Some Generalizations and Properties of Balancing Numbers

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SOME GENERALIZATIONS AND PROPERTIES OF BALANCING NUMBERS

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Certificate

This is to certify that the thesis titled ***Some Generalizations and Properties of Balancing Numbers*** which is being submitted by ***Sudhansu Sekhar Rout***, Roll No. 510MA106, for the award of the degree of Doctor of Philosophy from National Institute of Technology Rourkela, is a record of bonafide research work, carried out by him under my supervision. The results embodied in this thesis are new and have not been submitted to any other university or institution for the award of any degree or diploma.

Gopal Krishna Panda

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Contents

Abstract

The sequence of balancing numbers admits generalization in two different ways. The first way is through altering coefficients occurring in its binary recurrence sequence and the second way involves modification of its defining equation, thereby allowing more than one gap. The former generalization results in balancing-like numbers that enjoy all important properties of balancing numbers. The second generalization gives rise to gap balancing numbers and for each particular gap, these numbers are realized in multiple sequences. The definition of gap balancing numbers allow further generalization resulting in higher order gap balancing numbers but unlike gap balancing numbers, these numbers are scarce, the existence of these numbers are often doubtful. The balancing zeta function—a variant of Riemann zeta function—permits analytic continuation to the entire complex plane, while the series converges to irrational numbers at odd negative integers. The periods of balancing numbers modulo positive integers exhibits many wonderful properties. It coincides with the modulus of congruence if calculated modulo any power of two. There are three known primes such that the period modulo any one of these primes is equal to the period modulo its square. The sequence of balancing numbers remains stable modulo half of the primes, while modulo other half, the sequence is unstable.

Introduction

“Why are numbers beautiful? It’s like asking why is Beethoven’s Ninth Symphony beautiful. If you don’t see why, someone can’t tell you. I know numbers are beautiful. If they aren’t beautiful, nothing is." In this quote, the great Hungarian mathematician Paul Erdős did not want to listen anything or any question against the beauty of numbers.

The study of numbers and, in particular, number sequences has been a source of fascination to mathematician since ancient time. The most well-known and fascinating number sequence is the celebrated Fibonacci sequence discovered by the Italian mathematician Leonardo Pisano (1170-1250) who is better known by his nick-name Fibonacci. A problem in third section of his book *Liber Abaci*, published in 1202, led to the introduction of the Fibonacci numbers and the Fibonacci sequence for which Fibonacci is best remembered today [75]. The problem is:

“A certain man put a pair of rabbits in a place surrounded by walls. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?”

The resulting sequence is $1,1,2,3,5,8,13,21,34,55,\ldots$ (Fibonacci omitted the first term in Liber abaci). This sequence, in which each term is the sum of the two preceding terms, has proved extremely useful and appears in diversified areas of mathematics, science and engineering.

The Fibonacci numbers are usually called the nature’s numbering system. They appear everywhere in nature, from the leaf arrangement in plants to the pattern of the florets of a flower, the bracts of a pine cone or the scales of a pineapple. The Fibonacci numbers are, therefore, applicable to the growth of every living thing, including a single cell, a grain of wheat, a hive of bees, and even all of mankind [11, 26, 86].

Plants do not know about this sequence - they just grow in the most efficient ways. Many plants show the Fibonacci numbers in the arrangement of the leaves around the stem. Some pine cones and fir cones also show the numbers, as do daisies and sunflowers.

Sunflowers can contain the number 89 or even 144. Many other plants, such as succulents, also show the numbers. Some coniferous trees show these numbers in the bumps on their trunks. Palm trees show the numbers in the rings on their trunks [11, 16].

