

# ADAMS COMPLETION FOR $CW$ -COMPLEXES

*by*

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A thesis in conformity with the requirements for the  
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## **CERTIFICATE**

This is to certify that this thesis entitled “ADAMS COMPLETION FOR *CW*-COMPLEXES” which is being submitted by Mrs. Sandhya Rani Mohapatra, Ph.D. Student in Mathematics, Studentship Roll No. 510MA903, National Institute of Technology, Rourkela - 769 008 (India), for the award of the Degree of Doctor of Philosophy in Mathematics at National Institute of Technology, is a record of bonafide research work done by her under my supervision. The results embodied in the thesis are new and have not been submitted to any other University or Institution for the award of any Degree or Diploma.

To the best of my knowledge Mrs. Sandhya Rani Mohapatra bears a good moral character and is mentally and physically fit to get the degree.

Professor Akrur Behera  
Supervisor

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## **ABSTRACT**

The acyclic tower can be obtained through a general categorical completion process due to Adams. More precisely, the different stages of the acyclic tower of a space are obtained as the Adams completion of the space with respect to the carefully chosen sets of morphisms; it is done in the context of a Serre class of abelian groups. Postnikov-like approximation is obtained for a 1-connected based nilpotent space, in terms of Adams completion with respect to a suitable sets of morphisms, using the primary homotopy theory developed by Neisendonfer. Also Cartan-Whitehead decomposition is obtained for a 0-connected based nilpotent space, in terms of Adams cocompletion with respect to properly chosen sets of morphisms, using the primary homotopy theory developed by Neisendonfer. Under suitable assumption it is proved that weak fibration implies fibration and weak cofibration implies cofibration, as introduced by Bauer and Dugundji.

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## Chapter 0

### INTRODUCTION

It is to be emphasized that many algebraic and geometrical constructions in Algebraic Topology, Differential Topology, Differentiable Manifolds, Algebra, Analysis, Topology, etc., can be viewed as Adams completions or cocompletions of objects in suitable categories, with respect to carefully chosen sets of morphisms.

The notion of generalized completion (Adams completion) arose from a categorical completion process suggested by Adams [1, 3]. Originally this was considered for admissible categories and generalized homology (or cohomology) theories. Subsequently, this notion has been considered in a more general framework by Deleanu, Frei and Hilton [14], where an arbitrary category and an arbitrary set of morphisms of the category are considered; moreover they have also suggested the dual notion, namely the cocompletion (Adams cocompletion) of an object in a category.

It is to be noted that the notion of Adams completions or cocompletions arise via the category of fractions. We recall the abstract

definition of the category of fractions [21, 38]. Also for the explicit construction of the category of fractions we refer to [21, 38]. There are some set theoretic difficulties in constructing the category of fractions. These difficulties may be overcome by making some mild hypotheses and using Grothendieck universes. Precisely speaking if we start with a category belonging to a certain Grothendieck universe then the category of fractions with respect to a set of morphisms of the category belongs to a higher universe [38]. We note that the cases in which we are interested, will not present such difficulty. However, Nanda [35] has proved that if the set of morphisms admits a calculus of left (right) fractions then the category of fractions with respect to the set of morphisms of the category belongs to the same universe as to the universe that the category belongs. Also if the set of morphisms of the category admits a calculus of left (right) fractions then the category of fractions can be described nicely; this explicit construction is given in [38].

The central idea of this thesis is to investigate some cases showing how such algebraic and geometrical constructions are characterized in terms of Adams completions or cocompletions. We will deal with such cases involving  $CW$ -complexes and the concepts of calculus of left (right) fractions. In fact in each of the characterizations that we have undertaken in our study, the set of morphisms of the category has to admit either calculus of left fractions or calculus of right fractions.

In Chapter 1, we recall the definitions of Grothendieck universe, category of fractions, calculus of left (right) fractions [38] and generalized Adams completions (cocompletions) [14]. We state some results on the existence of global Adams completions (cocompletions) of an object in a cocomplete (complete) category with respect to a set of morphisms in the category [14]. Deleanu, Frei and Hilton [14] have shown that if the set of morphisms in the category is saturated then the Adams completion (cocompletion) of an object is characterized by a certain couniversal property. We state a stronger version of this result proved by Behera and Nanda [6] where the saturation assumption on the set of morphisms is dropped. We also state Behera and Nanda's result [6] that the canonical map from an object to its Adams completion (from Adams cocompletion to the object) is an element of the set of morphisms under very moderate assumption. These two results are fairly general in nature and applicable to most cases of interest.

Given an acyclic space, Dror [18] has given a general procedure for constructing a Postnikov-like tower of acyclic spaces which successively approximate the given space. In Chapter 2, it is shown that the acyclic tower can be obtained through a general categorical completion process due to Adams. More precisely, it is shown that if  $S_n$  denotes the set of all  $(n + 1)$ -equivalences in the homotopy category of based  $CW$ -complexes which induce isomorphisms in reduced integral homology, then the generalized Adams completion of an acyclic space with respect to  $S_n$  is

the  $n$ -stage of the acyclic tower; it is done in the context of a Serre class of abelian groups.

It is known that the different stages of the Postnikov approximation of a 1-connected space can be obtained as the Adams completions of the space with respect to suitable sets of morphisms [6]. In Chapter 3, Postnikov-like approximation is obtained for a 1-connected based nilpotent space, in terms of Adams completion, using the primary homotopy theory developed by Neisendonfer [36].

It is known that the different stages of the Cartan-Whitehead decomposition of a 0-connected space can be obtained as the Adams cocompletions of the space with respect to suitable sets of morphisms [5]. In Chapter 4, Cartan-Whitehead decomposition is obtained for a 0-connected based nilpotent space, in terms of Adams cocompletion, using the primary homotopy theory developed by Neisendonfer [36].

In Chapter 5, we recall the concepts of  $S$ -fibration, weak  $S$ -fibration,  $S$ -cofibration and weak  $S$ -cofibration from [4] and under suitable assumption we prove that weak  $S$ -fibration implies  $S$ -fibration and weak  $S$ -cofibration implies  $S$ -cofibration.

## Chapter 1

### PRE-REQUISITES

In this chapter we recall the definition of Adams completion (cocompletion) and state some results on the existence of global Adams completion (cocompletion) of an object in a category  $\mathcal{C}$ , with respect to a family of morphisms  $S$  in  $\mathcal{C}$ . We describe Deleanu, Frei and Hilton's characterization of Adams completion (cocompletion) in terms of its couniversal property. We also describe a stronger version of this result proved by Behera and Nanda [6]. We also state Behera and Nanda's result [6] that the canonical map from an object to its Adams completion is an element of the set of morphisms under very moderate assumption. Some more definitions and results are included in the relevant chapters. This chapter serves as the base and background for the subsequent chapters and we shall keep on referring back to it as and when required.

## 1.1 Category of fractions

In this section we recall the abstract definition of the category of fractions.

**1.1.1 Definition.** ([38], P. 266) A *Grothendieck universe* (or simply *universe*) is a collection  $\mathcal{U}$  of sets such that the following axioms are satisfied:

U (1): If  $\{X_i: i \in I\}$  is a family of sets belonging to  $\mathcal{U}$ , then  $\bigcup_{i \in I} X_i$  is an element of  $\mathcal{U}$ .

U (2): If  $x \in \mathcal{U}$ , then  $\{x\} \in \mathcal{U}$ .

U (3): If  $x \in X$  and  $X \in \mathcal{U}$  then  $x \in \mathcal{U}$ .

U (4): If  $X$  is a set belonging to  $\mathcal{U}$ , then  $P(X)$ , the power set of  $X$ , is an element of  $\mathcal{U}$ .

U (5): If  $X$  and  $Y$  are elements of  $\mathcal{U}$ , then  $\{X, Y\}$ , the ordered pair  $(X, Y)$  and  $X \times Y$  are elements of  $\mathcal{U}$ .

We fix a universe  $\mathcal{U}$  that contains  $\mathbb{N}$  the set of natural numbers (and so  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ ).

**1.1.2 Definition.** ([38], p. 267) A category  $\mathcal{C}$  is said to be a *small  $\mathcal{U}$ -category*,  $\mathcal{U}$  being a fixed Grothendieck universe, if the following conditions hold:

S (1) : The objects of  $\mathcal{C}$  form a set which is an element of  $\mathcal{U}$ .

S (2) : For each pair  $(X, Y)$  of objects of  $\mathcal{C}$ , the set  $\text{Hom}_{\mathcal{C}}(X, Y)$  is an element of  $\mathcal{U}$ .

**1.1.3 Definition.** ([38], p. 269) Let  $\mathcal{C}$  be any arbitrary category and  $S$  a set of morphisms of  $\mathcal{C}$ . A *category of fractions* of  $\mathcal{C}$  with respect to  $S$  is a category denoted by  $\mathcal{C}[S^{-1}]$  together with a functor

$$F_S : \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$$

having the following properties :

CF (1) : For each  $s \in S$ ,  $F_S(s)$  is an isomorphism in  $\mathcal{C}[S^{-1}]$ .

CF (2) :  $F_S$  is universal with respect to this property : If  $G : \mathcal{C} \rightarrow \mathcal{D}$  is a functor such that  $G(s)$  is an isomorphism in  $\mathcal{D}$ , for each  $s \in S$ , then there exists a unique functor  $H : \mathcal{C}[S^{-1}] \rightarrow \mathcal{D}$  such that  $G = HF_S$ . Thus we have the following commutative diagram:

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{F_S} & \mathcal{C}[S^{-1}] \\
 \downarrow G & & \swarrow H \\
 \mathcal{D} & & 
 \end{array}$$

**1.1.4 Remark.** For the explicit construction of the category  $\mathcal{C}[S^{-1}]$ , we refer to [38]. We content ourselves merely with the observation that the objects of  $\mathcal{C}[S^{-1}]$  are same as those of  $\mathcal{C}$  and in the case when  $S$  admits a calculus of left (right) fractions, the category  $\mathcal{C}[S^{-1}]$  can be described very nicely [21, 38].

## 1.2 Calculus of left (right) fractions

The concepts of calculus of left fractions and right fraction play a crucial role in constructing the category of fractions  $\mathcal{C}[S^{-1}]$ .

**1.2.1 Definition.** ([38], p. 258) A family of morphisms  $S$  in the category  $\mathcal{C}$  is said to admit a *calculus of left fractions* if

- (a)  $S$  is closed under finite compositions and contains identities of  $\mathcal{C}$ ,
- (b) any diagram

$$\begin{array}{ccc} & s & \\ & \longrightarrow & \\ X & & Y \\ & \downarrow f & \\ & Z & \end{array}$$

in  $\mathcal{C}$  with  $s \in S$  can be completed to a diagram

$$\begin{array}{ccc}
 & s & \\
 X & \longrightarrow & Y \\
 f \downarrow & & \downarrow g \\
 Z & \longrightarrow & W \\
 & t & 
 \end{array}$$

with  $t \in S$  and  $tf = gs$ ,

(c) given

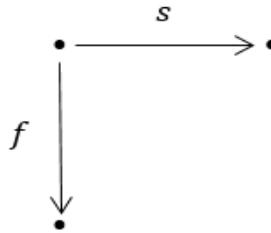
$$X \xrightarrow{s} Y \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Z \dashrightarrow^t W$$

with  $s \in S$  and  $fs = gs$ , there is a morphism  $t : Z \rightarrow W$  in  $S$  such that  $tf = tg$ .

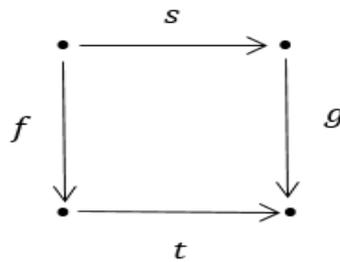
A simple characterization for a family of morphisms  $S$  to admit a calculus of left fractions is the following.

**1.2.2 Theorem.** ([14], Theorem 1.3, p. 67) *Let  $S$  be a closed family of morphisms of  $\mathcal{C}$  satisfying*

- (a) *if  $uv \in S$  and  $v \in S$ , then  $u \in S$ ,*
- (b) *every diagram*



in  $\mathcal{C}$  with  $s \in S$  can be embedded in a weak push-out diagram



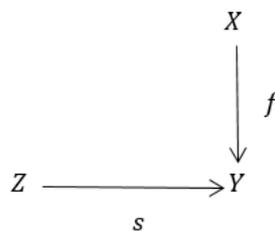
with  $t \in S$ .

Then  $S$  admits a calculus of left fractions.

The notion of a set of morphisms admitting a calculus of right fractions is defined dually.

**1.2.3 Definition.** ([38], p. 267) A family  $S$  of morphisms in a category  $\mathcal{C}$  is said to admit a *calculus of right fractions* if

(a) any diagram



in  $\mathcal{C}$  with  $s \in S$  can be completed to a diagram

$$\begin{array}{ccc}
 W & \xrightarrow{t} & X \\
 g \downarrow & & \downarrow f \\
 Z & \xrightarrow{s} & Y
 \end{array}$$

with  $t \in S$  and  $ft = sg$ ,

(b) given

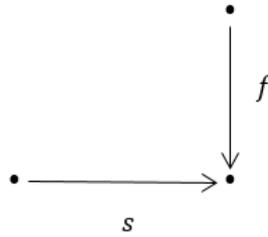
$$\begin{array}{ccccc}
 W & \xrightarrow{t} & X & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & Y & \xrightarrow{s} & Z
 \end{array}$$

with  $s \in S$  and  $sf = sg$ , there is a morphism  $t : W \rightarrow X$  in  $S$  such that  $ft = gt$ .

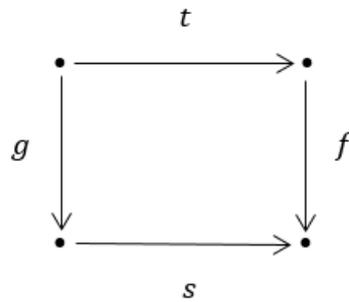
The analog of Theorem 1.2.2 follows immediately by duality.

**1.2.4 Theorem.** ([14], Theorem 1.3\*, p. 70) *Let  $S$  be a closed family of morphisms of  $\mathcal{C}$  satisfying*

- (a) *if  $vu \in S$  and  $v \in S$ , then  $u \in S$ ,*
- (b) *any diagram*



in  $\mathcal{C}$  with  $s \in S$ , can be embedded in a weak pull-back diagram



with  $t \in S$ .

Then  $S$  admits a calculus of right fractions.

**1.2.5 Remark.** There are some set-theoretic difficulties in constructing the category  $\mathcal{C}[S^{-1}]$ ; these difficulties may be overcome by making some mild hypotheses and using Grothendieck universes. Precisely speaking, the main logical difficulty involved in the construction of a category of fractions and its use, arises from the fact that if the category  $\mathcal{C}$  belongs to a particular universe, the category  $\mathcal{C}[S^{-1}]$  would, in general, belong to a higher universe ([38], Proposition 19.1.2). In most applications, however, it is necessary that we remain within the given initial universe. This logical difficulty can be overcome by making some kind of assumptions which would ensure that the category of fractions remains within the same

universe [13, 15]. Also the following theorem (Theorem 1.2.6) shows that if  $S$  admits a calculus of left (right) fractions, then the category of fractions  $\mathcal{C}[S^{-1}]$  remains within the same universe as to the universe to which the category  $\mathcal{C}$  belongs.

The following results will be used in our study.

**1.2.6 Theorem.** ([35], Proposition, p. 425) *Let  $\mathcal{C}$  be a small  $\mathcal{U}$ -category and  $S$  a set of morphisms of  $\mathcal{C}$  that admits a calculus of left (right) fractions. Then  $\mathcal{C}[S^{-1}]$  is a small  $\mathcal{U}$ -category.*

**1.2.7 Theorem.** ([38], Lemma 19.2.6, p. 261) *Let  $\mathcal{C}$  be an arbitrary category and  $S$  a set of morphisms of  $\mathcal{C}$ . Let  $\mathcal{C}[S^{-1}]$  denote the category of fractions of  $\mathcal{C}$  with respect to  $S$  and*

$$F_S : \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$$

*be the canonical functor. Let the following hold:*

- (a)  *$S$  consists of monomorphisms.*
- (b)  *$S$  admits a calculus of left fractions.*

*Then  $F_S$  is faithful.*

### 1.3 Adams completion and cocompletion

We recall the definitions of Adams completion and cocompletion.

