly, the above set of governing equations under (5.6) take a simple form. The above consideration for the case of unidirectional flow, the reduced equations allow us a compact form given by [69]:

$$\begin{aligned} \rho_t + u\rho_x + \rho u_x + \frac{m\rho u}{x} &= 0, \\ u_t + uu_x + \frac{1}{\rho}(p_x + h_x) &= 0, \\ p_t + up_x + \rho a^2\left(u_x + \frac{mu}{x}\right) &= 0, \\ h_t + uh_x + 2hu_x + \frac{2muh}{x} &= 0, \end{aligned} \tag{5.7}$$

where ρ, u, p and h are the density, the particle velocity, the pressure and the magnetic pressure respectively and h defined as $h = \frac{H^2}{8\pi}$ where H is the transverse magnetic field. The independent variable x is the spatial coordinate being either axial (cartesian coordinate with $m = 0$), cylindrical (radial coordinate with $m = 1$) or spherical (radial coordinate with $m = 2$).

In order to determine particular solutions, Lie symmetry analysis [11] is performed, Lie group of infinitesimal transformations which leave the system (5.7) invariant is derived as follows:

$\phi_1 = \alpha_1 + \alpha_2 t, \quad \phi_2 = \alpha_3 x, \quad \psi_1 = 0, \quad \psi_2 = (\alpha_3 - \alpha_2)u, \quad \psi_3 = 2(\alpha_3 - \alpha_2)p, \quad \psi_4 = 2(\alpha_3 - \alpha_2)h,$

which are used to construct the following infinitesimal generators

$$\begin{aligned} V_1 &= t\partial_t - u\partial_u - 2p\partial_p - 2h\partial_h, \\ V_2 &= x\partial_x + u\partial_u + 2p\partial_p + 2h\partial_h, \\ V_3 &= \partial_t. \end{aligned}$$

Using straight forward calculations as in [31], one can introduce a suitable invertible transformations obtained by the canonical variables associated with the commuting infinitesimal operators. These transformations consequently convert the given system of PDEs to an equivalent autonomous form. The autonomous system of PDEs help us to derive the non-trivial solutions for the original system of PDEs. This can be achieved by computing the solutions of the autonomous system. To this extent, we first try to obtain the corresponding autonomous system. It is observed that the infinitesimal generators V_1 and V_2 are commuting, i.e.,

$$[V_1, V_2] = V_1V_2 - V_2V_1 = 0.$$

This enables us that a 2-dimensional Abelian sub-algebra can be generated by the operators V_1 and V_2 which further transform the system (5.7) to an autonomous form by the invertible point transformations (see, [25]):

$$\tau = \ln t, \quad \eta = \ln x, \quad \rho = R, \; u = \frac{x}{t}U, \quad p = \frac{x^2}{t^2}P, \quad h = \frac{x^2}{t^2}H. \tag{5.8}$$

Substituting (5.8) in (5.7) we obtain the following autonomous form for the system (5.7)

$$\begin{aligned} R_\tau + UR_\eta + RU_\eta + (m+1)UR = 0, \\ U_\tau + UU_\eta + (U^2 - U) + \frac{1}{R}(P_\eta + H_\eta) + \frac{2}{R}(P + H) = 0, \\ P_\tau + UP_\eta + 2(U-1)P + \frac{\gamma P}{1 - bR}(U_\eta + (m+1)U) = 0, \\ H_\tau + UH_\eta + 2(U-1)H + HU_\eta + 2mUH = 0. \end{aligned} \tag{5.9}$$

For $U = 1$, the system (5.9) has the following particular solution

$$R = R_0 \exp\{-(m+1)\tau\}, \quad P = \frac{P_0 \exp\{-2(\eta+\tau)\}}{\exp(m+1)\tau - bR_0}^{\gamma}, \quad H = \frac{H_0 \exp(-2\eta)}{\exp(2m\tau)}, \tag{5.10}$$

which together with (5.8), produces a particular solution for (5.7) given by

$$\rho = \hat{\rho}\left(\frac{t}{t_0}\right)^{-(m+1)}, \quad u = \frac{x}{t}, \quad P = \frac{\hat{p}}{\left(\left(\frac{t}{t_0}\right)^{(m+1)} - b\hat{\rho}\right)^{\gamma}}, \quad h = \hat{h}\left(\frac{t}{t_0}\right)^{-2(m+1)}. \tag{5.11}$$

Here $\hat{\rho}$, $\hat{p}$ and $\hat{h}$ are reference constants that are assumed to be known. From (5.11), it is obvious that the particle velocity exhibits a linear dependency on spatial coordinate; indeed, such a state can be visualized in terms of an atmosphere filled with a gas which has spatially uniform pressure and density variations on account of the particle motion. The behavior and usefulness of such solution has been studied quite extensively in [21, 71, 90].

5.3 Evolution of characteristic shock

The governing system of quasilinear hyperbolic PDEs (5.7) can be written in a matrix form as

$$U_t + AU_x = f, \tag{5.12}$$

where $U = (\rho, u, p, h)^T$, $\quad f = (-\frac{m\rho u}{x}, 0, -\frac{m\rho a^2 u}{x}, -\frac{2muh}{x})^T$ and

$$A = \begin{pmatrix} u & \rho & 0 & 0 \\ 0 & u & \frac{1}{\rho} & \frac{1}{\rho} \\ 0 & \rho a^2 & u & 0 \\ 0 & 2h & 0 & u \end{pmatrix}.$$

The eigenvalues of A are given by

$$\lambda^{(1)} = (u + C), \qquad \lambda^{(2)} = u \text{ (double root)}, \qquad \lambda^{(3)} = (u - C), \tag{5.13}$$

with the corresponding left and right eigenvectors

$$\begin{aligned} l^{(1)} &= (0, \rho C, 1, 1); \qquad R^{(1)} = (\rho C^{-1}, 1, \rho a^2 C^{-1}, 2hC^{-1})^T, \\ l^{(2,1)} &= (a^2, 0, -1, 0); \qquad R^{(2,1)} = (1, 0, 0, 0)^T, \\ l^{(2,2)} &= (0, 0, -2h, \rho a^2); \qquad R^{(2,2)} = (0, 0, 1, -1)^T, \\ l^{(3)} &= (0, -\rho C, 1, 1); \qquad R^{(3)} = (-\rho C^{-1}, 1, -\rho a^2 C^{-1}, -2hC^{-1})^T, \end{aligned} \tag{5.14}$$

where $a = \left(\frac{\gamma p}{\rho(1-b\rho)}\right)^{\frac{1}{2}}$ is the sound speed and $C = (a^2 + c^2)^{\frac{1}{2}}$ the magneto-acoustic speed with $c = \left(\frac{2h}{\rho}\right)^{\frac{1}{2}}$ as alfven speed.

In case of multiple eigenvalues, the evolution of characteristic shock occurs if the corresponding characteristic field is linearly degenerate, i.e., $\nabla\lambda.R = 0$, where R is the right eigenvector corresponding to the eigenvalue λ of the matrix A and ∇ is the gradient operator with respect to U. The multiplicity of the eigenvalue $\lambda^{(2)} = u$ indicates that there exists a characteristic shock propagating with the speed $V = u$, and the shock curve coincides with a characteristic curve (see, [43]). The Rankine-Hugoniot jump conditions across the characteristic shock can be given as $[\rho] = \xi$, $[u] = 0$, $[p] = -\eta$, $[h] = \eta$ where ξ and η are unknown functions of t and are to be determined. Here $[U]$, defined as $[U] = U_* - U$, denotes the jump in U across the characteristic shock propagating with the speed $\lambda^{(2)} = u$, where U_* and U are the values of the variable U ahead and behind the shock, respectively.

The evolutionary law for ξ and η can be obtained by multiplying equation (5.12) by L, the left eigenvector of A corresponding to the eigenvalue $\lambda^{(2)} = u$, and then forming the jumps in the usual manner across the characteristic shock, i.e.,

$$L\frac{d[U]}{dt} + [L]\frac{dU_*}{dt} = L[f] + [L]f_* \tag{5.15}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$ denotes the material derivative following the shock.

Now usage of (5.12) and (5.14) in (5.15), yields the transport equations for the unknown functions ξ and η as follows:

$$\begin{aligned}\frac{d\xi}{dt} &= \frac{(\rho(\rho-\xi)(a^2-a_*^2)+2\rho\eta-2h\xi)}{(2(h-\eta)+(\rho-\xi)a_*^2)}\left(u_x+\frac{mu}{x}\right),\\ \frac{d\eta}{dt} &= \frac{(2\rho a^2(h-\eta)-h(\rho-\xi)a_*^2)}{(2(h-\eta)+(\rho-\xi)a_*^2)}\left(u_x+\frac{mu}{x}\right).\end{aligned} \tag{5.16}$$

5.4 Evolution of C^1 discontinuity

Let us assume that the initial data at (x_0, t_0) corresponding to the system (5.7) suffer a jump in the first order derivatives. The corresponding C^1 discontinuity propagating along the characteristic originating at (x_0, t_0) can be determined via $\frac{dx}{dt} = \lambda^{(1)}$. The transport equation for the C^1 discontinuity across the first characteristic of a hyperbolic system of equations is given by ([75]):

$$\begin{aligned}l^{(1)}\left(\frac{d\Lambda}{dt}+(U_x+\Lambda)(\nabla\lambda^{(1)})\Lambda\right)+((\nabla l^{(1)})\Lambda)^T\frac{dU}{dt}+(l^{(1)}\Lambda)((\nabla\lambda^{(1)})U_x+\lambda_x^{(1)})\\ -(\nabla(l^{(1)}f))\Lambda=0,\end{aligned} \tag{5.17}$$

where $\Lambda = \beta(t)R^{(1)}$ and $\nabla = (\frac{\partial}{\partial\rho}, \frac{\partial}{\partial u}, \frac{\partial}{\partial p}, \frac{\partial}{\partial h})$, Λ is the jump in U_x across the C^1 discontinuity wave with amplitude β. Substitution of (5.11), (5.13) together with (5.14) in (5.17) gives the Bernoulli type of equation with the non-dimensional variables $\tilde{t}$, $\tilde{x}$ and $\tilde{\beta}$ as follows:

$$\frac{d\tilde{\beta}}{d\tilde{t}}+\Psi_1(\tilde{x},\tilde{t})\beta_0\tilde{\beta}^2+\Psi_2(\tilde{x},\tilde{t})\tilde{\beta}=0, \qquad \frac{d\tilde{x}}{d\tilde{t}}=\frac{\tilde{x}}{\tilde{t}}+\Theta(\tilde{x},\tilde{t}), \tag{5.18}$$

where

$$\Psi_1(\tilde{x},\tilde{t}) = \frac{\gamma\hat{p}\tilde{t}^{3(m+1)}(4b\hat{\rho}+(\gamma-3)\tilde{t}^{(m+1)})-2\hat{h}(\tilde{t}^{(m+1)}-b\hat{\rho})^{(\gamma+2)}}{2\sqrt{\hat{\rho}}(\tilde{t}^{(m+1)}-b\hat{\rho})^{\frac{\gamma+3}{2}}(\gamma\hat{p}\tilde{t}^{3(m+1)}+2\hat{h}(\tilde{t}^{(m+1)}-b\hat{\rho})^{(\gamma+1)})^{\frac{1}{2}}}$$

$$\begin{aligned}\Psi_2(\tilde{x},\tilde{t}) = \frac{4}{\tilde{t}} &- \frac{(4+m)\{\gamma\hat{p}\tilde{t}^{3(m+1)}(2b\hat{\rho}+(\gamma-1)\tilde{t}^{(m+1)})+2\hat{h}(\tilde{t}^{(m+1)}-b\hat{\rho})^{(\gamma+2)}\}}{\tilde{t}(\tilde{t}^{(m+1)}-b\hat{\rho})(\gamma\hat{p}\tilde{t}^{3(m+1)}+2\hat{h}(\tilde{t}^{(m+1)}-b\hat{\rho})^{(\gamma+1)})}\\ &- \frac{m(\gamma\hat{p}\tilde{t}^{3(m+1)}+2\hat{h}(\tilde{t}^{(m+1)}-b\hat{\rho})^{(\gamma+1)})^{\frac{1}{2}}}{\tilde{x}\tilde{t}^{(m+1)}(\tilde{t}^{(m+1)}-b\hat{\rho})^{\frac{\gamma+1}{2}}}\end{aligned}$$

and

$$\Theta(\tilde{x},\tilde{t}) = \left(\frac{(\gamma\hat{p}\tilde{t}^{3(m+1)}+2\hat{h}(\tilde{t}^{(m+1)}-b\hat{\rho})^{(\gamma+1)})}{\hat{\rho}\tilde{t}^{(m+1)}(\tilde{t}^{(m+1)}-b\hat{\rho})^{(\gamma+1)}}\right)^{\frac{1}{2}}.$$

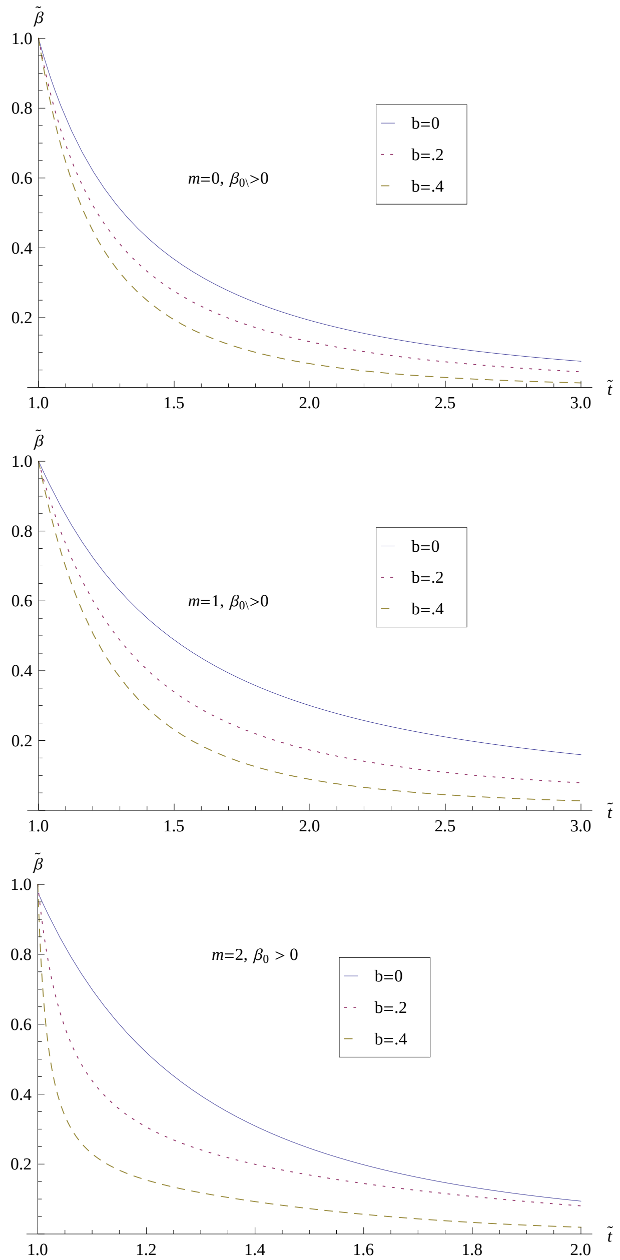

Figure 5.1: : Evolution of C^1 wave for $\beta_0 > 0$, influenced by the van der Waals excluded volume b for plane ($m = 0$), cylindrical ($m = 1$), and spherical ($m = 2$) flows.