To enumerate the Fibonacci sequence, one usually forgets the rabbit problem and defines the sequence recursively as $F_0 = 0,\ F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. A sequence $L_n,\ n = 0, 1, \ldots$ with similar recurrence $L_{n+1} = L_n + L_{n-1}$ with the initialisations $L_0 = 2,\ L_1 = 1$ is known as the Lucas sequence. This sequence was discovered by the French Mathematician Edouard Lucas (1842-1891) in 1870's. The Lucas sequence shares many interesting relationships with the Fibonacci sequence and occurs now and then while working on the Fibonacci sequence [46, 49, 87].

The Lucas sequence, however, in its most general form, is defined by the binary recurrence $x_{n+1} = Ax_n + Bx_{n-1}$ from which one can extract two independent number sequences (one is not a constant multiple of other) [24, 52, 59]. The first sequence corresponds to the initial conditions $x_0 = 0,\ x_1 = 1$ and the second sequence corresponds to $x_0 = 2,\ x_1 = A$. Any other sequence, obtained by virtue of the recurrence relation $x_{n+1} = Ax_n + Bx_{n-1}$, can be expressed as a linear combination of these two sequences.

As immediate generalizations of Fibonacci and Lucas sequences, taking $A = 2$ and $B = 1$ in the recurrence $x_{n+1} = Ax_n + Bx_{n-1}$, one arrives at two independent sequences defined by $P_0 = 0,\ P_1 = 1$ and for $n \geq 1$

$$P_{n+1} = 2P_n + P_{n-1}$$

and $Q_0 = 1,\ Q_1 = 1$ and for $n \geq 1$

$$Q_{n+1} = 2Q_n + Q_{n-1}.$$

The former is known as the Pell sequence while the latter is known as the associated Pell sequence. Some authors call the second sequence as the Lucas-Pell sequence.

The importance of the associated Pell and the Pell sequences lies in the fact that their ratios are successive convergents in the continued fraction representation of $\sqrt{2}$ [39, 67]. To be more precise, one can express $\sqrt{2}$ in a continued fraction as

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}$$

and its successive convergents are $1, 3/2, 7/5, 17/12, \ldots$ The denominator sequence $1, 2, 5, 12, \ldots$

and the numerator sequence $1,3,7,17,\ldots$ are nothing but the Pell and associated Pell sequences recursively described in the last paragraph.

The beauty of Pell and associated Pell sequences further lies in the fact that products of their terms with similar indices result in another interesting sequence known as the sequence of balancing numbers [13, 33]. The balancing numbers, however, did not come to limelight in this manner. In the year 1965, motivated by Adams ([1, p.27]), Finkelstein [33] introduced the concept of numerical centers which coincides with the concept of balancing numbers introduced and studied by Behera and Panda [13] in the year 1999. According to Behera and Panda, a balancing number is a natural number n for which there corresponds another natural number r, called its balancer, such that

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r).$$

If n is a balancing number then $8n^2+1$ is a perfect square [13], hence n^2 is a square triangular number and the square root of $8n^2+1$ is called a Lucas-balancing number. Since $8\cdot 1^2+1$ is a perfect square, it is customary to accept 1 as a balancing number though it does not satisfy the defining equation. The n^{th} balancing number is denoted by B_n and the balancing numbers satisfy the recurrence relation

$$B_{n+1}=6B_n-B_{n-1};\ n=2,3,\ldots$$

with initial condition $B_1=1$ and $B_2=6$. In view of

$$B_{n-1}=6B_n-B_{n+1},$$

$B_0=0$ and the above recurrence is preferably written as

$$B_{n+1}=6B_n-B_{n-1},\ n=1,2,\ldots$$

with initial conditions $B_0=0$ and $B_1=1$.