**1.3.1 Definition.** [14] Let  $\mathcal{C}$  be an arbitrary category and  $S$  a set of morphisms of  $\mathcal{C}$ . Let  $\mathcal{C}[S^{-1}]$  denote the category of fractions of  $\mathcal{C}$  with respect to  $S$  and

$$F : \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$$

be the canonical functor. Let  $\mathcal{S}$  denote the category of sets and functions.

Then for a given object  $Y$  of  $\mathcal{C}$ ,

$$\mathcal{C}[S^{-1}](-, Y) : \mathcal{C} \rightarrow \mathcal{S}$$

defines a contravariant functor. If this functor is representable by an object  $Y_S$  of  $\mathcal{C}$ , i.e.,

$$\mathcal{C}[S^{-1}](-, Y) \cong \mathcal{C}(-, Y_S)$$

then  $Y_S$  is called the (*generalized*) *Adams completion* of  $Y$  with respect to the set of morphisms  $S$  or simply the *S-completion* of  $Y$ . We shall often refer to  $Y_S$  as the *completion* of  $Y$  [14].

The above definition can be dualized as follows:

**1.3.2 Definition.** [14] Let  $\mathcal{C}$  be an arbitrary category and  $S$  a set of morphisms of  $\mathcal{C}$ . Let  $\mathcal{C}[S^{-1}]$  denote the category of fractions of  $\mathcal{C}$  with respect  $S$  and

$$F: \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$$

be the canonical functor. Let  $\mathcal{S}$  denote the category of sets and functions.

Then for a given object  $Y$  of  $\mathcal{C}$ ,

$$\mathcal{C}[S^{-1}](Y, -) : \mathcal{C} \rightarrow \mathcal{S}$$

defines a covariant functor. If this functor is representable by an object  $Y_S$  of  $\mathcal{C}$ , i.e.,

$$\mathcal{C}[S^{-1}](Y, -) \cong \mathcal{C}(Y_S, -)$$

then  $Y_S$  is called the (*generalized*) *Adams cocompletion* of  $Y$  with respect to the set of morphisms  $S$  or simply the  *$S$ -cocompletion of  $Y$* . We shall often refer to  $Y_S$  as the cocompletion of  $Y$  [14].

#### 1.4 Existence theorems

We recall some results on the existence of Adams completion and cocompletion. We state Deleanu's theorem [15] that under certain conditions, global Adams completion of an object always exists.

**1.4.1 Theorem.** ([15], Theorem 1; [35], Theorem 1) *Let  $\mathcal{C}$  be a cocomplete small  $\mathcal{U}$ -category ( $\mathcal{U}$  is a fixed Grothendieck universe) and  $S$  a set of morphisms of  $\mathcal{C}$  that admits a calculus of left fractions. Suppose that the following compatibility condition with coproduct is satisfied.*

(C) *If each  $s_i : X_i \rightarrow Y_i$ ,  $i \in I$ , is an element of  $S$ , where the index set  $I$  is an element of  $\mathcal{U}$ , then*

$$\coprod_{i \in I} s_i : \coprod_{i \in I} X_i \rightarrow \coprod_{i \in I} Y_i$$

*is an element of  $S$ .*

*Then every object  $X$  of  $\mathcal{C}$  has an Adams completion  $X_S$  with respect to the set of morphisms  $S$ .*

**1.4.2 Remark.** Deleanu's theorem quoted above has an extra condition to ensure that  $\mathcal{C}[S^{-1}]$  is again a small  $\mathcal{U}$ -category; in view of Theorem 1.2.6 the extra condition is not necessary.

Theorem 1.4.1 can be dualized as follows (it is also Theorem 2 in [35]).

**1.4.3 Theorem.** *Let  $\mathcal{C}$  be a complete small  $\mathcal{U}$ -category ( $\mathcal{U}$  is a fixed Grothendieck universe) and  $S$  a set of morphisms of  $\mathcal{C}$  that admits a calculus of right fractions. Suppose that the following compatibility condition with product is satisfied.*

(C) *If each  $s_i : X_i \rightarrow Y_i$ ,  $i \in I$ , is an element of  $S$ , where the index set  $I$  is an element of  $\mathcal{U}$ , then*

$$\prod_{i \in I} s_i : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$$

*is an element of  $S$ .*

*Then every object  $X$  of  $\mathcal{C}$  has an Adams cocompletion  $X_S$  with respect to the set of morphisms  $S$ .*

We will recall some more results on the existence of Adams completion and cocompletion in the relevant chapters.

## 1.5 Couniversal property

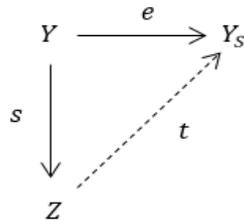
The concepts of Adams completion and cocompletion can be characterized in terms of a couniversal property.

**1.5.1 Definition.** [14] Given a set  $S$  of morphisms of  $\mathcal{C}$ , we define  $\bar{S}$ , the *saturation* of  $S$  as the set of all morphisms  $u$  in  $\mathcal{C}$  such that  $F(u)$  is an isomorphism in  $\mathcal{C}[S^{-1}]$ .  $S$  is said to be *saturated* if  $S = \bar{S}$ .

**1.5.2 Theorem.** ([14], Proposition 1.1, p. 63) A family  $S$  of morphisms of  $\mathcal{C}$  is *saturated* if and only if there exists a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  such that  $S$  is the collection of all morphisms  $f$  such that  $F(f)$  is invertible.

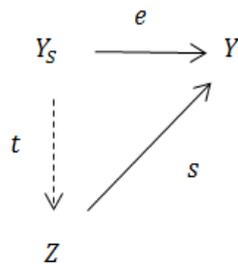
Deleanu, Frei and Hilton have shown that if the set of morphisms  $S$  is saturated then the Adams completion of a space is characterized by a certain couniversal property.

**1.5.3 Theorem.** ([14], Theorem 1.2, p. 63) *Let  $S$  be a saturated family of morphisms of  $\mathcal{C}$  admitting a calculus of left fractions. Then an object  $Y_S$  of  $\mathcal{C}$  is the  $S$ -completion of the object  $Y$  with respect to  $S$  if and only if there exists a morphism  $e : Y \rightarrow Y_S$  in  $S$  which is couniversal with respect to morphisms of  $S$ : given a morphism  $s : Y \rightarrow Z$  in  $S$  there exists a unique morphism  $t : Z \rightarrow Y_S$  in  $S$ , such that  $ts = e$ . In other words, the following diagram is commutative:*



Theorem 1.5.3 can be dualized as follows.

**1.5.4 Theorem.** ([14], Theorem 1.4, p. 68) *Let  $S$  be a saturated family of morphisms of  $\mathcal{C}$  admitting a calculus of right fractions. Then an object  $Y_S$  of  $\mathcal{C}$  is the  $S$ -cocompletion of the object  $Y$  with respect to  $S$  if and only if there exists a morphism  $e : Y_S \rightarrow Y$  in  $S$  which is couniversal with respect to morphisms of  $S$ : given a morphism  $s : Z \rightarrow Y$  in  $S$  there exists a unique morphism  $t : Y_S \rightarrow Z$  in  $S$  such that  $st = e$ . In other words, the following diagram is commutative:*



In most of applications, however, the set of morphisms  $S$  is not saturated. The following is a stronger version of Deleanu, Frei and Hilton's characterization of Adams completion in terms of a couniversal property.

**1.5.5 Theorem.** ([6], Theorem 1.2, p. 528) *Let  $S$  be a set of morphisms of  $\mathcal{C}$  admitting a calculus of left fractions. Then an object  $Y_S$  of  $\mathcal{C}$  is the  $S$ -*

completion of the object  $Y$  with respect to  $S$  if and only if there exists a morphism  $e : Y \rightarrow Y_S$  in  $\bar{S}$  which is couniversal with respect to morphisms of  $S$ : given a morphism  $s : Y \rightarrow Z$  in  $S$  there exists a unique morphism  $t : Z \rightarrow Y_S$  in  $\bar{S}$  such that  $ts = e$ . In other words, the following diagram is commutative:

$$\begin{array}{ccc}
 Y & \xrightarrow{e} & Y_S \\
 s \downarrow & \nearrow t & \\
 Z & & 
 \end{array}$$

Theorem 1.5.5 can be dualized as follows.

**1.5.6 Theorem.** ([5], Proposition 1.1, p. 224) *Let  $S$  be a set of morphisms of  $\mathcal{C}$  admitting a calculus of right fractions. Then an object  $Y_S$  of  $\mathcal{C}$  is the  $S$ -cocompletion of the object  $Y$  with respect to  $S$  if and only if there exists a morphism  $e : Y_S \rightarrow Y$  in  $\bar{S}$  which is couniversal with respect to morphisms of  $S$ : given a morphism  $s : Z \rightarrow Y$  in  $S$  there exists a unique morphism  $t : Y_S \rightarrow Z$  in  $\bar{S}$  such that  $st = e$ . In other words, the following diagram is commutative:*

$$\begin{array}{ccc}
 Y_S & \xrightarrow{e} & Y \\
 t \downarrow & \nearrow s & \\
 Z & & 
 \end{array}$$

For most of the application it is essential that the morphism  $e : Y \rightarrow Y_S$  ( $e : Y_S \rightarrow Y$ ) has to be in  $S$ ; this is the case when  $S$  is saturated and the results are as follows:

**1.5.7 Theorem.** ([14], Theorem 2.9, p. 76) *Let  $S$  be a saturated family of morphisms of  $\mathcal{C}$  and let every object of  $\mathcal{C}$  admit an  $S$ -completion. Then the morphism  $e : Y \rightarrow Y_S$  belongs to  $S$  and is universal for morphisms to  $S$ -complete objects and couniversal for morphisms in  $S$ .*

The above result can be dualized as follows.

**1.5.8 Theorem.** ([14], dual of Theorem 2.9, p. 76) *Let  $S$  be a saturated family of morphisms of  $\mathcal{C}$  and let every object of  $\mathcal{C}$  admit an  $S$ -cocompletion. Then the morphism  $e : Y_S \rightarrow Y$  belongs to  $S$  and is universal for morphisms to  $S$ -cocomplete objects and couniversal for morphisms in  $S$ .*

However, in many cases of practical interest  $S$  is not saturated. The following result shows that under some extra conditions on  $S$ , the morphism  $e : Y \rightarrow Y_S$  ( $e : Y_S \rightarrow Y$ ) always belongs to  $S$ .

**1.5.9 Theorem.** ([6], Theorem 1.3, p. 533) *Let  $S$  be a set of morphisms in a category  $\mathcal{C}$  admitting a calculus of left fractions. Let  $e : Y \rightarrow Y_S$  be the canonical morphism as defined in Theorem 1.5.5, where  $Y_S$  is the  $S$ -*

completion of  $Y$ . Furthermore, let  $S_1$  and  $S_2$  be sets of morphisms in the category  $\mathcal{C}$  which have the following properties:

- (a)  $S_1$  and  $S_2$  are closed under composition,
- (b)  $fg \in S_1$  implies that  $g \in S_1$ ,
- (c)  $fg \in S_2$  implies that  $f \in S_2$ ,
- (d)  $S = S_1 \cap S_2$ .

Then  $e \in S$ .

Theorem 1.5.9 can be dualized as follows:

**1.5.10 Theorem.** ([6], dual of Theorem 1.3, p.533) *Let  $S$  be a set of morphisms in a category  $\mathcal{C}$  admitting a calculus of right fractions. Let  $e : Y_S \rightarrow Y$  be the canonical morphism as defined in Theorem 1.5.6, where  $Y_S$  is the  $S$ -cocompletion of  $Y$ . Furthermore, let  $S_1$  and  $S_2$  be sets of morphisms in the category  $\mathcal{C}$  which have the following properties:*

- (a)  $S_1$  and  $S_2$  are closed under composition,
- (b)  $fg \in S_1$  implies that  $g \in S_1$ ,
- (c)  $fg \in S_2$  implies that  $f \in S_2$ ,
- (d)  $S = S_1 \cap S_2$ .

Then  $e \in S$ .

The following result has many applications.

**1.5.11 Theorem.** ([14], Theorem 2.10, p. 76) *Let  $S$  be a saturated family of morphisms of the category  $\mathcal{C}$ . Then the following three statements are equivalent:*

- (a) *Every object  $Y$  in  $\mathcal{C}$  admits an  $S$ -completion.*
- (b)  *$S$  admits a calculus of left fractions,  $\varinjlim P_Y$  exists for all  $Y$ , where  $P_Y : \mathcal{C}(Y; S) \rightarrow \mathcal{C}$ , and  $F_S$  commutes with  $\varinjlim P_Y$ .*
- (c)  *$S$  admits a calculus of left fractions,  $\varinjlim P_Y$  exists for all  $Y$  and  $F_S$  commutes with all colimits in  $\mathcal{C}$ .*

## 1.6 Modulo a Serre class $\mathcal{C}$ of abelian groups

In Chapter 2 we use “modulo a Serre class  $\mathcal{C}$  of abelian groups” [40] to obtain the mod- $\mathcal{C}$  acyclic decomposition of a 1-connected based CW-complex, with the help of a suitable set of morphisms. In order to make the exposition self-contained, we collect some relevant definitions and theorems involving Serre classes of abelian groups [40].

**1.6.1 Definition.** [40] A nonempty class  $\mathcal{C}$  of abelian groups is called a *Serre class of abelian groups* if and only if whenever the three-term sequence  $A \rightarrow B \rightarrow C$  of abelian groups is exact and  $A, C \in \mathcal{C}$ , then  $B \in \mathcal{C}$ .

The following theorem is easy to verify.

**1.6.2 Theorem.** ([40], Theorem 1, p. 504) *A class  $\mathcal{C}$  of abelian groups is a Serre class if and only if it has the following properties:*

- (a)  $\mathcal{C}$  contains a trivial group.
- (b) If  $A \in \mathcal{C}$  and  $A \approx A'$ , then  $A' \in \mathcal{C}$ ,
- (c) If  $A \subset B$  and  $B \in \mathcal{C}$ , then  $A \in \mathcal{C}$ ,
- (d) If  $A \subset B$  and  $B \in \mathcal{C}$ , then  $B/A \in \mathcal{C}$ ,
- (e) If  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is a short exact sequence, with  $A, C \in \mathcal{C}$ , then  $B \in \mathcal{C}$ .

**1.6.3 Examples.** ([40], p. 505) We list some examples of Serre class:

- (a) The class of all abelian groups.
- (b) The class of trivial groups.
- (c) The class of finitely generated abelian groups.
- (d) The class of finite abelian groups.
- (e) The class of torsion abelian groups.
- (f) The class of  $p$ -groups for a given prime  $p$ .
- (g) The class of groups having no element with order a positive power of a given prime  $p$ .

**1.6.4 Definition.** [40] Let  $\mathcal{C}$  be a Serre class of abelian groups and  $A, B \in \mathcal{C}$ . A homomorphism  $f : A \rightarrow B$

- (a) is a  $\mathcal{C}$ -monomorphism if  $\ker f \in \mathcal{C}$ ,
- (b) is a  $\mathcal{C}$ -epimorphism if  $\text{coker } f \in \mathcal{C}$ ,

- (c) is a  $\mathcal{C}$ -isomorphism if it is both a  $\mathcal{C}$ -monomorphism and  $\mathcal{C}$ -epimorphism.

**1.6.5 Theorem.** [40] Let  $A, B \in \mathcal{C}$  and  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be two homomorphisms. Then the following statements are true.

- (a) If  $gf$  is  $\mathcal{C}$ -monic, then so is  $f$ .  
 (b) If  $gf$  is  $\mathcal{C}$ -epic, then so is  $g$ .