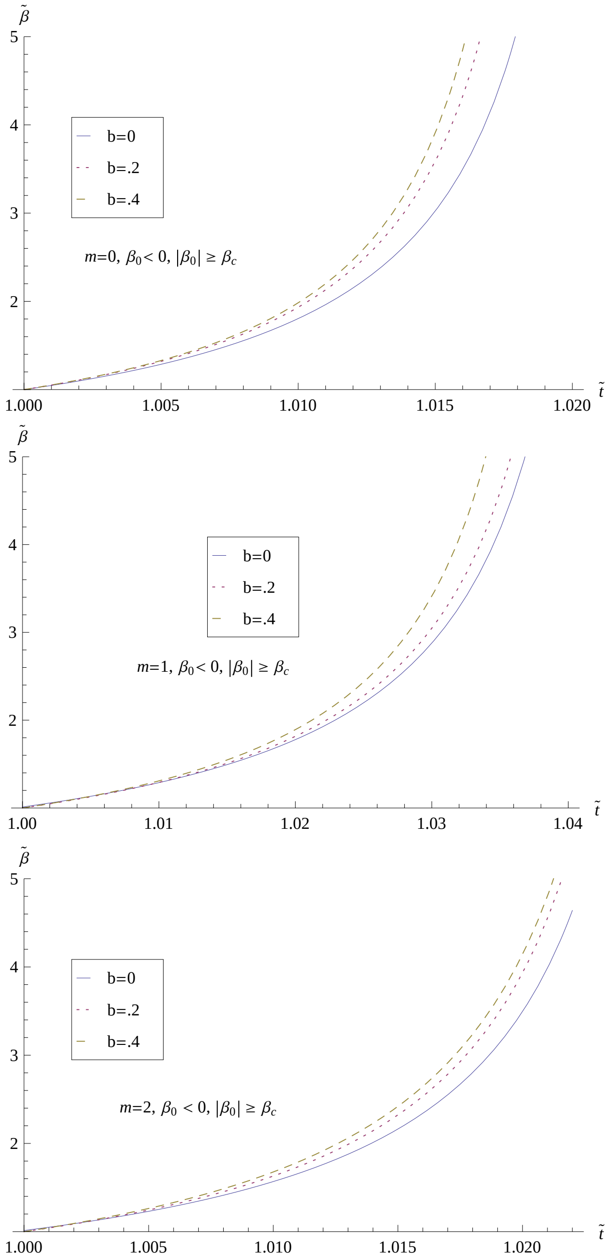

Figure 5.2: :: Evolution of C^1 wave for $\beta_0 < 0$ and $|\beta_0| \geq \beta_c$, influenced by the van der Waals excluded volume b for plane ($m = 0$), cylindrical ($m = 1$), and spherical ($m = 2$) flows.

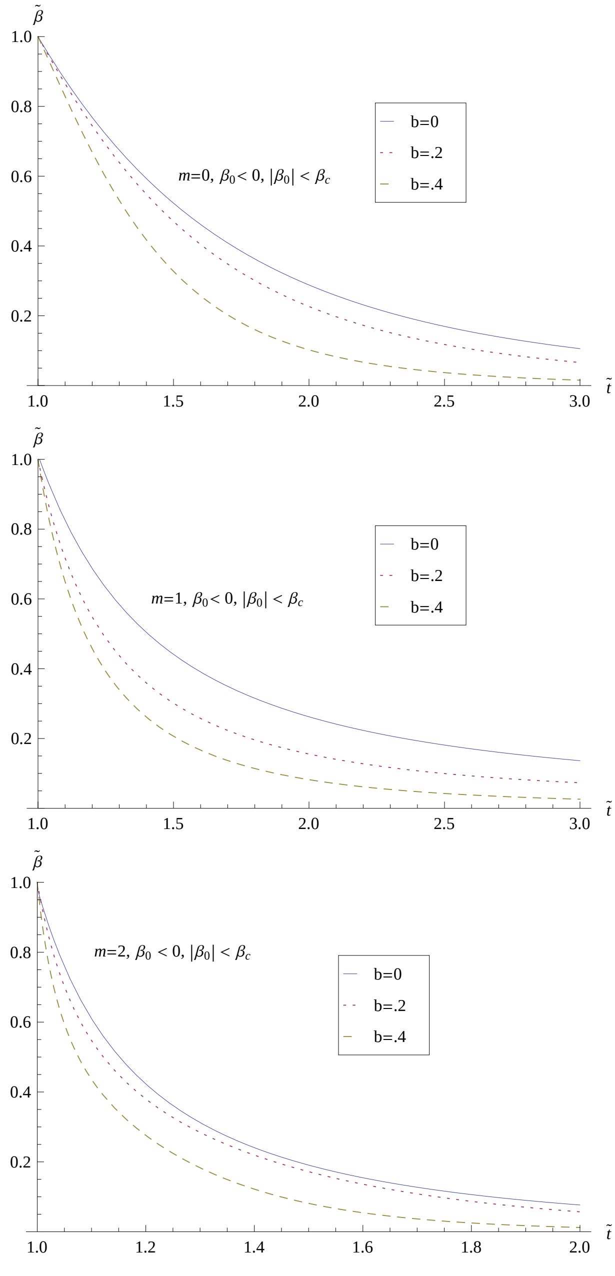

Figure 5.3: :: Evolution of C^1 wave for $\beta_0 < 0$ and $|\beta_0| < \beta_c$, influenced by the van der Waals excluded volume b for plane ($m = 0$), cylindrical ($m = 1$), and spherical ($m = 2$) flows.

The solution of (5.18) can be written in quadrature form as $\tilde{\beta}(\tilde{t}) = \dfrac{I(\tilde{t})}{1+\beta_0 J(\tilde{t})}$ where

$$I(\tilde{t}) = \exp\left(\int_{t_0}^{\tilde{t}} -\Psi_2(x(s),s)ds\right) \quad \text{and} \quad J(\tilde{t}) = \int_{t_0}^{\tilde{t}} \Psi_1(x(\tilde{t}'),\tilde{t}')\exp\left(\int_{t_0}^{\tilde{t}'} -\Psi_2(x(s),s)ds\right)d\tilde{t}'.$$

The functions $I(\tilde{t})$ and $J(\tilde{t})$ are finite and continuous in the interval $[1,\infty)$ and as $\tilde{t}\to\infty$, $I(\tilde{t})\to 0$ where as $J(\infty)<\infty$. For $\beta_0>0$, which corresponds to an expansion wave, $\tilde{\beta}(\tilde{t})\to 0$ as $\tilde{t}\to\infty$ indicating that the wave decays and dies out eventually which is well illustrated by Figure 5.1. However, for the case of compressive wave, i.e., $\beta_0<0$, there exists a positive quantity $\beta_c>0$, for a finite time t_c given by the solution $J(t_c)=\frac{1}{|\beta_0|}$, such that when $|\beta_0|\geq\beta_c$, $\tilde{\beta}(\tilde{t})$ increases from β_0 and terminates into a shock; the corresponding situation is noticed by the curve in Figure 5.2. Whenever $|\beta_0|<\beta_c$, $\tilde{\beta}(\tilde{t})$ initially decreases from β_0 and reaches to minimum at finite time which can be observed by the Figure 5.3.

5.5 Interaction of weak discontinuity with characteristic shock

In order to study the amplitudes of the reflected and transmitted weak discontinuities, we consider the conservative form of the original system (5.7), having the forms in the regions behind and ahead of the shock (i.e. to the left and to the right of the discontinuity curve, $\frac{dx}{dt}=V(=u)$ which propagates with the speed V).

$$\begin{aligned} G_t(x,t,U)+F_x(x,t,U) &= H(x,t,U),\\ G_t(x,t,U_*)+F_x(x,t,U_*) &= H(x,t,U_*), \end{aligned} \tag{5.19}$$

where $U=(\rho,u,p,h)^T$ and $U_*=(\rho-\xi,u,p+\eta,h-\eta)^T$ are the solution vectors to the left and just to the right of the shock curve; and G, F and H are given by

$$\begin{aligned} G &= \left(\rho,\rho u,\frac{(1-b\rho)p}{\gamma-1}+\frac{\rho u^2}{2}+h,h^{1/2}\right)^T,\\ F &= \left(\rho u,\rho u^2+p+h,\left(\frac{(1-b\rho)p}{\gamma-1}+\frac{\rho u^2}{2}+p+2h\right)u,h^{1/2}u\right)^T,\\ H &= \left(-\frac{m\rho u}{x},-\frac{m\rho u^2}{x},\frac{m(b\rho-\gamma)pu}{(\gamma-1)x}-\frac{m\rho u^3}{2x}-\frac{2mhu}{x},-\frac{mh^{1/2}u}{x}\right)^T. \end{aligned} \tag{5.20}$$

We now envisage the situation when the C^1 discontinuity wave encounters the shock wave at time $t = t_p$. Let $P(x_p, t_p)$ be the point at which the fastest C^1 discontinuity of (5.19), moving along the characteristic $\frac{dx}{dt} = \lambda^{(i)}$ and originating from the point (x_0, t_0) intersect the shock $\frac{dx}{dt} = V$. As in ([85]), the amplitudes of the incident, reflected and transmitted waves on the discontinuity line are respectively given by the relations

$$\Lambda_1(P) = \sum_{k=1}^{m_1} \pi_k(t_p) R_s^{(1,k)}, \quad \Lambda_i^{(R)}(P) = \sum_{k=1}^{m_i} \alpha_k^{(i)}(t_p) R_s^{(i,k)}, \quad \Lambda_i^{*(R)}(P) = \sum_{k=1}^{m_i^*} \beta_k^{(i)}(t_p) R_s^{*(i,k)}, \quad (5.21)$$

Now, the evolutionary equations to determine the jump in the acceleration of the shock $|[\dot{V}]|$, the coefficients of the amplitudes of reflected waves $\alpha_k^{(i)}$ and transmitted waves $\beta_k^{(i)}$ after interaction are given by the matrix equation (see, [75])

$$(G - G_*)_s |[\dot{V}]| + (\nabla G)_s \sum_{i=p-q+1}^{p} \left(\sum_{k=1}^{m_i} \alpha_k^{(i)} (V - \lambda^{(i)})^2 R_s^{(i,k)} \right) -$$

$$(\nabla^* G_*)_s \sum_{j=1}^{q} \left(\sum_{k=1}^{m_j} \beta_k^{(j)} (V - \lambda_*^{(j)})^2 R_s^{*(i,k)} \right) = -(\nabla G)_s \sum_{k=1}^{m_i} \pi_k (V - \lambda^{(1)})^2 R_s^{(1,k)}. \quad (5.22)$$

This is a system of n inhomogeneous algebraic equations and the subscript s refers the values evaluated at the shock. At $t = t_p$, the eigenvalues on the both sides of the shock are given by

$$\lambda^{(1)} = u + \left(\frac{\gamma p}{\rho(1 - b\rho)} + \frac{2h}{\rho} \right)^{1/2}, \qquad \lambda_*^{(1)} = u + \left(\frac{\gamma(p + \eta)}{(\rho - \xi)(1 - b(\rho - \xi))} + \frac{2(h - \eta)}{(\rho - \xi)} \right)^{1/2},$$

$$\lambda^{(2)} = u \qquad \lambda_*^{(2)} = u, \quad (5.23)$$

$$\lambda^{(3)} = u - \left(\frac{\gamma p}{\rho(1 - b\rho)} + \frac{2h}{\rho} \right)^{1/2}, \qquad \lambda_*^{(3)} = u - \left(\frac{\gamma(p + \eta)}{(\rho - \xi)(1 - b(\rho - \xi))} + \frac{2(h - \eta)}{(\rho - \xi)} \right)^{1/2}.$$