The n^{th} Lucas-balancing number is denoted by C_n and by definition, $C_n=\sqrt{8B_n^2+1}$. The Lucas-balancing numbers also satisfy a recurrence relation identical with that of balancing numbers; in particular, $C_0=1,\ C_1=3$ and

$$C_{n+1}=6C_n-C_{n-1};\ \ n=1,2,\ldots$$

It is seen in [70] that these numbers are very closely associated with balancing numbers just like Lucas numbers are associated with Fibonacci numbers. For example, $(n+1)^{\text{st}}$ balancing number is a linear combination of n^{th} balancing and n^{th} Lucas-balancing number, that is,

$$B_{n+1}=3B_n+C_n.$$

Further, the $(m+n)^{\text{th}}$ balancing number can be written as

$$B_{m+n} = B_m C_n + C_m B_n.$$

The de-Moivre's theorem for balancing numbers relates balancing and Lucas-balancing numbers as

$$(C_m + \sqrt{8}B_m)^n = C_{mn} + \sqrt{8}B_{mn}.$$

The Fibonacci numbers satisfy an important property. If m and n are positive integers and m divides n then F_m divides F_n. The converse of this result is also true, i.e., if F_m divides F_n then m divides n. A sequence possessing this property is called a divisibility sequence. The sequence of balancing numbers (henceforth we call balancing sequence) is also a divisibility sequence. Indeed, both the sequences satisfy the strong divisibility property, i.e., for positive integers m and n, $(F_m, F_n) = F_{(m,n)}$ and $(B_m, B_n) = B_{(m,n)}$ where (x, y) denotes the greatest common divisor of x and y. These properties are not limited to only second order linear recurrences. There do exist divisibility sequences associated with higher order linear recurrences [38]. A general characterization of divisibility sequences is presented by Bézivin et al. [15].

The sequence of balancing numbers sometimes behave like the sequence of natural numbers. In fact, the recurrence relation $x_{n+1} = 2x_n - x_{n-1}$ with initial conditions $x_0 = 0$ and $x_1 = 1$, which is similar to the recurrence relation for balancing numbers, generates all the natural numbers. Just like the sum of first n odd numbers is equal to n^2, i.e.,

$$1 + 3 + \cdots + (2n-1) = n^2,$$

the sum of first n odd balancing number is equal to the square of n^{th} balancing number, i.e.,

$$B_1 + B_3 + \cdots + B_{2n-1} = B_n^2.$$

Further, the sum formula of the first n even numbers is

$$2 + 4 + \cdots + 2n = n(n+1),$$

and the balancing numbers also enjoy a similar formula, namely,

$$B_2 + B_4 + \cdots + B_{2n} = B_n B_{n+1}.$$

The Fibonacci numbers, however, do not enjoy these wonderful properties.

The balancing numbers satisfy a very interesting identity. If n is a natural number then

$$B_{n+1} \cdot B_{n-1} = B_n^2 - 1$$

and more generally, for natural numbers m and n,

$$B_{n+m} \cdot B_{n-m} = B_n^2 - B_m^2. \tag{0.0.1}$$

In these two identities, the balancing sequence behaves like the identity function of their subscripts. The corresponding identities for Fibonacci numbers namely,

$$F_{n+1} \cdot F_{n-1} = F_n^2 + (-1)^n \quad \text{and} \quad F_{m+n} \cdot F_{m-n} = F_m^2 - (-1)^{m+n} F_n^2$$

do not look so nice. Equation (0.0.1) is also true for arbitrary binary recurrence sequence given by $G_{n+1} = AG_n - G_{n-1}$ with $G_0 = 0$ and $G_1 = 1$ [80].

The sequences of balancing numbers and balancers are very closely associated with two other types of number sequences, namely, the sequences of cobalancing numbers and cobalancers. According to Panda and Ray [72], the values of n satisfying the Diophantine equation

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r)$$

for some natural numbers r, are known as cobalancing numbers while for each cobalancing number n, the associated r is called a cobalancer. It is known that for each cobalancing number n, $8n^2 + 8n + 1$ is a perfect square and hence the pronic number $n(n+1)$ is triangular [72]. For each cobalancing number n, the square root of $8n^2 + 8n + 1$ is called a Lucas-cobalancing number.