**1.6.6 Theorem.** (The mod- $\mathcal{C}$  Five lemma) ([40], p. 519) Suppose that

$$\begin{array}{ccccccccc}
 A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
 f_1 \downarrow & & f_2 \downarrow & & f_3 \downarrow & & \downarrow & f_4 & \downarrow f_5 \\
 B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
 \end{array}$$

is a diagram of abelian groups and homomorphisms in  $\mathcal{C}$  in which the rows are exact and each square is commutative. Then the following hold:

- (a) If  $f_2, f_4$  are  $\mathcal{C}$ -monomorphisms and  $f_1$  is a  $\mathcal{C}$ -epimorphism, then  $f_3$  is a  $\mathcal{C}$ -monomorphism.  
 (b) If  $f_2, f_4$  are  $\mathcal{C}$ -epimorphisms and  $f_5$  is a  $\mathcal{C}$ -monomorphism, then  $f_3$  is a  $\mathcal{C}$ -epimorphism.

**1.6.7 Definition.** [40] A Serre class  $\mathcal{C}$  is called an *ideal of abelian groups* if  $A \in \mathcal{C}$  implies  $A \otimes B, \text{Tor}(A, B) \in \mathcal{C}$  for arbitrary  $B$ .

**1.6.8 Definition.** [40] A topological space  $X$  is said to be  $\mathcal{C}$ -acyclic if its integral homology groups  $H_n(X) \in \mathcal{C}$  for all  $n > 0$ .

**1.6.9 Definition.** [40] A Serre class  $\mathcal{C}$  of abelian groups is said to be an *acyclic class* if any space of type  $(\pi, 1)$  with  $\pi \in \mathcal{C}$  is  $\mathcal{C}$ -acyclic.

## 1.7 Primary homotopy theory.

In [36] Neisendorfer has studied primary homotopy theory in a very precise manner by giving complete proofs of most results so that the subject should be accessible to lay topologists. We briefly recall the primary homotopy theory.

**1.7.1 Definition.** [36] Let  $S^m$  denote the  $m$ -dimensional sphere. Suppose  $m \geq 2$ , and let  $k : S^m \rightarrow S^m$  denote a map of degree  $k$ . The space  $S^{m-1} \cup_k e^m$  is denoted by  $P^m(k)$  or  $P^m(\mathbb{Z}/k\mathbb{Z})$ . If  $m \geq 2$ , the  $m$ -th *mod  $k$  homotopy group* of  $X$  is  $[P^m(k); X]$ , denoted by  $\pi_m(X; \mathbb{Z}/k\mathbb{Z})$ . If  $m \geq 3$ ,  $\pi_m(X; \mathbb{Z}/k\mathbb{Z})$  is a group and  $m \geq 4$ ,  $\pi_m(X; \mathbb{Z}/k\mathbb{Z})$  is an abelian group.

If  $f : X \rightarrow Y$  is a map, then there are induced maps

$$f_* : \pi_m(X; \mathbb{Z}/k\mathbb{Z}) \rightarrow \pi_m(Y; \mathbb{Z}/k\mathbb{Z})$$

defined by  $f_*[g] = [fg]$ . If  $m \geq 3$ ,  $f_*$  is a homomorphism and, if  $m = 2$ ,  $f_*$  is compatible with the action of  $\pi_2$  [36].

**1.7.2 Definition.** [24] For a group  $G$ , the *lower central series*

$$\dots \subseteq \Gamma^{i+1}(G) \subseteq \Gamma^i(G) \dots \subseteq \Gamma^1(G)$$

of  $G$ , is defined by the setting

$$\Gamma^1(G) = G, \quad \Gamma^{i+1}(G) = [G, \Gamma^i(G)], \quad i \geq 1.$$

$G$  is said to be *nilpotent* if  $\Gamma^j(G) = \{1\}$  for  $j$  sufficiently large.

**1.7.3 Definition.** [24] A connected  $CW$ -complex  $X$  is said to *nilpotent* if  $\pi_1(X)$  is nilpotent and operates nilpotently on  $\pi_n(X)$  for every  $n \geq 2$ .

The following result will be used in our study.

**1.7.4 Theorem.** ([36], Corollary 3.10) *Let  $f : X \rightarrow Y$  be a map between simply connected spaces. Then*

$$f_* : \pi_i(X; \mathbb{Z}/k\mathbb{Z}) \rightarrow \pi_i(Y; \mathbb{Z}/k\mathbb{Z})$$

*is a bijection for all  $i \geq 2$  if and only if*

$$f_* : H_i(X; \mathbb{Z}/k\mathbb{Z}) \rightarrow H_i(Y; \mathbb{Z}/k\mathbb{Z})$$

*is a bijection for all  $i \geq 2$ .*

## Chapter 2

### CATEGORY OF FRACTIONS AND ACYCLIC SPACES

Given an acyclic space  $X$  (i.e. a topological space  $X$  with  $\tilde{H}_*(X) = 0$ , where  $\tilde{H}_*(X)$  denotes the reduced integral homology functor), Dror [18] has given a general procedure for constructing a tower of acyclic spaces (which he calls an acyclic decomposition of  $X$ ) which successively approximate  $X$ . An acyclic decomposition of a space is a Postnikov-like decomposition and, as Dror has shown, has many advantages, particularly in analyzing the homotopy structure of an acyclic space. The  $n$ -stage of an acyclic decomposition in Dror's construction is obtained via the acyclic functor from the corresponding  $n$ -stage of the Postnikov tower. The object of this chapter is to demonstrate that the acyclic decomposition tower can be obtained through a general categorical completion process which is due to Adams [1] and which has been developed by Deleanu, Frei and Hilton [14] in a much more general context. More precisely, it is shown that each stage of acyclic tower can be obtained as the generalized Adams completion of a certain sets of morphisms in the category of the based  $CW$ -complexes. The relationship between any stage of the acyclic

decomposition and the corresponding stage of the Postnikov decomposition is also clearly demonstrated within the frame work of generalized Adams completion. We formulate the results in the homotopy category of the based  $CW$ -complexes. There is no loss of generality in working in this category; for if  $X$  is an arbitrary topological space, then there is a  $CW$ -complex  $Y$  and a map  $f : Y \rightarrow X$  which is a weak homotopy equivalence and since the reduced integral homology functor satisfies the weak homotopy equivalence axiom ([42], p.181), it follows that if  $X$  is acyclic, so is  $Y$ . The problem is then to construct a tower of acyclic spaces in the homotopy category of the based  $CW$ -complexes whose inverse limit is  $Y$ . This we do using the notation of generalized Adams completion in the context of a Serre class of abelian groups [42], a brief summary of which is already given in Section 1.6 above. From now onwards, we assume that  $\mathcal{C}$  is a Serre class which is moreover an acyclic ideal of abelian groups.

## 2.1 The category $\widetilde{\mathcal{CW}}$

Let  $\mathcal{CW}$  denote the category of 1-connected based  $CW$ -complexes and based maps and let  $\widetilde{\mathcal{CW}}$  be the corresponding homotopy category. We assume that the underlying sets of the elements of  $\widetilde{\mathcal{CW}}$  are elements of  $\mathcal{U}$  where  $\mathcal{U}$  is a fixed Grothendieck universe. We now fix suitable sets of morphisms of  $\widetilde{\mathcal{CW}}$  as stated below :

- (a) Let  $S = \{\alpha : X \rightarrow Y \text{ in } \widetilde{\mathcal{CW}} \mid \alpha_* : \widetilde{H}_*(X) \rightarrow \widetilde{H}_*(Y) \text{ is a } \mathcal{C}\text{-isomorphism in reduced integral homology}\}$ .
- (b) A map  $\alpha : X \rightarrow Y$  in  $\widetilde{\mathcal{CW}}$  is called a mod- $\mathcal{C}$   $(n + 1)$ -equivalence if  $\alpha_* : \pi_*(X) \rightarrow \pi_*(Y)$  is a  $\mathcal{C}$ -isomorphism for  $m \leq n$  and a  $\mathcal{C}$ -epimorphisms for  $m = n + 1$ .
- Let  $S_n$  denote the set all a mod- $\mathcal{C}$   $(n + 1)$  equivalences in  $\widetilde{\mathcal{CW}}$ .
- (c) Let  $S_n^* = S_n \cap S$ .

Since the category  $\widetilde{\mathcal{CW}}$  as stated above is neither cocomplete nor small, Theorem 1.4.1 cannot be used to show the existence of Adams completion of an object in the category of  $\widetilde{\mathcal{CW}}$  with respect to the set of morphisms  $S$ . However, we have the following result (Theorem 2.1.1) which is essentially Theorem 4.7 [1] and Theorem 3.8 [3] (it is also a generalization of Theorem [13]).

**2.1.1 Theorem.** *Let  $\mathcal{U}$  be a fixed Grothendieck universe. Let  $\widetilde{\mathcal{C}}$  be the category defined as follows : the objects of  $\widetilde{\mathcal{C}}$  are the connected based CW-complexes whose underlying sets are elements of  $\mathcal{U}$ ; the morphisms of  $\widetilde{\mathcal{C}}$  are based homotopy classes of based-point preserving maps between such CW-complexes. Let  $S$  be a family of morphisms of  $\widetilde{\mathcal{C}}$  admitting a*

calculus of left fractions and satisfying the following axioms of compatibility with coproducts:

- (C) If  $s_i : X_i \rightarrow Y_i$  lies in  $S$  for each  $i \in I$ , where the index set  $I$  is an element of  $\mathcal{U}$ ; then

$$\bigvee_{i \in I} s_i : \bigvee_{i \in I} X_i \rightarrow \bigvee_{i \in I} Y_i$$

lies in  $S$ .

Assume that the family  $S$  and the object  $X$  of  $\tilde{\mathcal{C}}$  satisfy the condition:

- (\*) There exists a subset  $S_X$  of the set  $\{s : X \rightarrow X' \mid s \in S\}$  such that  $S_X$  is an element of the universe  $\mathcal{U}$  and for each  $s : X \rightarrow X'$ ,  $s \in S$ , there exist an  $s' : X \rightarrow X''$  in  $S_X$  and a morphism  $u : X' \rightarrow X''$  of  $\tilde{\mathcal{C}}$  rendering the following diagram commutative :

$$\begin{array}{ccc} X & & \\ \downarrow s & \searrow s' & \\ X' & \xrightarrow{u} & X'' \end{array}$$

Then the Adams completion  $X_S$  of  $X$  does exist.

**2.1.2 Note.** As remarked by Adams of page 34 of [3] this result remains valid if  $\tilde{\mathcal{C}}$  is the homotopy category of 1-connected based CW-complexes (whose underlying sets belong to  $\mathcal{U}$ ).

**2.1.3 Note.** It is to be emphasized that condition (\*) is essential in order to be able to apply E.H. Browns responsibility theorem to prove this result.

## 2.2 $S$ -completion in $\widetilde{\mathcal{CW}}$

We use Theorems 1.5.5, 1.5.9 and 2.1.1 to show that the Adams completion with respect to the set of morphisms  $S$  always exists in  $\widetilde{\mathcal{CW}}$ .

**2.2.1 Proposition.**  $S$  admits a calculus of left fractions.

**Proof.** Clearly  $S$  is closed under composition. We shall verify conditions (a) and (b) of Theorem 1.2.3. Only condition (b) is in question. For proving this condition it is enough to prove that every diagram

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & Z \\ \alpha \downarrow & & \\ Y & & \end{array}$$

in  $\widetilde{\mathcal{CW}}$  with  $\gamma \in S$ , can be embedded in a weak push-out diagram

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & Z \\ \alpha \downarrow & & \downarrow \beta \\ Y & \xrightarrow{\delta} & W \end{array}$$

with  $\delta \in S$ . Suppose  $\alpha = [f]$  and  $\gamma = [s]$ . Let  $i_f : X \rightarrow M_f$  be the usual inclusion of  $X$  into  $M_f$ , the reduced mapping cylinder of  $f$  and  $i_f$  is defined by  $i_f(x) = [0, x]$ . The map  $j : Y \rightarrow M_f$  is defined by  $j(y) = [y]$  and the map  $r : M_f \rightarrow Y$  is defined by

$$r([y]) = y, r([s, x]) = f(x).$$

Hence  $rj = 1_Y$ ,  $jr \simeq 1_{M_f}$  and  $ri_f = f$  [42]. Consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{s} & Z \\ i_f \downarrow & & \\ Y & & \end{array}$$

and form its push-out in  $\mathcal{CW}$ :

$$\begin{array}{ccc} X & \xrightarrow{s} & Z \\ i_f \downarrow & & \downarrow u \\ Y & \xrightarrow{t} & W \end{array}$$

Since  $i_f : X \rightarrow M_f$  is a cofibration so is  $u : Z \rightarrow W$ . Hence we have

the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i_f} & M_f & \xrightarrow{p} & C \\ s \downarrow & & \downarrow t & & \parallel \\ Z & \xrightarrow{u} & W & \xrightarrow{q} & C \end{array}$$

where  $C$  is the cokernel of  $i_f$ , as well as of  $u$ ;  $p$  and  $q$  are the usual projections. We consider the following commutative diagram

$$\begin{array}{ccccc} \cdots & \longrightarrow & \tilde{H}_{m+1}(C) & \longrightarrow & \tilde{H}_m(X) & \longrightarrow & \tilde{H}_{m-1}(M_f) \\ & & \parallel & & s_* \downarrow & & t_* \downarrow \\ \cdots & \longrightarrow & \tilde{H}_{m+1}(C) & \longrightarrow & \tilde{H}_m(Z) & \longrightarrow & \tilde{H}_{m-1}(W) \end{array}$$

$$\begin{array}{ccccccc} & & \longrightarrow & \tilde{H}_m(C) & \longrightarrow & \tilde{H}_{m-1}(X) & \longrightarrow \cdots \\ & & & \parallel & & s_* \downarrow & \\ & & \longrightarrow & \tilde{H}_m(C) & \longrightarrow & \tilde{H}_{m-1}(Z) & \longrightarrow \cdots \end{array}$$

It follows that

$$s_* : \tilde{H}_m(X) \rightarrow \tilde{H}_m(Z)$$

is a  $\mathcal{C}$ -isomorphism. By mod- $\mathcal{C}$  Five lemma we have that

$$t_* : \tilde{H}_m(M_f) \rightarrow \tilde{H}_m(Z)$$

is a  $\mathcal{C}$ -isomorphism and hence

$$t_* : \pi_m(M_f) \rightarrow \pi_m(W)$$

is a  $\mathcal{C}$ -isomorphism. Let  $\beta = [u]$  and  $\delta = [tj]$ . Since  $j$  is a mod- $\mathcal{C}$  homotopy equivalence,  $j_*$  is a  $\mathcal{C}$ -isomorphism of the corresponding homotopy groups; thus  $\delta \in S$ . We consider the following diagram in  $\mathcal{CW}$ :

$$\begin{array}{ccc} X & \xrightarrow{s} & Z \\ f \downarrow & & \downarrow u \\ Y & \xrightarrow{tj} & W \end{array}$$

We have

$$tj f = tj r i_f \simeq t 1_{M_f} i_f = t i_f = u s.$$

Taking the corresponding homotopy classes we get the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & Z \\ \alpha \downarrow & & \downarrow \beta \\ Y & \xrightarrow{\delta} & W \end{array}$$

in  $\widetilde{\mathcal{CW}}$ , with  $\delta \in S$ . This is indeed a weak push-out diagram in  $\widetilde{\mathcal{CW}}$ .