Thus, the discontinuity $V = \lambda^{(2)} = \lambda_*^{(2)}$ is a physical shock provided the following Lax evolutionary condition holds ([13]):

$$\lambda^{(3)} < u = \lambda^{(2)} < \lambda^{(1)} \text{ and } \lambda_*^{(3)} < u = \lambda_*^{(2)} = \lambda_*^{(2)} < \lambda_*^{(1)}.$$

This reveals that, when the incident wave with velocity $\lambda^{(1)}$ at $t = t_p$ intersects the characteristic shock, it yields a reflected wave with velocity $\lambda^{(3)}$ and a transmitted wave with velocity $\lambda_*^{(1)}$ along the characteristics issuing from the collision point. The reflection and transmission coefficients α and $\beta^{(1)}$ and the jump in acceleration of shock $|[\dot{V}]| = \dot{V}_{t_p^+} - \dot{V}_{t_p^-}$

can be determined from the algebraic equation (5.22)

$$(G-G_*)_s|[\dot{V}]| + (\nabla G)_s R_s^{(3)}(V-\lambda_s^{(3)})^2\alpha - (\nabla_* G_*)_s R_s^{*(1)}(V-\lambda_*^{(1)})^2\beta^{(1)} = -(\nabla G)_s R_s^{(1)}(V-\lambda_s^{(1)})^2\pi. \qquad (5.24)$$

Using (5.13), (5.14), (5.19) and (5.23) in (5.24), the balance equations for the unknowns α, $\beta^{(1)}$ and $|[\dot{V}]|$ can be written as the following system of algebraic equations:

$$\xi|[\dot{V}]| - C\rho\alpha - C_*(\rho-\xi)\beta^{(1)} = -C\rho\pi$$

$$\xi u|[\dot{V}]| - C\rho(u-C)\alpha - C_*(\rho-\xi)(u-C_*)\beta^{(1)} = -C\rho(u+C)\pi$$

$$\begin{aligned}&\{\xi\left(\frac{u^2}{2}-\frac{bp}{\gamma-1}\right)+\eta\left(1-\frac{(1-b(\rho-\xi))}{\gamma-1}\right)\}|[\dot{V}]|\\ &\qquad -\{C\rho\left(\frac{u^2}{2}-\frac{bp}{\gamma-1}\right)-C^2\rho u+\frac{Ca^2\rho(1-b\rho)}{\gamma-1}+2hC\}\alpha\\ &-\{C_*(\rho-\xi)\left(\frac{u^2}{2}-\frac{b(p+\eta)}{\gamma-1}\right)-C_*^2(\rho-\xi)u+\frac{C_*a_*^2(\rho-\xi)(1-b(\rho-\xi))}{\gamma-1}+2(h-\eta)C_*\}\beta^{(1)}\\ &\qquad = -\{C\rho\left(\frac{u^2}{2}-\frac{bp}{\gamma-1}\right)+C^2\rho u+\frac{Ca^2\rho(1-b\rho)}{\gamma-1}+2hC\}\pi\end{aligned}$$

$$\{h^{1/2}-(h-\eta)^{1/2}\}|[\dot{V}]| - Ch^{1/2}\alpha - (h-\eta)^{1/2}C_*\beta^{(1)} = -Ch^{1/2}\pi,$$

whose solution is given by

$$\begin{aligned}\alpha &= -\left(1-\frac{C_*}{C}\left(1-\frac{\xi}{\rho}\right)\left(1+\frac{C_*}{C}\left(1-\frac{\xi}{\rho}\right)\right)^{-1}\right)\pi(t_p),\\ \beta^{(1)} &= \frac{C_*}{C}\left(1+\frac{C_*}{C}\left(1-\frac{\xi}{\rho}\right)\right)^{-1}\pi(t_p),\\ |[\dot{V}]| &= -\frac{2\rho}{\xi}\left(C-C_*\left(1-\frac{\xi}{\rho}\right)\right)\left(1+\frac{C_*}{C}\left(1-\frac{\xi}{\rho}\right)\right)^{-1}\pi(t_p).\end{aligned} \qquad (5.25)$$

The coefficients α and $\beta^{(1)}$ determine the amplitude vectors $\Lambda^R = \alpha R_s^{(3)}(t_p)$ and $\Lambda^T = \beta R_s^{*(1)}(t_p)$ of the reflected and transmitted waves propagating along the characteristic fronts with the velocities $\lambda_s^{(3)}$ and $\lambda_*^{(1)}$, respectively. From (5.25), it is obvious that in the absence of the incident wave (*i.e.*, $\pi(t_p)=0$), the jump in shock acceleration vanishes and there are no reflected or transmitted waves. It can also be observed that an increase in the magnitude of the initial discontinuity π_0 associated with the incident wave, cause the α, $\beta^{(1)}$ and the jump in shock acceleration to increase in magnitude. Moreover, the parameters α, $\beta^{(1)}$ and $|[\dot{V}]|$ depend upon the ambient density on both sides of the characteristic shock and the shock will either accelerate or decelerate depending on whether the incident wave is compressive or expansive. These results are in agreement with the observations made in ([27]), that if the shock front is overtaken by a compression (respectively, expansion) wave, it is accelerated (respectively, decelerated), and consequently the strength of the shock increases (respectively, decreases).

5.6 Results and conclusions

The present chapter concerns with Lie group analysis and evolutionary behavior of a characteristic shock for a quasilinear hyperbolic system of PDEs describing a planar, or cylindrically or spherically symmetric flows in the presence of magnetic field and with the van der Waals gas equation of state. A particular solution to the governing system of PDEs is derived and the evolutionary equations for the jumps in density, pressure and magnetic pressure for the flow of gas are determined. The evolution of weak discontinuity through the particular solution is discussed and the following results are noticed;

1. For $\beta_0 > 0$ (*i.e.*, an expansion wave), as shown in the Figure 5.1, the presence of the van der Waals excluded volume enhances the decaying of an expansion wave.

2. For the case $\beta_0 < 0$ (*i.e.*, compression wave), the following two possibilities may occur:

   (a) In Figure 5.2, it has been observed that a compression wave culminates into shock after a finite time, only if the initial discontinuity associated with it exceeds a critical value, *i.e.*, $|\beta_0| \geq \beta_c$ and the presence of van der Waals excluded volume reduces the shock formation time as compared to what it would be in a corresponding ideal gas ($b = 0$).

   (b) In Figure 5.3, it is noticed that the presence of van der Waals excluded volume fastens the decaying rate of the compression wave and the wave dies out eventually.

Further, the results corresponding to interaction theory are used to study the existence of reflected and transmitted wave amplitudes. Accordingly, once an acceleration wave interacts the characteristic shock, the jump in shock acceleration together with the amplitudes of reflected and transmitted waves can be determined in terms of incident wave.

Chapter 6

The application of Lie groups to an isentropic drift-flux model of two-phase flows

6.1 Introduction

Mathematical analysis represents a fundamental tool to a wide range of two-phase flow applications. Such analysis depends on the theoretical and physical details of a particular two-phase flow configuration such as slug, stratified, bubbly flow and many more [33, 41, 100]. Unfortunately, these theoretical details are mainly limited to the hyperbolicity and conservativity characters of the resulting non-linear partial differential equations [93, 95, 98]. Within such equations, fundamental understanding and the ability to develop theoretical analysis are of great relevance for different two-phase flow models. These models include the drift-flux model [100], the six and seven equations model [59], the homogeneous mixture models [33, 41] and the fully hyperbolic and fully conservative models [84, 99]. When we attempt to investigate such models mathematically, however, we always impose many physical effects such as the interfacial pressure terms [96], the virtual mass [32] and other external force [41] to improve the mathematical properties of the model equations. As such, these models are quite complex to solve analytically or numerically because of the physics and thermodynamics of each phase, and due to the large number of waves related to the hyperbolic character. These facts motivate the research of more mathematical analysis of such models.

In view of the current status on the mathematical analysis of two-phase flow models, it is desirable to undertake a systematic investigation and develop a comprehensive fundamental analysis to such models. In this chapter, therefore, we are concerned with the application of Lie group analysis to the drift-flux model. Such a model is widely used in formulation of the simple equations for two-phase fluid flow problems. Further, the model is known to be relatively fast to compute, by its simplicity, transparency and accuracy in a wide real-world applications. These include thermal-hydraulic analysis of nuclear reactors, pipelines, wellbores and many more. The drift-flux model, however, requires a number of correlations data and analytical solutions are available for steady-state homogenous flows, see for example [100]. Within this framework, we propose an analytical solution for the unsteady state of the drift-flux model using the Lie group analysis. Lie group analysis is one of the most widely used mathematical methods for deriving analytical solutions of non-linear systems of partial differential equations with applications in different fields, see for example [8, 10, 11, 18, 19, 47, 67, 72, 86]. For instance, Lie group analysis was applied to derive analytical solutions to the Euler equations and magnetogasdynamics equations [69, 73]. One of the fundamental principles of such a theoretical method is the development of symmetry groups of the PDEs then reducing it to system of ODEs. The derivation is carried out by reducing the number of independent variables by using the invariant transformations leading to a set of solutions known as the similarity solutions. An important feature of this solution is that an evolution of weak discontinuity may be investigated within the solution curves. Using Lie group analysis then, we will investigate the similarity solutions for the isentropic drift-flux model of two-phase flows. Further, we will show that the model equations have general group generators using such analysis. This will lead a closed form solution to the model as we shall see.

This chapter proceeds as follows. In section 2, we present the system of PDEs governing the drift-flux model of two-phase flows. The Lie group analysis is performed and the symmetry group of transformations are derived in section 3. The investigation of the optimal Lie algebra from the Lie symmetry groups; reductions of the governing system of PDEs to the system of ODEs and similarity solutions for the given system of PDEs are presented in section 4. The discussion of the behaviour of weak discontinuity through one of the solution curves is placed in the section 5. Finally, conclusions and recommendations for further work are presented.

6.2 Mathematical formulation - The drift-flux model

For a gas (g) and a liquid (l) phase, the one-dimensional isentropic two-fluid model may be written as:

Mass conservation equations:

$$\frac{\partial}{\partial t}(\rho_g\alpha_g) + \frac{\partial}{\partial x}(\rho_g\alpha_g v_g) = 0, \tag{6.1}$$

$$\frac{\partial}{\partial t}(\rho_l\alpha_l) + \frac{\partial}{\partial x}(\rho_l\alpha_l v_l) = 0. \tag{6.2}$$

Momentum balance equations:

$$\frac{\partial}{\partial t}(\rho_g\alpha_g v_g) + \frac{\partial}{\partial x}(\rho_g\alpha_g v_g^2) + \frac{\partial}{\partial x}(\alpha_g p_g) - p^i\frac{\partial}{\partial x}(\alpha_g) = Q_g + M_g^i, \tag{6.3}$$

$$\frac{\partial}{\partial t}(\rho_l\alpha_l v_l) + \frac{\partial}{\partial x}(\rho_l\alpha_l v_l^2) + \frac{\partial}{\partial x}(\alpha_l p_l) - p^i\frac{\partial}{\partial x}(\alpha_l) = Q_l + M_l^i, \tag{6.4}$$

where ρ_k, p_k, v_k denote the density, pressure and fluid velocity of phase k, respectively, and p^i is the pressure at the gas liquid interface, α_k is the volume fraction which satisfying the relation

$$\alpha_g + \alpha_l = 1.$$

M_k^i represents interphasic momentum exchange terms with $M_g^i + M_l^i = 0$, *gravitational* and *wall friction* forces are represented by Q_k for both the phases separately, x and t denote the independent variables space and time respectively. Further, equations (6.3) and (6.4) contain non-conservative products. It is necessary, therefore, to write these equations in conservation PDEs form. Thus, adding (6.3) and (6.4) leads to the following mixture momentum

$$\frac{\partial}{\partial t}(\rho_g\alpha_g v_g + \rho_l\alpha_l v_l) + \frac{\partial}{\partial x}(\rho_g\alpha_g v_g^2 + \rho_l\alpha_l v_l^2 + \alpha_g p_g + \alpha_l p_l) = Q, \tag{6.5}$$

where $Q = Q_g + Q_l$ is the momentum source term that is due to the gravitational acceleration or the wall friction forces for each phase. These can be expressed as

$$Q_k = -\rho_k \alpha_k g \sin\theta,$$

and

$$Q_k = -f_k \frac{\rho_k |v_k| v_k}{2},$$

where g is the gravitational constant, θ is the angle of the flow direction with respect to the horizontal and the Blasius equation is commonly used for calculating f_k. Since the above model equations consist of phase and mixture equations, additional hydrodynamic relation is needed. This relation represents the relative velocity between phases and can be expressed in the following general form

$$u_r = v_g - v_l \tag{6.6}$$

which is also known as the slip relation. Such a relation was originally introduced by Zuber and Findlay and have been extensively used for different industrial applications (see, for example, [36, 53]). The above relation can be expressed algebraically for any particular purpose of interest as

$$v_g = K v_{_{mix}} + u_r = K \frac{\rho_g \alpha_g v_g + \rho_l \alpha_l v_l}{\rho_g \alpha_g + \rho_l \alpha_l} + u_r, \tag{6.7}$$

where K is a profile parameter and u_r is the drift velocity of gas relative to the liquid. Within the context of two-phase fluid flow problems, the above system of equations is known as the drift-flux model. The model has been widely described and investigated in the literature as mentioned previously for industrial and computational purposes. For a number of purposes, the system of equations (6.1)-(6.4) is closed by equations of state for both the gas and liquid phases. These equations can be written in the following general form

$$p_g = (\rho_g) \quad \text{and} \quad p_l = (\rho_l).$$

Based in the above drift-flux model and for the purposes of this chapter, we have neglected source terms appearing in the mixture momentum equation (6.5). We also assume no slip between the two phases. Consequently, $K = 1$ and $u_r = 0$ in (6.7), and $Q = 0$ in (6.5), respectively. Further, we can rewrite the model equations (6.1), (6.2) and (6.5) in a more common form by noting such assumptions and defining the following flow variables

$$\rho_1 = \rho_l \alpha_l, \;\; \rho_2 = \rho_g \alpha_g, \;\; P_2 = \alpha_g p_g \quad \text{and} \quad P_1 = \alpha_l p_l$$

Thus, the model equations can be written as

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \frac{\partial \rho_1 u}{\partial x} &= 0, \\ \frac{\partial \rho_2}{\partial t} + \frac{\partial \rho_2 u}{\partial x} &= 0, \\ \frac{\partial}{\partial t}[(\rho_1+\rho_2)u] + \frac{\partial}{\partial x}[(\rho_1+\rho_2)u^2 + p] &= 0, \end{aligned} \tag{6.8}$$

where

$$u = v_g = v_l \quad \text{and} \quad p = P_1 + P_2.$$

These equations will be used to investigate the theoretical properties and analytical solutions for a widely used two-phase flow model.

