ADAMS COMPLETION FOR *CW*-COMPLEXES

by

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CERTIFICATE

This is to certify that this thesis entitled "ADAMS COMPLETION FOR CW-COMPLEXES" which is being submitted by Mrs. Sandhya Rani Mohapatra, Ph.D. Student in Mathematics, Studentship Roll No. 510MA903, National Institute of Technology, Rourkela - 769 008 (India), for the award of the Degree of Doctor of Philosophy in Mathematics at National Institute of Technology, is a record of bonafide research work done by her under my supervision. The results embodied in the thesis are new and have not been submitted to any other University or Institution for the award of any Degree or Diploma.

To the best of my knowledge Mrs. Sandhya Rani Mohapatra bears a good moral character and is mentally and physically fit to get the degree.

Professor Akrur Behera
Supervisor

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ABSTRACT

The acyclic tower can be obtained through a general categorical completion process due to Adams. More precisely, the different stages of the acyclic tower of a space are obtained as the Adams completion of the space with respect to the carefully chosen sets of morphisms; it is done in the context of a Serre class of abelian groups. Postnikov-like approximation is obtained for a 1-connected based nilpotent space, in terms of Adams completion with respect to a suitable sets of morphisms, using the primary homotopy theory developed by Neisendonfer. Also Cartan-Whitehead decomposition is obtained for a 0-connected based nilpotent space, in terms of Adams cocompletion with respect to properly chosen sets of morphisms, using the primary homotopy theory developed by Neisendonfer. Under suitable assumption it is proved that weak fibration implies fibration and weak cofibration implies cofibration, as introduced by Bauer and Dugundji.

TABLE OF CONTENTS

Chapter 0 **INTRODUCTION** **1**

Chapter 1 **PRE-REQUISITES** 5

1.1 Category of fractions 6

1.2 Calculus of left (right) fractions 8

1.3 Adams completion and cocompletion 13

1.4 Existence theorems 15

1.5 Couniversal property 17

1.6 Modulo a Serre class $\mathcal{C}$ of abelian groups 22

1.7 Primary homotopy theory 25

Chapter 2 **CATEGORY OF FRACTIONS AND ACYCLIC SPACES** **27**

2.1 The category $\widetilde{\mathcal{CW}}$ 28

2.2 S-completion in $\widetilde{\mathcal{CW}}$ 31

2.3 S_n-completion in $\widetilde{\mathcal{CW}}$ 38

2.4 S_n^*-completion in $\widetilde{\mathcal{CW}}$ 40

2.5 $X_{S_n^*}$ and the n-stage of the acyclic tower in $\widetilde{\mathcal{CW}}$ 42

Chapter	3	**ADAMS COMPLETION AND 1-CONNECTED NILPOTENT SPACE**	**49**
	3.1	The category $\widetilde{\mathcal{N}}_1$	49
	3.2	Existence of Adams completion in $\widetilde{\mathcal{N}}_1$	57
	3.3	Primary approximation of a 1-connected based nilpotent space	58
Chapter	4	**PRIMARY DECOMPOSITION OF A 0-CONNECTED NILPOTENT SPACE**	**62**
	4.1	The category $\mathcal{N}_0$	63
	4.2	Existence of Adams cocompletion in $\widetilde{\mathcal{N}}_0$	69
	4.3	Primary decomposition of a 0-connected based nilpotent space	72
Chapter	5	**ADAMS COMPLETION AND S-FIBRATIONS**	**76**
	5.1	S-fibrations	77
	5.2	S-cofibrations	80
	5.3	Adams completion and S-fibrations	84
		REFERENCES	**86**
		LIST OF RESEARCH PAPERS PUBLISHED/ ACCEPTED / COMMUNICATED	**92**

Chapter 0

INTRODUCTION

It is to be emphasized that many algebraic and geometrical constructions in Algebraic Topology, Differential Topology, Differentiable Manifolds, Algebra, Analysis, Topology, etc., can be viewed as Adams completions or cocompletions of objects in suitable categories, with respect to carefully chosen sets of morphisms.

The notion of generalized completion (Adams completion) arose from a categorical completion process suggested by Adams [1, 3]. Originally this was considered for admissible categories and generalized homology (or cohomology) theories. Subsequently, this notion has been considered in a more general framework by Deleanu, Frei and Hilton [14], where an arbitrary category and an arbitrary set of morphisms of the category are considered; moreover they have also suggested the dual notion, namely the cocompletion (Adams cocompletion) of an object in a category.

It is to be noted that the notion of Adams completions or cocompletions arise via the category of fractions. We recall the abstract definition of the category of fractions [21, 38]. Also for the explicit construction of the category of fractions we refer to [21, 38]. There are some set theoretic difficulties in constructing the category of fractions. These difficulties may be overcome by making some mild hypotheses and using Grothendeick universes. Precisely speaking if we start with a category belonging to a certain Grothendeick universe then the category of fractions with respect to a set of morphisms of the category belongs to a higher universe [38]. We note that the cases in which we are interested, will not present such difficulty. However, Nanda [35] has proved that if the set of morphisms admits a calculus of left (right) fractions then the category of fractions with respect to the set of morphisms of the category belongs to the same universe as to the universe that the category belongs. Also if the set of morphisms of the category admits a calculus of left (right) fractions then the category of fractions can be described nicely; this explicit construction is given in [38].

The central idea of this thesis is to investigate some cases showing how such algebraic and geometrical constructions are characterized in terms of Adams completions or cocompletions. We will deal with such cases involving CW-complexes and the concepts of calculus of left (right) fractions. In fact in each of the characterizations that we have undertaken in our study, the set of morphisms of the category has to admit either calculus of left fractions or calculus of right fractions.

In Chapter 1, we recall the definitions of Grothendiek universe, category of fractions, calculus of left (right) fractions [38] and generalized Adams completions (cocompletions) [14]. We state some results on the existence of global Adams completions (cocompletions) of an object in a cocomplete (complete) category with respect to a set of morphisms in the category [14]. Deleanu, Frei and Hilton [14] have shown that if the set of morphisms in the category is saturated then the Adams completion (cocompletion) of an object is characterized by a certain couniversal property. We state a stronger version of this result proved by Behera and Nanda [6] where the saturation assumption on the set of morphisms is dropped. We also state Behera and Nanda's result [6] that the canonical map from an object to its Adams completion (from Adams cocompletion to the object) is an element of the set of morphisms under very moderate assumption. These two results are fairly general in nature and applicable to most cases of interest.

Given an acyclic space, Dror [18] has given a general procedure for constructing a Postnikov-like tower of acyclic spaces which successively approximate the given space. In Chapter 2, it is shown that the acyclic tower can be obtained through a general categorical completion process due to Adams. More precisely, it is shown that if S_n denotes the set of all $(n+1)$-equivalences in the homotopy category of based CW-complexes which induce isomorphisms in reduced integral homology, then the generalized Adams completion of an acyclic space with respect to S_n is the n-stage of the acyclic tower; it is done in the context of a Serre class of abelian groups.

It is known that the different stages of the Postnikov approximation of a 1-connected space can be obtained as the Adams completions of the space with respect to suitable sets of morphisms [6]. In Chapter 3, Postnikov-like approximation is obtained for a 1-connected based nilpotent space, in terms of Adams completion, using the primary homotopy theory developed by Neisendonfer [36].

It is known that the different stages of the Cartan-Whitehead decomposition of a 0-connected space can be obtained as the Adams cocompletions of the space with respect to suitable sets of morphisms [5]. In Chapter 4, Cartan-Whitehead decomposition is obtained for a 0-connected based nilpotent space, in terms of Adams cocompletion, using the primary homotopy theory developed by Neisendonfer [36].

In Chapter 5, we recall the concepts of S-fibration, weak S-fibration, S-cofibration and weak S-cofibration from [4] and under suitable assumption we prove that weak S-fibration implies S-fibration and weak S-cofibration implies S-cofibration.

Chapter 1

PRE-REQUISITES

In this chapter we recall the definition of Adams completion (cocompletion) and state some results on the existence of global Adams completion (cocompletion) of an object in a category $\mathcal{C}$, with respect to a family of morphisms S in $\mathcal{C}$. We describe Deleanu, Frei and Hilton's characterization of Adams completion (cocompletion) in terms of its couniversal property. We also describe a stronger version of this result proved by Behera and Nanda [6]. We also state Behera and Nanda's result [6] that the canonical map from an object to its Adams completion is an element of the set of morphisms under very moderate assumption. Some more definitions and results are included in the relevant chapters. This chapter serves as the base and background for the subsequent chapters and we shall keep on referring back to it as and when required.

1.1 Category of fractions

In this section we recall the abstract definition of the category of fractions.

1.1.1 Definition. ([38], P. 266) A *Grothendeick universe* (or simply *universe*) is a collection $\mathcal{U}$ of sets such that the following axioms are satisfied:

U (1): If $\{X_i : i \in I\}$ is a family of sets belonging to $\mathcal{U}$, then $\bigcup_{i \in \mathrm{I}} X_i$ is an element of $\mathcal{U}$.

U (2): If $x \in \mathcal{U}$, then $\{x\} \in \mathcal{U}$.

U (3): If $x \in X$ and $X \in \mathcal{U}$ then $x \in \mathcal{U}$.

U (4): If X is a set belonging to $\mathcal{U}$, then $P(X)$, the power set of X, is an element of $\mathcal{U}$.

U (5): If X and Y are elements of $\mathcal{U}$, then $\{X, Y\}$, the ordered pair (X, Y) and $X \times Y$ are elements of $\mathcal{U}$.

We fix a universe $\mathcal{U}$ that contains $\mathbb{N}$ the set of natural numbers (and so $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$).

1.1.2 Definition. ([38], p. 267) A category $\mathcal{C}$ is said to be a *small $\mathcal{U}$-category*, $\mathcal{U}$ being a fixed Grothendeick universe, if the following conditions hold:

S (1) : The objects of $\mathcal{C}$ form a set which is an element of $\mathcal{U}$.

S (2) : For each pair (X, Y) of objects of $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an element of $\mathcal{U}$.

1.1.3 Definition. ([38], p. 269) Let $\mathcal{C}$ be any arbitrary category and S a set of morphisms of $\mathcal{C}$. A *category of fractions* of $\mathcal{C}$ with respect to S is a category denoted by $\mathcal{C}[S^{-1}]$ together with a functor

$$F_S : \mathcal{C} \to \mathcal{C}[S^{-1}]$$

having the following properties :

CF (1) : For each $s \in S$, $F_S(s)$ is an isomorphism in $\mathcal{C}[S^{-1}]$.

CF (2) : F_S is universal with respect to this property : If $G : \mathcal{C} \to \mathcal{D}$ is a functor such that $G(s)$ is an isomorphism in $\mathcal{D}$, for each $s \in S$, then there exists a unique functor $H : \mathcal{C}[S^{-1}] \to \mathcal{D}$ such that $G = HF_S$. Thus we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F_S"] \arrow[d, "G"'] & \mathcal{C}[S^{-1}] \arrow[dl, dashed, "H"] \\
\mathcal{D} &
\end{tikzcd}$$

1.1.4 Remark. For the explicit construction of the category $\mathcal{C}[S^{-1}]$, we refer to [38]. We content ourselves merely with the observation that the objects of $\mathcal{C}[S^{-1}]$ are same as those of $\mathcal{C}$ and in the case when S admits a calculus of left (right) fractions, the category $\mathcal{C}[S^{-1}]$ can be described very nicely [21, 38].

1.2 Calculus of left (right) fractions

The concepts of calculus of left fractions and right fraction play a crucial role in constructing the category of fractions $\mathcal{C}[S^{-1}]$.

1.2.1 Definition. ([38], p. 258) A family of morphisms S in the category $\mathcal{C}$ is said to admit a *calculus of left fractions* if

(a) S is closed under finite compositions and contains identities of $\mathcal{C}$,

(b) any diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & Y \\ Z & \end{tikzcd}$$

in $\mathcal{C}$ with $s \in S$ can be completed to a diagram

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[t]{} & W \end{array}$$

with $t \in S$ and $tf = gs$,

(c) given

$$X \xrightarrow{s} Y \underset{g}{\overset{f}{\rightrightarrows}} Z \overset{t}{\dashrightarrow} W$$

with $s \in S$ and $fs = gs$, there is a morphism $t : Z \to W$ in S such that $tf = tg$.

A simple characterization for a family of morphisms S to admit a calculus of left fractions is the following.

1.2.2 Theorem. ([14], Theorem 1.3, p. 67) *Let S be a closed family of morphisms of $\mathcal{C}$ satisfying*

(a) *if $uv \in S$ and $v \in S$, then $u \in S$,*

(b) *every diagram*

$$\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, "f"'] & \bullet \\ \bullet & \end{tikzcd}$$

in $\mathcal{C}$ *with* $s \in S$ *can be embedded in a weak push-out diagram*

$$\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "t"'] & \bullet \end{tikzcd}$$

with $t \in S$.

Then S *admits a calculus of left fractions.*

The notion of a set of morphisms admitting a calculus of right fractions is defined dually.

1.2.3 Definition. ([38], p. 267) A family S of morphisms in a category $\mathcal{C}$ is said to admit a *calculus of right fractions* if

(a) any diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Z \arrow[r, "s"'] & Y \end{tikzcd}$$

in $\mathcal{C}$ with $s \in S$ can be completed to a diagram

$$\begin{tikzcd} W \arrow[r, "t"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Z \arrow[r, "s"'] & Y \end{tikzcd}$$

with $t \in S$ and $ft = sg$,

(b) given

$$\begin{tikzcd} W \arrow[r, dashed, "t"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "s"] & Z \end{tikzcd}$$

with $s \in S$ and $sf = sg$, there is a morphism $t : W \to X$ in S such that $ft = gt$.

The analog of Theorem 1.2.2 follows immediately by duality.

1.2.4 Theorem. ([14], Theorem1.3*, p. 70) *Let S be a closed family of morphisms of $\mathcal{C}$ satisfying*

(a) *if $vu \in S$ and $v \in S$, then $u \in S$,*

(b) *any diagram*

$$\begin{tikzcd} & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "s"'] & \bullet \end{tikzcd}$$

in $\mathcal{C}$ *with* $s \in S$, *can be embedded in a weak pull-back diagram*

$$\begin{tikzcd} \bullet \arrow[r, "t"] \arrow[d, "g"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "s"'] & \bullet \end{tikzcd}$$

with $t \in S$.