This completes the proof of the Proposition 2.2.1. ■

**2.2.2 Proposition.** *Let  $\{s_i : X_i \rightarrow Y_i, i \in I\}$  be a subset of  $S$ ; then*

$$\bigvee_{i \in I} s_i : \bigvee_{i \in I} X_i \rightarrow \bigvee_{i \in I} Y_i$$

*is an element of  $S$ , where the index set  $I$  is in  $\mathcal{U}$ .*

**Proof.** We consider the commutative diagram

$$\begin{array}{ccc} \bigoplus_{i \in I} \tilde{H}_*(X_i) & \xrightarrow[\cong]{\{\alpha_{i*}\}} & \tilde{H}_*\left(\bigvee_{i \in I} X_i\right) \\ \bigoplus_{i \in I} s_{i*} \downarrow \cong & & \downarrow \left(\bigvee_{i \in I} s_i\right)_* \\ \bigoplus_{i \in I} \tilde{H}_*(Y_i) & \xrightarrow[\{\beta_{i*}\}]{\cong} & \tilde{H}_*\left(\bigvee_{i \in I} Y_i\right) \end{array}$$

where

$$\alpha_i : X_i \rightarrow \bigvee_{i \in I} X_i \quad \text{and} \quad \beta_i : Y_i \rightarrow \bigvee_{i \in I} Y_i$$

are the canonical inclusions. Note that each horizontal row is an isomorphism, hence a  $\mathcal{C}$ -isomorphism. Moreover, since each  $s_{i^*}$  is a  $\mathcal{C}$ -isomorphism, so is  $\bigoplus_{i \in I} s_{i^*}$ , and hence from the commutativity of the diagram it follows that  $(\bigvee_{i \in I} s_i)_*$  is also a  $\mathcal{C}$ -isomorphism. Thus  $\bigvee_{i \in I} s_i \in S$ . This completes the proof of Proposition 2.2.2. ■

For a fixed object  $X$  of the category  $\widetilde{\mathcal{CW}}$  and for each integer  $n \geq 1$ , let  $A_n = \{i_Y : X \hookrightarrow Y \mid (Y, X) \text{ is a relative } CW\text{-complex of dimension } n\}$ .

Let  $\mathcal{U}$  be a fixed Grothendieck universe such that the category of  $CW$ -complexes and homotopy classes of maps between them is a  $\mathcal{U}$ -category. Since  $S^1$  can be given the structure of a  $CW$ -complex,  $[S^1, *; S^1, *] \cong \mathbb{Z}$  is an element of  $\mathcal{U}$ , and it follows from the axioms of Grothendieck universe that  $\mathbb{Z}^+$ , the set of positive integers, is also an element of  $\mathcal{U}$ . We shall use this fact in proving the following results.

**2.2.3 Lemma.**  $A_1$  is an element of  $\mathcal{U}$ .

**Proof.** Let  $(Y, X)$  be a relative  $CW$ -complex of dimension 1. Then the set  $\{e_\alpha\}$  of attaching maps is an element of  $\mathcal{P}(\text{Map}(S^0, Y))$  where  $\mathcal{P}$  denotes the power set and  $\text{Map}(S^0, Y)$  is the set of all continuous maps between  $S^0$  and  $Y$ . Conversely every element of  $\mathcal{P}(\text{Map}(S^0, Y))$  will

give rise to a  $CW$ -complex  $(Y, X)$  of dimension of 1, which is unique up to homeomorphism. It is thus clear that

$$A_1 \approx \mathcal{P}(\text{Map}(S^0, Y)).$$

Since  $\text{Map}(S^0, Y) \in \mathcal{U}$ , it follows from the axioms of a Grothendieck universe that  $\mathcal{P}(\text{Map}(S^0, Y)) \in \mathcal{U}$ . This completes the proof of Lemma 2.2.3. ■

We can now use the induction to prove the following.

**2.2.4 Lemma.** *For each integer  $n \geq 1$ ,  $A_n$  is an element of  $\mathcal{U}$ .*

**Proof.** We can assume inductively that for every  $k \leq n - 1$ ,  $A_k$  is an element of  $\mathcal{U}$ . Let  $(Y_j, X)$  be a relative  $CW$ -complex of dimension  $n - 1$  and let  $i_j : X \hookrightarrow Y_j$  denote the inclusion; then  $i_j \in A_{n-1}$ . We can obtain a relative  $CW$ -complex of dimension  $n$  by a set of attaching maps

$$\{e_j\} \in \mathcal{P}(\text{Map}(S^{n-1}, Y_j)).$$

It follows easily that

$$A_n \approx \bigcup_{i_j \in A_{n-1}} \mathcal{P}(\text{Map}(S^{n-1}, Y_j)).$$

Since

$$\text{Map}(S^{n-1}, Y_j) \in \mathcal{U}$$

by assumption and  $A_{n-1} \in \mathcal{U}$ , by the inductive hypothesis, it follows that  $A_n \in \mathcal{U}$ . This completes the proof of the Lemma 2.2.4. ■

**2.2.5 Proposition.** *For a given object  $X$  of the category  $\widetilde{\mathcal{CW}}$  there exists a subset  $S_X$  of the set  $\{s : X \rightarrow X' \mid s \in S\}$  such that  $S_X$  is an element of the universe  $\mathcal{U}$  and for each  $s : X \rightarrow X', s \in S$ , there exist an  $s' : X \rightarrow X''$  in  $S_X$  and a morphism  $u : X' \rightarrow X''$  of  $\widetilde{\mathcal{CW}}$  rendering the following diagram is commutative:*

$$\begin{array}{ccc}
 X & & \\
 \downarrow s & \searrow s' & \\
 X' & \xrightarrow{u} & X''
 \end{array}$$

**Proof.** Given  $X$  in  $\widetilde{\mathcal{CW}}$ , let

$$S_X = \{s : X \rightarrow Y \mid (Y, X) \text{ is a relative } \widetilde{\mathcal{CW}}\text{-complex}\}.$$

We show that  $S_X$  is an element of  $\mathcal{U}$ ; we prove this by using induction and exploiting the properties of Grothendieck universe. Clearly

$$S_X = \bigcup_{n \geq 1} A_n.$$

Since each  $A_n \in \mathcal{U}$ , it follows that

$$S_X = \bigcup_{n \geq 1} A_n$$

is an element of  $\mathcal{U}$ . Moreover, for any  $s : X \rightarrow X'$  in  $S$ , the required factorization can be obtained by taking  $s' = s$  and  $u = 1_{X'}$ .

This completes the proof of the Proposition 2.2.5. ■

Hence from the Propositions 2.2.1, 2.2.2 and 2.2.5 it follows that all the conditions of Theorem 2.1.1 are satisfied and so by Theorem 1.5.5 we obtain the following theorem.

**2.2.6 Theorem.** *Every object  $X$  of the category  $\widetilde{\mathcal{CW}}$  has an Adams completion  $X_S$  with respect to the set of morphisms  $S$  and there exists a morphism  $e : X \rightarrow X_S$  in  $\bar{S}$  which is couniversal with respect to morphisms in  $S$ .*

**2.2.7 Proposition.** *The morphism as constructed in Proposition 2.2.6 is in  $S$ .*

**Proof.** Let  $S_1$  be the set of all morphisms  $\alpha : X \rightarrow Y$  in  $\widetilde{\mathcal{CW}}$  such that  $\alpha_* : \tilde{H}_*(X) \rightarrow \tilde{H}_*(Y)$  is a  $\mathcal{C}$ -monomorphism and  $S_2$  be the set of all morphisms  $\alpha : X \rightarrow Y$  in  $\widetilde{\mathcal{CW}}$  such that  $\alpha_* : \tilde{H}_*(X) \rightarrow \tilde{H}_*(Y)$  is a  $\mathcal{C}$ -epimorphism. Clearly

$$(a) \quad S = S_1 \cap S_2,$$

$$(b) \quad S_1 \text{ and } S_2 \text{ satisfy all conditions of Theorem 1.5.9;}$$

hence  $e \in S$ . This completes the proof of the Proposition 2.2.7. ■

### 2.3 $S_n$ -completion in $\widetilde{\mathcal{CW}}$

Behera and Nanda [6] have already shown that the Adams completion with respect to the set of morphisms  $S_n$ -always exists in  $\widetilde{\mathcal{CW}}$ ; however we state the necessary results which will be needed in the next section.

**2.3.1 Proposition.** ([6], Proposition 2.1, p. 535)  $S_n$ -admits a calculus of left fractions.

**2.3.2 Proposition.** ([6], Proposition 2.2, p. 537) Let  $s_i : X_i \rightarrow Y_i$ ,  $i \in I$  be a subset of  $S_n$ ; then

$$\bigvee_{i \in I} s_i : \bigvee_{i \in I} X_i \rightarrow \bigvee_{i \in I} Y_i$$

is an element of  $S_n$ , where the index set  $I$  is an element of  $\mathcal{U}$ .

**2.3.3 Proposition.** ([6], Proposition 2.3, p. 538) For a given object  $X$  of the category  $\widetilde{\mathcal{CW}}$  there exists a subset  $S_X$  of the set  $\{s : X \rightarrow X' \mid s \in S_n\}$  such that  $S_X$  is an element of the universe  $\mathcal{U}$  and for each  $s : X \rightarrow X'$ ,  $s \in S_n$ , there exist an  $s' \in S_X$  and a morphism  $u$  of  $\widetilde{\mathcal{CW}}$  rendering the following diagram commutative :

$$\begin{array}{ccc} X & & \\ \downarrow s & \searrow s' & \\ X' & \xrightarrow{u} & X'' \end{array}$$

Hence from Propositions 2.3.1, 2.3.2 and 2.3.3 it follows that all the conditions of Theorem 2.1.1 are satisfied and so by Theorem 1.5.5 the following result holds.

**2.3.4 Theorem.** ([6], Theorem 2.5, p. 540) Every object  $X$  of the category  $\widetilde{\mathcal{CW}}$  has an Adams completion  $X_{S_n}$  with respect to the set of

morphisms  $S_n$  and there exists a morphism  $e : X \rightarrow X_{S_n}$  in  $\bar{S}_n$  which is couniversal with respect to morphisms in  $S_n$ .

**2.3.5 Proposition.** ([6], Proposition 2.6, p. 540) *The morphism  $e : X \rightarrow X_{S_n}$  as constructed in Theorem 2.3.4 is in  $S_n$ .*

## 2.4 $S_n^*$ -completion in $\widetilde{\mathcal{CW}}$

In a similar way we can show the Adams completion with respect to the set of morphisms  $S_n^*$  always exists in  $\widetilde{\mathcal{CW}}$ .

**2.4.1 Proposition.**  $S_n^*$  admits a calculus of left fractions.

**Proof.** Since the weak push-out diagrams in Propositions 2.2.1 and 2.3.1 are the same, the result follows easily. ■

**2.4.2 Proposition.** Let  $s_i : X_i \rightarrow Y_i$ ,  $i \in I$ , be a subset of  $S_n^*$ ; then

$$\bigvee_{i \in I} s_i : \bigvee_{i \in I} X_i \rightarrow \bigvee_{i \in I} Y_i$$

is an element of  $S_n^*$ , where the index set  $I$  is an element of  $\mathcal{U}$ .

**Proof.** The proof follows from the proofs of Propositions 2.2.2 and 2.3.2. ■

**2.4.3 Proposition.** For a given object  $X$  of the category  $\widetilde{\mathcal{CW}}$  there exists a subset  $S_X$  of the set  $\{s : X \rightarrow X' \mid s \in S_n^*\}$  such that  $S_X$  is an

element of the universe  $\mathcal{U}$  and for each  $s : X \rightarrow X'$ ,  $s \in S_n^*$ , there exist an  $s' : X \rightarrow X''$  in  $S_X$  and a morphism  $u : X' \rightarrow X''$  of  $\widetilde{\mathcal{CW}}$  rendering the following diagram commutative:

$$\begin{array}{ccc}
 X & & \\
 \downarrow s & \searrow s' & \\
 X' & \xrightarrow{u} & X''
 \end{array}$$

**Proof.** The proof follows from the proofs of Propositions 2.2.5 and 2.3.3. ■

Hence from Propositions 2.4.1, 2.4.2 and 2.4.3 it follows that all conditions of Theorem 2.1.1 are satisfied and so by Theorem 1.5.5 we obtain the following theorem.

**2.4.4 Theorem.** *Every subset  $X$  of the category  $\widetilde{\mathcal{CW}}$  has an Adams completion  $X_{S_n^*}$  with respect to the set of morphisms  $S_n^*$  and there exists a morphism  $e_n^* : X \rightarrow X_{S_n^*}$  in  $\widetilde{S}_n^*$  which is couniversal with respect to morphisms in  $S_n^*$ .*

**2.4.5 Proposition.** *The morphism  $e_n^* : X \rightarrow X_{S_n^*}$ , as constructed in Theorem 2.4.4 is in  $S_n^*$ .*

**Proof.** The proof follows from the proofs of Propositions 2.2.7 and 2.3.5. ■

## 2.5 $X_{S_n^*}$ and the $n$ -stage of the Acyclic Tower in $\widetilde{\mathcal{CW}}$

For simplicity we denote the  $S_n^*$ -completion of a space  $X$  by  $X_n^*$  instead of  $X_{S_n^*}$  and the map  $X \rightarrow X_n^*$  by  $e_n^*$  which arises via Proposition 2.4.5. We shall show that  $X_n^*$  is the  $n$ -stage of the acyclic decomposition of an acyclic space in the sense of Dror [18].

We recall the definition of acyclic decomposition of an acyclic space.

**2.5.1 Acyclic decomposition.** [18] Let  $X$  be an acyclic space in  $\widetilde{\mathcal{CW}}$ . By an *acyclic decomposition* of  $X$  we mean a tower of fibrations

$$(*) \quad \lim_{\leftarrow} X^n = X \rightarrow \dots \rightarrow X^n \rightarrow X^{n-1} \rightarrow \dots \rightarrow X^0 = (\text{pt.})$$

where the  $X^n$ 's are in  $\widetilde{\mathcal{CW}}$  and such that for each  $n \geq 0$ :

- (a) the  $n$ -stage,  $X^n$ , is acyclic;
- (b) the  $n$ -stage,  $X^n$ , is  $j$ -simple for all  $j \geq n$  (i.e.,  $\pi_1(X^n)$  acts trivially on  $\pi_j(X^n)$ ); and
- (c) the fibre of  $X^n \rightarrow X^{n-1}$  is  $(n-1)$ -connected.

**2.5.2 Note.** It is clear that once we get a tower of acyclic spaces such that for each  $n$ ,  $X^n \rightarrow X^{n-1}$  is an  $n$ -equivalence, then this can be converted to a fibration with  $(n-1)$ -connected fibre; thus giving rise to a tower of fibrations.

The following result on acyclic functor will be needed.

**2.5.3 Theorem.** ([18], Theorem 2.1) *There exists a functor  $A : \widetilde{\mathcal{CW}} \rightarrow \widetilde{\mathcal{CW}}$  and for any  $K \in \widetilde{\mathcal{CW}}$ , a natural transformation  $a : AK \rightarrow K$  such that:*

- (i)  $AK$  is acyclic for all  $K$ .
- (ii) The map  $a : AK \rightarrow K$  is, up to homotopy, universal for maps of acyclic complexes into  $K$ .
- (iii) Let  $n$  be  $1 \leq n \leq \infty$ . If  $\widetilde{H}_j(K) \cong 0$  for all  $1 \leq n \leq \infty$ , then the fibre of  $a : AK \rightarrow K$  is  $(n-1)$ -connected. In particular, if  $K$  is acyclic then  $a$  is an equivalence.
- (iv) If  $K$  is  $j$ -simple for some  $j \geq 1$  then so is  $AK$ .
- (v) If  $p : E \rightarrow B$  is a (Kan)-fibre then so is  $Ap$ .
- (vi) Let  $K_\infty$  be the inverse limit of the tower:

$$K_\infty = \varprojlim K_n \rightarrow \cdots \rightarrow K_n \xrightarrow{p_n} K_{n-1} \rightarrow \cdots \rightarrow K_1.$$

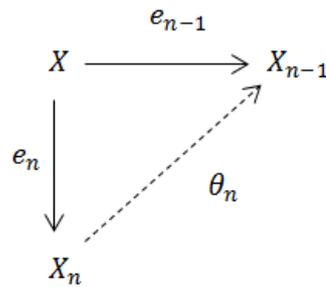
Assume that for any  $s$ -skeleton  $(K_n)_s$  the restriction  $p_n|_{(K_n)_s}$  is an isomorphism for  $n$  big enough. Then  $AK_\infty = \varprojlim AK_n$ . ■

We will show that the spaces as obtained in Theorem 2.4.4 satisfy the conditions of an acyclic decomposition of a space.