6.3 Symmetry analysis

Analytical solutions of the drift-flux model presented in the previous section are available for simple cases. In the current chapter we will develop some analytical solutions based on Lie group analysis as mentioned earlier. To process and simplify such analytical treatment, we will assume that the current drift-flux model is closed by isentropic laws of equations of state for both the gas and the liquid of the form

$$p = \kappa_1 \rho_1^{\gamma_1} + \kappa_2 \rho_2^{\gamma_2},$$

where κ_1, κ_2 are positive constants and γ_1, γ_2 are adiabatic constants lies between 1 and 2, which depends on both phases. For the sake of simplicity and the purpose of the present chapter we assume that the adiabatic constants are equal, that is, $\gamma_1 = \gamma_2 = \gamma$. As a result, the simplest isentropic two-phase flow model (6.8) can be expressed as follows

$$\begin{aligned} \rho_{1t} + u\rho_{1x} + \rho_1 u_x &= 0, \\ \rho_{2t} + u\rho_{2x} + \rho_2 u_x &= 0, \\ u_t + uu_x + \frac{\kappa_1 \gamma \rho_1^{\gamma-1}}{(\rho_1+\rho_2)}\rho_{1x} + \frac{\kappa_2 \gamma \rho_2^{\gamma-1}}{(\rho_1+\rho_2)}\rho_{2x} &= 0. \end{aligned} \tag{6.9}$$

Here we investigate the most general Lie group of transformations which leaves the system (6.9) invariant. Now, we consider one parameter Lie group of transformations with the independent variables x and t, and with the dependent variables $\rho_1,\ \rho_2,\ u$ for the current problem as

$$\begin{aligned}
\tilde{x} &= x + \epsilon\xi^x(x, t, \rho_1, \rho_2, u) + O(\epsilon^2),\\
\tilde{t} &= t + \epsilon\xi^t(x, t, \rho_1, \rho_2, u) + O(\epsilon^2),\\
\tilde{u} &= u + \epsilon\eta^u(x, t, \rho_1, \rho_2, u) + O(\epsilon^2),\\
\tilde{\rho_1} &= \rho_1 + \epsilon\eta^{\rho_1}(x, t, \rho_1, \rho_2, u) + O(\epsilon^2),\\
\tilde{\rho_2} &= \rho_2 + \epsilon\eta^{\rho_2}(x, t, \rho_1, \rho_2, u) + O(\epsilon^2).
\end{aligned}$$

Since the system of the isentropic no-slip drift-flux two-phase flow model has at most first-order derivatives, we consider the first order prolongation for the system of PDEs (6.9) and by the straightforward analysis mentioned in [11, 80, 86], we obtained the infinitesimal transformations as follows:

$$\xi^t = \alpha_1 + \alpha_2 t, \tag{6.10}$$

$$\xi^x = \alpha_3 + \alpha_4 t + \alpha_5 x, \tag{6.11}$$

$$\eta^{\rho_1} = \frac{2(\alpha_5 - \alpha_2)}{\gamma - 1}\rho_1, \tag{6.12}$$

$$\eta^{\rho_2} = \frac{2(\alpha_5 - \alpha_2)}{\gamma - 1}\rho_2, \tag{6.13}$$

$$\eta^u = \alpha_4 + (\alpha_5 - \alpha_2)u, \tag{6.14}$$

where $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4$ and α_5 are arbitrary constants.

6.4 Similarity reduced form of isentropic drift-flux model of two-phase flows

In this section, the symmetry group properties obtained in the previous section are used to reduce the system of PDEs governed by isentropic drift-flux model of two-phase flow to a system of ODEs. In literature there are two types of reduction methods; one of them is to analyze the relations between the parameters of the symmetry group and the other is to find the optimal system of the Lie algebra of the symmetry group [64]. At this point, we have to apply both of these methods to see the differences between them.

Primarily, we will try to find the optimal system of the Lie algebra L_5 having the symmetry group (6.10-6.14). For that purpose, we consider one parameter Lie group of transformations (6.10-6.14) and the corresponding infinitesimal generators are constructed as

$$\begin{aligned} V_1 &= t\frac{\partial}{\partial t} - 2M\rho_1\frac{\partial}{\partial \rho_1} - 2M\rho_2\frac{\partial}{\partial \rho_2} - u\frac{\partial}{\partial u} \\ V_2 &= \frac{\partial}{\partial t} \\ V_3 &= t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \\ V_4 &= x\frac{\partial}{\partial x} + 2M\rho_1\frac{\partial}{\partial \rho_1} + 2M\rho_2\frac{\partial}{\partial \rho_2} + u\frac{\partial}{\partial u} \\ V_5 &= \frac{\partial}{\partial x} \end{aligned}$$

where $M = 1/(\gamma - 1)$. To obtain the similarity-reduced forms of the given system of PDEs using the symmetry groups, the Lie algebra which occurs by the virtue of the corresponding infinitesimal generators of the groups must be solvable. Thereby, the set of these generators has to be closed under the following commutator table by using Lie bracket defined as

$$[V_i, V_j] = V_iV_j - V_jV_i, \qquad (i, j = 1, 2, 3, 4, 5).$$

and the commutator table is constructed as below . This commutator table shows that the set

Table 6.1: The commutator table

$[*, *]$	V_1	V_2	V_3	V_4	V_5
V_1	0	$-V_2$	V_3	0	0
V_2	V_2	0	V_5	0	0
V_3	$-V_3$	$-V_5$	0	V_3	0
V_4	0	0	$-V_3$	0	$-V_5$
V_5	0	0	0	V_5	0

of infinitesimal generators becomes a closed Lie algebra under the Lie bracket operations and also this Lie algebra is solvable. For this reason, the symmetry group (6.10-6.14) can be used to obtain the similarity-reduced forms of the system (6.9). For the similarity-reduced forms of the system, the infinitesimal generators are used to reconstruct the adjoint representation of a Lie group on its Lie algebra. If V generates one parameter subgroup $\{\exp(\varepsilon V)\}$, then we let V be the vector field on its Lie algebra ϱ generating the corresponding one-parameter subgroup of adjoint representations (see, [64]):

$$Ad(\exp(\varepsilon V))\omega = \omega - \varepsilon[V,\omega] + \frac{\varepsilon^2}{2!}[V,[V,\omega]] - \cdots\cdots \qquad (\omega\epsilon\varrho).$$

Using the above definition, the adjoint representation table of the L_5 having the symmetry group (6.10-6.14) constructed in Table 2, where V_1, V_2, V_3, V_4 and V_5 are the infinitesimal generators of the symmetry group (6.10-6.14).

Table 6.2: Adjoint representation table of the infinitesimal generators of the symmetry group

$Ad(\exp(\varepsilon *)*)$	V_1	V_2	V_3	V_4	V_5
V_1	V_1	$e^{\varepsilon}V_2$	$e^{-\varepsilon}V_3$	V_4	V_5
V_2	$V_1-\varepsilon V_2$	V_2	$V_3-\varepsilon V_5$	V_4	V_5
V_3	$V_1+\varepsilon V_3$	$V_2+\varepsilon V_5$	V_3	$V_4-\varepsilon V_3$	V_5
V_4	V_1	V_2	$e^{\varepsilon}V_3$	V_4	$e^{\varepsilon}V_3$
V_5	V_1	V_2	V_3	$V_4-\varepsilon V_5$	V_5

Given a nonzero vector $V = \beta_1V_1 + \beta_2V_2 + \beta_3V_3 + \beta_4V_4 + \beta_5V_5$, our aim is to simplify as many of the coefficients β_i as possible through judicious applications of adjoint maps to V. For this purpose, we assume that $\beta_i \neq 0$.

Suppose first that $\beta_5 \neq 0$. Scaling V if necessary we can assume that $\beta_5 = 1$. From the table-2, if we act on a V by $Ad(\exp(\varepsilon V_2))$

$$\begin{aligned}
\upsilon' &= Ad(\exp(\varepsilon V_2)V) = V - \varepsilon[V_2,V] + \frac{\varepsilon^2}{2!}[V_2,[V_2,V]] - \cdots\cdots \\
\upsilon' &= V - \varepsilon(\beta_1V_2 + \beta_3V_5) + \frac{\varepsilon^2}{2!}[V_2,\beta_1V_2+\beta_3V_5] - \cdots\cdots \\
\upsilon' &= V - \varepsilon\beta_1V_2 - \varepsilon\beta_3V_5 \\
\upsilon' &= \beta_1V_1 + (\beta_2-\varepsilon\beta_1)V_2 + \beta_3V_3 + \beta_4V_4 + (1-\varepsilon\beta_3)\beta_5V_5
\end{aligned}$$

for $\varepsilon = 1/\beta_3$ and $\beta_2 = \beta_1/\beta_3$, the coefficients of V_2 and V_5 can make vanish such that $\upsilon' = \beta_1V_1 + \beta_3V_3 + \beta_4V_4$. Next let us now act on υ' by $Ad(\exp(\varepsilon V_1))$

$$\begin{aligned}
\upsilon'' &= Ad(\exp(\varepsilon V_1)\upsilon') = \upsilon' - \varepsilon[V_1,\upsilon'] + \frac{\varepsilon^2}{2!}[V_1,[V_1,\upsilon']] - \cdots\cdots \\
\upsilon'' &= \upsilon' - \varepsilon(\beta_3V_3) + \frac{\varepsilon^2}{2!}[V_1,\beta_3V_3] - \cdots\cdots \\
\upsilon'' &= \beta_1V_1 + \beta_3e^{-\varepsilon}V_3 + \beta_4V_4
\end{aligned}$$

Since we assume that $\varepsilon = \ln(1/\beta_3)$ where $\beta_3e^{-\varepsilon} = \kappa \in \{-1,0,1\}$, then all reductions acting by V with the assumption $\beta_5 \neq 0$ are equivalent to the reductions acting by $\upsilon'' = \beta_1V_1 + \kappa V_3 + \beta_4V_4$.

Now let us suppose that $\beta_3 = 0$, $\beta_2 \neq 0$ and we can assume that $\beta_2 = 1$, then we act on V by $Ad(\exp(\varepsilon V_2))$ which lead to

$$\begin{aligned}
\omega &= Ad(\exp(\varepsilon V_2)V) = V - \varepsilon[V_2, V] + \frac{\varepsilon^2}{2!}[V_2, [V_2, V]] - \cdots\cdots \\
\omega &= V - \varepsilon(\beta_1 V_2) \\
\omega &= V - \varepsilon\beta_1 V_2 - \varepsilon\beta_3 V_5 \\
\omega &= \beta_1 V_1 + (\beta_2 - \varepsilon\beta_1)V_2 + \beta_4 V_4 + \beta_5 V_5
\end{aligned}$$

Next, for $\varepsilon = 1/\beta_1$ we can act on ω by $Ad(\exp(\varepsilon V_5))$

$$\begin{aligned}
\omega' &= Ad(\exp(\varepsilon V_5)\omega) = \omega - \varepsilon[V_5, \omega] + \frac{\varepsilon^2}{2!}[V_5, [V_5, \omega]] - \cdots\cdots \\
\omega' &= \omega - \varepsilon(\beta_5 V_5) \\
\omega' &= \beta_1 V_1 + \beta_4 V_4 + (1 - \varepsilon\beta_5)V_5.
\end{aligned}$$

It is easy to make vanish the coefficient of V_5 for $\varepsilon = 1/\beta_5$. Thus all reductions acting by V with the assumption $\beta_3 = 0$, $\beta_2 \neq 0$ are equivalent to the reductions acting by $\omega' = \beta_1 V_1 + \beta_4 V_4$.

Next, we suppose $\beta_3 = \beta_4 = \beta_5 = 0$. If we act on V by $Ad(\exp(\varepsilon V_1))$, produces the following equations

$$\begin{aligned}
\bar{\upsilon} &= Ad(\exp(\varepsilon V_1)V) = V - \varepsilon[V_1, V] + \frac{\varepsilon^2}{2!}[V_1, [V_1, V]] - \cdots\cdots \\
\bar{\upsilon} &= V - \varepsilon(-\beta_2 V_2) + \frac{\varepsilon^2}{2!}[V_1, -\beta_2 V_2] \\
\bar{\upsilon} &= V - \varepsilon\beta_1 V_2 - \varepsilon\beta_3 V_5 \\
\bar{\upsilon} &= \beta_1 V_1 + \beta_2 e^{\varepsilon} V_2
\end{aligned}$$

and for $\varepsilon = \beta_2$, $\beta_2 e^{\varepsilon} = \kappa \in \{-1, 0, 1\}$, we obtain $\bar{\upsilon} = \beta_1 V_1 + \kappa V_2$ which means that all the reductions acting by V are equivalent to the reductions acting by $\bar{\upsilon}$.