The n^{th} cobalancing number is denoted by b_n and unlike balancing and Lucas-balancing numbers, they satisfy a non-homogeneous binary recurrence

$$b_{n+1} = 6b_n - b_{n-1} + 2,\ b_0 = b_1 = 0.$$

Indeed, the initial condition $b_1 = 0$ is not at par with the definition of cobalancing numbers, but since 0 is the first non negative integer such that $8 \cdot 0^2 + 8 \cdot 0 + 1$ is a perfect square, it is accepted as the first cobalancing number, just like 1 is accepted as the first balancing number. However, unlike balancing numbers which are alternately odd and even, the cobalancing numbers are all even.

The n^{th} Lucas-cobalancing number is denoted by c_n, and the sequence of Lucas-cobalancing numbers satisfy a recurrence relation identical with that of balancing numbers. More precisely, $c_{n+1} = 6c_n - c_{n-1}$ with $c_1 = 1,\ c_2 = 7$ that holds for $n \geq 2$. The Lucas-cobalancing numbers are also, in some identities, related to cobalancing numbers just like Lucas-balancing numbers are associated with balancing numbers. For example,

$$b_{n+1} = 3b_n + c_n + 1,\ n = 2, 3, \ldots$$

The cobalancing numbers, comparable to the identity $B_{n+1} \cdot B_{n-1} = B_n^2 - 1$ on balancing numbers, satisfy

$$b_{n+1} \cdot b_{n-1} = (b_n - 1)^2 - 1$$

that holds for $n \geq 2$. However, in case of cobalancing numbers, it is an open problem to find an identity similar to $B_{n+m} \cdot B_{n-m} = B_n^2 - B_m^2$.

The balancing numbers and cobalancing numbers have very close relationships. The most surprising result proved by Panda and Ray [72, Theorem 6.1], states that all cobalancing numbers are balancers and all cobalancers are balancing numbers. More precisely, the n^{th} cobalancing number is the n^{th} balancer while the n^{th} balancing number is the $(n+1)^{\text{st}}$ cobalancer. However, twice the sum of first n balancing numbers is equal to the $(n+1)^{\text{st}}$ cobalancing number.

The balancing numbers, cobalancing numbers, Lucas-balancing numbers and the Lucas-cobalancing numbers are all related to the Pell and associated Pell numbers by several means. The following results were proved by Panda and Ray in [73]. The sum of any two consecutive cobalancing number is either a perfect square or 2 less than a perfect square, and the set of ceiling functions of square roots of such sums constitutes the Pell sequence. The union of Lucas-balancing and Lucas-cobalancing numbers is the associated Pell sequence. Every even Pell number is twice of a balancing number and every odd Pell number, reduced by one, is twice of a cobalancing number. The sum of first $2n-1$ Pell numbers is equal to the sum of n^{th} balancing number and the associated balancer. In addition, every balancing number is product of a Pell number and an associated Pell number.

The theory of balancing and cobalancing numbers has been generalized in many directions. Panda introduced the concept of sequence balancing and cobalancing numbers using any arbitrary sequence $\{a_m\}_{m=1}^{\infty}$ of real numbers [69]. A term a_n of such a sequence is called a sequence balancing number if

$$a_1 + a_2 + \cdots + a_{n-1} = a_{n+1} + \cdots + a_{n+r}$$

for some natural number r. Similarly, a_n is known as a sequence cobalancing number if

$$a_1 + a_2 + \cdots + a_n = a_{n+1} + \cdots + a_{n+r}$$

for some natural number r. It is known that there is no sequence balancing number in the Fibonacci sequence while 1 is the only sequence cobalancing number in this sequence [69]. So far as the sequence of odd natural numbers is concerned, there do exist infinitely many sequences balancing numbers; however, no sequence cobalancing number exists in this sequence. The sequence balancing numbers in the odd natural numbers are nothing but the even terms of the associated Pell sequence, while the sequence balancing numbers in the even natural numbers are twice the balancing numbers that are the even terms of the Pell sequence.