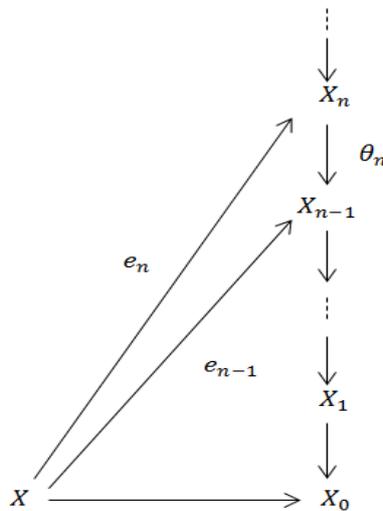
**2.5.4 Theorem.** *The space  $X_n^*$ , the  $S_n^*$ -completion of  $X$ , is the  $n$ -stage of the acyclic decomposition of  $X$  in  $\widetilde{\mathcal{CW}}$ .*

**Proof.** We recall from [6] every object  $X$  of  $\widetilde{\mathcal{CW}}$  has an  $S_n$ -completion denoted by  $X_n$  - which is precisely the  $n$ -th mod- $\mathcal{C}$  Postnikov section of  $X$ , i.e.,  $\pi_j(X) \rightarrow \pi_j(X_n)$  is a  $\mathcal{C}$ -isomorphism for  $j \leq n$  and  $\pi_j(X_n) = 0$  for  $j > n$ .

Moreover, if  $e_n : X \rightarrow X_n$  denotes the map that arises via Theorem 2.4.4 then  $e_n \in S_n$ . Since  $e_n$  is also an element of  $S_{n-1}$ , it follows from the couniversal property of the map  $e_{n-1} : X \rightarrow X_{n-1}$  that there is a map  $\theta_n : X_n \rightarrow X_{n-1}$  such that  $\theta_n e_n = e_{n-1}$ , i.e., the following diagram



is commutative. Thus we get a tower of spaces as shown in the following diagram.



Also we can assume that the maps  $\theta_n$ 's are all fibrations [6].

We have the same sort of situation with respect to the families of morphisms  $S_n^*$ 's; for, corresponding to each  $n \geq 0$ , we get a space  $X_n^*$ , the  $S_n^*$ -completion of  $X$ . Moreover, the map  $e_n^* : X \rightarrow X_n^*$  being a mod- $\mathcal{C}$   $(n+1)$ -equivalence, is also a mod- $\mathcal{C}$   $n$ -equivalence and hence belongs to  $S_{n-1}^*$ . The couniversal property of  $e_{n-1}^*$  then implies that there is a map  $\varphi_n : X_n^* \rightarrow X_{n-1}^*$  such that  $\varphi_n e_n^* = e_{n-1}^*$ , i.e., the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{e_{n-1}^*} & X_{n-1}^* \\
 e_n^* \downarrow & \nearrow \varphi_n & \\
 X_n^* & & 
 \end{array}$$

commutes. We may assume that the maps  $\varphi_n$ 's are all fibrations. It follows from Proposition 2.4.5 that  $\tilde{H}_n(X) \approx \tilde{H}_n(X_n^*)$  and therefore, if  $X$  is acyclic, so is  $X_n^*$  for each  $n$ . It is also equally clear that the fibre of  $\varphi_n : X_n^* \rightarrow X_{n-1}^*$  is  $(n-1)$ -connected. Thus, conditions (a) and (c) of an acyclic tower are satisfied by the spaces  $\{X_n^*\}$ .

To show directly that condition (b) is also satisfied seems to be difficult. We, therefore, take the help of Dror's acyclic functor  $A$  [18] to show that  $X_n^*$  and  $AX_n^*$  (which is the  $n$ -stage of the Dror's acyclic tower) are homotopically equivalent.

Consider the commutative diagram

$$\begin{array}{ccc}
 AX & \xrightarrow{Ae_n^*} & AX_n^* \\
 \downarrow f & \nearrow g_n & \downarrow f_n \\
 X & \xrightarrow{e_n^*} & X_n^*
 \end{array}$$

Since  $X$  is acyclic and  $X_n^*$  is the  $n$ -th mod- $\mathcal{C}$  Postnikov section of  $X$ , we have  $H_i(X_n^*) = 0$  for  $i \leq n + 1$ . It follows from Theorem 2.5.3 (iii) that  $f_n^*$  is a mod- $\mathcal{C}$   $(n + 1)$ -equivalence and that  $f$  is a mod- $\mathcal{C}$  homotopy equivalence, showing that

$$(Ae_n^*)_* : \pi_m(AX) \rightarrow \pi_m(AX_n^*)$$

is a  $\mathcal{C}$ -isomorphism for  $m \leq n$ . This together with the fact that  $A\varphi_n$  is a mod- $\mathcal{C}$   $n$ -equivalence, from the following commutative diagram

$$\begin{array}{ccc}
 AX & \xrightarrow{Ae_n^*} & AX_n^* \\
 \searrow Ae_{n-1}^* & & \downarrow A\varphi_n \\
 & & AX_{n-1}^*
 \end{array}$$

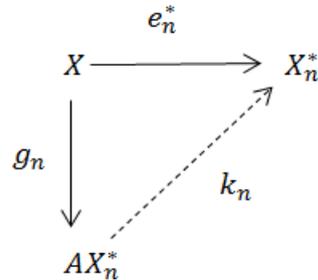
it follows that that  $Ae_{n-1}^*$  is a mod- $\mathcal{C}$   $n$ -equivalence. Thus we have a map

$$g_n = (Ae_n^*)f^{-1} : X \xrightarrow{f^{-1}} AX \xrightarrow{Ae_n^*} AX_n^*$$

which is a mod- $\mathcal{C}$   $(n + 1)$ -equivalence.

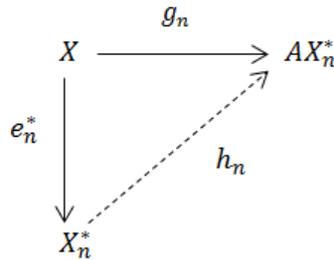
By the couniversal property of  $e_n^* : X \rightarrow X_n^*$ , with respect to all mod- $\mathcal{C}$   $(n + 1)$ -equivalences which induce reduced integral homology

$\mathcal{C}$ -isomorphisms, there exists a unique map  $k_n : AX_n^* \rightarrow X_n^*$  as shown in the diagram

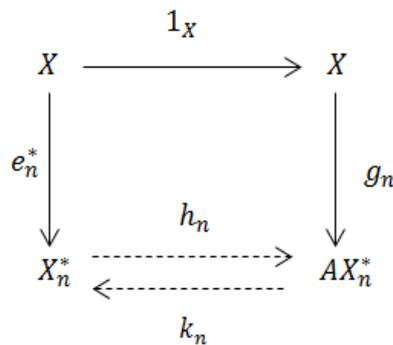


such that  $k_n g_n = e_n^*$ .

Also by the couniversal property of  $g_n : X \rightarrow AX_n^*$  with respect to all maps from  $X$  to acyclic spaces (Theorem 2.5.3) there exists a unique map  $h_n : X_n^* \rightarrow AX_n^*$  as shown in the diagram,



such that  $h_n e_n^* = g_n$ . We now collect the relevant spaces and maps:



The diagram above brings out the relationship between  $X_n^*$  and  $AX_n^*$ .

Thus

$$e_n^* = k_n g_n = k_n h_n e_n^*$$

implying that

$$k_n h_n = 1_{X_n^*}$$

and

$$g_n = h_n e_n^* = h_n k_n g_n$$

implying that

$$h_n k_n = 1_{AX_n^*};$$

$h_n$  must, therefore, be a homotopy equivalence. This completes the proof that  $X_n^*$  is the  $n$ -stage of the acyclic decomposition. ■

## Chapter 3

### ADAMS COMPLETION AND 1-CONNECTED NILPOTENT SPACE

Behera and Nanda [6] have described the  $\text{mod-}\mathcal{C}$  Postnikov approximation of a 1-connected space where  $\mathcal{C}$  is a Serre class of abelian groups, with the help of a suitable set of morphisms in the category of 1-connected based  $CW$ -complexes. Following the arguments of the work done by Behera and Nanda we study the primary decomposition of a 1-connected based nilpotent space using primary homotopy theory as given in [36]. Neisendorfer [36] has studied primary homotopy theory in detail; we have given a brief description of this theory in Section 1.7.

#### 3.1 The category $\tilde{\mathcal{N}}_1$

Let  $\mathcal{N}_1$  denote the category of 1-connected based nilpotent spaces and based maps and let  $\tilde{\mathcal{N}}_1$  be the corresponding homotopy category. We assume that the underlying sets of the elements of  $\tilde{\mathcal{N}}_1$  are elements of  $\mathcal{U}$  where  $\mathcal{U}$  is a fixed Grothendieck universe. We now fix suitable sets of morphisms of  $\tilde{\mathcal{N}}_1$ .

A map  $\alpha : X \rightarrow Y$  in  $\tilde{\mathcal{N}}_1$  is called a mod  $k$   $(n + 1)$ -equivalence if

$$\alpha_* : \pi_m(X; \mathbb{Z}/k\mathbb{Z}) \rightarrow \pi_m(Y; \mathbb{Z}/k\mathbb{Z})$$

is an isomorphism for  $m \leq n$  and an epimorphism for  $m = n + 1$ . Let  $S_n$  denote the set of all mod  $k$   $(n + 1)$ -equivalences in  $\tilde{\mathcal{N}}_1$ .

**3.1.1 Proposition.**  $S_n$  admits a calculus of left fractions.

**Proof.** Clearly  $S_n$  is closed under composition. We shall verify conditions (a) and (b) of Theorem 1.2.3. Only condition (b) is in question. For proving this condition it is enough to prove that every diagram

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & Z \\ \alpha \downarrow & & \\ Y & & \end{array}$$

in  $\tilde{\mathcal{N}}_1$  with  $\gamma \in S_n$ , can be embedded in a weak push-out diagram

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & Z \\ \alpha \downarrow & & \downarrow \beta \\ Y & \xrightarrow{\delta} & W \end{array}$$

with  $\delta \in S_n$ . Suppose  $\alpha = [f]$  and  $\gamma = [s]$ . Let  $i_f, M_f, j, r$  be as in the proof Theorem 2.2.1 and we consider the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{s} & Z \\
 i_f \downarrow & & \\
 M_f & & 
 \end{array}$$

and form its push-out in  $\tilde{\mathcal{N}}_1$

$$\begin{array}{ccc}
 X & \xrightarrow{s} & Z \\
 i_f \downarrow & & \downarrow u \\
 M_f & \xrightarrow{t} & W
 \end{array}$$

since  $i_f : X \rightarrow M_f$  is a cofibration so is  $u : Z \rightarrow W$ . Hence we have the following commutative diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{i_f} & M_f & \xrightarrow{p} & C \\
 s \downarrow & & \downarrow t & & \parallel \\
 Z & \xrightarrow{u} & W & \xrightarrow{q} & C
 \end{array}$$

where  $C$  is the cokernel of  $i_f$ , as well as of  $u$ ;  $p$  and  $q$  are the usual projections. We consider the exact homology sequence



$$\begin{array}{ccc}
 X & \xrightarrow{s} & Z \\
 f \downarrow & & \downarrow u \\
 Y & \xrightarrow{tj} & W
 \end{array}$$

We have

$$tj f = tj r i_f \simeq t 1_{M_f} = t i_f = u s.$$

Taking the corresponding homotopy classes we get the following commutative diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\gamma} & Z \\
 \alpha \downarrow & & \downarrow \beta \\
 Y & \xrightarrow{\delta} & W
 \end{array}$$

in  $\tilde{\mathcal{N}}_1$ , with  $\delta \in S_n$ . This is indeed a weak push-out diagram in  $\tilde{\mathcal{N}}_1$ . This completes the proof of the Proposition 3.2.1. ■

**3.1.2 Proposition.** Let  $\{s_i: X_i \rightarrow Y_i, i \in I\}$  be a subset of  $S_n$ ; then

$$\bigvee_{i \in I} s_i : \bigvee_{i \in I} X_i \rightarrow \bigvee_{i \in I} Y_i$$

is an element of  $S_n$ , where the index set  $I$  is in  $\mathcal{U}$ .

**Proof.** We consider the following commutative diagram

$$\begin{array}{ccc}
\bigoplus_{i \in I} H_*(X_i; \mathbb{Z}/k\mathbb{Z}) & \xrightarrow[\cong]{\{\alpha_{i*}\}} & H_*\left(\bigvee_{i \in I} X_i; \mathbb{Z}/k\mathbb{Z}\right) \\
\bigoplus_{i \in I} s_{i*} \downarrow \cong & & \downarrow (\bigvee_{i \in I} s_i)_* \\
\bigoplus_{i \in I} H_*(Y_i; \mathbb{Z}/k\mathbb{Z}) & \xrightarrow[\{\beta_{i*}\}]{\cong} & H_*\left(\bigvee_{i \in I} Y_i; \mathbb{Z}/k\mathbb{Z}\right)
\end{array}$$

where

$$\alpha_i : X_i \rightarrow \bigvee_{i \in I} X_i \quad \text{and} \quad \beta_i : Y_i \rightarrow \bigvee_{i \in I} Y_i$$

are the canonical inclusions. We note that each horizontal row is an isomorphism. Moreover, since each  $s_{i*}$  is an isomorphism in dimension  $\leq n$  and an epimorphism in dimension  $n + 1$ , so is  $\bigoplus_{i \in I} s_{i*}$ , and from the commutativity of the diagram it follows that  $(\bigvee_{i \in I} s_i)_*$  is also an isomorphism in dimension  $\leq n$  and an epimorphism in dimension  $n + 1$ . Thus  $\bigvee_{i \in I} s_i \in S_n$ . This completes the proof of Proposition 3.2.2.  $\blacksquare$

Let  $\mathcal{U}$  be a fixed Grothendieck universe such that the category of nilpotent  $CW$ -complexes and homotopy classes of maps between them is a  $\mathcal{U}$ -category. Since  $S^1$  can be given the structure of a  $CW$ -complex,  $[S^1, S^1] \cong \mathbb{Z}$  is an element of  $\mathcal{U}$ , and it follows from the axioms of a Grothendieck universe that  $\mathbb{Z}^+$ , the set of positive integers, is also an element of  $\mathcal{U}$  [6]. We shall use this fact in proving the following result.

**3.1.3 Proposition.** *For a given object  $X$  of the category  $\tilde{\mathcal{N}}_1$  there exists a subset  $S_X$  of the set  $\{s : X \rightarrow X' \mid s \in S_n\}$  such that  $S_X$  is an element of the universe  $\mathcal{U}$  and for each  $s : X \rightarrow X', s \in S_n$  there exist*

an  $s' : X \rightarrow X''$  in  $S_X$  and a morphism  $u : X' \rightarrow X''$  in  $\tilde{\mathcal{N}}_1$  rendering the following diagram commutative,

$$\begin{array}{ccc} X & \xrightarrow{s} & X' \\ s' \downarrow & & \swarrow u \\ & & X'' \end{array}$$

i.e.,  $us = s'$ .

**Proof.** Given any  $X$  in  $\tilde{\mathcal{N}}_1$ , let  $S_X = \{s : X \rightarrow Y \mid (Y, X) \text{ is a relative nilpotent } CW\text{-complex in } \dim \geq n + 3\}$ . Clearly  $S_X \subset S_n$ . For any  $s : X \rightarrow Y$  in  $S_n$ , we can find a nilpotent  $CW$ -complex  $Z$ , such that  $(Z, X)$  has cells in  $\dim \geq n + 3$  and there is a map  $u : Z \rightarrow Y$  which is a mod  $k$   $(n + 1)$ -equivalence extending  $s : X \rightarrow Y$ . Let  $v : Y \rightarrow Z$  be the mod  $k$  homotopy inverse of  $u : Z \rightarrow Y$  and  $s' : X \rightarrow Z$  be the usual inclusion. Then we have the following homotopy commutative diagram,

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ s' \downarrow & & \swarrow v \\ & & Z \end{array}$$

i.e.,  $vs \simeq s'$ .