Similarly, for $\beta_3 = \beta_2 = 0$, $\beta_4 \neq 0$ and we act on V by $Ad(\exp(\varepsilon V_4))$, leading to the following equations

$$\begin{aligned}
\bar{\omega} &= Ad(\exp(\varepsilon V_4)V) = V - \varepsilon[V_4, V] + \frac{\varepsilon^2}{2!}[V_4, [V_4, V]] - \cdots\cdots \\
\bar{\omega} &= V - \varepsilon(-\beta_5 V_5) + \frac{\varepsilon^2}{2!}[V_4, -\beta_5 V_5] \\
\bar{\omega} &= \beta_1 V_1 + \beta_4 V_4 + \beta_5 e^{\varepsilon} V_5.
\end{aligned}$$

Choosing $\beta_5 e^{\varepsilon} = \kappa \in \{-1, 0, 1\}$ and $\varepsilon = \ln \beta_5$, we can say that all the reductions acting on V by $Ad(\exp(\varepsilon V_4))$ is equivalent to the reductions acting by $\bar{\omega}$ and denoted by $\bar{\omega} = \beta_1 V_1 + \beta_4 V_4 + \kappa V_5$.

Finally, suppose that $\beta_3 = \beta_4 = \beta_5 = 0$ and acting on V by $Ad(\exp(\varepsilon V_2))$, we obtain

$$\begin{aligned}
\vartheta &= Ad(\exp(\varepsilon V_2)V) = V - \varepsilon[V_2, V] + \frac{\varepsilon^2}{2!}[V_2, [V_2, V]] - \cdots\cdots \\
\vartheta &= V - \varepsilon(-\beta_5 V_5) + \frac{\varepsilon^2}{2!}[V_4, -\beta_5 V_5] \\
\vartheta &= V - \varepsilon\beta_1 V_2 \\
\vartheta &= \beta_1 V_1 + (\beta_2 - \varepsilon\beta_1)V_2.
\end{aligned}$$

It is clear that the coefficient of V_2 can be vanish for $\varepsilon = \beta_2/\beta_1$ such that $\vartheta = \beta_1 V_1$. Thus, in this case all the reductions acting by V are equivalent to the reductions acting by $\varepsilon = \beta_2/\beta_1$.

An optimal system of Lie algebra L_5 having the symmetry group (6.10-6.14) can be defined by the following subalgebras

$$\begin{aligned}
L^a_{1,1} &= \beta_1 V_1, \qquad L^\kappa_{1,2} = \beta_1 V_1 + \kappa V_2, \qquad L^a_{1,3} = \beta_1 V_1 + \beta_4 V_4, \\
L^\kappa_{1,4} &= \beta_1 V_1 + \beta_4 V_4 + \kappa V_5, \qquad L^\kappa_{1,5} = \beta_1 V_1 + \kappa V_3 + \beta_4 V_4.
\end{aligned} \tag{6.15}$$

Now the subalgebras (6.15) is directly applicable to the classification of group-invariant solutions.

6.4.1 The reduction by finding the optimal system of the Lie groups and similarity solutions

In this section, we will try to reduce for each subalgebra in the optimal system of L_5 to obtain the reduced forms of the system (6.9) and try to obtain similarity solutions for all the possible cases. For this purpose, we need to write the characteristic equation in the following form:

The reduction for $L^a_{1,1}$:

$$\frac{dt}{\beta_1 t} = \frac{dx}{0} = \frac{d\rho_1}{-2M\beta_1\rho_1} = \frac{d\rho_2}{-2M\beta_1\rho_2} = \frac{du}{-\beta_1 u} \tag{6.16}$$

Then the similarity independent variable is $\xi = x$, and the similarity forms of the dependent variables R_1, R_2 and U which are obtained by integrating the characteristic equation as

$$\rho_1 = t^{-2M}R_1, \qquad \rho_2 = t^{-2M}R_2, \qquad u = t^{-1}U. \tag{6.17}$$

Using (6.17) in (6.9) we can obtain the reduced system of ODEs as

$$\begin{aligned} U\frac{dR_1}{d\xi} + R_1\frac{dU}{d\xi} - 2MR_1 = 0,\\ U\frac{dR_2}{d\xi} + R_2\frac{dU}{d\xi} - 2MR_2 = 0,\\ U\frac{dU}{d\xi} + \frac{\kappa_1\gamma R_1^{\gamma-1}}{R_1+R_2}\frac{dR_1}{d\xi} + \frac{\kappa_2\gamma R_2^{\gamma-1}}{R_1+R_2}\frac{dR_2}{d\xi} - U = 0, \end{aligned} \tag{6.18}$$

which can be solved for $\kappa_1 C_1^\gamma = -\kappa_2 C_2^\gamma$ and obtain the solution of the system (6.9) as

$$\rho_1 = C_1 t^{\frac{-2}{\gamma-1}} x^{\frac{3-\gamma}{\gamma-1}}, \qquad \rho_2 = C_2 t^{\frac{-2}{\gamma-1}} x^{\frac{3-\gamma}{\gamma-1}}, \qquad u = \frac{x}{t}.$$

Here C_1 and C_2 are integration constants.

Similarly applying the same procedure to each subalgebra of the optimal system, we see that all the similarity forms assimilate to each other and also we obtain similar reduced forms of the system (6.9). The similarity variables and reduced system of equations are given below for each subalgebras of the optimal system.

For $L_{1,2}^{\kappa}$: The similarity variables and the similarity forms are

$$\xi = x, \qquad \rho_1 = (\beta_1 t+\kappa)^{-2M}R_1, \qquad \rho_2 = (\beta_1 t+\kappa)^{-2M}R_2, \qquad u = (\beta_1 t+\kappa)^{-1}U, \tag{6.19}$$

which consequently reduce (6.9) to a system of ODEs as follows

$$\begin{aligned} U\frac{dR_1}{d\xi} + R_1\frac{dU}{d\xi} - 2M\beta_1R_1 = 0,\\ U\frac{dR_2}{d\xi} + R_2\frac{dU}{d\xi} - 2M\beta_1R_2 = 0,\\ U\frac{dU}{d\xi} + \frac{\kappa_1\gamma R_1^{\gamma-1}}{R_1+R_2}\frac{dR_1}{d\xi} + \frac{\kappa_2\gamma R_2^{\gamma-1}}{R_1+R_2}\frac{dR_2}{d\xi} - \beta_1U = 0. \end{aligned} \tag{6.20}$$

For $U = \beta_1\xi$ we solve the above equations with $\kappa_1 C_3^\gamma = -\kappa_2 C_4^\gamma$ and obtain the solution as

$$R_1 = C_3\xi^{2M-1}, \qquad R_2 = C_4\xi^{2M-1}.$$

From which the solution for the given system of PDEs is discovered as

$$\rho_1 = C_3(\beta_1 t+\kappa)^{\frac{-2}{\gamma-1}} x^{\frac{3-\gamma}{\gamma-1}}, \qquad \rho_2 = C_4(\beta_1 t+\kappa)^{\frac{-2}{\gamma-1}} x^{\frac{3-\gamma}{\gamma-1}}, \qquad u = \frac{\beta_1 x}{(\beta_1 t+\kappa)}.$$

For $L_{1,3}^{a}$: The similarity variable and the similarity forms:

$$\xi = xt^{\frac{-\beta_4}{\beta_1}}, \qquad \rho_1 = t^{\frac{2M(\beta_4-\beta_1)}{\beta_1}}R_1, \qquad \rho_2 = t^{\frac{2M(\beta_4-\beta_1)}{\beta_1}}R_2, \qquad u = t^{\frac{(\beta_4-\beta_1)}{\beta_1}}U, \tag{6.21}$$

and the reduced equations:

$$\begin{aligned}(U - \frac{\beta_4}{\beta_1}\xi)\frac{dR_1}{d\xi} + R_1\frac{dU}{d\xi} + \frac{2M(\beta_4-\beta_1)}{\beta_1}R_1 = 0,\\ (U - \frac{\beta_4}{\beta_1}\xi)\frac{dR_2}{d\xi} + R_2\frac{dU}{d\xi} + \frac{2M(\beta_4-\beta_1)}{\beta_1}R_2 = 0,\\ (U - \frac{\beta_4}{\beta_1}\xi)\frac{dU}{d\xi} - \frac{(\beta_4-\beta_1)}{\beta_1}U + \frac{\kappa_1\gamma R_1^{\gamma-1}}{R_1+R_2}\frac{dR_1}{d\xi} + \frac{\kappa_2\gamma R_2^{\gamma-1}}{R_1+R_2}\frac{dR_2}{d\xi} = 0.\end{aligned} \tag{6.22}$$

Considering $U = \xi$, we can solve the above equations and obtain the solution as

$$R_1 = C_5\xi^{2M - \frac{\beta_1}{\beta_1-\beta_4}}, \qquad R_2 = C_6\xi^{2M - \frac{\beta_1}{\beta_1-\beta_4}},$$

for $\kappa_1 C_5^\gamma = -\kappa_2 C_6^\gamma$. By backward substitution yields the solution of the system (6.9) as

$$\begin{aligned}\rho_1 &= C_5 t^{(\frac{\beta_4}{\beta_1-\beta_4} - \frac{2}{\gamma-1})} x^{(\frac{2}{\gamma-1} - \frac{\beta_1}{\beta_1-\beta_4})},\\ \rho_2 &= C_6 t^{(\frac{\beta_4}{\beta_1-\beta_4} - \frac{2}{\gamma-1})} x^{(\frac{2}{\gamma-1} - \frac{\beta_1}{\beta_1-\beta_4})},\\ u &= \frac{x}{t}.\end{aligned}$$

For $L^\kappa_{1,4}$: This case yield the similarity variable and the similarity forms as:

$$\xi = (\beta_4 x + \kappa)t^{\frac{-\beta_4}{\beta_1}}, \quad \rho_1 = t^{\frac{2M(\beta_4-\beta_1)}{\beta_1}}R_1, \quad \rho_2 = t^{\frac{2M(\beta_4-\beta_1)}{\beta_1}}R_2, \quad u = t^{\frac{(\beta_4-\beta_1)}{\beta_1}}U, \tag{6.23}$$

and the corresponding reduced equations are

$$\begin{aligned}(U - \frac{\xi}{\beta_1})\frac{dR_1}{d\xi} + R_1\frac{dU}{d\xi} + \frac{2M(\beta_4-\beta_1)}{\beta_4\beta_1}R_1 = 0,\\ (U - \frac{\xi}{\beta_1})\frac{dR_2}{d\xi} + R_2\frac{dU}{d\xi} + \frac{2M(\beta_4-\beta_1)}{\beta_4\beta_1}R_2 = 0,\\ (U - \frac{\xi}{\beta_1})\frac{dU}{d\xi} + \frac{\kappa_1\gamma R_1^{\gamma-1}}{R_1+R_2}\frac{dR_1}{d\xi} + \frac{\kappa_2\gamma R_2^{\gamma-1}}{R_1+R_2}\frac{dR_2}{d\xi} = 0.\end{aligned} \tag{6.24}$$

$R_1 = C_7$ and $R_2 = C_8$, we get $U = \frac{\xi}{\beta_1}$ which inturn produces the solution of (6.9) as

$$\rho_1 = \frac{C_7}{t}, \qquad \rho_2 = \frac{C_8}{t}, \qquad u = (x + \frac{\kappa}{\beta_1})t^{-1}.$$

For $L^\kappa_{1,5}$: We obtain the similarity variable and similarity forms for a particular case $\beta_1 = \beta_4 = 1$ as:

$$\frac{x}{t} = \ln t^\kappa + \xi, \qquad \rho_1 = R_1, \qquad \rho_2 = R_2, \qquad u = \ln t^\kappa + U,$$

which consequently reduces (6.9) as follow:

$$\begin{aligned}(U-\kappa-\xi)\frac{dR_1}{d\xi}+R_1\frac{dU}{d\xi}&=0,\\(U-\kappa-\xi)\frac{dR_2}{d\xi}+R_2\frac{dU}{d\xi}&=0,\\(U-\kappa-\xi)\frac{dU}{d\xi}+\frac{\kappa}{\xi}+\frac{\kappa_1\gamma R_1^{\gamma-1}}{R_1+R_2}\frac{dR_1}{d\xi}+\frac{\kappa_2\gamma R_2^{\gamma-1}}{R_1+R_2}\frac{dR_2}{d\xi}&=0,\end{aligned}\qquad(6.25)$$

which can be solved numerically.

6.4.2 The reduction by analyzing the relations between the Lie group parameters

We can obtain the reduced forms of the system (6.9) by analyzing the relations between the parameters of the symmetry group (6.10-6.14) to show the difference between the two methods. To find similarity variables and new dependent variables, i.e., similarity forms, we consider the characteristic equations associated with the (6.10-6.14) as follow:

$$\frac{dt}{\xi^t}=\frac{dx}{\xi^x}=\frac{d\rho_1}{\eta^{\rho_1}}=\frac{d\rho_2}{\eta^{\rho_2}}=\frac{du}{\eta^u},$$

which could be written as

$$\frac{dt}{\alpha_1+\alpha_2 t}=\frac{dx}{\alpha_3+\alpha_4 t+\alpha_5 x}=\frac{d\rho_1}{\frac{2(\alpha_5-\alpha_2)}{\gamma-1}\rho_1}=\frac{d\rho_2}{\frac{2(\alpha_5-\alpha_2)}{\gamma-1}\rho_2}=\frac{du}{\alpha_4+(\alpha_5-\alpha_2)u}.\qquad(6.26)$$

One can solve the above characteristic equations considering the following cases.