As a particular case, Panda called the sequence balancing and cobalancing numbers of the sequence $\{n^k\}_{n=1}^{\infty}$, higher order balancing and cobalancing numbers respectively [69]. The case $k = 1$ corresponds to balancing and cobalancing numbers respectively. For $k = 2$, the higher order balancing and cobalancing numbers are known as balancing squares and cobalancing squares. Similarly, for $k = 3$, these are known as balancing cubes and cobalancing cubes. Panda in [69] also proved that there does not exist any balancing cube or cobalancing cube and conjectured that no higher order balancing or cobalancing numbers exists when $k > 1$.

Bérczes, Liptai and Pink [14] studied the existence of sequence balancing numbers in $\{R_n\}_{n=1}^{\infty}$, which is a sequence defined by means of a binary recurrence $R_{n+1} = AR_n + BR_{n-1}$ with $A, B \neq 0$ and $|R_0| + |R_1| > 0$. They proved that if $A^2 + 4B > 0$ and $(A, B) \neq (0, 1)$, no sequence balancing number exists.

Kovács, Liptai and Olajos [50] considered the problem of finding sequence balancing numbers in arithmetic progressions. Starting with two co-prime positive integers a and b with $b \geq 0$, they call $an + b$ an (a, b)-type balancing numbers if

$$(a+b) + (2a+b) + \cdots + ((n-1)a+b) = ((n+1)a+b) + \cdots + ((n+r)a+b)$$

holds for some natural number r. The sequence of (a, b)-type balancing numbers coincides with the balancing sequence when $a = 1$ and $b = 0$. Denoting the n^{th} (a, b)-type balancing number by $B_n^{(a,b)}$, they proved that the $8[B_n^{(a,b)}]^2 + a^2 - 4ab - 4b^2$ is a perfect square for each n. In a similar manner, the (a, b)-type cobalancing numbers was introduced and studied by Kovács et al [50].

The concept of higher order balancing numbers was generalized by many authors. Liptai et al. [55] considered the problem of finding n such that

$$1^k + 2^k + \cdots + (n-1)^k = (n+1)^l + \cdots + (n+r)^l$$

for some natural number r, where k and l are also natural numbers. They call any such n, if exists, a (k, l)-power numerical centre or (k, l)- balancing number and proved that if $k > 1$, then (k, l)-balancing numbers exist for finitely many values of $l \geq 1$. For example, when $(k, l) = (2, 1)$, there are three known $(2, 1)$-balancing numbers, namely $5, 13$ and 36. In an earlier paper [33], Finkelstein conjectured that if $k > 1$, there exist no (k, k)-power numerical centre and proved the case $k = 3$ in [77] using a result from Ljunggren [56] and Cassels [21]. In 2005, Ingram [41] proved the case $k = 5$. Observe that the concept of $(1, 1)$ numerical centre is equivalent to balancing numbers [13] while, for $k > 1$, the

(k,k)-power numerical centres are nothing but n^{th} order balancing numbers [69].

In another generalization of higher order balancing numbers, Behera et al. [12] considered the problem of finding the quadruple (n,r,k,l) in positive integers with $n \geq 2$ satisfying the equation

$$F_1^k + F_2^k + \cdots + F_{n-1}^k = F_{n+1}^l + F_{n+2}^l + \cdots + F_{n+r}^l.$$

In this connection, they conjectured that the only quadruple satisfying the above equation is $(4,3,8,2)$. Subsequently, Alvarado et al. [3] confirmed that this conjecture is true. Irmak [43], studied the equation

$$B_1^k + B_2^k + \cdots + B_{n-1}^k = B_{n+1}^l + B_{n+2}^l + \cdots + B_{n+r}^l$$

in powers of balancing numbers and established that no quadruple (n,r,k,l) in positive integers with $n \geq 2$ satisfies the equation.