Then S *admits a calculus of right fractions.*

1.2.5 Remark. There are some set-theoretic difficulties in constructing the category $\mathcal{C}[S^{-1}]$; these difficulties may be overcome by making some mild hypotheses and using Grothendeick universes. Precisely speaking, the main logical difficulty involved in the construction of a category of fractions and its use, arises from the fact that if the category $\mathcal{C}$ belongs to a particular universe, the category $\mathcal{C}[S^{-1}]$ would, in general belongs to a higher universe ([38], Proposition 19.1.2). In most applications, however, it is necessary that we remain within the given initial universe. This logical difficulty can be overcome by making some kind of assumptions which would ensure that the category of fractions remains within the same universe [13, 15]. Also the following theorem (Theorem 1.2.6) shows that if S admits a calculus of left (right) fractions, then the category of fractions $\mathcal{C}[S^{-1}]$ remains within the same universe as to the universe to which the category $\mathcal{C}$ belongs.

The following results will be used in our study.

1.2.6 Theorem. ([35], Proposition, p. 425) *Let $\mathcal{C}$ be a small $\mathcal{U}$-category and S a set of morphisms of $\mathcal{C}$ that admits a calculus of left (right) fractions. Then $\mathcal{C}[S^{-1}]$ is a small $\mathcal{U}$-category.*

1.2.7 Theorem. ([38], Lemma 19.2.6, p. 261) *Let $\mathcal{C}$ be an arbitrary category and S a set of morphisms of $\mathcal{C}$. Let $\mathcal{C}[S^{-1}]$ denote the category of fractions of $\mathcal{C}$ with respect to S and*

$$F_S : \mathcal{C} \to \mathcal{C}[S^{-1}]$$

be the canonical functor. Let the following hold:

(a) *S consists of monomorphisms.*

(b) *S admits a calculus of left fractions.*

Then F_S is faithful.

1.3 Adams completion and cocompletion

We recall the definitions of Adams completion and cocompletion.

1.3.1 Definition. [14] Let $\mathcal{C}$ be an arbitrary category and S a set of morphisms of $\mathcal{C}$. Let $\mathcal{C}[S^{-1}]$ denote the category of fractions of $\mathcal{C}$ with respect to S and

$$F : \mathcal{C} \to \mathcal{C}[S^{-1}]$$

be the canonical functor. Let $\mathcal{S}$ denote the category of sets and functions. Then for a given object Y of $\mathcal{C}$,

$$\mathcal{C}[S^{-1}](-, Y) : \mathcal{C} \to \mathcal{S}$$

defines a contravariant functor. If this functor is representable by an object Y_S of $\mathcal{C}$, i.e.,

$$\mathcal{C}[S^{-1}](-, Y) \cong \mathcal{C}(-, Y_S)$$

then Y_S is called the (*generalized*) *Adams completion of* Y with respect to the set of morphisms S or simply the S-*completion* of Y, We shall often refer to Y_S as the *completion* of Y [14].

The above definition can be dualized as follows:

1.3.2 Definition. [14] Let $\mathcal{C}$ be an arbitrary category and S a set of morphisms of $\mathcal{C}$. Let $\mathcal{C}[S^{-1}]$ denote the category of fractions of $\mathcal{C}$ with respect S and

$$F : \mathcal{C} \to \mathcal{C}[S^{-1}]$$

be the canonical functor. Let $\mathcal{S}$ denote the category of sets and functions. Then for a given object Y of $\mathcal{C}$,

$$\mathcal{C}[S^{-1}](Y, -) : \mathcal{C} \to \mathcal{S}$$

defines a covariant functor. If this functor is representable by an object Y_S of $\mathcal{C}$, i.e.,

$$\mathcal{C}[S^{-1}](Y,-) \cong \mathcal{C}(Y_S,-)$$

then Y_S is called the (*generalized*) *Adams cocompletion* of Y with respect to the set of morphisms S or simply the *S-cocompletion of Y*. We shall often refer to Y_S as the cocompletion of Y [14].

1.4 Existence theorems

We recall some results on the existence of Adams completion and cocompletion. We state Deleanu's theorem [15] that under certain conditions, global Adams completion of an object always exists.

1.4.1 Theorem. ([15], Theorem 1; [35], Theorem 1) *Let $\mathcal{C}$ be a cocomplete small $\mathcal{U}$-category ($\mathcal{U}$ is a fixed Grothendeick universe) and S a set of morphisms of $\mathcal{C}$ that admits a calculus of left fractions. Suppose that the following compatibility condition with coproduct is satisfied.*

(C) *If each $s_i : X_i \to Y_i$, $i \in I$, is an element of S, where the index set I is an element of $\mathcal{U}$*, then

$$\coprod_{i\in I} s_i \;:\; \coprod_{i\in I} X_i \;\to\; \coprod_{i\in I} Y_i$$

is an element of S.

Then every object X of $\mathcal{C}$ has an Adams completion X_S with respect to the set of morphisms S.

1.4.2 Remark. Deleanu's theorem quoted above has an extra condition to ensure that $\mathcal{C}[S^{-1}]$ is again a small $\mathcal{U}$-category; in view of Theorem 1.2.6 the extra condition is not necessary.

Theorem 1.4.1 can be dualized as follows (it is also Theorem 2 in [35]).

1.4.3 Theorem. *Let* $\mathcal{C}$ *be a complete small* $\mathcal{U}$*-category* ($\mathcal{U}$ *is a fixed Grothendeick universe*) *and* S *a set of morphisms of* $\mathcal{C}$ *that admits a calculus of right fractions. Suppose that the following compatibility condition with product is satisfied.*

(C) *If each* $s_i : X_i \to Y_i$, $i \in I$, *is an element of* S, *where the index set* I *is an element of* $\mathcal{U}$, *then*

$$\prod_{i \in I} s_i : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$$

is an element of S.

Then every object X *of* $\mathcal{C}$ *has an Adams cocompletion* X_S *with respect to the set of morphisms* S.

We will recall some more results on the existence of Adams completion and cocompletion in the relevant chapters.

1.5 Couniversal property

The concepts of Adams completion and cocompletion can be characterized in terms of a couniversal property.

1.5.1 Definition. [14] Given a set S of morphisms of $\mathcal{C}$, we define $\bar{S}$, the *saturation* of S as the set of all morphisms u in $\mathcal{C}$ such that $F(u)$ is an isomorphism in $\mathcal{C}[S^{-1}]$. S is said to be *saturated* if $S = \bar{S}$.

1.5.2 Theorem. ([14], Proposition 1.1, p. 63) A family S of morphisms of $\mathcal{C}$ is *saturated* if and only if there exists a functor $F : \mathcal{C} \to \mathcal{D}$ such that S is the collection of all morphisms f such that $F(f)$ is invertible.

Deleanu, Frei and Hilton have shown that if the set of morphisms S is saturated then the Adams completion of a space is characterized by a certain couniversal property.

1.5.3 Theorem. ([14], Theorem 1.2, p. 63) *Let S be a saturated family of morphisms of $\mathcal{C}$ admitting a calculus of left fractions. Then an object Y_S of $\mathcal{C}$ is the S-completion of the object Y with respect to S if and only if there exists a morphism $e : Y \to Y_S$ in S which is couniversal with respect to morphisms of S: given a morphism $s : Y \to Z$ in S there exists a unique morphism $t : Z \to Y_S$ in S, such that $ts = e$. In other words, the following diagram is commutative*:

$$\begin{tikzcd} Y \arrow[r, "e"] \arrow[d, "s"'] & Y_S \\ Z \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

Theorem 1.5.3 can be dualized as follows.

1.5.4 Theorem. ([14], Theorem 1.4, p. 68) *Let S be a saturated family of morphisms of $\mathcal{C}$ admitting a calculus of right fractions. Then an object Y_S of $\mathcal{C}$ is the S-cocompletion of the object Y with respect to S if and only if there exists a morphism $e : Y_S \to Y$ in S which is couniversal with respect to morphisms of S: given a morphism $s : Z \to Y$ in S there exists a unique morphism $t : Y_S \to Z$ in S such that $st = e$. In other words, the following diagram is commutative*:

$$\begin{tikzcd} Y_S \arrow[r, "e"] \arrow[d, dashed, "t"'] & Y \\ Z \arrow[ur, "s"'] & \end{tikzcd}$$

In most of applications, however, the set of morphisms S is not saturated. The following is a stronger version of Deleanu, Frei and Hilton's characterization of Adams completion in terms of a couniversal property.

1.5.5 Theorem. ([6], Theorem 1.2, p. 528) *Let S be a set of morphisms of $\mathcal{C}$ admitting a calculus of left fractions. Then an object Y_S of $\mathcal{C}$ is the S-*

completion of the object Y *with respect to* S *if and only if there exists a morphism* $e : Y \to Y_S$ *in* $\bar{S}$ *which is couniversal with respect to morphisms of* S *: given a morphism* $s : Y \to Z$ *in* S *there exists a unique morphism* $t : Z \to Y_S$ *in* $\bar{S}$ *such that* $ts = e$. *In other words, the following diagram is commutative:*

$$\begin{tikzcd} Y \arrow[r, "e"] \arrow[d, "s"'] & Y_S \\ Z \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

Theorem 1.5.5 can be dualized as follows.

1.5.6 Theorem. ([5], Proposition 1.1, p. 224) *Let* S *be a set of morphisms of* $\mathcal{C}$ *admitting a calculus of right fractions. Then an object* Y_S *of* $\mathcal{C}$ *is the* S*-cocompletion of the object* Y *with respect to* S *if and only if there exists a morphism* $e : Y_S \to Y$ *in* $\bar{S}$ *which is couniversal with respect to morphisms of* S*: given a morphism* $s : Z \to Y$ *in* S *there exists a unique morphism* $t : Y_S \to Z$ *in* $\bar{S}$ *such that* $st = e$. *In other words, the following diagram is commutative:*

$$\begin{tikzcd} Y_S \arrow[r, "e"] \arrow[d, dashed, "t"'] & Y \\ Z \arrow[ur, "s"'] & \end{tikzcd}$$

For most of the application it is essential that the morphism $e : Y \to Y_S$ $(e : Y_S \to Y)$ has to be in S; this is the case when S is saturated and the results are as follows:

1.5.7 Theorem. ([14], Theorem 2.9, p. 76) *Let S be a saturated family of morphisms of $\mathcal{C}$ and let every object of $\mathcal{C}$ admit an S-completion. Then the morphism $e : Y \to Y_S$ belongs to S and is universal for morphisms to S-complete objects and couniversal for morphisms in S.*

The above result can be dualized as follows.

1.5.8 Theorem. ([14], dual of Theorem 2.9, p. 76) *Let S be a saturated family of morphisms of $\mathcal{C}$ and let every object of $\mathcal{C}$ admit an S-cocompletion. Then the morphism $e : Y_S \to Y$ belongs to S and is universal for morphisms to S-cocomplete objects and couniversal for morphisms in S.*

However, in many cases of practical interest S is not saturated. The following result shows that under some extra conditions on S, the morphism $e : Y \to Y_S$ $(e : Y_S \to Y)$ always belongs to S.

1.5.9 Theorem. ([6], Theorem 1.3, p. 533) *Let S be a set of morphisms in a category $\mathcal{C}$ admitting a calculus of left fractions. Let $e : Y \to Y_S$ be the canonical morphism as defined in Theorem* 1.5.5, *where Y_S is the S-*

completion of Y. *Furthermore, let* S_1 *and* S_2 *be sets of morphisms in the category* $\mathcal{C}$ *which have the following properties*:

(a) S_1 *and* S_2 *are closed under composition,*

(b) $fg \in S_1$ *implies that* $g \in S_1$,

(c) $fg \in S_2$ *implies that* $f \in S_2$,

(d) $S = S_1 \cap S_2$.

Then $e \in S$.

Theorem 1.5.9 can be dualized as follows:

1.5.10 Theorem. ([6], dual of Theorem 1.3, p.533) *Let* S *be a set of morphisms in a category* $\mathcal{C}$ *admitting a calculus of right fractions. Let* $e : Y_S \to Y$ *be the canonical morphism as defined in Theorem* 1.5.6, *where* Y_S *is the* S*-cocompletion of* Y. *Furthermore, let* S_1 *and* S_2 *be sets of morphisms in the category* $\mathcal{C}$ *which have the following properties*:

(a) S_1 *and* S_2 *are closed under composition,*

(b) $fg \in S_1$ *implies that* $g \in S_1$,

(c) $fg \in S_2$ *implies that* $f \in S_2$,

(d) $S = S_1 \cap S_2$.

Then $e \in S$.

The following result has many applications.

1.5.11 Theorem. ([14], Theorem 2.10, p. 76) *Let* S *be a saturated family of morphisms of the category* $\mathcal{C}$. *Then the following three statements are equivalent*:

(a) *Every object* Y *in* $\mathcal{C}$ *admits an* S*-completion*.

(b) S *admits a calculus of left fractions,* $\varinjlim P_Y$ *exists for all* Y, *where* $P_Y : \mathcal{C}(Y;S) \to \mathcal{C}$, *and* F_S *commutes with* $\varinjlim P_Y$.

(c) S *admits a calculus of left fractions,* $\varinjlim P_Y$ *exists for all* Y *and* F_S *commutes with all colimits in* $\mathcal{C}$.

1.6 Modulo a Serre class $\mathcal{C}$ of abelian groups

In Chapter 2 we use "modulo a Serre class $\mathcal{C}$ of abelian groups" [40] to obtain the mod-$\mathcal{C}$ acyclic decomposition of a 1-connected based CW-complex, with the help of a suitable set of morphisms. In order to make the exposition self-contained, we collect some relevant definitions and theorems involving Serre classes of abelian groups [40].

1.6.1 Definition. [40] A nonempty class $\mathcal{C}$ of abelian groups is called a *Serre class of abelian groups* if and only if whenever the three-term sequence $A \to B \to C$ of abelian groups is exact and $A, C \in \mathcal{C}$, then $B \in \mathcal{C}$.

The following theorem is easy to verify.

1.6.2 Theorem. ([40], Theorem 1, p. 504) *A class $\mathcal{C}$ of abelian groups is a Serre class if and only if it has the following properties*:

(a) $\mathcal{C}$ *contains a trivial group*.

(b) *If* $A \in \mathcal{C}$ *and* $A \approx A'$, *then* $A' \in \mathcal{C}$,

(c) *If* $A \subset B$ *and* $B \in \mathcal{C}$, *then* $A \in \mathcal{C}$,

(d) *If* $A \subset B$ *and* $B \in \mathcal{C}$, *then* $B/A \in \mathcal{C}$,

(e) *If* $0 \to A \to B \to C \to 0$ *is a short exact sequence*, *with* $A, C \in \mathcal{C}$, *then* $B \in \mathcal{C}$.