Case A: $\alpha_1\neq 0$ and $\alpha_2\neq 0$.

In this case, the similarity variable and the new dependent variables are obtained as follows

$$\begin{aligned}\rho_1&=R_1\ t^{\dfrac{2(\alpha_5-\alpha_2)}{\alpha_2(\gamma-1)}},\\\rho_2&=R_2\ t^{\dfrac{2(\alpha_5-\alpha_2)}{\alpha_2(\gamma-1)}},\\u&=\frac{U}{\alpha_5-\alpha_2}\ t^{\dfrac{(\alpha_5-\alpha_2)}{\alpha_2(\gamma-1)}}-\frac{\alpha_4}{\alpha_5-\alpha_2},\\\xi&=\frac{\alpha_4 t+(\alpha_5-\alpha_2)\left(x+\frac{\alpha_3}{\alpha_5}\right)}{t^{\dfrac{\alpha_5}{\alpha_2}}}.\end{aligned}\qquad(6.27)$$

Using (6.27) in (6.9), we obtain the following reduced system of ODEs

$$\begin{aligned}
\left(U-\frac{\alpha_5}{\alpha_2}\xi\right)\frac{dR_1}{d\xi}+R_1\frac{dU}{d\xi}+\frac{2(\alpha_5-\alpha_2)}{\alpha_2(\gamma-1)}R_1 &= 0,\\
\left(U-\frac{\alpha_5}{\alpha_2}\xi\right)\frac{dR_2}{d\xi}+R_2\frac{dU}{d\xi}+\frac{2(\alpha_5-\alpha_2)}{\alpha_2(\gamma-1)}R_2 &= 0, \qquad (6.28)\\
\left(U-\frac{\alpha_5}{\alpha_2}\xi\right)\frac{dU}{d\xi}+\frac{\kappa_1\gamma R_1^{\gamma-1}}{R_1+R_2}\frac{dR_1}{d\xi}+\frac{\kappa_2\gamma R_2^{\gamma-1}}{R_1+R_2}\frac{dR_2}{d\xi}+\frac{(\alpha_5-\alpha_2)}{\alpha_2}U &= 0.
\end{aligned}$$

The solution of (6.28) can be obtained as

$$R_1=C_9,\qquad R_2=C_{10}\qquad \text{and}\qquad U=\frac{2(\alpha_2-\alpha_5)}{\alpha_2(\gamma-1)}\xi,$$

where C_9 and C_{10} are arbitrary integration constants and can lead to the solution of (6.9) with

$$\alpha_5=\frac{\gamma-3}{\gamma-1}\quad\text{and}\quad \alpha_2=\frac{-2}{\gamma-1},$$

as below

$$\rho_1=\frac{C_9}{t},\qquad \rho_2=\frac{C_{10}}{t}\qquad\text{and}\qquad u=\left(\frac{x+\frac{\gamma-1}{\gamma-3}\alpha_3}{t}\right).$$

Case B: $\alpha_1=0$ and $\alpha_2=0$.

The similarity variable and the dependent variables associated to this case are

$$\begin{aligned}
\xi &= t,\\
\rho_1 &= R_1\ (\alpha_3+\alpha_4 t+\alpha_5 x)^{\frac{2}{\gamma-1}},\\
\rho_2 &= R_2\ (\alpha_3+\alpha_4 t+\alpha_5 x)^{\frac{2}{\gamma-1}}, \qquad (6.29)\\
u &= \frac{U\ (\alpha_3+\alpha_4 t+\alpha_5 x)}{\alpha_5}-\frac{\alpha_4}{\alpha_5},
\end{aligned}$$

which reduces (6.9) to a system of ODEs as

$$\begin{aligned}
\frac{dR_1}{d\xi}+\frac{\gamma+1}{\gamma-1}R_1U &= 0,\\
\frac{dR_2}{d\xi}+\frac{\gamma+1}{\gamma-1}R_2U &= 0, \qquad (6.30)\\
\frac{dU}{d\xi}+\frac{2\alpha_5^2\kappa_1\gamma R_1^{\gamma}}{R_1+R_2}\frac{dR_1}{d\xi}+\frac{2\alpha_5^2\kappa_2\gamma R_2^{\gamma}}{R_1+R_2}\frac{dR_2}{d\xi}+U^2 &= 0.
\end{aligned}$$

For $U=\frac{1}{\xi}$ and $\kappa_1C_{11}^{\gamma+1}=-\kappa_2C_{12}^{\gamma+1}$, we get the solution of (6.30) as

$$R_1=\frac{C_{11}}{\xi^{\frac{\gamma+1}{\gamma-1}}},\qquad R_2=\frac{C_{12}}{\xi^{\frac{\gamma+1}{\gamma-1}}}\qquad\text{and}\qquad U=\frac{1}{\xi},$$

where C_{11} and C_{12} are integration constants. Finally, the corresponding solution of (6.9) is found to be

$$\begin{aligned} u &= \frac{\alpha_3+\alpha_5 x}{\alpha_5 t}, \\ \rho_1 &= \frac{C_{11}(\alpha_3+\alpha_4 t+\alpha_5 x)^{\frac{2}{\gamma-1}}}{t^{\frac{\gamma+1}{\gamma-1}}}, \\ \rho_2 &= \frac{C_{12}(\alpha_3+\alpha_4 t+\alpha_5 x)^{\frac{2}{\gamma-1}}}{t^{\frac{\gamma+1}{\gamma-1}}}. \end{aligned}$$

Case C: $\alpha_1 = 0$, $\alpha_2 = 0$ and $\alpha_4 = 0$.

The similarity variable and the dependent variables are

$$\begin{aligned} \xi &= t, \quad u = U\ (\alpha_3+\alpha_5 x), \quad \rho_1 = R_1\ (\alpha_3+\alpha_5 x)^{\dfrac{2}{\gamma-1}}, \\ \rho_2 &= R_2\ (\alpha_3+\alpha_5 x)^{\dfrac{2}{\gamma-1}}. \end{aligned} \tag{6.31}$$

Substitution of (6.31) in (6.9) yields the following reduced system of ODEs

$$\begin{aligned} \frac{dR_1}{d\xi} + \alpha_5\left(\frac{\gamma+1}{\gamma-1}\right) R_1 U &= 0, \\ \frac{dR_2}{d\xi} + \alpha_5\left(\frac{\gamma+1}{\gamma-1}\right) R_2 U &= 0, \\ \frac{dU}{d\xi} + \frac{2\alpha_5^2\kappa_1\gamma R_1^{\gamma}}{(\gamma-1)(R_1+R_2)} + \frac{2\alpha_5^2\kappa_2\gamma R_2^{\gamma}}{(\gamma-1)(R_1+R_2)} + U^2 &= 0. \end{aligned} \tag{6.32}$$

Assuming $U = \frac{1}{\xi}$ and $\kappa_1 C_{13}^{\gamma+1} = -\kappa_2 C_{14}^{\gamma+1}$ we obtain the following solution for (6.32)

$$R_1 = C_{13}\xi^{\dfrac{-\alpha_5(\gamma+1)}{(\gamma-1)}}, \qquad R_2 = C_{14}\xi^{\dfrac{-\alpha_5(\gamma+1)}{(\gamma-1)}} \qquad \text{and} \qquad U = \frac{1}{\xi}. \tag{6.33}$$

Using (6.33) together with system (6.31) we find the solution of (6.9) to be

$$\begin{aligned} u &= \frac{(\alpha_3+\alpha_5 x)}{t}, \\ \rho_1 &= C_{13} t^{\dfrac{-\alpha_5(\gamma+1)}{(\gamma-1)}} (\alpha_3+\alpha_5 x)^{\dfrac{2}{\gamma-1}}, \\ \rho_2 &= C_{14} t^{\dfrac{-\alpha_5(\gamma+1)}{(\gamma-1)}} (\alpha_3+\alpha_5 x)^{\dfrac{2}{\gamma-1}}. \end{aligned}$$

Case D: $\alpha_1 = 0$, $\alpha_2 = 0$ and $\alpha_5 = 0$.

These yields the following similarity and new dependent variables

$$\xi = t, \qquad \rho_1 = R_1, \qquad \rho_2 = R_2 \qquad \text{and} \qquad u = U + \frac{\alpha_4 x}{\alpha_3+\alpha_4 t}. \tag{6.34}$$

Substituting these new dependent variables in (6.9), we obtain the following system of ODEs

$$\begin{aligned} \frac{dR_1}{d\xi} + \frac{\alpha_4}{\alpha_3 + \alpha_4 \xi} R_1 &= 0, \\ \frac{dR_2}{d\xi} + \frac{\alpha_4}{\alpha_3 + \alpha_4 \xi} R_2 &= 0, \\ \frac{dU}{d\xi} + \frac{\alpha_4}{\alpha_3 + \alpha_4 \xi} U &= 0. \end{aligned} \tag{6.35}$$

The solution of (6.35) is given by

$$R_1 = \frac{C_{15}}{\alpha_3 + \alpha_4 \xi}, \qquad R_2 = \frac{C_{16}}{\alpha_3 + \alpha_4 \xi} \qquad \text{and} \qquad U = \frac{C_{17}}{\alpha_3 + \alpha_4 \xi}, \tag{6.36}$$

where C_{15}, C_{16} and C_{17} are arbitrary integration constants. Combining (6.34) and (6.36) produces the following solution for (6.9)

$$u = \frac{C_{17} + \alpha_4 x}{\alpha_3 + \alpha_4 t}, \qquad \rho_1 = \frac{C_{15}}{\alpha_3 + \alpha_4 t}, \qquad \rho_2 = \frac{C_{16}}{\alpha_3 + \alpha_4 t}. \tag{6.37}$$

Case E: $\alpha_2 = \alpha_4 = \alpha_5 = 0$.

This case produces the similarity and the new dependent variables as follows

$$u = U, \qquad \rho_1 = R_1, \qquad \rho_2 = R_2, \qquad \xi = x - \frac{\alpha_3}{\alpha_1} t, \tag{6.38}$$

and the corresponding reduced system of ODEs is

$$\begin{aligned} \left(U - \frac{\alpha_3}{\alpha_1}\right) \frac{dR_1}{d\xi} + R_1 \frac{dU}{d\xi} &= 0, \\ \left(U - \frac{\alpha_3}{\alpha_1}\right) \frac{dR_2}{d\xi} + R_2 \frac{dU}{d\xi} &= 0, \\ \left(U - \frac{\alpha_3}{\alpha_1}\right) \frac{dU}{d\xi} + \frac{\kappa_1 \gamma R_1^{\gamma-1}}{R_1 + R_2} \frac{dR_1}{d\xi} + \frac{\kappa_2 \gamma R_2^{\gamma-1}}{R_1 + R_2} \frac{dR_2}{d\xi} &= 0. \end{aligned} \tag{6.39}$$

Considering

$$R_1 = C_{18}\ \xi \quad \text{and} \quad R_2 = C_{19}\ \xi,$$

we obtain

$$U = \frac{\alpha_3}{\alpha_1},$$

where C_{18} and C_{19} are constants such that $\kappa_1 C_{18}^{\gamma-1} = -\kappa_2 C_{19}^{\gamma-1}$. Then the corresponding solution to system (6.9) is obtained as

$$\rho_1 = C_{18}\left(x - \frac{\alpha_3}{\alpha_1} t\right), \qquad \rho_2 = C_{19}\left(x - \frac{\alpha_3}{\alpha_1} t\right) \qquad \text{and} \qquad u = \frac{\alpha_3}{\alpha_1}.$$

Case F: $\alpha_2 = \alpha_3 = \alpha_5 = 0$.

In this case the similarity and the new dependent variables are

$$\xi = \frac{\alpha_4}{2\alpha_1} t^2 - x, \qquad \rho_1 = R_1, \qquad \rho_2 = R_2 \qquad \text{and} \qquad u = \frac{\alpha_4}{\alpha_1} t + U. \tag{6.40}$$

Which consequently reduces system (6.9) into the following system of ODEs

$$\begin{aligned} U\frac{dR_1}{d\xi}+R_1\frac{dU}{d\xi} &= 0,\\ U\frac{dR_2}{d\xi}+R_2\frac{dU}{d\xi} &= 0,\\ -U\frac{dU}{d\xi}+\frac{\kappa_1\gamma R_1^{\gamma-1}}{R_1+R_2}\frac{dR_1}{d\xi}+\frac{\kappa_2\gamma R_2^{\gamma-1}}{R_1+R_2}\frac{dR_2}{d\xi}+\frac{\alpha_4}{\alpha_1} &= 0. \end{aligned} \tag{6.41}$$

The above system of equations can be solved for $U=\sqrt{\xi}$ and $2\alpha_4=\alpha_1$

$$R_1=\frac{C_{20}}{\sqrt{\xi}},\qquad R_2=\frac{C_{21}}{\sqrt{\xi}}\qquad \text{and}\qquad U=\sqrt{\xi},$$

where C_{20}, C_{21} are constants of integration with $\kappa_1 C_{20}^{\gamma-1}=-\kappa_2 C_{21}^{\gamma-1}$. This produces the following solution of (6.9) as

$$\rho_1=\frac{C_{20}}{(\frac{1}{4}t^2-x)^{\frac{1}{2}}},\quad \rho_2=\frac{C_{21}}{(\frac{1}{4}t^2-x)^{\frac{1}{2}}},\quad u=\frac{1}{2}t+(\frac{1}{4}t^2-x)^{\frac{1}{2}}.$$

Case G: $\alpha_2=\alpha_5$ and $\alpha_4=0$.