In order to show that  $S_X \in \mathcal{U}$ , let  $A_K = \{s : X \rightarrow Y \mid (Y, X) \text{ has cells } e^m \text{ such that } n + 3 \leq \dim(e^m) \leq m + k + 2\}$ . We have

$$S_X = \bigcup_{k \geq 1} A_k.$$

We inductively show that  $A_k \in \mathcal{U}$ ,  $k \geq 1$ . For  $k = 1$ , we have  $A_1 = \{s : X \rightarrow Y \mid (Y, X) \text{ is a relative nilpotent } CW\text{-complex having cells in } \dim n + 3 \text{ only}\}$ . Thus

$$Y = X \cup_{\varphi_i} P^{m+3}(k),$$

where  $\varphi_i : P^{m+2}(k) \rightarrow X$  and  $i \in I$  for some index set  $I \in \mathcal{U}$ . Thus every family

$$\{\varphi_i : P^{m+2}(k) \rightarrow X\} \subset [P^{m+2}(k); X]$$

determines a space  $Y$  such that  $(Y, X)$  is a relative nilpotent  $CW$ -complex having cells in  $\dim n + 3$  only. Thus

$$A_1 \cong \mathcal{P}[P^{m+2}(k); X]$$

where  $\mathcal{P}$  denotes the power set. Since

$$[P^{m+2}(k); X] \in \mathcal{U},$$

it follows from the axioms of Grothendieck universe that

$$\mathcal{P}[P^{m+2}(k); X] \in \mathcal{U};$$

thus  $A_1 \in \mathcal{U}$ .

Assume that  $A_k \in \mathcal{U}$ . For showing that  $A_{k+1} \in \mathcal{U}$ , let  $s : X \rightarrow Y$  be in  $A_k$ . Thus  $(Y, X)$  is a relative nilpotent  $CW$ -complex having cells  $e^m$  such that  $n + 3 \leq \dim(e^m) \leq n + k + 2$ . Let  $\{\varphi_i : i \in I\}$  be a family of maps with

$$\varphi_i : S_i^{n+k+2} \rightarrow Y$$

for some index set  $I$ . Thus the inclusion map

$$X \hookrightarrow Y \cup_{\varphi_i} e_i^{n+k+3}$$

is in  $A_{k+1}$ . Moreover, every map  $s : X \rightarrow Z$  of  $A_{k+1}$  arises in this way; therefore

$$A_{k+1} = \bigcup_Y \mathcal{P}[P^{n+k+2}(k); Y]$$

where the union is taken over all  $Y$  such that  $s : X \rightarrow Y$  is in  $A_k$ . Since  $A_k \in \mathcal{U}$  and

$$\mathcal{P}[P^{m+k+1}(k); X] \in \mathcal{U},$$

we have  $A_{k+1} \in \mathcal{U}$ . Similarly since the set of positive integers is an element of the universe  $\mathcal{U}$ , so is the union

$$\bigcup_{k \geq 1} A_k = S_X.$$

This completes the proof of Proposition 3.2.4. ■

### 3.2 Existence of Adams completion in $\widetilde{\mathcal{N}}_1$

We shall use Theorem 2.1.1 in order to obtain the existence of Adams completion in the category  $\widetilde{\mathcal{N}}_1$ .

From Propositions 3.1.1, 3.1.2 and 3.1.3 it follows that the conditions of Theorem 2.1.1 are satisfied and so by Theorem 1.5.5 we obtain the following result.

**3.2.1 Theorem.** *Every object  $X$  of the category  $\widetilde{\mathcal{N}}_1$  has an Adams completion  $X_{S_n}$  with respect to the set of morphisms  $S_n$  and there exists a morphism  $e_n : X \rightarrow X_{S_n}$  in  $\widetilde{\mathcal{S}}_n$ , which is couniversal with respect to morphisms in  $S_n$ .*

**3.2.2 Proposition.** *The morphism  $e_n : X \rightarrow X_{S_n}$  as constructed in Theorem 3.2.3 is in  $S_n$ .*

**Proof.** Let

$$S_n^1 = \{f : X \rightarrow Y \text{ in } \bar{\mathcal{N}}_1 \mid f_* : \pi_n(X; \mathbb{Z}/k\mathbb{Z}) \rightarrow \pi_m(Y; \mathbb{Z}/k\mathbb{Z}) \text{ is a monomorphism for } m \leq n\}$$

and

$$S_n^2 = \{f : X \rightarrow Y \text{ in } \bar{\mathcal{N}}_1 \mid f_* : \pi_n(X; \mathbb{Z}/k\mathbb{Z}) \rightarrow \pi_m(Y; \mathbb{Z}/k\mathbb{Z}) \text{ is an epimorphism for } m \leq n + 1\}.$$

Clearly,

$$(i) \quad S_n = S_n^1 \cap S_n^2,$$

$$(ii) \quad S_n^1 \text{ and } S_n^2 \text{ satisfy all the conditions of Theorem 1.5.9;}$$

hence  $e_n \in S_n$ . This completes the proof of Proposition 3.2.4. ■

### 3.3 Primary approximation of a 1-connected based nilpotent space.

We can obtain a primary approximation of a 1-connected based nilpotent space with the help of the sets of morphism  $S_n$ .

**3.3.1 Theorem.** Let  $X$  be a 1-connected based nilpotent space. Then for  $n \geq 3$ , there exist 1-connected based nilpotent spaces  $X_n$ , maps  $e_n : X \rightarrow X_n$  and fibrations  $p_{n+1} : X_{n+1} \rightarrow X_n$  such that

- (a)  $e_{n_*} : \pi_m(X; \mathbb{Z}/k\mathbb{Z}) \rightarrow \pi_m(X_n; \mathbb{Z}/k\mathbb{Z})$  is an isomorphism for  $m \leq n$  and  $\pi_m(X; \mathbb{Z}/k\mathbb{Z}) = 0$  for  $m > n$ ,
- (b)  $e_n = p_{n+1} \circ e_{n+1}$ .

**Proof.** For each  $n \geq 3$ , let  $X_n$  be the  $S_n$ -completion of  $X$  and  $e_n : X \rightarrow X_n$  be the canonical map as constructed in Proposition 3.2.3. Since  $e_{n+1} \in S_{n+1}$ , it follows that  $e_{n+1} \in S_n$ . By the couniversal property of  $e_n : X \rightarrow X_n$  (Theorem 1.5.5), we have a map  $p_{n+1} : X_{n+1} \rightarrow X_n$  making the diagram commutative,

$$\begin{array}{ccc}
 X & \xrightarrow{e_n} & X_n \\
 e_{n+1} \downarrow & \nearrow p_{n+1} & \\
 X_{n+1} & & 
 \end{array}$$

i.e.,  $e_n = p_{n+1} \circ e_{n+1}$ . The maps  $\{p_n\}$  can be replaced by fibrations in the usual manner. Thus (b) holds.

Since  $e_n \in S_n$ ,

$$e_{n_*} : \pi_m(X; \mathbb{Z}/k\mathbb{Z}) \rightarrow \pi_m(X_n; \mathbb{Z}/k\mathbb{Z})$$

is an isomorphism for  $m \leq n$ . In order to show that  $\pi_m(X; \mathbb{Z}/k\mathbb{Z}) = 0$  for  $m > n$ , consider the map

$$f : P^m(k) \rightarrow X_n, \quad m > n,$$

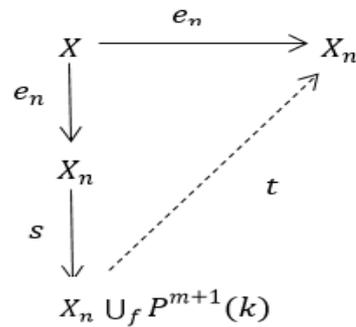
and let  $s$  denote the inclusion

$$X_n \hookrightarrow X_n \cup_f P^{m+1}(k).$$

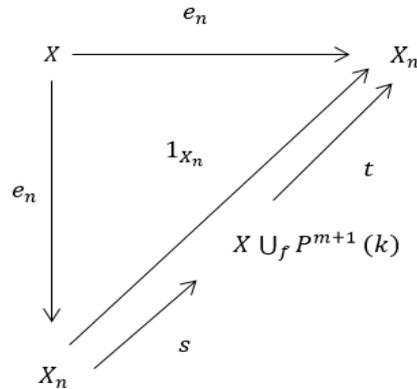
Clearly  $s$  is a mod  $k$   $m$ -equivalence; since  $m > n$ ,  $s$  is a mod  $k$   $(n + 1)$ -equivalence. Hence  $s \in S_n$  and  $se_n \in S_n$ . By the couniversal property of  $e_n$  we have a map

$$t : X_n \cup_f P^{m+1}(k) \rightarrow X_n$$

which makes the diagram

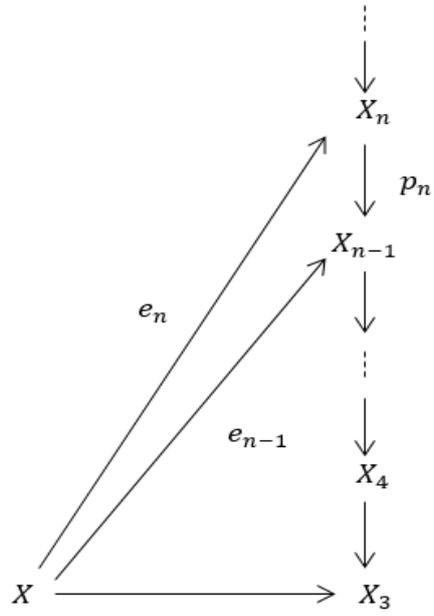


commutative:  $tse_n = e_n$ . Now we consider the following diagram



By the uniqueness condition of the couniversal property of  $e_n$ , we have  $ts = 1_{X_n}$ . Thus we have  $f \simeq 0$ ; so that  $\pi_m(X; \mathbb{Z}/k\mathbb{Z}) = 0$  for  $m > n$ . Thus (a) holds.

We have a tower of spaces as shown in the following diagram:



This completes the proof of Theorem 3.4.1. ■

## Chapter 4

### PRIMARY DECOMPOSITION OF A 0-CONNECTED NILPOTENT SPACE

As stated earlier we note that Deleanu, Frei and Hilton have developed the notion of generalized Adams completion in a categorical context; they have also studied the dual notion, namely the Adams cocompletion of an object in a category [14]. Behera and Nanda [5] have shown that the different stages of the Cartan-Whitehead decomposition of a 0-connected space are the Adams cocompletion of a space with respect to suitable sets of morphisms. In [36], Neisendonfer has studied the primary homotopic theory in an exhaustive manner. The central idea of this chapter is to study how Cartan-Whitehead decomposition of a 0-connected nilpotent space is characterized in terms of its Adams cocompletions; it is done using the primary homotopy theory developed by Neisendonfer.

#### 4.1 The category $\mathcal{N}_0$ .

Let  $\mathcal{N}_0$  denote the category of 0-connected based nilpotent spaces and based maps and let  $\tilde{\mathcal{N}}_0$  be the corresponding homotopy category. We assume that the underlying sets of the elements of  $\tilde{\mathcal{N}}_0$  are the elements of  $\mathcal{U}$ , where  $\mathcal{U}$  is a fixed Grothendieck universe. We now fix suitable sets of morphisms of  $\tilde{\mathcal{N}}_0$ .

Let  $S_n$  denote the set of all maps  $\alpha$  in  $\tilde{\mathcal{N}}_0$  having the following property:  $\alpha : X \rightarrow Y$  is in  $S_n$  if and only if

$$\alpha_* : \pi_m(X; \mathbb{Z}/k\mathbb{Z}) \rightarrow \pi_m(Y; \mathbb{Z}/k\mathbb{Z})$$

is an isomorphism for  $m > n$  and a monomorphism for  $m = n$ .

##### 4.1.1 Proposition. $S_n$ admits a calculus of right fractions.

**Proof.** Clearly  $S_n$  is closed under composition. We shall verify conditions (a) and (b) of Theorem 1.2.4. Only condition (b) is in question. For proving this condition it is enough to prove that every diagram

$$\begin{array}{ccc} & A & \\ & \downarrow \alpha & \\ C & \xrightarrow{\gamma} & B \end{array}$$

in  $\tilde{\mathcal{N}}_0$  with  $\gamma \in S_n$ , can be embedded in a weak pull-back diagram

$$\begin{array}{ccc}
 & \delta & \\
 D & \longrightarrow & A \\
 \beta \downarrow & & \downarrow \alpha \\
 C & \xrightarrow{\gamma} & B
 \end{array}$$

with  $\delta \in \mathcal{S}_n$ . Suppose  $\alpha = [f]$  and  $\gamma = [s]$ . We replace  $f$  and  $s$  by fibrations  $f'$  and  $s'$  respectively; we have

$$f = f'r : A \xrightarrow{r} P_f \xrightarrow{f'} B$$

and

$$s = s't : C \xrightarrow{t} P_s \xrightarrow{s'} B$$

where  $r$  and  $t$  are homotopy equivalences and  $P_f$  and  $P_s$  are mapping tracks of  $f$  and  $s$  respectively. Let  $\bar{r}$  and  $\bar{t}$  be the homotopy inverses of  $r$  and  $t$  respectively. Let  $D$  be the usual pull-back of  $f'$  and  $s'$  and

$$p : D \rightarrow P_f, \quad q : D \rightarrow P_s$$

be the respective projections. Let  $\delta = [\bar{r}p]$  and  $\beta = [\bar{t}q]$ . Thus

$$\begin{aligned}
 \alpha\delta &= [f][\bar{r}p] = [f\bar{r}p] \\
 &= [f'r\bar{r}p] = [f'p] \\
 &= [s'q] = [s't\bar{t}q] \\
 &= [s\bar{t}q] = [s][\bar{t}q] = \gamma\beta.
 \end{aligned}$$

Moreover, if  $\alpha\mu = \gamma\lambda$ , let  $u : U \rightarrow A$ ,  $v : U \rightarrow C$  be in the classes  $\mu, \lambda$  respectively so that

$$fu \simeq sv \quad \text{or} \quad f'ru \simeq sv.$$

Let

$$F : U \times I \rightarrow B$$

be a homotopy with

$$F_0 = f'ru \quad \text{and} \quad F_1 = sv.$$

Since  $f'$  is a fibration there exists a homotopy

$$G : U \times I \rightarrow P_f$$

such that

$$f'G_t = F_t \quad \text{and} \quad G_0 = ru.$$

Thus

$$f'G_1 = F_1 = sv = s'tv.$$

By the pull-back property of  $D$  there exists a map  $k : U \rightarrow D$  such that

$$pk = G_1 \simeq ru \quad \text{and} \quad qk = tv.$$

Thus if  $\rho = [k]$ , then

$$\delta\rho = [\bar{r}p][k] = [\bar{r}pk] = [\bar{r}ru] = [u] = \mu$$

and

$$\beta\rho = [\bar{t}q][k] = [\bar{t}qk] = [\bar{t}tv] = [v] = \lambda.$$

It remains to be shown that  $\delta \in S_n$ . We assume that the map  $\alpha : A \rightarrow B$  is a fibration with fibre  $Q$ . We note that  $Q$  is also the fibre of  $\beta : D \rightarrow C$ . We have the following commutative diagram

$$\begin{array}{ccccc}
\cdots \longrightarrow \pi_{m+1}(C; \mathbb{Z}/k\mathbb{Z}) & \longrightarrow & \pi_m(Q; \mathbb{Z}/k\mathbb{Z}) & \longrightarrow & \pi_m(D; \mathbb{Z}/k\mathbb{Z}) \\
& & \downarrow \gamma_* & & \parallel & & \downarrow \delta_* \\
\cdots \longrightarrow \pi_{m+1}(B; \mathbb{Z}/k\mathbb{Z}) & \longrightarrow & \pi_m(Q; \mathbb{Z}/k\mathbb{Z}) & \longrightarrow & \pi_m(A; \mathbb{Z}/k\mathbb{Z}) \\
& & & & & & \\
& & & & \longrightarrow \pi_m(C; \mathbb{Z}/k\mathbb{Z}) & \longrightarrow & \pi_{m-1}(Q; \mathbb{Z}/k\mathbb{Z}) & \longrightarrow \cdots \\
& & & & \downarrow \gamma_* & & \parallel \\
& & & & \longrightarrow \pi_m(B; \mathbb{Z}/k\mathbb{Z}) & \longrightarrow & \pi_{m-1}(Q; \mathbb{Z}/k\mathbb{Z}) & \longrightarrow \cdots
\end{array}$$

By Five Lemma  $\delta_*$  is an isomorphism for  $m > n$  and a monomorphism for  $m = n$ , showing  $\delta \in S_n$ . This completes the proof of Proposition 4.1.1. ■

In fact, the set  $S_n$  admits a strong calculus of right fractions. A set  $S$  of morphisms of a small  $\mathcal{V}$ -category  $\mathcal{C}$ ,  $\mathcal{V}$  being a Grothendieck universe, admits a strong calculus of right fractions [38] if

- (i)  $S$  admits a calculus of right fractions,
- (ii) for any set  $\{s_i : B_i \rightarrow A, i \in I, I \text{ is a } \mathcal{V}\text{-set}\}$ , there exists a commutative completion  $\{f_i : C \rightarrow B_i, i \in I\}$  such that  $s_i f_i \in S$  for every  $i \in I$ .