1.6.3 Examples. ([40], p. 505) We list some examples of Serre class:

(a) The class of all abelian groups.

(b) The class of trivial groups.

(c) The class of finitely generated abelian groups.

(d) The class of finite abelian groups.

(e) The class of torsion abelian groups.

(f) The class of p-groups for a given prime p.

(g) The class of groups having no element with order a positive power of a given prime p.

1.6.4 Definition. [40] Let $\mathcal{C}$ be a Serrre class of abelian groups and $A, B \in \mathcal{C}$. A homomorphism $f : A \to B$

(a) is a $\mathcal{C}$-*monomorphism* if $\ker f \in \mathcal{C}$,

(b) is a $\mathcal{C}$-*epimorphism* if $\operatorname{coker} f \in \mathcal{C}$,

(c) is a $\mathcal{C}$-*isomorphism* if it is both a $\mathcal{C}$-monomorphism and $\mathcal{C}$-epimorphism.

1.6.5 Theorem. [40] *Let* $A, B \in \mathcal{C}$ *and* $f : A \to B$ *and* $g : B \to C$ *be two homomorphisms. Then the following statements are true.*

(a) *If* gf *is* $\mathcal{C}$*-monic, then so is* f.

(b) *If* gf *is* $\mathcal{C}$*-epic, then so is* g.

1.6.6 Theorem. (The mod-$\mathcal{C}$ Five lemma) ([40], p. 519) *Suppose that*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & A_4 \arrow[r] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \end{tikzcd}$$

is a diagram of abelian groups and homomorphisms in $\mathcal{C}$ *in which the rows are exact and each square is commutative. Then the following hold*:

(a) *If* f_2, f_4 *are* $\mathcal{C}$*-monomorphisms and* f_1 *is a* $\mathcal{C}$*-epimorphism, then* f_3 *is a* $\mathcal{C}$*-monomorphism.*

(b) *If* f_2, f_4 *are* $\mathcal{C}$*-epimorphisms and* f_5 *is a* $\mathcal{C}$*-monomorphism, then* f_3 *is a* $\mathcal{C}$*-epimorphism.*

1.6.7 Definition. [40] A Serre class $\mathcal{C}$ is called an *ideal of abelian groups* if $A \in \mathcal{C}$ implies $A \otimes B$, $\operatorname{Tor}(A, B) \in \mathcal{C}$ for arbitrary B.

1.6.8 Definition. [40] A topological space X is said to be $\mathcal{C}$-*acyclic* if its integral homology groups $H_n(X) \in \mathcal{C}$ for all $n > 0$.

1.6.9 Definition. [40] A Serre class $\mathcal{C}$ of abelian groups is said to be an *acyclic class* if any space of type $(\pi, 1)$ with $\pi \in \mathcal{C}$ is $\mathcal{C}$-acyclic.

1.7 Primary homotopy theory.

In [36] Neisendorfer has studied primary homotopy theory in a very precise manner by giving complete proofs of most results so that the subject should be accessible to lay topologists. We briefly recall the primary homotopy theory.

1.7.1 Definition. [36] Let S^m denote the m-dimensional sphere. Suppose $m \geq 2$, and let $k : S^m \to S^m$ denote a map of degree k. The space $S^{m-1} \cup_k e^m$ is denoted by $P^m(k)$ or $P^m(\mathbb{Z}/k\mathbb{Z})$. If $m \geq 2$, the m-th *mod* k *homotopy group* of X is $[P^m(k); X]$, denoted by $\pi_m(X; \mathbb{Z}/k\mathbb{Z})$. If $m \geq 3$, $\pi_m(X; \mathbb{Z}/k\mathbb{Z})$ is a group and $m \geq 4$, $\pi_m(X; \mathbb{Z}/k\mathbb{Z})$ is an abelian group.

If $f : X \longrightarrow Y$ is a map, then there are induced maps

$$f_* : \pi_m(X; \mathbb{Z}/k\mathbb{Z}) \to \pi_m(Y; \mathbb{Z}/k\mathbb{Z})$$

defined by $f_*[\,g\,] = [\,fg\,]$. If $m \geq 3$, f_* is a homomorphism and, if $m = 2$, f_* is compatible with the action of π_2 [36].

1.7.2 Definition. [24] For a group G, the *lower central series*

$$\cdots \subseteq \Gamma^{i+1}(G) \subseteq \Gamma^{i}(G) \cdots \subseteq \Gamma^{1}(G)$$

of G, is defined by the setting

$$\Gamma^{1}(G) = G, \ \Gamma^{i+1}(G) = [G, \Gamma^{i}(G)], \ i \geq 1.$$

G is said to be *nilpotent* if $\Gamma^{j}(G) = \{1\}$ for j sufficiently large.

1.7.3 Definition. [24] A connected CW-complex X is said to *nilpotent* if $\pi_1(X)$ is nilpotent and operates nilpotently on $\pi_n(X)$ for every $n \geq 2$.

The following result will be used in our study.

1.7.4 Theorem. ([36], Corollary 3.10) *Let $f : X \to Y$ be a map between simply connected spaces. Then*

$$f_* : \ \pi_i(X; \mathbb{Z}/k\mathbb{Z}) \to \pi_i(Y; \mathbb{Z}/k\mathbb{Z})$$

is a bijection for all $i \geq 2$ *if and only if*

$$f_* : \ H_i(X; \mathbb{Z}/k\mathbb{Z}) \to H_i(Y; \mathbb{Z}/k\mathbb{Z})$$

is a bijection for all $i \geq 2$.

Chapter 2

CATEGORY OF FRACTIONS AND ACYCLIC SPACES

Given an acyclic space X (i.e. a topological space X with $\widetilde{H}_*(X) = 0$, where $\widetilde{H}_*(X)$ denotes the reduced integral homology functor), Dror [18] has given a general procedure for constructing a tower of acyclic spaces (which he calls an acyclic decomposition of X) which successively approximate X. An acyclic decomposition of a space is a Postnikov-like decomposition and, as Dror has shown, has many advantages, particularly in analyzing the homotopy structure of an acyclic space. The n-stage of an acyclic decomposition in Dror's construction is obtained via the acyclic functor from the corresponding n-stage of the Postnikov tower. The object of this chapter is to demonstrate that the acyclic decomposition tower can be obtained through a general categorical completion process which is due to Adams [1] and which has been developed by Deleanu, Frei and Hilton [14] in a much more general context. More precisely, it is shown that each stage of acyclic tower can be obtained as the generalized Adams completion of a certain sets of morphisms in the category of the based CW-complexes. The relationship between any stage of the acyclic decomposition and the corresponding stage of the Postnikov decomposition is also clearly demonstrated within the frame work of generalized Adams completion. We formulate the results in the homotopy category of the based CW-complexes. There is no loss of generality in working in this category; for if X is an arbitrary topological space, then there is a CW-complex Y and a map $f : Y \to \mathrm{X}$ which is a weak homotopy equivalence and since the reduced integral homology functor satisfies the weak homotopy equivalence axiom ([42], p.181), it follows that if X is acyclic, so is Y. The problem is then to construct a tower of acyclic spaces in the homotopy category of the based CW-complexes whose inverse limit is Y. This we do using the notation of generalized Adams completion in the context of a Serre class of abelian groups [42], a brief summary of which is already given in Section 1.6 above. From now onwards, we assume that $\mathcal{C}$ is a Serre class which is moreover an acyclic ideal of abelian groups.

2.1 The category $\widetilde{\mathcal{CW}}$

Let $\mathcal{CW}$ denote the category of 1-connected based CW-complexes and based maps and let $\widetilde{\mathcal{CW}}$ be the corresponding homotopy category. We assume that the underlying sets of the elements of $\widetilde{\mathcal{CW}}$ are elements of $\mathcal{U}$ where $\mathcal{U}$ is a fixed Grothendeick universe. We now fix suitable sets of morphisms of $\widetilde{\mathcal{CW}}$ as stated below :

(a) Let $S = \{\alpha : X \to Y$ in $\widetilde{\mathcal{CW}} \mid \alpha_* : \widetilde{H}_* (X) \to \widetilde{H}_* (Y)$ is a $\mathcal{C}$-isomorphisms in reduced integral homology$\}$.

(b) A map $\alpha : X \to Y$ in $\widetilde{\mathcal{CW}}$ is called a mod-$\mathcal{C}$ $(n+1)$-equivalence if $\alpha_* : \pi_*(X) \to \pi_*(Y)$ is a $\mathcal{C}$-isomorphism for $m \leq n$ and a $\mathcal{C}$-epimorphisms for $m = n+1$.

Let S_n denote the set all a mod-$\mathcal{C}$ $(n+1)$ equivalences in $\widetilde{\mathcal{CW}}$.

(c) Let $S_n^* = S_n \cap S$.

Since the category $\widetilde{\mathcal{CW}}$ as stated above is neither cocomplete nor small, Theorem 1.4.1 cannot be used to show the existence of Adams completion of an object in the category of $\widetilde{\mathcal{CW}}$ with respect to the set of morphisms S. However, we have the following result (Theorem 2.1.1) which is essentially Theorem 4.7 [1] and Theorem 3.8 [3] (it is also a generalization of Theorem [13]).

2.1.1 Theorem. *Let $\mathcal{U}$ be a fixed Grothendieck universe. Let $\tilde{\mathcal{C}}$ be the category defined as follows : the objects of $\tilde{\mathcal{C}}$ are the connected based CW-complexes whose underlying sets are elements of $\mathcal{U}$; the morphisms of $\tilde{\mathcal{C}}$ are based homotopy classes of based-point preserving maps between such CW-complexes. Let S be a family of morphisms of $\tilde{\mathcal{C}}$ admitting a* *calculus of left fractions and satisfying the following axioms of compatibility with coproducts* :

(C) *If* $s_i : X_i \to Y_i$ *lies in* S *for each* $i \in I$, *where the index set* I *is an element of* $\mathcal{U}$; *then*

$$\vee_{i\in I} s_i : \vee_{i\in I} X_i \to \vee_{i\in I} Y_i$$

lies in S.

Assume that the family S *and the object* X *of* $\tilde{C}$ *satisfy the condition* :

(*) *There exists a subset* S_X *of the set* $\{s : X \to X' \mid s \in S\}$ *such that* S_X *is an element of the universe* $\mathcal{U}$ *and for each* $s : X \to X'$, $s \in S$, *there exist an* $s' : X \to X''$ *in* S_X *and a morphism* $u : X' \to X''$ *of* $\tilde{C}$ *rendering the following diagram commutative* :

$$\begin{tikzcd}
X \arrow[d, "s"'] \arrow[dr, dashed, "s'"] & \\
X' \arrow[r, "u"'] & X''
\end{tikzcd}$$

Then the Adams completion X_S *of* X *does exist.*

2.1.2 Note. As remarked by Adams of page 34 of [3] this result remains valid if $\tilde{C}$ is the homotopy category of 1-connected based CW-complexes (whose underlying sets belong to $\mathcal{U}$).

2.1.3 Note. It is to be emphasized that condition $(*)$ is essential in order to be able to apply E.H. Browns responsibility theorem to prove this result.

2.2 $\mathcal{S}$-completion in $\widetilde{\mathcal{CW}}$

We use Theorems 1.5.5, 1.5.9 and 2.1.1 to show that the Adams completion with respect to the set of morphisms S always extsts in $\widetilde{\mathcal{CW}}$.

2.2.1 Proposition. S *admits a calculus of left fractions*.

Proof. Clearly S is closed under composition. We shall verify conditions (a) and (b) of Theorem 1.2.3. Only condition (b) is in question. For proving this condition it is enough to prove that every diagram

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & Z \\ Y & \end{tikzcd}$$

in $\widetilde{\mathcal{CW}}$ with $\gamma \in S$, can be embedded in a weak push-out diagram

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & Z \arrow[d, "\beta"] \\ Y \arrow[r, "\delta"'] & W \end{tikzcd}$$

with $\delta \in S$. Suppose $\alpha = [f]$ and $\gamma = [s]$. Let $i_f : X \to M_f$ be the usual inclusion of X into M_f, the reduced mapping cylinder of f and i_f is defined by $i_f(x) = [0, x]$. The map $j : Y \to M_f$ is defined by $j(y) = [y]$ and the map $r : M_f \to Y$ is defined by

$$r([y]) = y,\ r([s, x]) = f(x).$$

Hence $rj = 1_Y$, $jr \simeq 1_{M_f}$ and $ri_f = f$ [42]. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "i_f"'] & Z \\ Y & \end{tikzcd}$$

and form its push-out in $\widetilde{\mathcal{CW}}$:

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "i_f"'] & Z \arrow[d, "u"] \\ Y \arrow[r, "t"'] & W \end{tikzcd}$$

Since $i_f : X \to M_f$ is a cofibration so is $u : Z \to W$. Hence we have the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "i_f"] \arrow[d, "s"'] & M_f \arrow[r, "p"] \arrow[d, "t"] & C \arrow[d, equal] \\ Z \arrow[r, "u"'] & W \arrow[r, "q"'] & C \end{tikzcd}$$

where C is the cokernel of i_f, as well as of u; p and q are the usual projections. We consider the following commutative diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow \widetilde{H}_{m+1}(C) & \longrightarrow & \widetilde{H}_m(X) & \longrightarrow & \widetilde{H}_{m-1}(M_f) \\
\| & & \downarrow s_* & & \downarrow t_* \\
\cdots \longrightarrow \widetilde{H}_{m+1}(C) & \longrightarrow & \widetilde{H}_m(Z) & \longrightarrow & \widetilde{H}_{m-1}(W)
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow \widetilde{H}_m(C) & \longrightarrow & \widetilde{H}_{m-1}(X) & \longrightarrow \cdots \\
\| & & \downarrow s_* & \\
\longrightarrow \widetilde{H}_m(C) & \longrightarrow & \widetilde{H}_{m-1}(Z) & \longrightarrow \cdots
\end{array}$$

It follows that

$$s_* \; : \; \widetilde{H}_m(X) \to \widetilde{H}_m(Z)$$

is a $\mathcal{C}$-isomorphism. By mod-$\mathcal{C}$ Five lemma we have that

$$t_* : \widetilde{H}_m\big(M_f\big) \to \widetilde{H}_m(Z)$$

is a $\mathcal{C}$-isomorphism and hence

$$t_* : \pi_m(M_f) \to \pi_m(W)$$

is a $\mathcal{C}$-isomorphism. Let $\beta = [u]$ and $\delta = [tj]$. Since j is a mod-$\mathcal{C}$ homotopy equivalence, j_* is a $\mathcal{C}$-isomorphism of the corresponding homotopy groups; thus $\delta \in S$. We consider the following diagram in $\mathcal{CW}$:

$$\begin{array}{ccc}
X & \xrightarrow{\;s\;} & Z \\
f \downarrow & & \downarrow u \\
Y & \xrightarrow[\;tj\;]{} & W
\end{array}$$

We have

$$tjf = tjri_f \simeq t1_{M_f}i_f = ti_f = us.$$

Taking the corresponding homotopy classes we get the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & Z \arrow[d, "\beta"] \\ Y \arrow[r, "\delta"'] & W \end{tikzcd}$$

in $\widetilde{\mathcal{CW}}$, with $\delta \in S$. This is indeed a weak push-out diagram in $\widetilde{\mathcal{CW}}$. This completes the proof of the Proposition 2.2.1. ■

2.2.2 Proposition. *Let* $\{s_i : X_i \to Y_i, i \in I\}$ *be a subset of* S; *then*

$$\vee_{i\in I} s_i : \vee_{i\in I} X_i \to \vee_{i\in I} Y_i$$

is an element of S, *where the index set* I *is in* $\mathcal{U}$.