Komatsu and Szalay [48] studied the existence of sequence balancing numbers involving binomial coefficients. They considered the problem of finding x and $y \geq x+2$ satisfying the Diophantine equation

$$\binom{0}{k} + \binom{1}{k} + \cdots + \binom{x-1}{k} = \binom{x+1}{l} + \cdots + \binom{y-1}{l}$$

with given positive integers k and l and solved completely when $1 \leq k,\ l \leq 3$.

The definition of balancing and cobalancing numbers [13,33,72], as we have already discussed, involves balancing sums of natural numbers. Behera and Panda, after the introduction of balancing numbers in [13], considered the problem of balancing products of natural numbers. They called a positive integer n a product balancing number if

$$1 \cdot 2 \cdots (n-1) = (n+1) \cdots (n+r)$$

for some natural number r. They also identified 7 as the first product balancing number. However, they could not provide a second product balancing number.

In [78], Szakács observed that if n is a product balancing number then none of $(n+1),(n+2),\ldots,(n+r)$ is a prime and proved that no product balancing number other than 7 exists. Indeed, he used a different name multiplying balancing numbers for product balancing numbers. Szakács [78] also defined multiplying cobalancing number as a positive integer n satisfying

$$1 \cdot 2 \cdots n = (n+1) \cdots (n+r)$$

for some natural number r and proved that no multiplying cobalancing number exists.

Parallel to the definition of (k,l)-power numerical centre by Liptai et al., Szakács [78] defined a (k,l)-power multiplying balancing number n as a solution of the Diophantine equation

$$1^k \cdot 2^k \cdots (n-1)^k = (n+1)^l \cdot (n+2)^l \cdots (n+r)^l$$

which holds for some natural number r. He proved that only one (k,l)-power multiplying balancing number corresponding to $k=l$ exists and is precisely $n=7$.

There are many interesting papers on balancing numbers. Liptai [53,54] investigated the presence of balancing numbers in Fibonacci and Lucas sequences. In both these papers, Liptai first showed that there are only finitely many common solutions of the pair of Pell's equations $x^2-8y^2=1$ and $x^2-5y^2=\pm 4$ using a method of Baker and Davenport [8]. He proved that there is no balancing number in Fibonacci and Lucas sequence. Szalay [79] got the same conclusion by converting the above Pell's equations into a family of Thue equations [7,83].

As we already know, $x \geq 1$ is a balancing number if and only if $8x^2+1$ is a perfect square. The balancing numbers are alternately even and odd, the first number 1 is a square while the second, third and fourth balancing numbers $6,35$ and 204 are not squares. Thus a natural question is "Is there any perfect square balancing number other than the first one?" In the paper [71], Panda answered this question in negative by proving that the equation $8x^4+1=y^2$ has no solution in positive integers other than 1.

In [50], Kovács, Liptai and Olajos studied the possibility of balancing numbers expressible as product of consecutive integers. They proved that the equation

$$B_n = x(x+1)\cdots(x+k-1)$$

has only finitely many solutions when $k \geq 2$. They obtained all solutions when $k=2,3,4$ and also certain small solutions corresponding to $k=6,8$. Later on, Tengely [82] showed that the above equation has no solution when $k=5$ with $m \geq 0$ and $x \in \mathbb{Z}$ using the ideas described in [17].

A Diophantine n-tuple is a set $\{x_1,x_2,\ldots,x_n\}$ of positive integers such that the product of any two increased by 1 is a perfect square. However, Diophantus who introduced this concept, initially considered rational quadruples [25,68]. Fermat was first to find the integer quadruple $\{1,3,8,120\}$. It is known that there are infinitely many Diophantine quadruples and there it has been conjectured that there are no Diophantine quintuples. In [27], Dujella showed that there can be at most finitely many Diophantine quintuples.