The similarity and the dependent variables for this case are found to be

$$\xi=\frac{\alpha_3+\alpha_2 x}{\alpha_1+\alpha_2 t},\qquad \rho_1=R_1,\qquad \rho_2=R_2\qquad \text{and}\qquad u=U. \tag{6.42}$$

Substituting (6.42) in (6.9) we obtain the following system of ODEs

$$\begin{aligned} (U-\xi)\frac{dR_1}{d\xi}+R_1\frac{dU}{d\xi} &= 0,\\ (U-\xi)\frac{dR_2}{d\xi}+R_2\frac{dU}{d\xi} &= 0,\\ (U-\xi)\frac{dU}{d\xi}+\frac{\kappa_1\gamma R_1^{\gamma-1}}{R_1+R_2}\frac{dR_1}{d\xi}+\frac{\kappa_2\gamma R_2^{\gamma-1}}{R_1+R_2}\frac{dR_2}{d\xi} &= 0. \end{aligned} \tag{6.43}$$

The above system can be solved by assuming

$$U=\frac{\xi}{2}\quad \text{and}\quad \kappa_1 C_{22}^{\gamma-1}+\kappa_2 C_{23}^{\gamma-1}=\frac{C_{22}+C_{23}}{12}.$$

As a result we find

$$R_1=C_{22}\xi,\qquad R_2=C_{23}\xi\qquad \text{and}\qquad U=\frac{\xi}{2},$$

where C_{22}, C_{23} are arbitrary integration constants. This gives the solution of (6.9) as

$$\rho_1=C_{22}\left(\frac{\alpha_3+\alpha_2 x}{\alpha_1+\alpha_2 t}\right),\quad \rho_2=C_{23}\left(\frac{\alpha_3+\alpha_2 x}{\alpha_1+\alpha_2 t}\right)\quad \text{and}\quad u=\frac{\alpha_3+\alpha_2 x}{2(\alpha_1+\alpha_2 t)}.$$

Case H: $\alpha_3 = \alpha_4 = \alpha_5 = 0$.

Here the corresponding similarity and dependent variables are

$$\begin{aligned} \xi &= x, \quad u = U \ (\alpha_1 + \alpha_2 t)^{-1}, \quad \rho_1 = R_1 \ (\alpha_1 + \alpha_2 t)^{-\dfrac{2}{\gamma - 1}}, \\ \rho_2 &= R_2 \ (\alpha_1 + \alpha_2 t)^{-\dfrac{2}{\gamma - 1}}, \end{aligned} \tag{6.44}$$

and the corresponding reduced system of ODEs is

$$\begin{aligned} U\frac{dR_1}{d\xi} + R_1\frac{dU}{d\xi} - \frac{2}{\gamma - 1}\alpha_2 R_1 &= 0, \\ U\frac{dR_2}{d\xi} + R_2\frac{dU}{d\xi} - \frac{2}{\gamma - 1}\alpha_2 R_2 &= 0, \\ U\frac{dU}{d\xi} + \frac{\kappa_1\gamma R_1^{\gamma-1}}{R_1 + R_2}\frac{dR_1}{d\xi} + \frac{\kappa_2\gamma R_2^{\gamma-1}}{R_1 + R_2}\frac{dR_2}{d\xi} - \alpha_2 U &= 0. \end{aligned} \tag{6.45}$$

The system (6.45) is solved for $U = \alpha_2\xi$ and we get

$$R_1 = C_{24}\xi^{-\dfrac{(\gamma - 3)}{(\gamma - 1)}}, \qquad R_2 = C_{25}\xi^{-\dfrac{(\gamma - 3)}{(\gamma - 1)}} \qquad \text{and} \qquad U = \alpha_2\xi, \tag{6.46}$$

where C_{24} and C_{25} are integration constants with $\kappa_1 C_{24}^{\gamma-1} = -\kappa_2 C_{25}^{\gamma-1}$.

From (6.44) and (6.46), we obtain the following solution for system (6.9) as

$$\begin{aligned} u &= \alpha_2 x(\alpha_1 + \alpha_2 t)^{-1}, \qquad \rho_1 = C_{24}x^{-\dfrac{(\gamma - 3)}{(\gamma - 1)}}(\alpha_1 + \alpha_2 t)^{-\dfrac{2}{\gamma - 1}}, \\ \rho_2 &= C_{25}x^{-\dfrac{(\gamma - 3)}{(\gamma - 1)}}(\alpha_1 + \alpha_2 t)^{-\dfrac{2}{\gamma - 1}}. \end{aligned}$$

Case I: $\alpha_5 = 0$, $\alpha_1 = \alpha_3$ and $\alpha_2 = \alpha_4$.

The similarity and new dependent variables are derived as

$$\begin{aligned} \xi &= x - t, \\ u &= U \ (\alpha_1 + \alpha_2 t)^{-1} + 1, \\ \rho_1 &= R_1 \ (\alpha_1 + \alpha_2 t)^{-\dfrac{2}{\gamma - 1}}, \\ \rho_2 &= R_2 \ (\alpha_1 + \alpha_2 t)^{-\dfrac{2}{\gamma - 1}}. \end{aligned} \tag{6.47}$$

Substituting these new dependent variables in (6.9) leads to the following new system of ODEs

$$\begin{aligned} U\frac{dR_1}{d\xi} + R_1\frac{dU}{d\xi} - \frac{2\alpha_2}{\gamma-1}R_1 &= 0, \\ U\frac{dR_2}{d\xi} + R_2\frac{dU}{d\xi} - \frac{2\alpha_2}{\gamma-1}R_2 &= 0, \\ U\frac{dU}{d\xi} + \frac{\kappa_1\gamma R_1^{\gamma-1}}{R_1+R_2}\frac{dR_1}{d\xi} + \frac{\kappa_2\gamma R_2^{\gamma-1}}{R_1+R_2}\frac{dR_2}{d\xi} - \alpha_2 U &= 0. \end{aligned} \tag{6.48}$$

We solve the above system of ODEs by using $U = \alpha_2\xi$ to obtain

$$R_1 = C_{26}\xi^{\frac{(3-\gamma)}{(\gamma-1)}}, \qquad R_2 = C_{27}\xi^{\frac{(3-\gamma)}{(\gamma-1)}} \qquad \text{and} \qquad U = \alpha_2\xi, \tag{6.49}$$

where C_{26} and C_{27} are arbitrary integration constants with $\kappa_1 C_{26}^{\gamma-1} = -\kappa_2 C_{27}^{\gamma-1}$.

Combining (6.47) and (6.49) we obtain the solution for (6.9) as follows

$$\begin{aligned} u &= \alpha_2(x-t)(\alpha_1+\alpha_2 t)^{-1} + 1, \\ \rho_1 &= C_{26}(x-t)^{\frac{(3-\gamma)}{(\gamma-1)}}(\alpha_1+\alpha_2 t)^{-\frac{2}{\gamma-1}}, \\ \rho_2 &= C_{27}(x-t)^{\frac{(3-\gamma)}{(\gamma-1)}}(\alpha_1+\alpha_2 t)^{-\frac{2}{\gamma-1}}. \end{aligned}$$

Case J: $\alpha_1 \neq 0$ and $\alpha_2 = 0$.

This case yield the similarity and the dependent variables as follows

$$\begin{aligned} u &= \frac{U}{\alpha_5}\exp\left(\frac{\alpha_5}{\alpha_1}t\right) - \frac{\alpha_4}{\alpha_5}, \\ \rho_1 &= R_1\exp\left(\frac{2\alpha_5}{\alpha_1(\gamma-1)}t\right), \\ \rho_2 &= R_2\exp\left(\frac{2\alpha_5}{\alpha_1(\gamma-1)}t\right), \\ \xi &= \left(x + \left(\frac{\alpha_3+\alpha_4 t}{\alpha_5} - \frac{\alpha_4}{\alpha_1}\right)\right)\exp\left(\frac{-\alpha_5}{\alpha_1}t\right). \end{aligned} \tag{6.50}$$

Substitution of the variables from (6.50) in (6.9) we obtain

$$\begin{aligned} \left(\frac{U}{\alpha_5} - \frac{\alpha_5}{\alpha_2}\xi\right)\frac{dR_1}{d\xi} + \frac{R_1}{\alpha_5}\frac{dU}{d\xi} + \frac{\alpha_5}{\alpha_1(\gamma-1)}R_1 &= 0, \\ \left(\frac{U}{\alpha_5} - \frac{\alpha_5}{\alpha_2}\xi\right)\frac{dR_2}{d\xi} + \frac{R_2}{\alpha_5}\frac{dU}{d\xi} + \frac{\alpha_5}{\alpha_1(\gamma-1)}R_2 &= 0, \\ \left(\frac{U}{\alpha_5^2} - \frac{\alpha_5}{\alpha_2}\xi\right)\frac{dU}{d\xi} + \frac{\kappa_1\gamma R_1^{\gamma-1}}{R_1+R_2}\frac{dR_1}{d\xi} + \frac{\kappa_2\gamma R_2^{\gamma-1}}{R_1+R_2}\frac{dR_2}{d\xi} + \frac{U}{\alpha_1} &= 0, \end{aligned}$$

which can be solved numerically.

At this point, it is convenient to comment on the reduction processes due to the above results. The reduced forms and the reduced equations (6.27)-(6.48) are similar in that the reduced forms and reduced equations (6.17)-(6.25) are obtained by finding the optimal system of the symmetry group, but the method of analyzing the relations between the Lie group parameters can provide a different similarity forms in (6.50). Therefore, one can conclude that it is fair to say that the second method is more effective than the first one.

6.5 Evolution of weak discontinuity

The matrix form of the governing hyperbolic system is

$$U_t + QU_x = 0, \tag{6.51}$$

where $U = \Big(\rho_1, \rho_2, u\Big)^T$ is a column vector with superscript T denoting transposition, while Q is Jacobian matrix found to be

$$\begin{pmatrix} u & 0 & \rho_1 \\ 0 & u & \rho_2 \\ \frac{b^2}{\rho_1} & \frac{c^2}{\rho_2} & u \end{pmatrix}, \tag{6.52}$$

with $b_1^2 = \frac{\kappa_1 \gamma \rho_1^\gamma}{(\rho_1+\rho_2)}$ and $c_1^2 = \frac{\kappa_2 \gamma \rho_2^\gamma}{(\rho_1+\rho_2)}$. Further, the eigenvalues of Q are

$$\lambda_1 = u - w_1, \qquad \lambda_2 = u \qquad \text{and} \qquad \lambda_3 = u + w_1,$$

where $w_1 = \sqrt{c_1^2 + b_1^2}$ with the corresponding left and right eigenvectors

$$\begin{aligned} l_1 &= \left(\frac{b_1^2}{\rho_1}, \frac{c_1^2}{\rho_1}, -w_1\right), & r_1 &= (\rho_1, \rho_2, -w_1)^T, \\ l_2 &= (\rho_2, -\rho_1, 0), & r_2 &= \left(\frac{c_1^2}{\rho_1}, -\frac{b_1^2}{\rho_1}, 0\right)^T, \\ l_3 &= \left(\frac{b_1^2}{\rho_1}, \frac{c_1^2}{\rho_1}, w_1\right), & r_3 &= (\rho_1, \rho_2, w_1)^T. \end{aligned} \tag{6.53}$$

The transport equation for the weak discontinuity across the third characteristic of a hyperbolic system of equations is given by [89]:

$$l_3\left(\frac{d\Lambda}{dt} + (U_x + \Lambda)(\nabla\lambda_3)\Lambda\right) + ((\nabla l_3)\Lambda)^T \frac{dU}{dt} + (l_3\Lambda)((\nabla\lambda_3)U_x + (\lambda_3)_x) = 0, \tag{6.54}$$

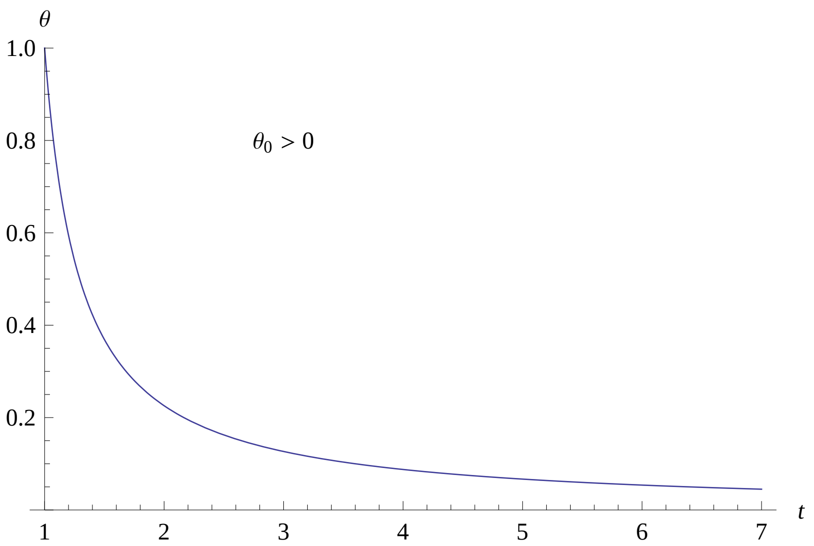

Figure 6.1: The behavior of θ with t for $\theta_0 > 0$.

where Λ, which denotes the jump in U_x across the weak discontinuity, is a collinear to the right eigenvector r_3, i.e., $\Lambda = \theta(t) r_3$ with $\theta(t)$ as the amplitude of the weak discontinuity wave and $\nabla = \left(\frac{\partial}{\partial \rho_1}, \frac{\partial}{\partial \rho_2}, \frac{\partial}{\partial u}\right)$.