Proof. We consider the commutative diagram

$$\begin{array}{ccc} \bigoplus_{i\in I} \widetilde{H}_*(X_i) & \underset{\cong}{\overset{\{\alpha_{i*}\}}{\rightarrow}} & \widetilde{H}_*\left(\bigvee_{i\in I} X_i\right) \\ \bigoplus_{i\in I} s_{i*} \downarrow\cong & & \downarrow (\bigvee_{i\in I} s_i)_* \\ \bigoplus_{i\in I} \widetilde{H}_*(Y_i) & \underset{\{\beta_{i*}\}}{\overset{\cong}{\rightarrow}} & \widetilde{H}_*\left(\bigvee_{i\in I} Y_i\right) \end{array}$$

where

$$\alpha_i : X_i \to \vee_{i\in I} X_i \quad \text{and} \quad \beta_i : Y_i \to \vee_{i\in I} Y_i$$

are the canonical inclusions. Note that each horizontal row is an isomorphism, hence a $\mathcal{C}$-isomorphism. Moreover, since each s_{i*} is a $\mathcal{C}$-isomorphism, so is $\bigoplus_{i\in I} s_{i*}$, and hence from the commutativity of the diagram it follows that $(\bigvee_{i\in I} s_i)_*$ is also a $\mathcal{C}$-isomorphism. Thus $\bigvee_{i\in I} s_i \in S$. This completes the proof of Proposition 2.2.2. ■

For a fixed object X of the category $\widetilde{\mathcal{CW}}$ and for each integer $n \geq 1$, let $A_n = \{\, i_Y : X \hookrightarrow Y \mid (Y,X)$ is a relative CW-complex of dimension $n\}$.

Let $\mathcal{U}$ be a fixed Grothendeick universe such that the category of CW-complexes and homotopy classes of maps between them is a $\mathcal{U}$-category. Since S^1 can be given the structure of a CW-complex, $[\,S^1,*; S^1,*] \cong \mathbb{Z}$ is an element of $\mathcal{U}$, and it follows from the axioms of Grothendeick universe that $\mathbb{Z}^+$, the set of positive integers, is also an element of $\mathcal{U}$. We shall use this fact in proving the following results.

2.2.3 Lemma. A_1 *is an element of* $\mathcal{U}$.

Proof. Let (Y,X) be a relative CW-complex of dimension 1. Then the set $\{e_\alpha\}$ of attaching maps is an element of $\mathcal{P}(\mathrm{Map}(S^0,Y))$ where $\mathcal{P}$ denotes the power set and $\mathrm{Map}(S^0,Y)$ is the set of all continuous maps between S^0 and Y. Conversely every element of $\mathcal{P}(\mathrm{Map}(S^0,Y))$ will give rise to a CW-complex (Y, X) of dimension of 1, which is unique up to homeomorphism. It is thus clear that

$$A_1 \approx \mathcal{P}(\mathrm{Map}(S^0, Y)).$$

Since $\mathrm{Map}(S^0,\ Y) \in \mathcal{U}$, it follows from the axioms of a Grothendeick universe that $\mathcal{P}(\mathrm{Map}(S^0, Y)) \in \mathcal{U}$. This completes the proof of Lemma 2.2.3. ■

We can now use the induction to prove the following.

2.2.4 Lemma. *For each integer* $n \geq 1$, A_n *is an element of* $\mathcal{U}$.

Proof. We can assume inductively that for every $k \leq n-1$, A_k is an element of $\mathcal{U}$. Let (Y_j, X) be a relative CW-complex of dimension $n-1$ and let $i_j : X \hookrightarrow Y_j$ denote the inclusion; then $i_j \in A_{n-1}$. We can obtain a relative CW-complex of dimension n by a set of attaching maps

$$\{e_j\} \in \mathcal{P}(\mathrm{Map}(S^{n-1}, Y_j)).$$

It follows easily that

$$A_n \approx \bigcup_{i_j \in A_{n-1}} \mathcal{P}\left(\mathrm{Map}\left(S^{n-1}, Y_j\right)\right).$$

Since

$$\mathrm{Map}\left(S^{n-1}, Y_j\right) \in \mathcal{U}$$

by assumption and $A_{n-1} \in \mathcal{U}$, by the inductive hypothesis, it follows that $A_n \in \mathcal{U}$. This completes the proof of the Lemma 2.2.4. ■

2.2.5 Proposition. *For a given object* X *of the category* $\widetilde{\mathcal{CW}}$ *there exists a subset* S_X *of the set* $\{s : X \to X' \mid s \in S\}$ *such that* S_X *is an element of the universe* $\mathcal{U}$ *and for each* $s : X \longrightarrow X'$, $s \in S$, *there exist an* $s' : X \to X''$ *in* S_X *and a morphism* $u : X' \to X''$ *of* $\widetilde{\mathcal{CW}}$ *rendering the following diagram is commutative* :

$$\begin{tikzcd} X \arrow[d, "s"'] \arrow[dr, dashed, "s'"] & \\ X' \arrow[r, "u"'] & X'' \end{tikzcd}$$

Proof. Given X in $\widetilde{\mathcal{CW}}$, let

$$S_X = \{s : X \longrightarrow Y \mid (Y, X) \text{ is a relative } \widetilde{\mathcal{CW}}\text{-complex}\}.$$

We show that S_X is an element of $\mathcal{U}$; we prove this by using induction and exploiting the properties of Grothendeick universe. Clearly

$$S_X = \cup_{n \geq 1} A_n.$$

Since each $A_n \in \mathcal{U}$, it follows that

$$S_X = \cup_{n \geq 1} A_n$$

is an element of $\mathcal{U}$. Moreover, for any $s : X \to X'$ in S, the required factorization can be obtained by taking $s' = s$ and $u = 1_{X'}$. This completes the proof of the Proposition 2.2.5. ■

Hence from the Propositions 2.2.1, 2.2.2 and 2.2.5 it follows that all the conditions of Theorem 2.1.1 are satisfied and so by Theorem 1.5.5 we obtain the following theorem.

2.2.6 Theorem. *Every object* X *of the category* $\widetilde{\mathcal{CW}}$ *has an Adams completion* X_S *with respect to the set of morphisms* S *and there exists a morphism* $e : X \longrightarrow X_S$ *in* $\bar{S}$ *which is couniversal with respect to morphisms in* S.

2.2.7 Proposition. *The morphism as constructed in Proposition* 2.2.6 *is in* S.

Proof. Let S_1 be the set of all morphisms $\alpha : X \rightarrow Y$ in $\widetilde{\mathcal{CW}}$ such that $\alpha_* : \widetilde{H}_*(X) \rightarrow \widetilde{H}_*(Y)$ is a $\mathcal{C}$-monomorphism and S_2 be the set of all morphisms $\alpha : X \rightarrow Y$ in $\widetilde{\mathcal{CW}}$ such that $\alpha_* : \widetilde{H}_*(X) \rightarrow \widetilde{H}_*(Y)$ is a $\mathcal{C}$-epimorphism. Clearly

(a) $S = S_1 \cap S_2$,

(b) S_1 and S_2 satisfy all conditions of Theorem 1.5.9;

hence $e \in S$. This completes the proof of the Proposition 2.2.7. ■

2.3 S_n-completion in $\widetilde{\mathcal{CW}}$

Behera and Nanda [6] have already shown that the Adams completion with respect to the set of morphisms S_n-always exists in $\widetilde{\mathcal{CW}}$; however we state the necessary results which will be needed in the next section.

2.3.1 Proposition. ([6], Proposition 2.1, p. 535) *S_n-admits a calculus of left fractions.*

2.3.2 Proposition. ([6], Proposition 2.2, p. 537) *Let $s_i : X_i \to Y_i$, $i \in I$ be a subset of S_n; then*

$$\vee_{i \in I} s_i : \vee_{i \in I} X_i \to \vee_{i \in I} Y_i$$

is an element of S_n, where the index set I is an element of $\mathcal{U}$.

2.3.3 Proposition. ([6], Proposition 2.3, p. 538) *For a given object X of the category $\widetilde{\mathcal{CW}}$ there exists a subset S_X of the set $\{s : X \to X' \mid s \in S_n\}$ such that S_X is an element of the universe $\mathcal{U}$ and for each $s : X \to X'$, $s \in S_n$, there exist an $s' \in S_X$ and a morphism u of $\widetilde{\mathcal{CW}}$ rendering the following diagram commutative:*

$$\begin{tikzcd}
X \arrow[d, "s"'] \arrow[dr, "s'"] & \\
X' \arrow[r, "u"'] & X''
\end{tikzcd}$$

Hence from Propositions 2.3.1, 2.3.2 and 2.3.3 it follows that all the conditions of Theorem 2.1.1 are satisfied and so by Theorem 1.5.5 the following result holds.

2.3.4 Theorem. ([6], Theorem 2.5, p. 540) *Every object X of the category $\widetilde{\mathcal{CW}}$ has an Adams completion X_{S_n} with respect to the set of* *morphisms* S_n *and there exists a morphism* $e : X \to X_{S_n}$ *in* $\bar{S}_n$ *which is couniversal with respect to morphisms in* S_n.

2.3.5 Proposition. ([6], Proposition 2.6, p. 540) *The morphisms as* $e : X \to X_{S_n}$ *as constructed in Theorem* 2.3.4 *is in* S_n.

2.4 S_n^*-completion in $\widetilde{\mathcal{CW}}$

In a similar way we can show the Adams completion with respect to the set of morphisms S_n^* always exists in $\widetilde{\mathcal{CW}}$.

2.4.1 Proposition. S_n^* *admits a calculus of left fractions.*

Proof. Since the weak push-out diagrams in Propositions 2.2.1 and 2.3.1 are the same, the result follows easily. ■

2.4.2 Proposition. *Let* $s_i : X_i \to Y_i$, $i \in I$, *be a subset of* S_n^*; *then*

$$\vee_{i\in I} s_i : \vee_{i\in I} X_i \to \vee_{i\in I} Y_i$$

is an element of S_n^*, *where the index set* I *is an element of* $\mathcal{U}$.

Proof. The proof follows from the proofs of Propositions 2.2.2 and 2.3.2. ■

2.4.3 Proposition. *For a given object* X *of the category* $\widetilde{\mathcal{CW}}$ *there exists a subset* S_X *of the set* $\{s : X \to X' \mid s \in S_n^*\}$ *such that* S_X *is an* *element of the universe* $\mathcal{U}$ *and for each* $s : X \to X'$, $s \in S_n^*$, *there exist an* $s' : X \to X''$ *in* S_X *and a morphism* $u : X' \to X''$ *of* $\widetilde{\mathcal{CW}}$ *rendering the following diagram commutative*:

$$\begin{tikzcd} X \arrow[d, "s"'] \arrow[dr, "s'"] & \\ X' \arrow[r, "u"'] & X'' \end{tikzcd}$$

Proof. The proof follows from the proofs of Propositions 2.2.5 and 2.3.3. ■

Hence from Propositions 2.4.1, 2.4.2 and 2.4.3 it follows that all conditions of Theorem 2.1.1 are satisfied and so by Theorem 1.5.5 we obtain the following theorem.

2.4.4 Theorem. *Every subset* X *of the category* $\widetilde{\mathcal{CW}}$ *has an Adams completion* $X_{S_n^*}$ *with respect to the set of morphisms* S_n^* *and there exists a morphism* $e_n^* : X \to X_{S_n^*}$ *in* $\bar{S}_n^*$ *which is couniversal with respect to morphisms in* S_n^*.

2.4.5 Proposition. *The morphism* $e_n^* : X \to X_{S_n^*}$, as *constructed in Theorem* 2.4.4 *is in* S_n^*.

Proof. The proof follows from the proofs of Propositions 2.2.7 and 2.3.5. ■

2.5 $X_{S_n^*}$ and the n-stage of the Acyclic Tower in $\widetilde{\mathcal{CW}}$

For simplicity we denote the S_n^*-completion of a space X by X_n^* instead of $X_{S_n^*}$ and the map $X \to X_n^*$ by e_n^* which arises via Proposition 2.4.5. We shall show that X_n^* is the n-stage of the acyclic decomposition of an acyclic space in the sense of Dror [18].

We recall the definition of acyclic decomposition of an acyclic space.

2.5.1 Acyclic decomposition. [18] Let X be an acyclic space in $\widetilde{\mathcal{CW}}$. By an *acyclic decomposition* of X we mean a tower of fibrations

(*) $\varprojlim X^n = X \to \cdots \to X^n \to X^{n-1} \to \cdots \to X^0 = (\text{pt.})$

where the X^n's are in $\widetilde{\mathcal{CW}}$ and such that for each $n \geq 0$:

(a) the n- stage, X^n, is acyclic;

(b) the n- stage, X^n, is j-simple for all $j \geq n$ (i.e., $\pi_1(X^n)$ acts trivially on $\pi_j(X^n)$); and

(c) the fibre of $X^n \to X^{n-1}$ is $(n-1)$-connected.

2.5.2 Note. It is clear that once we get a tower of acyclic spaces such that for each n, $X^n \to X^{n-1}$ is an n-equivalence, then this can be converted to a fibration with $(n-1)$-connected fibre; thus giving rise to a tower of fibrations.

The following result on acyclic functor will be needed.

2.5.3 Theorem. ([18], Theorem 2.1) *There exists a functor* $A : \widetilde{\mathcal{CW}} \to \widetilde{\mathcal{CW}}$ *and for any* $K \in \widetilde{\mathcal{CW}}$, *a natural transformation* $a : AK \to K$ *such that*:

(i) AK *is acyclic for all* K.