**4.1.2 Proposition.**  $S_n$  admits a strong calculus of right fractions.

**Proof.** Let  $\{s_i : Y_i \rightarrow X, i \in I\}$  be a given set of morphisms in  $\widetilde{\mathcal{N}}_0$  with every  $s_i \in S_n$  and  $I \in \mathcal{U}$ . We have a map from  $X \rightarrow P^n X$ , where  $P^n X$  denote the Postnikov decomposition of  $X$  ([24], proof of Proposition 1.1). Convert this into a fibration:

$$X_n \xrightarrow{u_n} X \rightarrow P^n X,$$

$X_n$  being its fibre. Considering the homotopy exact sequence of this fibration we get

$$\pi_m(X_n; \mathbb{Z}/k\mathbb{Z}) = 0$$

for  $m \leq n$  and

$$\pi_m(X_n; \mathbb{Z}/k\mathbb{Z}) \cong \pi_m(X; \mathbb{Z}/k\mathbb{Z})$$

for  $m > n$ . Thus  $u_n \in S_n$ . Since  $\pi_1(X_n; \mathbb{Z}/k\mathbb{Z}) = 0$  we have a map (lifting)  $f_i : X_n \rightarrow Y_i$  such that  $s_i f_i = u_n$ , i.e., the following diagram commutes.

$$\begin{array}{ccc} & & Y_i \\ & \nearrow f_i & \downarrow s_i \\ X_n & \xrightarrow{u_n} & X \end{array}$$

This completes the proof of the Proposition 4.1.2. ■

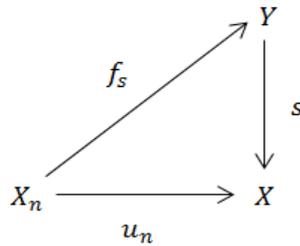
**4.1.3 Remark.** We note that the map  $u_n : X_n \rightarrow X$  is independent of the index  $i$ .

**4.1.4 Proposition.** For a given object  $X$  of the category  $\tilde{\mathcal{N}}_0$ , there exists a subset  $S_X$  of the set  $\{s : X' \rightarrow X \mid s \in S_n\}$  such that  $S_X$  is an element of the universe  $\mathcal{U}$  and for each  $s : X' \rightarrow X$ ,  $s \in S_n$ , there exists an  $s' : X'' \rightarrow X$  in  $S_X$  and a morphism  $u : X'' \rightarrow X'$  in  $\tilde{\mathcal{N}}_0$  such that  $su = s'$ , i.e., the following diagram is commutative.

$$\begin{array}{ccc}
 X' & \xrightarrow{s} & X \\
 \uparrow u & & \nearrow s' \\
 X'' & & 
 \end{array}$$

**Proof.** For a given object  $X$  in  $\tilde{\mathcal{N}}_0$ , let  $S_X$  denote the set of morphisms  $S_X = \{s : Y \rightarrow X \mid s \in S_n, Y \text{ is an object of } \tilde{\mathcal{N}}_0\}$ . We assert that  $S_X$  is an element of  $\mathcal{U}$ . For any object  $Y$  of  $\tilde{\mathcal{N}}_0$ , let  $S_{Y,X} = \{s : Y \rightarrow X, s \in S_n\}$ . It is clear that  $S_X = \bigcup_Y S_{Y,X}$  and  $S_{Y,X} = S_n \cap \text{Mor}_{\tilde{\mathcal{N}}_0}(Y, X)$ . Since  $\tilde{\mathcal{N}}_0$  is a small  $\mathcal{U}$ -category  $\text{Mor}_{\tilde{\mathcal{N}}_0}(Y, X)$  belongs to  $\tilde{\mathcal{N}}_0$  and so does  $S_{Y,X}$ , being a subset of  $\text{Mor}_{\tilde{\mathcal{N}}_0}(Y, X)$ . Therefore, the set  $S_X$ , being a union of sets all belonging to  $\mathcal{U}$  and indexed by the objects  $Y$  of  $\tilde{\mathcal{N}}_0$  (which is a subset of  $\mathcal{U}$ ) is itself in  $\mathcal{U}$ .

In view of Proposition 4.1.2 and Remark 4.1.3, we have the following commutative diagram



where  $s \in S_X$  is arbitrary,  $u_n$  is the map as constructed in Proposition 4.1.2 and  $f_s$  is the lifting of  $u_n$  corresponding to  $s$ . This completes the proof of the Proposition 4.1.4. ■

**4.1.5 Corollary.**  $u_n \in S_n$  and with respect to any  $s \in S_X$ ,  $u_n$  has couniversal property.

## 4.2 Existence of Adams cocompletion in $\tilde{\mathcal{N}}_0$

The following theorem shows that under certain conditions the Adams cocompletion of an object in the category  $\tilde{\mathcal{N}}_0$  always exists; the theorem is essentially the dual of Theorem 4.7 [1] and dual of Theorem 3.8 [3] (it is also a generalization of the dual of the Theorem in [13]).

**4.2.1 Theorem.** Let  $\mathcal{U}$  be a fixed Grothendieck universe. Let  $\tilde{\mathcal{C}}$  be the category defined as follows: the objects of  $\tilde{\mathcal{C}}$  are connected based nilpotent spaces whose underlying sets are elements of  $\mathcal{U}$ ; the morphisms of  $\tilde{\mathcal{C}}$  are based homotopy classes of based-point preserving maps between such nilpotent spaces. Let  $S$  be a family of morphisms of

$\tilde{\mathcal{C}}$  admitting a calculus of right fraction and satisfying the following axioms of compatibility with products :

- (P) If  $s_i : X_i \rightarrow Y_i$  lies in  $S$  for each  $i \in I$ , where the index set  $I$  is an element of  $\mathcal{U}$ , then

$$\prod_{i \in I} s_i : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$$

lies in  $S$ .

Assume that the family  $S$  and the object  $X$  of  $\tilde{\mathcal{C}}$  satisfy the condition :

- (\*) There exists a subset  $S_X$  of the set  $\{s : X' \rightarrow X \mid s \in S\}$  such that  $S_X$  is an element of the universe  $\mathcal{U}$  and for each  $s : X' \rightarrow X$ ,  $s \in S$ , there exist an  $s' : X'' \rightarrow X$  in  $S_X$  and a morphism  $u : X'' \rightarrow X'$  of  $\tilde{\mathcal{C}}$  rendering the following diagram is commutative :

$$\begin{array}{ccc} X' & \xrightarrow{s} & X \\ \uparrow u & & \nearrow s' \\ X'' & & \end{array}$$

Then the Adams cocompletion  $X_S$  of  $X$  does exist.

**4.2.2 Remark.** It is to be emphasized that condition (\*) is essential in order to be able to apply E.H. Brown's representability theorem to prove this result.

**4.2.3 Remark.** By the dual of Remark 1 ([15], P.36) we may replace the assumptions in Theorem 4.2.1 that the family  $S$  admits a calculus of right fractions and satisfies axiom (P) by the following assumption:

“ $S$  admits a strong calculus of right fractions.”

Furthermore Theorem 4.2.1 asserts the existence of a local Adams  $S$ -cocompletion. If condition  $(*)$  holds for each object  $X$  of  $\tilde{\mathcal{C}}$ , then, of course, we get the existence of a global Adams  $S$ -cocompletion. The outline of the proof of this fact can be obtained by following the dual arguments as stated in Remark 1 ([15], P.36).

From the Propositions 4.1.1, 4.1.2 and 4.1.4 and Remark 4.1.3, we note that the conditions of Theorem 4.2.1 are satisfied and so by Theorem 1.5.6 we obtain the following theorem.

**4.2.4 Theorem.** *Every object  $X$  of the category  $\tilde{\mathcal{N}}_0$  has an Adams cocompletion  $X_{S_n}$  with respect to the set of morphisms  $S_n$  and there exists a morphism  $e_n : X_{S_n} \rightarrow X$  in  $\bar{S}_n$  which is couniversal with respect to morphisms in  $S_n$ .*

**4.2.5 Proposition.** *The morphism  $e_n : X_{S_n} \rightarrow X$  as constructed in Theorem 4.2.4 is in  $S_n$ .*

**Proof.** We take

$$S_n^1 = \{ \alpha : X \rightarrow Y \text{ in } \tilde{\mathcal{N}}_0 \mid \alpha_* : \pi_m(X; \mathbb{Z}/k\mathbb{Z}) \rightarrow \pi_m(Y; \mathbb{Z}/k\mathbb{Z}) \}$$

is a monomorphism for  $m \geq n$

and

$$S_n^2 = \{\alpha : X \rightarrow Y \text{ in } \tilde{\mathcal{N}}_0 \mid \alpha_* : \pi_m(X; \mathbb{Z}/k\mathbb{Z}) \rightarrow \pi_m(Y; \mathbb{Z}/k\mathbb{Z})$$

is an epimorphism for  $m \geq n + 1\}$ .

Clearly

$$(i) \quad S_n = S_n^1 \cap S_n^2 \text{ and}$$

$$(ii) \quad S_n^1 \text{ and } S_n^2 \text{ satisfy all conditions of Theorem 1.5.10;}$$

hence  $e \in S_n$ . This completes the proof of the Proposition 4.2.5. ■

The above results will be used to obtain a tower for the 0-connected nilpotent spaces with the help of the sets of morphisms  $S_n$ .

### 4.3. Primary decomposition of a 0-connected based nilpotent space.

We obtain the primary decomposition of the 0-connected based nilpotent space with the help of the set of morphisms  $S_n$  as described below. In the process, starting from a 0-connected based nilpotent space  $X$  we get a tower of spaces,

$$\cdots \rightarrow X_{n+1} \xrightarrow{\theta_{n+1}} X_n \rightarrow \cdots \rightarrow X_1 \xrightarrow{\theta_1} X_0$$

and the direct limit of this tower gives us a space which in some sense is the Cartan-Whitehead decomposition of  $X$ .

**4.3.1 Proposition.**  $X_n$ , as constructed in the proof of Proposition 4.1.2, is homotopically equivalent to  $X_{S_n}$ , as constructed in Theorem 4.2.4.

**Proof.** By the couniversal property of  $u_n : X_n \rightarrow X$  we have a commutative diagram,

$$\begin{array}{ccc}
 X_n & \xrightarrow{u_n} & X \\
 \downarrow s & & \nearrow e_n \\
 X_{S_n} & & 
 \end{array}$$

i.e.,  $e_n s = u_n$ . Also by the couniversal property of  $e_n : X_{S_n} \rightarrow X$  we have a commutative diagram,

$$\begin{array}{ccc}
 X_{S_n} & \xrightarrow{e_n} & X \\
 \downarrow t & & \nearrow u_n \\
 X_n & & 
 \end{array}$$

i.e.,  $u_n t = e_n$ . Thus

$$u_n = e_n s = u_n t s$$

implies

$$t s = 1_{X_n}$$

and

$$e_n = u_n t = e_n s t$$

implies

$$s t = 1_{X_{S_n}}$$

and the required homeomorphism between  $X_n$  and  $X_{S_n}$  is obtained. ■

**4.3.2. Theorem.** *Let  $X$  be a 0-connected based nilpotent space. Then for  $n \geq 3$ , there exist 0-connected based nilpotent spaces  $X_n$ , maps  $e_n : X_n \rightarrow X$  and fibrations  $\theta_{n+1} : X_{n+1} \rightarrow X_n$  such that*

(a)  $e_{n*} : \pi_m(X_n; \mathbb{Z}/k\mathbb{Z}) \rightarrow \pi_m(X; \mathbb{Z}/k\mathbb{Z})$  is an isomorphism for  $m > n$  and  $\pi_m(X_n; \mathbb{Z}/k\mathbb{Z}) = 0$  for  $m \leq n$ ,

(b)  $e_{n+1} = e_n \circ \theta_{n+1}$ .

**Proof.** For each integer  $n \geq 3$ , let  $X_n$  be the  $S_n$ -completion of  $X$  and  $e_n : X_n \rightarrow X$  be the canonical map as stated in Theorem 4.2.5. Since  $e_n \in S_n \subset S_{n+1}$ , it follows from the couniversal property of  $e_{n+1}$  that there exists a unique morphism  $\theta_{n+1} : X_{n+1} \rightarrow X_n$  such that  $e_{n+1} = e_n \circ \theta_{n+1}$ , i.e., the following diagram is commutative:

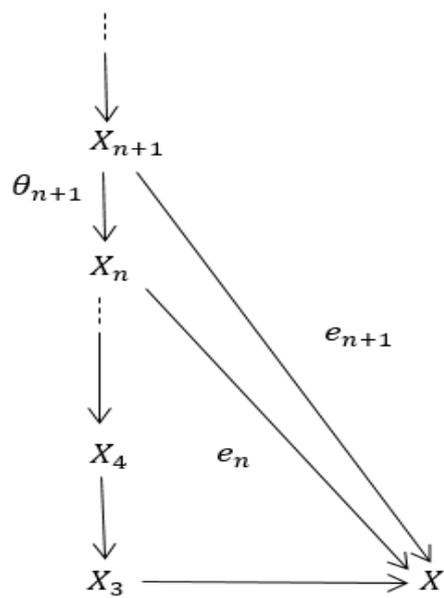
$$\begin{array}{ccc}
 X_{n+1} & \xrightarrow{e_{n+1}} & X \\
 \theta_{n+1} \downarrow & \nearrow e_n & \\
 X_n & & 
 \end{array}$$

The maps  $\theta_n$  can of course be replaced by fibrations in the usual manner.

Since  $e_n \in S_n$ ,

$$e_{n*} : \pi_m(X_n; \mathbb{Z}/k\mathbb{Z}) \rightarrow \pi_m(X; \mathbb{Z}/k\mathbb{Z})$$

is an isomorphism for  $m > n$ ; it is already proved in Proposition 4.1.2 that  $\pi_m(X_n; \mathbb{Z}/k\mathbb{Z}) = 0$  for  $m \leq n$ . Thus we have a tower of spaces:



This completes the proof of the theorem 4.3.2. ■

## Chapter 5

### ADAMS COMPLETION AND $S$ -FIBRATIONS

Let  $\mathcal{C}$  be any small  $\mathcal{U}$ -category; where  $\mathcal{U}$  is a fixed Grothendieck universe. Let  $S$  be a set of morphisms in the category  $\mathcal{C}$  and  $F_S : \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$  be the canonical functor. For convenience we write  $F_S = F$ . Bauer and Dugundji [4] have introduced the concept of  $S$ -fibration, weak  $S$ -fibration,  $S$ -cofibration and weak  $S$ -cofibration in the category  $\mathcal{C}$  and have explored the properties of these concepts.