Now, substituting (6.37) and (6.56) along with Λ in (6.54) gives the following Bernoulli type equation for the amplitude $\theta(t)$

$$\frac{d\theta}{dt} + \Psi_1(x,t)\theta^2 + \Psi_2(x,t)\theta = 0, \qquad \frac{dx}{dt} = u + w_1, \tag{6.55}$$

where

$$\Psi_1(x,t) = \frac{\gamma^{\frac{1}{2}}}{(C_5 + C_6)^{\frac{1}{2}}(\alpha_3 + \alpha_4 t)^{\frac{\gamma-1}{2}}} \left(\frac{(\gamma+2)(\kappa_1 C_5^{\gamma} + \kappa_2 C_6^{\gamma})}{2} - \frac{(\kappa_1 C_5^{\gamma+1} + \kappa_2 C_6^{\gamma+1})}{(C_5 + C_6)(\kappa_1 C_5^{\gamma} + \kappa_2 C_6^{\gamma})^{\frac{1}{2}}} \right)$$

$$\begin{aligned} \Psi_2(x,t) \;=\; & \frac{2\alpha_4}{(\alpha_3 + \alpha_4 t)} - \frac{(\gamma-1)\kappa_1 \alpha_4 C_5^{\gamma}}{2(\kappa_1 C_5^{\gamma} + \kappa_2 C_6^{\gamma})(\alpha_3 + \alpha_4 t)} - \frac{\alpha_4 \kappa_2 C_6^{\gamma} \left(C_5 \left((\gamma-2) + (\gamma-1)\frac{C_5}{C_6} \right) - C_6 \right)}{2(C_5 + C_6)(\kappa_1 C_5^{\gamma} + \kappa_2 C_6^{\gamma})(\alpha_3 + \alpha_4 t)} \\ & - \left(\frac{\gamma^{\frac{1}{2}}(C_5 + C_6)^{\frac{1}{2}}(\alpha_3 + \alpha_4 t)^{\frac{\gamma-1}{2}}}{4(\kappa_1 C_5^{\gamma} + \kappa_2 C_6^{\gamma})^{\frac{1}{2}}} - \frac{(\kappa_1 C_5^{\gamma+1} + \kappa_2 C_6^{\gamma+1})(\alpha_3 + \alpha_4 t)^{\frac{\gamma}{2}}}{2\gamma^{\frac{1}{2}}(C_5 + C_6)(\kappa_1 C_5^{\gamma} + \kappa_2 C_6^{\gamma})^{\frac{3}{2}}} \right) \left(\frac{\alpha_4(\alpha_4 x + C_7)}{(\alpha_3 + \alpha_4 t)^2} \right) \end{aligned}$$

Integrating (6.55) yields the wave amplitude θ as

$$\theta(t) = \frac{I(t)}{(1 + \theta_0 J(t))} \tag{6.56}$$

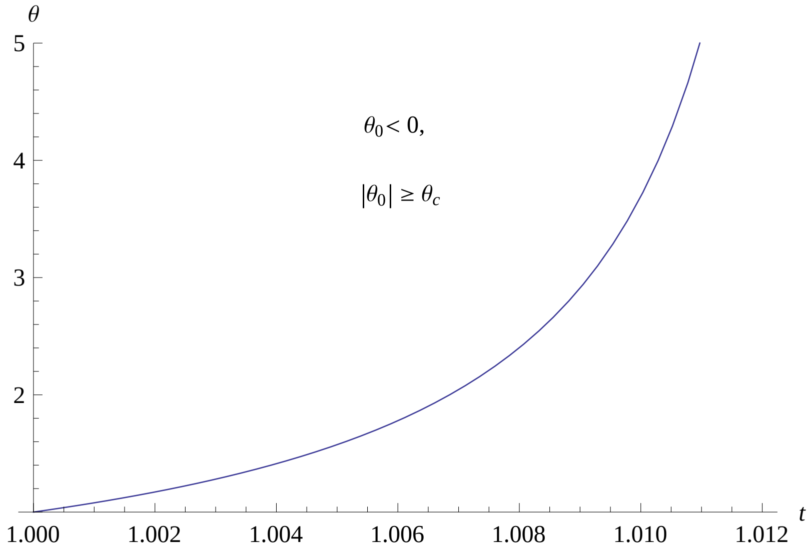

Figure 6.2: The behavior of θ with t for $\theta_0 < 0$ and $|\theta_0| \geq \theta_c$.

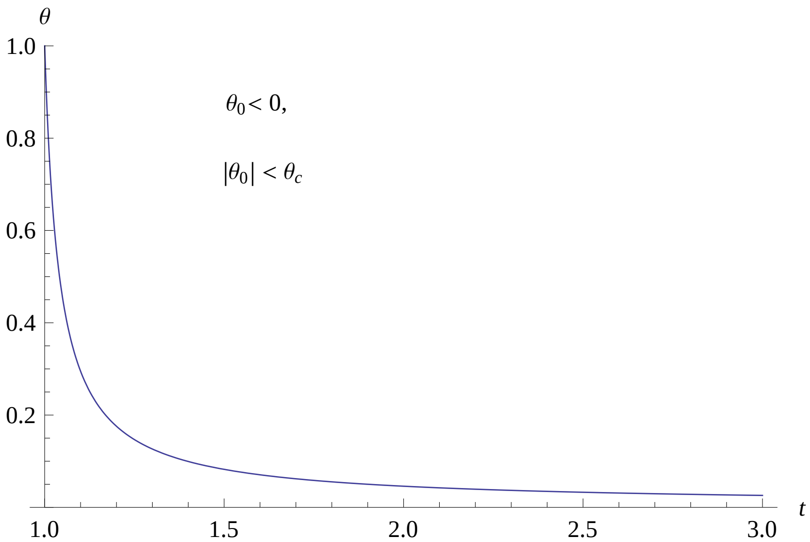

Figure 6.3: The behavior of θ with t for $\theta_0 < 0$ and $|\theta_0| < \theta_c$.

where

$$I(t) = \exp\left(\int_{t_0}^{t} -\Psi_1(x(s), s)ds\right)$$

and

$$J(t) = \int_{t_0}^{t} \Psi_2(x(t^{'}), t^{'}) \exp\left(\int_{t_0}^{t^{'}} -\Psi_1(x(s), s)ds\right) dt^{'}.$$

In the functions Ψ_1 and Ψ_2, we find that both integrals $I(t)$ and $J(t)$ are finite and continuous on $[t_0, \infty)$. With the initial conditions $\theta = \theta_0$ and $x = x_0$ at $t = t_0$, we studied the behavior of the weak discontinuity which is well observed in the Figures 6.1- 6.3. For $\theta_0 > 0$ and $t \to \infty$, it is clear that $I(t) \to 0$ where as $J(\infty) < \infty$, which give rise to an expansion wave and the wave decays and dies out eventually, the corresponding situation is shown in Figure 6.1. However, when $\theta_0 < 0$, which corresponds to a compressive wave, there exists a positive quantity $\theta_c > 0$, for a finite time t_c given by the solution of $J(t_c) = \frac{1}{|\theta_0|}$ such that, when $|\theta_0| \geq \theta_c$, $\theta(t)$ increases from θ_0 and terminates into a shock; the corresponding situation is illustrated by the curve in Figure 6.2. In Figure 6.3, it can be observed that, for $|\theta_0| < \theta_c$, $\theta(t)$ initially decreases from θ_0 and reaches to minimum at finite time.

6.6 Conclusions

In the present study, different similarity variables and reduction forms are obtained by finding the optimal system of the Lie groups and analyzing the group parameters. Those similarity variables and similarity forms are applied to transform the governing system of PDEs of the drift-flux model to a system of ODEs. Further, the reduced system of ODEs is solved analytically. These analytical solutions play an important role to a better understanding of qualitative features of two-phase flow equations. In this context, analytical solutions of non-linear differential equations graphically demonstrate and allow unraveling the mechanisms of many complex non-linear phenomena such as spatial localization of transfer processes, multiplicity or absence of steady states under various conditions, existence of peaking regimes. The behavior of weak discontinuity has been also discussed across the solution curve which is well illustrated by Figures 6.1- 6.3. For $\theta_0 > 0$ or $\theta_0 < 0$ and $|\theta_0| < \theta_c$, in both the cases the wave decays and dies out eventually. This is observed clearly in Figure 6.1 and Figure 6.3. For $\theta_0 < 0$ and $|\theta_0| \geq \theta_c$, Figure 6.2 demonstrate that weak discontinuity culminates into a shock after finite time.

However, analytical solutions in two-phase flow equations are only one way to study the drift-flux model with a clear physical meaning. Further investigation of the theoretical properties of two-phase flow models remain a key requirement as they provide the much-needed validation data for the numerical and analytical methods.

Chapter 7

Conclusions

This chapter presents the summary of the contributed results made in this thesis, followed by the future scopes for possible extensions of the present work.

7.1 Summary of the results

The work carried out in this thesis has achieved many interesting and good results which are concerned with solving hyperbolic PDEs by Lie group analysis method. The exact solutions as well as the similarity solutions obtained for the system of PDEs which are governed by many important physical phenomena. These exact solutions to mathematical equations play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. Exact solutions of nonlinear differential equations graphically demonstrate and allow unraveling the mechanisms of many complex nonlinear phenomena such as spatial localization of transfer processes, multiplicity or absence of steady states under various conditions, existence of peaking regimes, and many others. Furthermore, simple solutions are often used in teaching many courses as specific examples illustrating basic tenets of a theory that admit mathematical formulation such as heat and mass transfer theory, hydrodynamics, gas dynamics, wave theory and other fields. Even those special exact solutions that do not have a clear physical meaning can be used as test problems to verify the consistency and estimate errors of various numerical, asymptotic, and approximate analytical methods.

The chapter wise results of this thesis with some important observations are precisely highlighted below:

- In chapter 1, a short background history of Lie symmetries is provided and the motivation behind our interest is also stated.

- In chapter 2, the unsteady simple flow of an isentropic, inviscid and perfectly conducting compressible fluid, subjected to a transverse magnetic field is considered. Some exact solutions are obtained using Lie group analysis. The impact of magnetic field on the behavior of weak discontinuity and shock formation time is observed through the illustrated graphs.

- In chapter 3, Lie symmetry analysis is performed for an unsteady flow of inviscid and perfectly conducting compressible fluid, in the presence of magnetic field and some particular solutions are discovered. Furthermore, with the solutions in hand we discussed the evolution of weak discontinuity. Then, the behavior of weak discontinuity and shock formation in the presence of magnetic field is noticed.

- In chapter 4, we derived some exact solutions for the one dimensional unsteady flow of an ideal isentropic, inviscid and perfectly conducting compressible fluid, subject to a transverse magnetic field for the magnetogasdynamics system. By using Lie group theory, the full one-parameter infinitesimal transformations group leaving the equations of motion invariant is derived. The symmetry generators are used for constructing similarity variables which leads the system of PDEs to a reduced system of ODEs; in some cases, the reduced system of ODEs is solved exactly. Further, using the exact solution, we discuss the evolutionary behavior of weak discontinuity.

- In chapter 5, a particular exact solution to a quasilinear hyperbolic system of PDEs governing unsteady planar and radially symmetric motion of an inviscid perfectly conducting and non-ideal gas in which the effects of significant magnetic field is derived. The evolution of characteristic shock and the corresponding interaction with the weak discontinuity is studied. The amplitudes of reflected wave, transmitted wave and the jump in shock acceleration influenced by the incident wave after interaction are evaluated. Finally, the influence of van der Waals excluded volume in the behavior of weak discontinuity is completely characterized.

- In chapter 6, we consider a system of quasilinear PDEs governing no-slip drift-flux model for multi phase flows, which is used to investigate the theoretical properties and analytical solutions for a widely used two-phase flow model. Different similarity variables and reduction forms are obtained by finding the optimal system of the Lie groups and analyzing the group parameters. Then given system of PDEs is reduced to system of ODEs and in some cases, the reduced system of ODEs is solved analytically. Then comparison between the above said two methods is presented by observing the similarity variables and the reduced forms. Further, discussed the behavior of weak discontinuity along the solution curve.

7.2 Future scopes

In future we want to study some of the important issues in the context of wave interactions in quasilinear hyperbolic systems. In this regard, we identify the following:

1. To determine an optimal list of inequivalent subalgebras of the maximal Lie invariance algebra of quasilinear PDEs.

2. Obtaining exact solutions to the hyperbolic system of PDEs which satisfy the Rankine Hugoniot jump conditions and the stability conditions.

3. Interaction of weak discontinuities and classical waves such as rarefaction waves and contact discontinuity.

4. Interactions between classical waves and non-classical waves such as delta shock waves and singular shock waves.

5. Solutions of initial and boundary value problems of second order PDEs by using Lie group analysis.

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List of papers accepted/communicated

1. Bira and T. Raja Sekhar, Lie group analysis and propagation of weak discontinuity in one-dimensional ideal isentropic magnetogasdynamics, **Applicable Analysis (Taylor & Francis)** 93 (2014), no. 12, 2598-2607.

2. B. Bira and T. Raja Sekhar, Symmetry group analysis and exact solutions of isentropic magnetogasdynamics, **Indian Journal of Pure and Applied Mathematics (Springer)** 44 (2013), no. 2, 153-165.

3. B. Bira and T. Raja Sekhar, Exact solutions to magnetogasdynamics using Lie point symmetries, **Meccanica (Springer)** 48 (2013), no. 5, 1023-1029.

4. B. Bira and T. Raja Sekhar, Collision of characteristic shock with weak discontinuity in non-ideal magnetogasdynamics, (Communicated).

5. B. Bira and T. Raja Sekhar, The application of Lie groups to an isentropic drift-flux model of two-phase flows, (Communicated).