(ii) *The map* $a : AK \to K$ *is, up to homotopy, universal for maps of acyclic complexes into* K.

(iii) *Let* n *be* $1 \leq n \leq \infty$. *If* $\widetilde{H}_j(K) \cong 0$ *for all* $1 \leq n \leq \infty$, *then the fibre of* $a : AK \to K$ *is* $(n-1)$-*connected. In particular, if* K *is acyclic then* a *is an equivalence.*

(iv) *If* K *is* j-*simple for some* $j \geq 1$ *then so is* AK.

(v) *If* $p : E \to B$ *is a* (Kan)-*fibre then so is* Ap.

(vi) *Let* K_∞ *be the inverse limit of the tower*:

$$K_\infty = \varprojlim K_n \to \cdots \to K_n \overset{p_n}{\to} K_{n-1} \to \cdots \to K_1.$$

Assume that for any s-*skeleton* $(K_n)_s$ *the restriction* $p_n|_{(K_n)_s}$ *is an isomorphism for* n *big enough. Then* $AK_\infty = \varprojlim AK_n$. ■

We will show that the spaces as obtained in Theorem 2.4.4 satisfy the conditions of an acyclic decomposition of a space.

2.5.4 Theorem. *The space* X_n^*, the S_n^*-*completion of* X, *is the* n-*stage of the acyclic decomposition of* X *in* $\widetilde{\mathcal{CW}}$.

Proof. We recall from [6] every object X of $\widetilde{\mathcal{CW}}$ has an S_n-completion denoted by X_n - which is precisely the n-th mod-$\mathcal{C}$ Postnikov section of X, i.e., $\pi_j(X) \rightarrow \pi_j(X_n)$ is a $\mathcal{C}$-isomorphism for $j \leq n$ and $\pi_j\,(X^n) = 0$ for $j > n$.

Moreover, if $e_n : X \rightarrow X_n$ denotes the map that arises via Theorem 2.4.4 then $e_n \in S_n$. Since e_n is also an element of S_{n-1}, it follows from the couniversal property of the map $e_{n-1} : X \rightarrow X_{n-1}$ that there is a map $\theta_n : X_n \rightarrow X_{n-1}$ such that $\theta_n e_n = e_{n-1}$, i.e., the following diagram

$$\begin{tikzcd}
X \arrow[r, "e_{n-1}"] \arrow[d, "e_n"'] & X_{n-1} \\
X_n \arrow[ur, dashed, "\theta_n"'] &
\end{tikzcd}$$

is commutative. Thus we get a tower of spaces as shown in the following diagram.

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_n \arrow[d, "\theta_n"] \\
 & X_{n-1} \arrow[d] \\
 & \vdots \arrow[d] \\
 & X_1 \arrow[d] \\
X \arrow[uuuur, "e_n"] \arrow[uuur, "e_{n-1}"'] \arrow[r] & X_0
\end{tikzcd}$$

Also we can assume that the maps θ_n's are all fibrations [6].

We have the same sort of situation with respect to the families of morphisms S_n^*'s; for, corresponding to each $n \geq 0$, we get a space X_n^*, the S_n^*-completion of X. Moreover, the map $e_n^* : X \to X_n^*$ being a mod-$\mathcal{C}$ $(n+1)$-equivalence, is also a mod-$\mathcal{C}$ n-equivalence and hence belongs to S_{n-1}^*. The couniversal property of e_{n-1}^* then implies that there is a map $\varphi_n : X_n^* \to X_{n-1}^*$ such that $\varphi_n e_n^* = e_{n-1}^*$, i.e., the diagram

$$\begin{tikzcd} X \arrow[r, "e_{n-1}^*"] \arrow[d, "e_n^*"'] & X_{n-1}^* \\ X_n^* \arrow[ur, "\varphi_n"'] & \end{tikzcd}$$

commutes. We may assume that the maps φ_n's are all fibrations. It follows from Proposition 2.4.5 that $\widetilde{H}_n(X) \approx \widetilde{H}_n(X_n^*)$ and therefore, if X is acyclic, so is X_n^* for each n. It is also equally clear that the fibre of $\varphi_n : X_n^* \to X_{n-1}^*$ is $(n-1)$-connected. Thus, conditions (a) and (c) of an acyclic tower are satisfied by the spaces $\{X_n^*\}$.

To show directly that condition (b) is also satisfied seems to be difficult. We, therefore, take the help of Dror's acyclic functor A [18] to show that X_n^* and AX_n^* (which is the n-stage of the Dror's acyclic tower) are homotopically equivalent.

Consider the commutative diagram

$$\begin{tikzcd}
AX \arrow[r, "Ae_n^*"] \arrow[d, "f"'] & AX_n^* \arrow[d, "f_n"] \\
X \arrow[r, "e_n^*"'] \arrow[ur, dashed, "g_n"] & X_n^*
\end{tikzcd}$$

Since X is acyclic and X_n^* is the n-th mod-$\mathcal{C}$ Postnikov section of X, we have $H_i(X_n^*) = 0$ for $i \leq n+1$. It follows from Theorem 2.5.3 (iii) that f_n^* is a mod-$\mathcal{C}$ $(n+1)$-equivalence and that f is a mod-$\mathcal{C}$ homotopy equivalence, showing that

$$(Ae_n^*)_* \ : \ \pi_m(AX) \ \rightarrow \ \pi_m(AX_n^*)$$

is a $\mathcal{C}$-isomorphism for $m \leq n$. This together with the fact that $A\varphi_n$ is a mod-$\mathcal{C}$ n-equivalence, from the following commutative diagram

$$\begin{tikzcd}
AX \arrow[r, "Ae_n^*"] \arrow[dr, "Ae_{n-1}^*"'] & AX_n^* \arrow[d, "A\varphi_n"] \\
& AX_{n-1}^*
\end{tikzcd}$$

it follows that that Ae_{n-1}^* is a mod-$\mathcal{C}$ n-equivalence. Thus we have a map

$$g_n = (Ae_n^*)f^{-1} \ : \ X \ \xrightarrow{f^{-1}} \ AX \ \xrightarrow{Ae_n^*} \ AX_n^*$$

which is a mod-$\mathcal{C}$ $(n+1)$-equivalence.

By the couniversal property of $e_n^* : X \rightarrow X_n^*$, with respect to all mod-$\mathcal{C}$ $(n+1)$-equivalences which induce reduced integral homology

$\mathcal{C}$-isomorphisms, there exists a unique map $k_n : AX_n^* \to X_n^*$ as shown in the diagram

$$\begin{tikzcd} X \arrow[r, "e_n^*"] \arrow[d, "g_n"'] & X_n^* \\ AX_n^* \arrow[ur, dashed, "k_n"'] & \end{tikzcd}$$

such that $k_n g_n = e_n^*$.

Also by the couniversal property of $g_n : X \to AX_n^*$ with respect to all maps from X to acyclic spaces (Theorem 2.5.3) there exists a unique map $h_n : X_n^* \to AX_n^*$ as shown in the diagram,

$$\begin{tikzcd} X \arrow[r, "g_n"] \arrow[d, "e_n^*"'] & AX_n^* \\ X_n^* \arrow[ur, dashed, "h_n"'] & \end{tikzcd}$$

such that $h_n e_n^* = g_n$. We now collect the relevant spaces and maps:

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "e_n^*"'] & X \arrow[d, "g_n"] \\ X_n^* \arrow[r, dashed, shift left, "h_n"] & AX_n^* \arrow[l, dashed, shift left, "k_n"] \end{tikzcd}$$

The diagram above brings out the relationship between X_n^* and AX_n^*.

Thus

$$e_n^* = k_n g_n = k_n h_n e_n^*$$

implying that

$$k_n h_n = 1_{X_n^*}$$

and

$$g_n = h_n e_n^* = h_n k_n g_n$$

implying that

$$h_n k_n = 1_{AX_n^*};$$

h_n must, therefore, be a homotopy equivalence. This completes the proof that X_n^* is the n-stage of the acyclic decomposition. ■

Chapter 3

ADAMS COMPLETION AND 1-CONNECTED NILPOTENT SPACE

Behera and Nanda [6] have described the mod-$\mathcal{C}$ Postnikov approximation of a 1-connected space where $\mathcal{C}$ is a Serre class of abelian groups, with the help of a suitable set of morphisms in the category of 1-connected based CW-complexes. Following the arguments of the work done by Behera and Nanda we study the primary decomposition of a 1-connected based nilpotent space using primary homotopy theory as given in [36]. Neisendorfer [36] has studied primary homotopy theory in detail; we have given a brief description of this theory in Section 1.7.

3.1 The category $\widetilde{\mathcal{N}}_1$

Let $\mathcal{N}_1$ denote the category of 1-connected based nilpotent spaces and based maps and let $\widetilde{\mathcal{N}}_1$ be the corresponding homotopy category. We assume that the underlying sets of the elements of $\widetilde{\mathcal{N}}_1$ are elements of $\mathcal{U}$ where $\mathcal{U}$ is a fixed Grothendeick universe. We now fix suitable sets of morphisms of $\widetilde{\mathcal{N}}_1$.

A map $\alpha : X \to Y$ in $\widetilde{\mathcal{N}_1}$ is called a mod k $(n+1)$-*equivalence* if

$$\alpha_* : \pi_m(X; \mathbb{Z}/k\mathbb{Z}) \to \pi_m(Y; \mathbb{Z}/k\mathbb{Z})$$

is a isomorphism for $m \leq n$ and an epimorphism for $m = n+1$. Let S_n denote the set of all mod k $(n+1)$-equivalences in $\widetilde{\mathcal{N}_1}$.

3.1.1 Proposition. S_n *admits a calculus of left fractions.*

Proof. Clearly S_n is closed under composition. We shall verify conditions (a) and (b) of Theorem 1.2.3. Only condition (b) is in question. For proving this condition it is enough to prove that every diagram

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & Z \\ Y & \end{tikzcd}$$

in $\widetilde{\mathcal{N}_1}$ with $\gamma \in S_n$, can be embedded in a weak push-out diagram

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & Z \arrow[d, "\beta"] \\ Y \arrow[r, "\delta"'] & W \end{tikzcd}$$

with $\delta \in S_n$. Suppose $\alpha = [f]$ and $\gamma = [s]$. Let i_f , M_f, j, r be as in the proof Theorem 2.2.1 and we consider the diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "i_f"'] & Z \\ M_f & \end{tikzcd}$$

and form its push-out in $\widetilde{\mathcal{N}}_1$

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "i_f"'] & Z \arrow[d, "u"] \\ M_f \arrow[r, "t"'] & W \end{tikzcd}$$

since $i_f : X \to M_f$ is a cofibration so is $u : Z \to W$. Hence we have the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "i_f"] \arrow[d, "s"'] & M_f \arrow[r, "p"] \arrow[d, "t"] & C \arrow[d, equal] \\ Z \arrow[r, "u"'] & W \arrow[r, "q"'] & C \end{tikzcd}$$

where C is the cokernel of i_f, as well as of u; p and q are the usual projections. We consider the exact homology sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H_{m+1}(C;\mathbb{Z}/k\mathbb{Z}) \arrow[r] \arrow[d, equal] & H_m(X;\mathbb{Z}/k\mathbb{Z}) \arrow[r] \arrow[d, "s_*"'] & H_m(M_f;\mathbb{Z}/k\mathbb{Z}) \arrow[d, "t_*"'] \\
\cdots \arrow[r] & H_{m+1}(C;\mathbb{Z}/k\mathbb{Z}) \arrow[r] & H_m(Z;\mathbb{Z}/k\mathbb{Z}) \arrow[r] & H_m(W;\mathbb{Z}/k\mathbb{Z})
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r] & H_m(C;\mathbb{Z}/k\mathbb{Z}) \arrow[r] \arrow[d, equal] & H_{m-1}(X;\mathbb{Z}/k\mathbb{Z}) \arrow[r] \arrow[d, "s_*"'] & \cdots \\
\arrow[r] & H_m(C;\mathbb{Z}/k\mathbb{Z}) \arrow[r] & H_{m-1}(Z;\mathbb{Z}/k\mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}$$

where H_* denotes the singular homology functor. Since

$$s_* : \pi_m(X;\ \mathbb{Z}/k\mathbb{Z}) \ \rightarrow\ \pi_m(Z;\ \mathbb{Z}/k\mathbb{Z})$$

is an isomorphism for $m \le n$ and an epimorphism for $m = n + 1$, it follows from Theorem 1.7.4 that

$$s_* : H_m(X;\ \mathbb{Z}/k\mathbb{Z}) \ \rightarrow\ H_m(Z,;\mathbb{Z}/k\mathbb{Z})$$

is an isomorphism for $m \le n$ and an epimorphism for $m = n + 1$. From Five lemma we obtain that

$$t_* : H_m(M_f;\mathbb{Z}/k\mathbb{Z}) \rightarrow H_m(Z,;\mathbb{Z}/k\mathbb{Z})$$

is an isomorphism for $m \le n$ and an epimorphism for $m = n + 1$. Hence

$$t_* : \pi_m(M_f;\mathbb{Z}/k\mathbb{Z}) \ \rightarrow\ \pi_m(W;\ \mathbb{Z}/k\mathbb{Z})$$

is an isomorphism for $m \le n$ and an epimorphism for $m = n + 1$. Let $\beta = [u]$ and $\delta = [tj]$. Since j is a homotopy equivalence, j_* is an isomorphism of the corresponding homotopy groups; thus $\delta \in S_n$. We consider the following diagram in $\mathcal{N}_1$:

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & Z \arrow[d, "u"] \\ Y \arrow[r, "tj"'] & W \end{tikzcd}$$

We have

$$tjf = tjri_f \simeq t1_{M_f} = ti_f = us.$$

Taking the corresponding homotopy classes we get the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & Z \arrow[d, "\beta"] \\ Y \arrow[r, "\delta"'] & W \end{tikzcd}$$

in $\widetilde{\mathcal{N}}_1$, with $\delta \in S_n$. This is indeed a weak push-out diagram in $\widetilde{\mathcal{N}}_1$. This completes the proof of the Proposition 3.2.1. ■

3.1.2 Proposition. *Let* $\{s_i : X_i \to Y_i,\ i \in I\}$ *be a subset of* S_n; *then*

$$\vee_{i\in I}\, s_i : \vee_{i\in I} X_i \to \vee_{i\in I} Y_i$$

is an element of S_n, *where the index set* I *is in* $\mathcal{U}$.