There are some other advantages over the assumption that the set of morphisms  $S$  admits a calculus of left (right) fractions. In this chapter we study some cases showing how the assumption that  $S$  admits a calculus of left (right) fractions helps us to prove that weak  $S$ -fibration implies  $S$ -fibration and weak  $S$ -cofibration implies  $S$ -cofibration.

## 5.1 $S$ -fibrations

Each class  $S$  of morphisms in a category  $\mathcal{C}$  determines a concept of fibration (and cofibration) in  $\mathcal{C}$ . We recall the concepts of  $S$ -fibration and weak  $S$ -fibration from [4].

**5.1.1 Definition.** [4] Let  $S$  be a subset of morphisms of  $\mathcal{C}$ . A morphism  $p : E \rightarrow B$  in  $\mathcal{C}$  is called an  $S$ -fibration if for each diagram

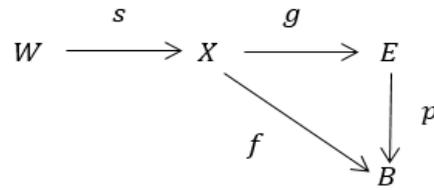
$$\begin{array}{ccccc} W & \xrightarrow{s} & X & \xrightarrow{g} & E \\ & & & \searrow f & \downarrow p \\ & & & & B \end{array}$$

with  $s \in S$  and  $pgs = fs$ , there exists a morphism  $g' : X \rightarrow E$  in  $\mathcal{C}$

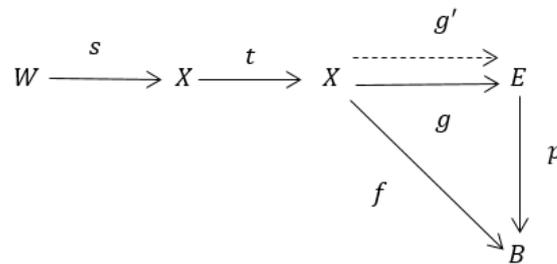
$$\begin{array}{ccccc} W & \xrightarrow{s} & X & \xrightarrow{g'} & E \\ & & & \searrow g & \downarrow p \\ & & & & B \end{array}$$

such that  $gs = g's$  and  $pg' = f$ .

**5.1.2 Definition.** [4] Let  $S$  be a subset of morphisms of  $\mathcal{C}$ . A morphism  $p : E \rightarrow B$  in  $\mathcal{C}$  is called a *weak  $S$ -fibration* if for each diagram



with  $s \in S$  and  $pgs = fs$ , there exists a morphism  $g' : X \rightarrow E$  in  $\mathcal{C}$  and a morphism  $t : X \rightarrow X$  with  $t \in S$

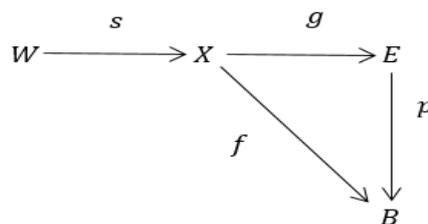


such that  $gs = g's$ ,  $ts = s$  and  $pg' = ft$ .

The following result is elementary in nature.

**5.1.3 Proposition.** *S-fibration implies weak S-fibration.*

**Proof:** Let  $p : E \rightarrow B$  be an  $S$ -fibration in the category  $\mathcal{C}$ . In order to show that  $p : E \rightarrow B$  is also a weak  $S$ -fibration consider an arbitrary diagram



with  $s \in S$  and  $pgs = fs$ . Since  $p : E \rightarrow B$  is a  $S$ -fibration, there exists a morphism  $g' : X \rightarrow E$  in  $\mathcal{C}$ ,

$$\begin{array}{ccccc}
 W & \xrightarrow{s} & X & \xrightarrow{g'} & E \\
 & & \searrow f & & \downarrow p \\
 & & & & B
 \end{array}$$

(Note: In the original image, the arrow from X to E is dashed, and the arrow from X to B is labeled f. The arrow from W to X is labeled s. The arrow from E to B is labeled p. The arrow from X to E is labeled g'. The arrow from X to B is labeled f.)

such that  $gs = g's$  and  $pg' = f$ . Considering  $t = 1_X : X \rightarrow X$ , we can have  $gs = g's$  and  $pg' = f1_X = ft$ . This completes the proof of the Proposition 5.1.3. ■

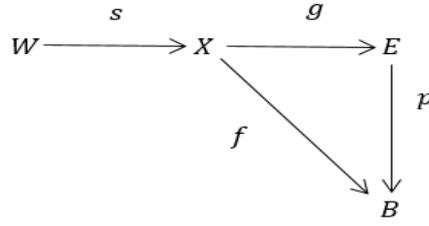
Under some moderate assumptions on the set  $S$ , it can be proved that weak  $S$ -fibration always implies  $S$ -fibration.

**5.1.4 Proposition.** *Let  $S$  be the set of morphisms in  $\mathcal{C}$ . Let  $F : \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$  be the canonical functor. Suppose the following conditions hold:*

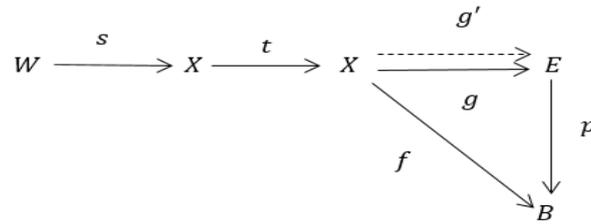
- (a)  $p : E \rightarrow B$  is a weak  $S$ -fibration.
- (b)  $S$  admits a calculus of left fractions.
- (c)  $S$  consists of monomorphisms.

*Then  $p : E \rightarrow B$  is an  $S$ -fibration.*

**Proof:** For showing  $p : E \rightarrow B$  is a fibration consider the diagram



with  $s \in S$  and  $pgs = fs$ . Since  $s \in S$ ,  $pgs = fs$  and  $p : E \rightarrow B$  is a weak fibration, there exist a morphism  $g' : X \rightarrow E$  and  $t : X \rightarrow X$  with  $t \in S$  such that the following diagram commutes



i.e.,  $g's = gs$ ,  $ts = s$  and  $pg' = ft$ . It is enough to prove that  $pg' = f$ . Since  $pg' = ft$  we have  $pg's = fts = fs$ . Since  $F$  is a covariant functor, we have  $F(pg's) = F(fs)$ , i.e.,  $F(p)F(g')F(s) = F(f)F(s)$ . Since  $F(s)$  is an isomorphism in  $\mathcal{C}[S^{-1}]$  we have  $F(p)F(g') = F(f)$ , i.e.,  $F(pg') = F(f)$ . By Theorem 1.2.7,  $F$  is faithful. Hence we have  $pg' = f$ . This completes the proof of the Proposition 5.1.4. ■

## 5.2 $S$ -cofibrations

The dual concepts of  $S$ -fibration and weak  $S$ -fibration are  $S$ -cofibration and weak  $S$ -cofibration. We recall these concepts from [4].

**5.2.1 Definition.** [4] Let  $S$  be an arbitrary set of morphisms in a category  $\mathcal{C}$ . A morphism  $j : B \rightarrow E \in \mathcal{C}$  is called an  $S$ -*cofibration* if for each diagram

$$\begin{array}{ccccc} & & E & \xrightarrow{g} & X & \xrightarrow{s} & W \\ & & \uparrow j & & \nearrow f & & \\ & & B & & & & \end{array}$$

with  $s \in S$  and  $sgj = sf$  there exists a morphism  $g' : E \rightarrow X$

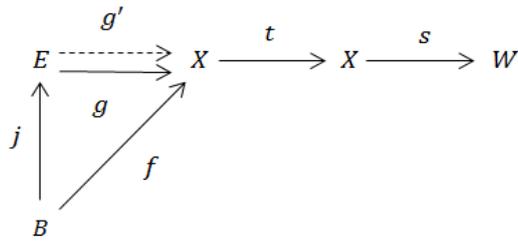
$$\begin{array}{ccccc} & & E & \xrightarrow{g'} & X & \xrightarrow{s} & W \\ & & \uparrow j & & \nearrow f & & \\ & & B & & & & \end{array}$$

in  $\mathcal{C}$  such that  $g'j = f$  and  $sg = sg'$ .

**5.2.2 Definition.** [4] A morphism  $j : B \rightarrow E \in \mathcal{C}$  is called a *weak S-cofibration* if for each diagram

$$\begin{array}{ccccc} & & E & \xrightarrow{g} & X & \xrightarrow{s} & W \\ & & \uparrow j & & \nearrow f & & \\ & & B & & & & \end{array}$$

with  $s \in S$  and  $sgj = sf$  there exists a morphism  $g' : E \rightarrow X$  and  $t : X \rightarrow X$  with  $t \in S$

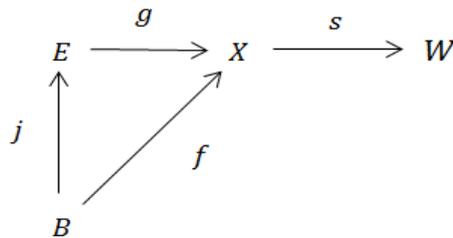


such that  $st = s$ ,  $g'j = tf$  and  $sg = sg'$ .

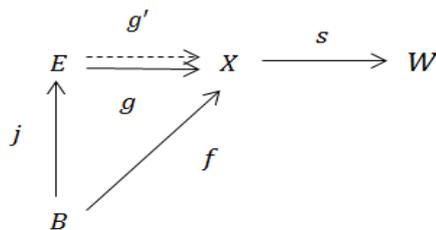
The following result is elementary in nature.

**5.2.3 Proposition.** *S-cofibration implies weak S-cofibration.*

**Proof.** Let  $j : B \rightarrow E$  be an  $S$ -cofibration in the category  $\mathcal{C}$ . In order to show that  $j : B \rightarrow E$  is also a weak  $S$ -cofibration consider an arbitrary diagram



with  $s \in S$  and  $sgj = sf$ . Since  $j : B \rightarrow E$  is an  $S$ -cofibration, there exists a morphism  $g' : E \rightarrow X$



in  $\mathcal{C}$  such that  $g'j = f$  and  $sg = sg'$ . Considering  $t = 1_X : X \rightarrow X$ , we can have  $st = s$ ,  $g'j = tf$  and  $sg = sg'$ . ■

Under some moderate assumptions on the set  $S$ , it can be proved that weak  $S$ -cofibration always implies  $S$ -cofibration.

**5.2.4 Proposition.** *Let  $S$  be the set of morphisms in  $\mathcal{C}$ . Let  $F_S = F : \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$  be the canonical functor. Suppose the following conditions hold:*

- (a)  $j : B \rightarrow E$  is a weak  $S$ -cofibration.
- (b)  $S$  admits a calculus of left fractions.
- (c)  $S$  consists of monomorphisms.

*Then  $j : B \rightarrow E$  is an  $S$ -cofibration.*

**Proof.** For showing that  $j : B \rightarrow E$  is an  $S$ -cofibration, consider an arbitrary diagram

$$\begin{array}{ccccc}
 & & g & & \\
 & & \longrightarrow & & \\
 E & \xrightarrow{\quad} & X & \xrightarrow{\quad} & W \\
 \uparrow j & & \nearrow f & & \\
 B & & & & 
 \end{array}$$

with  $s \in S$  and  $sgj = sf$ . Since  $s \in S$  and  $sgj = sf$  and  $j : B \rightarrow E$  is a weak  $S$ -cofibration, there exist a morphism  $g' : E \rightarrow X$  and  $t : X \rightarrow X$  with  $t \in S$  such that the following diagram commutes

$$\begin{array}{ccccc}
 & & g' & & \\
 & & \dashrightarrow & & \\
 E & \xrightarrow{g} & X & \xrightarrow{t} & X & \xrightarrow{s} & W \\
 & \nearrow f & & & & & \\
 j \uparrow & & & & & & \\
 B & & & & & & 
 \end{array}$$

i.e.,  $st = s$ ,  $g'j = tf$  and  $sg = sg'$ . It is enough to prove that  $g'j = f$ . Since  $g'j = tf$  we have  $sg'j = stf = sf$ . Since  $F$  is a covariant functor we have  $F(sg'j) = F(sf)$ , i.e.,  $F(s)F(g')F(j) = F(s)F(f)$ . Since  $F(s)$  is an isomorphism in  $\mathcal{C}[S^{-1}]$  we have  $F(g')F(j) = F(f)$ , i.e.,  $F(g'j) = F(f)$ . By Theorem 1.2.7,  $F$  is faithful. Hence we have  $g'j = f$ . This completes the proof of the Proposition 5.2.4.  $\blacksquare$

### 5.3 Adams completion and $S$ -fibrations

In [4], Bauer and Dugundji have examined the notion of  $S$ -fibration in the category  $\mathcal{T}$ , the category of topological spaces and continuous functions; under suitable choice of the set  $S$  they have shown that a map  $p : E \rightarrow B$  is an  $S$ -fibration if and only if it is a Hurewicz fibration. In this section, under reasonable assumptions we show that a morphism  $p : E \rightarrow B$  in a category  $\mathcal{C}$  is an  $S$ -fibration if and only if it is a weak  $S$ -fibration.

**5.3.1 Theorem.** *Let  $S$  be a saturated family of morphisms of a category  $\mathcal{C}$  and let every object in  $\mathcal{C}$  admit an Adams completion. Let  $S$  consist of monomorphisms. Then  $\{\text{weak } S\text{-fibrations}\} = \{S\text{-fibrations}\}$ .*

Proof. The proof follows from Theorem 1.5.11, Propositions 5.1.3 and 5.1.4. ■

The following is a direct consequence of Theorem 5.3.1.

**5.3.2 Corollary.** *Let  $\bar{S}$  be the saturation of a family of morphisms of a category  $\mathcal{C}$  and let every object in  $\mathcal{C}$  admit an  $\bar{S}$ -completion. Let  $S$  consist of monomorphisms. Then  $\{\text{weak } \bar{S}\text{-fibrations}\} = \{\bar{S}\text{-fibrations}\}$ .*

**5.3.3 Note.** In the presence of the conditions of Proposition 4.2.5, we have  $\{\text{weak } S\text{-fibrations}\} = \{S\text{-fibrations}\}$ .

**5.3.4 Note.** If  $S$  contains only the identities of the category  $\mathcal{C}$ , then  $\{\text{weak } S\text{-fibrations}\} = \{S\text{-fibrations}\}$  ([4], Remark 1); this is so because  $S$  satisfies the conditions of Propositions 4.2.5.

**5.3.5 Remark.** Everything which has been obtained for  $S$ -fibration and weak  $S$ -fibration can be dualized in the usual fashion to yield the corresponding results for  $S$ -cofibration and weak  $S$ -cofibration [4].

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**LIST OF RESEARCH PAPERS PUBLISHED/ACCEPTED/COMMUNICATED**

1. Akrur Behera and Sandhya Rani Mohapatra : *Category of Fractions and Acyclic Spaces* : Indian Journal of Mathematics, 54(2) (2012), 143 - 158.
2. Akrur Behera and Sandhya Rani Mohapatra : *Primary Decomposition of a 0-connected Nilpotent Space* : Communicated to Indian Journal of Mathematics.
3. Akrur Behera and Sandhya Rani Mohapatra : *Adams Completion and 1-connected Nilpotent Space* : Communicated to Indian Journal of Pure and Applied Mathematics.
4. Akrur Behera and Sandhya Rani Mohapatra : *On S-fibration* : Communicated to Bull. Inst. Math., Acad. Sin., Rpub. of China.

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