Proof. We consider the following commutative diagram

$$\begin{array}{ccc} \bigoplus_{i\in I} H_*(X_i;\mathbb{Z}/k\mathbb{Z}) & \underset{\cong}{\overset{\{\alpha_{i*}\}}{\rightarrow}} & H_*\left(\bigvee_{i\in I} X_i;\mathbb{Z}/k\mathbb{Z}\right) \\ \bigoplus_{i\in I} s_{i*} \downarrow\cong & & \downarrow (\bigvee_{i\in I} s_i)_* \\ \bigoplus_{i\in I} H_*(Y_i;\mathbb{Z}/k\mathbb{Z}) & \underset{\{\beta_{i*}\}}{\overset{\cong}{\rightarrow}} & H_*\left(\bigvee_{i\in I} Y_i\,;\mathbb{Z}/k\mathbb{Z}\right) \end{array}$$

where

$$\alpha_i : X_i \to \bigvee_{i\in I} X_i \quad \text{and} \quad \beta_i : Y_i \to \bigvee_{i\in I} Y_i$$

are the canonical inclusions. We note that each horizontal row is an isomorphism. Moreover, since each s_{i*} is an isomorphism in dimension $\leq n$ and an epimorphism in dimension $n+1$, so is $\bigoplus_{i\in I} s_{i*}$, and from the commutativity of the diagram it follows that $(\bigvee_{i\in I} s_i)_*$ is also an isomorphim in dimension $\leq n$ and an epimorphism in dimension $n+1$. Thus $\bigvee_{i\in I} s_i \in S_n$. This completes the proof of Proposition 3.2.2. ■

Let $\mathcal{U}$ be a fixed Grothendeick universe such that the category of nilpotent CW-complexes and homotopy classes of maps between them is a $\mathcal{U}$-category. Since S^1 can be given the structure of a CW-complex, $[S^1, S^1] \cong Z$ is an element of $\mathcal{U}$, and it follows from the axioms of a Grothendeick universe that $\mathbb{Z}^+$, the set of positive integers, is also an element of $\mathcal{U}$ [6]. We shall use this fact in proving the following result.

3.1.3 Proposition. *For a given object X of the category $\widetilde{\mathcal{N}}_1$ there exists a subset S_X of the set $\{\, s : X \to X' \mid s \in S_n\}$ such that S_X is an element of the universe $\mathcal{U}$ and for each $s : X \to X'$, $s \in S_n$ there exist* *an* $s' : X \to X''$ *in* S_X *and a morphism* $u : X' \to X''$ *in* $\widetilde{\mathcal{N}}_1$ *rendering the following diagram commutative,*

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "s'"'] & X' \arrow[dl, "u"] \\ X'' & \end{tikzcd}$$

$i.e.,\ us = s'.$

Proof. Given any X in $\widetilde{\mathcal{N}}_1$, let $S_X = \{s : X \to Y \mid (Y, X)$ is a relative nilpotent CW-complex in dim $\geq\ n+3\}$. Clearly $S_X \subset S_n$. For any $s : X \to Y$ in S_n, we can find a nilpotent CW-complex Z, such that (Z, X) has cells in dim $\geq n+3$ and there is a map $u : Z \to Y$ which is a mod k $(n+1)$-equivalence extending $s : X \to Y$. Let $v : Y \to Z$ be the mod k homotopy inverse of $u : Z \to Y$ and $s' : X \to Z$ be the usual inclusion. Then we have the following homotopy commutative diagram,

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "s'"'] & Y \arrow[dl, "v"] \\ Z & \end{tikzcd}$$

i.e., $vs \simeq s'$.

In order to show that $S_X \in \mathcal{U}$, let $A_K = \{s : X \to Y \mid (Y, X)$ has cells e^m such that $n+3 \leq \dim(e^m) \leq m+k+2\}$. We have

$$S_X = \bigcup_{k\geq 1} A_k.$$

We inductively show that $A_k \in \mathcal{U}$, $k \geq 1$. For $k = 1$, we have $A_1 = \{s : X \to Y \mid (Y, X)$ is a relative nilpotent CW-complex having cells in dim $n+3$ only$\}$. Thus

$$Y = X \cup_{\varphi_i} P^{m+3}(k),$$

where $\varphi_i : P^{m+2}(k) \to X$ and $i \in I$ for some index set $I \in \mathcal{U}$. Thus every family

$$\{\varphi_i : P^{m+2}(k) \to X\} \subset [P^{m+2}(k); X]$$

determines a space Y such that (Y, X) is a relative nilpotent CW-complex having cells in dim $n+3$ only. Thus

$$A_1 \cong \mathcal{P}[P^{m+2}(k);\ X]$$

where $\mathcal{P}$ denotes the power set. Since

$$[P^{m+2}(k); X] \in \mathcal{U},$$

it follows from the axioms of Grothendeick universe that

$$\mathcal{P}[P^{m+2}(k); X] \in \mathcal{U};$$

thus $A_1 \in \mathcal{U}$.

Assume that $A_k \in \mathcal{U}$. For showing that $A_{k+1} \in \mathcal{U}$, let $s : X \to Y$ be in A_k. Thus (Y, X) is a relative nilpotent CW-complex having cells e^m such that $n+3 \leq \dim(e^m) \leq n+k+2$. Let $\{\varphi_i : i \in I\}$ be a family of maps with

$$\varphi_i : S_i^{n+k+2} \to Y$$

for some index set I. Thus the inclusion map

$$X \hookrightarrow Y \cup_{\varphi_i} e_i^{n+k+3}$$

is in A_{k+1}. Moreover, every map $s : X \to Z$ of A_{k+1} arises in this way; therefore

$$A_{k+1} = \cup_Y \mathcal{P}\,[P^{n+k+2}(k); Y]$$

where the union is taken over all Y such that $s : X \to Y$ is in A_k. Since $A_k \in \mathcal{U}$ and

$$\mathcal{P}[P^{m+k+1}(k); X] \in \mathcal{U},$$

we have $A_{k+1} \in \mathcal{U}$. Similarly since the set of positive integers is an element of the universe $\mathcal{U}$, so is the union

$$\cup_{k\geq 1} A_k = S_X.$$

This completes the proof of Proposition 3.2.4. ∎

3.2 Existence of Adams completion in $\widetilde{\mathcal{N}}_1$

We shall use Theorem 2.1.1 in order to obtain the existence of Adams completion in the category $\widetilde{\mathcal{N}}_1$.

From Propositions 3.1.1, 3.1.2 and 3.1.3 it follows that the conditions of Theorem 2.1.1 are satisfied and so by Theorem 1.5.5 we obtain the following result.

3.2.1 Theorem. *Every object* X *of the category* $\bar{\mathcal{N}}_1$ *has an Adams completion* X_{S_n} *with respect to the set of morphisms* S_n *and there exists a morphism* $e_n : X \to X_{S_n}$ *in* $\bar{S}_n$, which *is couniversal with respect to morphisms in* S_n.

3.2.2 Proposition. *The morphism* $e_n : X \to X_{S_n}$ *as constructed in Theorem* 3.2.3 *is in* S_n.

Proof. Let

$$S_n^1 = \{ f : X \to Y \text{ in } \bar{\mathcal{N}}_1 \mid f_* : \pi_n(X; \mathbb{Z}/k\mathbb{Z}) \to \pi_m(Y; \mathbb{Z}/k\mathbb{Z}) \text{ is a monomorphism for } m \leq n\}$$

and

$$S_n^2 = \{ f : X \to Y \text{ in } \bar{\mathcal{N}}_1 \mid f_* : \pi_n(X; \mathbb{Z}/k\mathbb{Z}) \to \pi_m(Y; \mathbb{Z}/k\mathbb{Z}) \text{ is an epimorphism for } m \leq n+1\}.$$

Clearly,

(i) $S_n = S_n^1 \cap S_n^2$,

(ii) S_n^1 and S_n^2 satisfy all the conditions of Theorem 1.5.9;

hence $e_n \in S_n$. This completes the proof of Proposition 3.2.4. ■

3.3 Primary approximation of a 1-connected based nilpotent space.

We can obtain a primary approximation of a 1-connected based nilpotent space with the help of the sets of morphism S_n.

3.3.1 Theorem. Let X be a 1-connected based nilpotent space. Then for $n \geq 3$, there exist 1-connected based nilpotent spaces X_n, maps $e_n : X \to X_n$ and fibrations $p_{n+1} : X_{n+1} \to X_n$ such that

(a) $e_{n_*} : \pi_m(X;\mathbb{Z}/k\mathbb{Z}) \to \pi_m(X_n;\mathbb{Z}/k\mathbb{Z})$ is an isomorphism for $m \leq n$ and $\pi_m(X;\mathbb{Z}/k\mathbb{Z}) = 0$ for $m > n$,

(b) $e_n = p_{n+1} \circ e_{n+1}$.

Proof. For each $n \geq 3$, let X_n be the S_n-completion of X and $e_n : X \to X_n$ be the canonical map as constructed in Proposition 3.2.3. Since $e_{n+1} \in S_{n+1}$, it follows that $e_{n+1} \in S_n$. By the couniversal property of $e_n : X \to X_n$ (Theorem 1.5.5), we have a map $p_{n+1} : X_{n+1} \to X_n$ making the diagram commutative,

$$\begin{tikzcd} X \arrow[r, "e_n"] \arrow[d, "e_{n+1}"'] & X_n \\ X_{n+1} \arrow[ur, dashed, "p_{n+1}"'] & \end{tikzcd}$$

i.e., $e_n = p_{n+1} \circ e_{n+1}$. The maps $\{p_n\}$ can be replaced by fibrations in the usual manner. Thus (b) holds.

Since $e_n \in S_n$,

$$e_{n_*} : \pi_m(X;\mathbb{Z}/k\mathbb{Z}) \to \pi_m(X_n;\mathbb{Z}/k\mathbb{Z})$$

is an isomorphism for $m \leq n$. In order to show that $\pi_m(X;\mathbb{Z}/k\mathbb{Z}) = 0$ for $m > n$, consider the map

$$f : P^m(k) \to X_n, \quad m > n,$$

and let s denote the inclusion

$$X_n \hookrightarrow X_n \cup_f P^{m+1}(k).$$

Clearly s is a mod k m-equivalence; since $m > n$, s is a mod k $(n+1)$-equivalence. Hence $s \in S_n$ and $se_n \in S_n$. By the couniversal property of e_n we have a map

$$t : X_n \cup_f P^{m+1}(k) \to X_n$$

which makes the diagram

$$\begin{tikzcd}
X \arrow[r, "e_n"] \arrow[d, "e_n"'] & X_n \\
X_n \arrow[d, "s"'] & \\
X_n \cup_f P^{m+1}(k) \arrow[uur, dashed, "t"'] &
\end{tikzcd}$$

commutative: $tse_n = e_n$. Now we consider the following diagram

$$\begin{tikzcd}
X \arrow[rr, "e_n"] \arrow[dd, "e_n"'] & & X_n \\
& X \cup_f P^{m+1}(k) \arrow[ur, "t"'] & \\
X_n \arrow[uurr, "1_{X_n}"] \arrow[ur, "s"'] & &
\end{tikzcd}$$

By the uniqueness condition of the couniversal property of e_n, we have $ts = 1_{X_n}$. Thus we have $f \simeq 0$; so that $\pi_m(X; \mathbb{Z}/k\mathbb{Z}) = 0$ for $m > n$. Thus (a) holds.

We have a tower of spaces as shown in the following diagram:

$$
\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_n \arrow[d, "p_n"] \\
 & X_{n-1} \arrow[d] \\
 & \vdots \arrow[d] \\
 & X_4 \arrow[d] \\
X \arrow[r] \arrow[uuuur, "e_n"] \arrow[uuur, "e_{n-1}"'] & X_3
\end{tikzcd}
$$

This completes the proof of Theorem 3.4.1. ■

Chapter 4

PRIMARY DECOMPOSITION OF A 0-CONNECTED NILPOTENT SPACE

As stated earlier we note that Deleanu, Frei and Hilton have developed the notion of generalized Adams completion in a categorical context; they have also studied the dual notion, namely the Adams cocompletion of an object in a category [14]. Behera and Nanda [5] have shown that the different stages of the Cartan-Whitehead decomposition of a 0-connected space are the Adams cocompletion of a space with respect to suitable sets of morphisms. In [36], Neisendonfer has studied the primary homotopic theory in an exhaustive manner. The central idea of this chapter is to study how Cartan-Whitehead decomposition of a 0-connected nilpotent space is characterized in terms of its Adams cocompletions; it is done using the primary homotopy theory developed by Neisendonfer.

4.1 The category $\mathcal{N}_0$.

Let $\mathcal{N}_0$ denote the category of 0-connected based nilpotent spaces and based maps and let $\widetilde{\mathcal{N}}_0$ be the corresponding homotopy category. We assume that the underlying sets of the elements of $\widetilde{\mathcal{N}}_0$ are the elements of $\mathcal{U}$, where $\mathcal{U}$ is a fixed Grothendeick universe. We now fix suitable sets of morphisms of $\widetilde{\mathcal{N}}_0$.

Let S_n denote the set of all maps α in $\widetilde{\mathcal{N}}_0$ having the following property: $\alpha : X \to Y$ is in S_n if and only if

$$\alpha_* : \pi_m(X; \mathbb{Z}/k\mathbb{Z}) \to \pi_m(Y; \mathbb{Z}/k\mathbb{Z})$$

is an isomorphism for $m > n$ and a monomorphism for $m = n$.

4.1.1 Proposition. S_n *admits a calculus of right fractions.*

Proof. Clearly S_n is closed under composition. We shall verify conditions (a) and (b) of Theorem 1.2.4. Only condition (b) is in question. For proving this condition it is enough to prove that every diagram

$$\begin{tikzcd} & A \arrow[d, "\alpha"] \\ C \arrow[r, "\gamma"'] & B \end{tikzcd}$$

in $\widetilde{\mathcal{N}}_0$ with $\gamma \in S_n$, can be embedded in a weak pull-back diagram

$$\begin{tikzcd} D \arrow[r, "\delta"] \arrow[d, "\beta"'] & A \arrow[d, "\alpha"] \\ C \arrow[r, "\gamma"'] & B \end{tikzcd}$$

with $\delta \in S_n$. Suppose $\alpha = [f]$ and $\gamma = [s]$. We replace f and s by fibrations f' and s' respectively; we have

$$f = f'r : A \xrightarrow{r} P_f \xrightarrow{f'} B$$

and

$$s = s't \ : \ C \xrightarrow{t} P_s \xrightarrow{s'} B$$

where r and t are homotopy equivalences and P_f and P_s are mapping tracks of f and s respectively. Let $\bar{r}$ and $\bar{t}$ be the homotopy inverses of r and t respectively. Let D be the usual pull-back of f' and s' and

$$p \ : D \to P_f, \quad q : D \to P_s$$

be the respective projections. Let $\delta = [\bar{r}p]$ and $\beta = [\bar{t}p]$. Thus

$$\begin{aligned} \alpha\delta &= [f][\bar{r}p] = [f\bar{r}p] \\ &= [f'r\bar{r}p] = [f'p] \\ &= [s'q] = [s't\bar{t}q] \\ &= [s\bar{t}q] = [s][\bar{t}q] = \gamma\beta. \end{aligned}$$

Moreover, if $\alpha\mu = \gamma\lambda$, let $u : U \to A$, $v : U \to C$ be in the classes μ, λ respectively so that

$$fu \simeq sv \quad \text{or} \quad f'ru \simeq sv.$$

Let

$$F : U \times I \to B$$

be a homotopy with

$$F_0 = f'ru \quad \text{and} \quad F_1 = sv.$$

Since f' is a fibration there exists a homotopy

$$G : U \times I \to P_f$$

such that

$$f'G_t = F_t \quad \text{and} \quad G_0 = ru.$$

Thus

$$f'G_1 = F_1 = sv = s'tv.$$

By the pull-back property of D there exists a map $k : U \to D$ such that

$$pk = G_1 \simeq ru \quad \text{and} \quad qk = tv.$$

Thus if $\rho = [k]$, then

$$\delta\rho = [\bar{r}p][k] = [\bar{r}pk] = [\bar{r}ru] = [u] = \mu$$

and

$$\beta\rho = [\bar{t}q][k] = [\bar{t}qk] = [\bar{t}tv] = [v] = \lambda.$$

It remains to be shown that $\delta \in S_n$. We assume that the map $\alpha : A \to B$ is a fibration with fibre Q. We note that Q is also the fibre of $\beta : D \to C$. We have the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{m+1}(C;\mathbb{Z}/k\mathbb{Z}) \arrow[r] \arrow[d, "\gamma_*"] & \pi_m(Q;\mathbb{Z}/k\mathbb{Z}) \arrow[r] \arrow[d, equal] & \pi_m(D;\mathbb{Z}/k\mathbb{Z}) \arrow[d, "\delta_*"] \\
\cdots \arrow[r] & \pi_{m+1}(B;\mathbb{Z}/k\mathbb{Z}) \arrow[r] & \pi_m(Q;\mathbb{Z}/k\mathbb{Z}) \arrow[r] & \pi_m(A;\mathbb{Z}/k\mathbb{Z})
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r] & \pi_m(C;\mathbb{Z}/k\mathbb{Z}) \arrow[r] \arrow[d, "\gamma_*"] & \pi_{m-1}(Q;\mathbb{Z}/k\mathbb{Z}) \arrow[r] \arrow[d, equal] & \cdots \\
\arrow[r] & \pi_m(B;\mathbb{Z}/k\mathbb{Z}) \arrow[r] & \pi_{m-1}(Q;\mathbb{Z}/k\mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}$$

By Five Lemma δ_* is an isomorphism for $m > n$ and a monomorphism for $m = n$, showing $\delta \in S_n$. This completes the proof of Proposition 4.1.1. ∎

In fact, the set S_n admits a strong calculus of right fractions. A set S of morphisms of a small $\mathcal{V}$-category $\mathcal{C}$, $\mathcal{V}$ being a Grothendieck universe, *admits a strong calculus of right fractions* [38] if

(i) S admits a calculus of right fractions,

(ii) for any set $\{s_i : B_i \to A,\ i \in I,\ I$ is a $\mathcal{V}$-set$\}$, there exists a commutative completion $\{f_i : C \to B_i,\ i \in I\}$ such that $s_i f_i \in S$ for every $i \in I$.

4.1.2 Proposition. S_n *admits a strong calculus of right fractions.*

Proof. Let $\{s_i : Y_i \to X,\ i \in I\}$ be a given set of morphisms in $\widetilde{\mathcal{N}}_0$ with every $s_i \in S_n$ and $I \in \mathcal{U}$. We have a map from $X \to P^nX$, where P^nX denote the Postnikov decomposition of X ([24], proof of Proposition 1.1). Convert this into a fibration:

$$X_n \xrightarrow{u_n} X \to P^nX,$$

X_n being its fibre. Considering the homotopy exact sequence of this fibration we get

$$\pi_m(X_n; \mathbb{Z}/k\mathbb{Z}) = 0$$

for $m \leq n$ and

$$\pi_m(X_n; \mathbb{Z}/k\mathbb{Z}) \cong \pi_m(X; \mathbb{Z}/k\mathbb{Z})$$

for $m > n$. Thus $u_n \in S_n$. Since $\pi_1(X_n; \mathbb{Z}/k\mathbb{Z}) = 0$ we have a map (lifting) $f_i : X_n \to Y_i$ such that $s_i f_i = u_n$, i.e., the following diagram commutes.

$$\begin{tikzcd} & Y_i \arrow[d, "s_i"] \\ X_n \arrow[ur, dashed, "f_i"] \arrow[r, "u_n"'] & X \end{tikzcd}$$

This completes the proof of the Proposition 4.1.2. ∎

4.1.3 Remark. We note that the map $u_n : X_n \to X$ is independent of the index i.

4.1.4 Proposition. *For a given object* X *of the category* $\widetilde{\mathcal{N}}_0$, *there exists a subset* S_X *of the set* $\{s : X' \to X \mid s \in S_n\}$ *such that* S_X *is an element of the universe* $\mathcal{U}$ *and for each* $s : X' \to X$, $s \in S_n$, *there exists an* $s' : X'' \to X$ *in* S_X *and a morphism* $u : X'' \to X'$ *in* $\widetilde{\mathcal{N}}_0$ *such that* $su = s'$, i.e., the following diagram is commutative.

$$\begin{tikzcd}
X' \arrow[r, "s"] & X \\
X'' \arrow[u, "u"] \arrow[ur, dashed, "s'"'] &
\end{tikzcd}$$

Proof. For a given object X in $\widetilde{\mathcal{N}}_0$, let S_X denote the set of morphisms $S_X = \{s : Y \to X \mid s \in S_n,\ Y \text{ is an object of } \widetilde{\mathcal{N}}_0\}$. We assert that S_X is an element of $\mathcal{U}$. For any object Y of $\widetilde{\mathcal{N}}_0$, let $S_{Y,X} = \{s : Y \to X,\ s \in S_n\}$. It is clear that $S_X = \cup_Y S_{Y,X}$ and $S_{Y,X} = S_n \cap \mathrm{Mor}_{\widetilde{\mathcal{N}}_0}(Y, X)$. Since $\widetilde{\mathcal{N}}_0$ is a small $\mathcal{U}$-category $\mathrm{Mor}_{\widetilde{\mathcal{N}}_0}(Y, X)$ belongs to $\widetilde{\mathcal{N}}_0$ and so does $S_{Y,X}$, being a subset of $\mathrm{Mor}_{\widetilde{\mathcal{N}}_0}(Y, X)$. Therefore, the set S_X, being a union of sets all belonging to $\mathcal{U}$ and indexed by the objects Y of $\widetilde{\mathcal{N}}_0$ (which is a subset of $\mathcal{U}$) is itself in $\mathcal{U}$.

In view of Proposition 4.1.2 and Remark 4.1.3, we have the following commutative diagram

$$\begin{tikzcd} & Y \arrow[d, "s"] \\ X_n \arrow[ur, "f_s"] \arrow[r, "u_n"'] & X \end{tikzcd}$$

where $s \in S_X$ is arbitrary, u_n is the map as constructed in Proposition 4.1.2 and f_s is the lifting of u_n corresponding to s. This completes the proof of the Proposition 4.1.4. ■

4.1.5 Corollary. $u_n \in S_n$ *and with respect to any* $s \in S_X$, u_n *has couniversal property.*

4.2 Existence of Adams cocompletion in $\widetilde{\mathcal{N}}_0$

The following theorem shows that under certain conditions the Adams cocompletion of an object in the category $\widetilde{\mathcal{N}}_0$ always exists; the theorem is essentially the dual of Theorem 4.7 [1] and dual of Theorem 3.8 [3] (it is also a generalization of the dual of the Theorem in [13]).

4.2.1 Theorem. *Let* $\mathcal{U}$ *be a fixed Grothendieck universe. Let* $\tilde{\mathcal{C}}$ *be the category defined as follows : the objects of* $\tilde{\mathcal{C}}$ *are connected based nilpotent spaces whose underlying sets are elements of* $\mathcal{U}$; *the morphisms of* $\tilde{\mathcal{C}}$ *are based homotopy classes of based-point preserving maps between such nilpotent spaces. Let* S *be a family of morphisms of*

$\tilde{\mathcal{C}}$ admitting a calculus of right fraction and satisfying the following axioms of compatibility with products:

(P) *If $s_i : X_i \to Y_i$ lies in S for each $i \in I$, where the index set I is an element of $\mathcal{U}$, then*

$$\prod_{i \in I} s_i \ : \ \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$$

lies in S.

Assume that the family S and the object X of $\tilde{\mathcal{C}}$ satisfy the condition:

$(*)$ *There exists a subset S_X of the set $\{s : X' \to X \mid s \in S\}$ such that S_X is an element of the universe $\mathcal{U}$ and for each $s : X' \to X$, $s \in S$, there exist an $s' : X'' \to X$ in S_X and a morphism $u : X'' \to X'$ of $\tilde{\mathcal{C}}$ rendering the following diagram is commutative*:

$$\begin{tikzcd}
X' \arrow[r, "s"] & X \\
X'' \arrow[u, "u"] \arrow[ur, dashed, "s'"'] &
\end{tikzcd}$$

Then the Adams cocompletion X_S of X does exist.

4.2.2 Remark. It is to be emphasized that condition $(*)$ is essential in order to be able to apply E.H. Brown's representability theorem to prove this result.

4.2.3 Remark. By the dual of Remark 1 ([15], P.36) we may replace the assumptions in Theorem 4.2.1 that the family S admits a calculus of right fractions and satisfies axiom (P) by the following assumption:

"S admits a strong calculus of right fractions."

Furthermore Theorem 4.2.1 asserts the existence of a local Adams S-cocompletion. If condition $(*)$ holds for each object X of $\tilde{\mathcal{C}}$, then, of course, we get the existence of a global Adams S-cocompletion. The outline of the proof of this fact can be obtained by following the dual arguments as stated in Remark 1 ([15], P.36).

From the Propositions 4.1.1, 4.1.2 and 4.1.4 and Remark 4.1.3, we note that the conditions of Theorem 4.2.1 are satisfied and so by Theorem 1.5.6 we obtain the following theorem.

4.2.4 Theorem. *Every object* X *of the category* $\widetilde{\mathcal{N}}_0$ *has an Adams cocompletion* X_{S_n} *with respect to the set of morphisms* S_n *and there exists a morphism* $e_n : X_{S_n} \to X$ *in* $\bar{S}_n$ *which is couniversal with respect to morphisms in* S_n.

4.2.5 Proposition. *The morphism* $e_n : X_{S_n} \to X$ *as constructed in Theorem* 4.2.4 *is in* S_n.

Proof. We take

$$S_n^1 = \{\alpha : X \to Y \text{ in } \widetilde{\mathcal{N}}_0 \mid \alpha_* : \pi_m(X; \mathbb{Z}/k\mathbb{Z}) \to \pi_m(Y; \mathbb{Z}/k\mathbb{Z})$$

is a monomorphism for $m \geq n\}$

and

$$S_n^2 = \{\alpha : X \to Y \text{ in } \widetilde{\mathcal{N}}_0 \mid \alpha_* : \pi_m(X;\mathbb{Z}/k\mathbb{Z}) \to \pi_m(Y;\mathbb{Z}/k\mathbb{Z})$$

is an epimorphism for $m \geq n+1\}$.

Clearly

(i) $S_n = S_n^1 \cap S_n^2$ and

(ii) S_n^1 and S_n^2 satisfy all conditions of Theorem 1.5.10;

hence $e \in S_n$. This completes the proof of the Proposition 4.2.5. ■

The above results will be used to obtain a tower for the 0-connected nilpotent spaces with the help of the sets of morphisms S_n.

4.3. Primary decomposition of a 0-connected based nilpotent space.

We obtain the primary decomposition of the 0-connected based nilpotent space with the help of the set of morphisms S_n as described below. In the process, starting from a 0-connected based nilpotent space X we get a tower of spaces,

$$\cdots \to X_{n+1} \xrightarrow{\theta_{n+1}} X_n \to \cdots \to X_1 \xrightarrow{\theta_1} X_0$$

and the direct limit of this tower gives us a space which in some sense is the Cartan-Whitehead decomposition of X.

4.3.1 Proposition. X_n, *as constructed in the proof of Proposition* 4.1.2, *is homotopically equivalent to* X_{S_n}, *as constructed in Theorem* 4.2.4.

Proof. By the couniversal property of $u_n : X_n \to X$ we have a commutative diagram,

$$\begin{tikzcd} X_n \arrow[r, "u_n"] \arrow[d, dashed, "s"'] & X \\ X_{S_n} \arrow[ur, "e_n"'] & \end{tikzcd}$$

i.e., $e_n s = u_n$. Also by the couniversal property of $e_n : X_{S_n} \to X$ we have a commutative diagram,

$$\begin{tikzcd} X_{S_n} \arrow[r, "e_n"] \arrow[d, dashed, "t"'] & X \\ X_n \arrow[ur, "u_n"'] & \end{tikzcd}$$

i.e., $u_n t = e_n$. Thus

$$u_n = e_n s = u_n t s$$

implies

$$ts = 1_{X_n}$$

and

$$e_n = u_n t = e_n s t$$

implies

$$st = 1_{X_{S_n}}$$

and the required homeomorphism between X_n and X_{S_n} is obtained. ■

4.3.2. Theorem. *Let* X *be a* 0*-connected based nilpotent space. Then for* $n \geq 3$, *there exist* 0*-connected based nilpotent spaces* X_n, *maps* $e_n : X_n \to X$ *and fibrations* $\theta_{n+1} : X_{n+1} \to X_n$ *such that*

(a) $e_{n_*} : \pi_m(X_n; \mathbb{Z}/k\mathbb{Z}) \to \pi_m(X; \mathbb{Z}/k\mathbb{Z})$ *is an isomorphism for* $m > n$ *and* $\pi_m(X_n; \mathbb{Z}/k\mathbb{Z}) = 0$ *for* $m \leq n$,

(b) $e_{n+1} = e_n \circ \theta_{n+1}$.

Proof. For each integer $n \geq 3$, let X_n be the S_n-completion of X and $e_n : X_n \to X$ be the canonical map as stated in Theorem 4.2.5. Since $e_n \in S_n \subset S_{n+1}$, it follows from the couniversal property of e_{n+1} that there exists a unique morphism $\theta_{n+1} : X_{n+1} \to X_n$ such that $e_{n+1} = e_n \circ \theta_{n+1}$, i.e., the following diagram is commutative:

$$\begin{tikzcd} X_{n+1} \arrow[r, "e_{n+1}"] \arrow[d, dashed, "\theta_{n+1}"'] & X \\ X_n \arrow[ur, "e_n"'] & \end{tikzcd}$$

The maps θ_n can of course be replaced by fibrations in the usual manner. Since $e_n \in S_n$,

$$e_{n_*} : \pi_m(X_n; \mathbb{Z}/k\mathbb{Z}) \to \pi_m(X; \mathbb{Z}/k\mathbb{Z})$$

is an isomorphism for $m > n$; it is already proved in Proposition 4.1.2 that $\pi_m(X_n; \mathbb{Z}/k\mathbb{Z}) = 0$ for $m \leq n$. Thus we have a tower of spaces:

$$
\begin{tikzcd}
\vdots \arrow[d] & \\
X_{n+1} \arrow[d, "\theta_{n+1}"'] \arrow[ddddr, "e_{n+1}"] & \\
X_n \arrow[dddr, "e_n"'] & \\
\vdots \arrow[d] & \\
X_4 \arrow[d] & \\
X_3 \arrow[r] & X
\end{tikzcd}
$$

This completes the proof of the theorem 4.3.2. ■

Chapter 5

ADAMS COMPLETION AND AND S-FIBRATIONS

Let $\mathcal{C}$ be any small $\mathcal{U}$-category; where $\mathcal{U}$ is a fixed Grothendeick universe. Let S be a set of morphisms in the category $\mathcal{C}$ and $F_S : \mathcal{C} \to \mathcal{C}[S^{-1}]$ be the canonical functor. For convenience we write $F_S = F$. Bauer and Dugundji [4] have introduced the concept of S-fibration, weak S-fibration, S-cofibration and weak S-cofibration in the category $\mathcal{C}$ and have explored the properties of these concepts.

There are some other advantages over the assumption that the set of morphisms S admits a calculus of left (right) fractions. In this chapter we study some cases showing how the assumption that S admits a calculus of left (right) fractions helps us to prove that weak S-fibration implies S-fibration and weak S-cofibration implies S-cofibration.

5.1 S-fibrations

Each class S of morphisms in a category $\mathcal{C}$ determines a concept of fibration (and cofibration) in $\mathcal{C}$. We recall the concepts of S-fibration and weak S-fibration from [4].

5.1.1 Definition. [4] Let S be a subset of morphisms of $\mathcal{C}$. A morphism $p : E \longrightarrow B$ in $\mathcal{C}$ is called an *S-fibration* if for each diagram

$$\begin{tikzcd} W \arrow[r, "s"] & X \arrow[r, "g"] \arrow[dr, "f"'] & E \arrow[d, "p"] \\ & & B \end{tikzcd}$$

with $s \in S$ and $pgs = fs$, there exists a morphism $g' : X \rightarrow E$ in $\mathcal{C}$

$$\begin{tikzcd} W \arrow[r, "s"] & X \arrow[r, dashed, shift left, "g'"] \arrow[r, shift right, "g"'] \arrow[dr, "f"'] & E \arrow[d, "p"] \\ & & B \end{tikzcd}$$

such that $gs = g's$ and $pg' = f$.

5.1.2 Definition. [4] Let S be a subset of morphisms of $\mathcal{C}$. A morphism $p : E \longrightarrow B$ in C is called a *weak S-fibration* if for each diagram

$$\begin{tikzcd} W \arrow[r, "s"] & X \arrow[r, "g"] \arrow[dr, "f"'] & E \arrow[d, "p"] \\ & & B \end{tikzcd}$$

with $s \in S$ and $pgs = fs$, there exists a morphism $g' : X \to E$ in $\mathcal{C}$ and a morphism $t : X \to X$ with $t \in S$

$$\begin{tikzcd} W \arrow[r, "s"] & X \arrow[r, "t"] & X \arrow[r, dashed, shift left, "g'"] \arrow[r, shift right, "g"'] \arrow[dr, "f"'] & E \arrow[d, "p"] \\ & & & B \end{tikzcd}$$

such that $gs = g's$, $ts = s$ and $pg' = ft$.

The following result is elementary in nature.

5.1.3 Proposition. *S-fibration implies weak S-fibration.*

Proof: Let $p : E \to B$ be an S-fibration in the category $\mathcal{C}$. In order to show that $p : E \to B$ is also a weak S-fibration consider an arbitrary diagram

$$\begin{tikzcd} W \arrow[r, "s"] & X \arrow[r, "g"] \arrow[dr, "f"'] & E \arrow[d, "p"] \\ & & B \end{tikzcd}$$

with $s \in S$ and $pgs = fs$. Since $p : E \to B$ is a S-fibration, there exists a morphism $g' : X \to E$ in $\mathcal{C}$,

$$\begin{tikzcd} W \arrow[r, "s"] & X \arrow[r, dashed, shift left, "g'"] \arrow[r, shift right, "g"'] \arrow[dr, "f"'] & E \arrow[d, "p"] \\ & & B \end{tikzcd}$$

such that $gs = g's$ and $pg' = f$. Considering $t = 1_X : X \to X$, we can have $gs = g's$ and $pg' = f1_X = ft$. This completes the proof of the Proposition 5.1.3. ■

Under some moderate assumptions on the set S, it can be proved that weak S-fibration always implies S-fibration.

5.1.4 Proposition. *Let* S *be the set of morphisms in* $\mathcal{C}$. *Let* $F : \mathcal{C} \to \mathcal{C}[S^{-1}]$ be the canonical functor. *Suppose the following conditions hold*:

(a) $p : E \to B$ *is a weak* S*-fibration*.

(b) S *admits a calculus of left fractions*.

(c) S consists of monomorphisms.

Then $p : E \to B$ *is an* S*-fibration*.

Proof: For showing $p : E \to B$ is a fibration consider the diagram

$$\begin{tikzcd}
W \arrow[r, "s"] & X \arrow[r, "g"] \arrow[dr, "f"'] & E \arrow[d, "p"] \\
& & B
\end{tikzcd}$$

with $s \in S$ and $pgs = fs$. Since $s \in S$, $pgs = fs$ and $p : E \to B$ is a weak fibration, there exist a morphism $g' : X \to E$ and $t : X \to X$ with $t \in S$ such that the following diagram commutes

$$\begin{tikzcd}
W \arrow[r, "s"] & X \arrow[r, "t"] & X \arrow[r, dashed, shift left, "g'"] \arrow[r, shift right, "g"'] \arrow[dr, "f"'] & E \arrow[d, "p"] \\
& & & B
\end{tikzcd}$$

i.e., $g's = gs$, $ts = s$ and $pg' = ft$. It is enough to prove that $pg' = f$. Since $pg' = ft$ we have $pg's = fts = fs$. Since F is a covariant functor, we have $F(pg's) = F(fs)$, i.e., $F(p)F(g')F(s) = F(f)F(s)$. Since $F(s)$ is an isomorphism in $\mathcal{C}[S^{-1}]$ we have $F(p)F(g') = F(f)$, i.e., $F(pg') = F(f)$. By Theorem 1.2.7, F is faithful. Hence we have $pg' = f$. This completes the proof of the Proposition 5.1.4. ■

5.2 S-cofibrations

The dual concepts of S-fibration and weak S-fibration are S-cofibration and weak S-cofibration. We recall these concepts from [4].

5.2.1 Definition. [4] Let S be an arbitrary set of morphisms in a category $\mathcal{C}$. A morphism $j : B \to E \in \mathcal{C}$ is called an S-*cofibration* if for each diagram

$$\begin{tikzcd} E \arrow[r, "g"] & X \arrow[r, "s"] & W \\ B \arrow[u, "j"] \arrow[ur, "f"'] & & \end{tikzcd}$$

with $s \in S$ and $sgj = sf$ there exists a morphism $g' : E \to X$

$$\begin{tikzcd} E \arrow[r, dashed, shift left, "g'"] \arrow[r, shift right, "g"'] & X \arrow[r, "s"] & W \\ B \arrow[u, "j"] \arrow[ur, "f"'] & & \end{tikzcd}$$

in $\mathcal{C}$ such that $g'j = f$ and $sg = sg'$.

5.2.2 Definition. [4] A morphism $j : B \to E \in \mathcal{C}$ is called a *weak* S-*cofibration* if for each diagram

$$\begin{tikzcd} E \arrow[r, "g"] & X \arrow[r, "s"] & W \\ B \arrow[u, "j"] \arrow[ur, "f"'] & & \end{tikzcd}$$

with $s \in S$ and $sgj = sf$ there exists a morphism $g' : E \to X$ and $t : X \to X$ with $t \in S$

$$\begin{tikzcd}
E \arrow[r, dashed, shift left, "g'"] \arrow[r, shift right, "g"'] & X \arrow[r, "t"] & X \arrow[r, "s"] & W \\
B \arrow[u, "j"] \arrow[ur, "f"'] & & &
\end{tikzcd}$$

such that $st = s$, $g'j = tf$ and $sg = sg'$.

The following result is elementary in nature.

5.2.3 Proposition. *S-cofibration implies weak S-cofibration.*

Proof. Let $j : B \to E$ be an S-cofibration in the category $\mathcal{C}$. In order to show that $j : B \to E$ is also a weak S-cofibration consider an arbitrary diagram

$$\begin{tikzcd}
E \arrow[r, "g"] & X \arrow[r, "s"] & W \\
B \arrow[u, "j"] \arrow[ur, "f"'] & &
\end{tikzcd}$$

with $s \in S$ and $sgj = sf$. Since $j : B \to E$ is an S-cofibration, there exists a morphism $g' : E \to X$

$$\begin{tikzcd}
E \arrow[r, dashed, shift left, "g'"] \arrow[r, shift right, "g"'] & X \arrow[r, "s"] & W \\
B \arrow[u, "j"] \arrow[ur, "f"'] & &
\end{tikzcd}$$

in $\mathcal{C}$ such that $g'j = f$ and $sg = sg'$. Considering $t = 1_X : X \to X$, we can have $st = s$, $g'j = tf$ and $sg = sg'$. ■

Under some moderate assumptions on the set S, it can be proved that weak S-cofibration always implies S-cofibration.

5.2.4 Proposition. *Let* S *be the set of morphisms in* $\mathcal{C}$. *Let* $F_S = F : \mathcal{C} \to \mathcal{C}[S^{-1}]$ be the canonical functor. *Suppose the following conditions hold*:

(a) $j : B \to E$ *is a weak* S*-cofibration.*

(b) S *admits a calculus of left fractions.*

(c) S consists of monomorphisms.

Then $j : B \to E$ *is an* S*-cofibration.*

Proof. For showing that $j : B \to E$ is an S-cofibration, consider an arbitrary diagram

$$\begin{tikzcd}
E \arrow[r, "g"] & X \arrow[r, "s"] & W \\
B \arrow[u, "j"] \arrow[ur, "f"'] & &
\end{tikzcd}$$

with $s \in S$ and $sgj = sf$. Since $s \in S$ and $sgj = sf$ and $j : B \to E$ is a weak S-cofibration, there exist a morphism $g' : E \to X$ and $t : X \to X$ with $t \in S$ such that the following diagram commotes

$$\begin{tikzcd}
E \arrow[r, dashed, shift left, "g'"] \arrow[r, shift right, "g"'] & X \arrow[r, "t"] & X \arrow[r, "s"] & W \\
B \arrow[u, "j"] \arrow[ur, "f"'] & & &
\end{tikzcd}$$

i.e., $st = s$, $g'j = tf$ and $sg = sg'$. It is enough to prove that $g'j = f$. Since $g'j = tf$ we have $sg'j = stf = sf$. Since F is a covariant functor we have $F(sg'j) = F(sf)$, i.e., $F(s)F(g')F(j) = F(s)F(f)$. Since $F(s)$ is an isomorphism in in $\mathcal{C}[S^{-1}]$ we have $F(g')F(j) = F(f)$, i.e., $F(g'j) = F(f)$. By Theorem 1.2.7, F is faithful. Hence we have $g'j = f$. This completes the proof of the Proposition 5.2.4. ■

5.3 Adams completion and S-fibrations

In [4], Bauer and Dugundji have examined the notion of S-fibration in the category $\mathcal{T}$, the category of topological spaces and continuous functions; under suitable choice of the set S they have shown that a map $p : E \to B$ is an S-fibration if and only if it is a Hurewicz fibration. In this section, under reasonable assumptions we show that a morphism $p : E \to B$ in a category $\mathcal{C}$ is an S-fibration if and only if it is a weak S-fibration.

5.3.1 Theorem. *Let S be a saturated family of morhpisms of a category $\mathcal{C}$ and let every object in $\mathcal{C}$ admit an Adams completion. Let S consist of monomorphisms. Then $\{$weak S-fibrations$\} = \{S$-fibrations$\}$.*

Proof. The proof follows from Theorem 1.5.11, Propositions 5.1.3 and 5.1.4. ■

The following is a direct consequence of Theorem 5.3.1.

5.3.2 Corollary. *Let $\bar{S}$ be the saturation of a family of morhpisms of a category $\mathcal{C}$ and let every object in $\mathcal{C}$ admit an $\bar{S}$-completion. Let S consist of monomorphisms. Then* $\{$*weak $\bar{S}$-fibrations*$\} = \{$*$\bar{S}$-fibrations*$\}$.

5.3.3 Note. In the presence of the conditions of Proposition 4.2.5, we have $\{$*weak S-fibrations*$\} = \{$*S-fibrations*$\}$.

5.3.4 Note. If S contains only the identities of the category $\mathcal{C}$, then $\{$*weak S-fibrations*$\} = \{$*S-fibrations*$\}$ ([4], Remark 1); this is so because S satisfies the conditions of Propositions 4.2.5.

5.3.5 Remark. Everything which has been obtained for S-fibration and weak S-fibration can be dualized in the usual fashion to yield the corresponding results for S-cofibration and weak S-cofibration [4].

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1. Akrur Behera and Sandhya Rani Mohapatra : *Category of Fractions and Acyclic Spaces* : Indian Journal of Mathematics, 54(2) (2012), 143 - 158.

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3. Akrur Behera and Sandhya Rani Mohapatra : *Adams Completion and* 1*-connected Nilpotent Space* : Communicated to Indian Journal of Pure and Applied Mathematics.

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