Fuch, Luca and Szalay [35] considered a variation of the Diophantine n-tuple. Let $\psi = \{a_n\}_{n=1}^{\infty}$ be any integer sequence. They examined the problem of finding a set of three integers a, b, c such that all of $ab+1$, $ac+1$ and $bc+1$ are members of ψ. They call any such set a Diophantine triple. When $a_n = 2^n + 1$, there are infinitely many such triples. They showed that only similar sequences have infinitude of solutions. In [57, 58], Luca and Szalay proved that no such triple exists in Fibonacci and Lucas sequences. Alp, Irmak and Szalay [2] ascertained the absence of any such triple in the balancing sequence.

As we have already mentioned, the balancing sequence, like the sequence of Fibonacci numbers, is a divisibility sequence. Indeed, it is a strong divisibility sequence. The sum of first n odd terms of the balancing sequence is equal to square of the n^{th} balancing numbers, a property satisfied by the sequence of natural numbers. This property makes the balancing sequence a natural sequence [69]. The balancing sequence also satisfies the interesting identity $B_{m+n} = B_m C_n + C_m B_n$, a property resembling the trigonometric identity $\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y$. A natural question is therefore, " Is there any other integer sequence that satisfies all the above properties?" The answer to this question is yes. There do exists infinitely many sequences with these properties. We call them balancing-like sequences and their terms as balancing-like numbers. They can be obtained from the class of binary recurrences

$$x_{n+1} = Ax_n - x_{n-1};\ x_0 = 0,\ x_1 = 1$$

where $A > 2$ is a natural number. The balancing sequence is a member of this class corresponding to $A = 6$. Chapter 2 is dedicated to the study of the integer sequences obtained from the binary recurrences

$$x_{n+1} = Ax_n - Bx_{n-1};\ x_0 = 0,\ x_1 = 1,$$

where A and B are natural numbers such that $A^2 - 4B > 0$.

As we already know, a natural number n is a cobalancing number if

$$1+2+\cdots+n = (n+1)+(n+2)+\cdots+m$$

for some natural number m [72] while n is a balancing number if

$$1+2+\cdots+(n-1) = (n+1)+(n+2)+\cdots+m$$

for some m [13]. It is clear that while defining cobalancing number all natural numbers from 1 to m are used, while in the definition of balancing numbers, a number is deleted (and hence a gap is created) in the first m natural numbers to balance numbers on its both sides. A natural question is therefore, "Is it possible to delete two consecutive numbers from the list of first m natural numbers so that the sum to the left of these deleted numbers is equal to the sum to the right?" The answer is yes and we call the sum of these two deleted numbers a 2-gap or g_2-balancing number. The theory of g_2-balancing numbers is quite impressive, these numbers partition into two disjoint classes, the numbers in one class are associated with the solution of an almost Pythagorean equation. In Chapter 3, we present a detailed discussion on g_2-balancing numbers.

A further generalization of g_2-balancing numbers is possible by deleting more than two consecutive numbers from the list of first m natural numbers so that the sum to the left of these deleted numbers is equal to the sum to the right. If k numbers are removed then it leads to the definition of k-gap balancing numbers (also called the g_k-balancing numbers). However, to maintain integral values, when k is odd, the k-gap balancing number is taken as the median of the deleted numbers; when k is even, twice the median of the deleted numbers is called a k-gap balancing number. Preparing a list of g_k-balancing numbers requires solving the generalized Pell's equations $y^2 - 2x^2 = 2k^2 - 1$ or $y^2 - 8x^2 = 2k^2 - 1$ according as k is even or odd. Since these equations can be completely solved for any given value of k, it is possible to enumerate all the g_k-balancing numbers; however, for arbitrary k, two classes of solutions, but not all the solutions, can be obtained. A detailed theory of g_k-balancing numbers is presented in Chapter 4.

As we have already discussed, given any real sequence $\{a_m\}_{m=1}^{\infty}$, a term a_n of this sequence is called a sequence balancing number if

$$a_1 + a_2 + \cdots + a_{n-1} = a_{n+1} + \cdots + a_{n+r}$$

for some natural number r. If $a_m = m$ for all m, the sequence balancing numbers are nothing but balancing numbers, but if $a_m = m^k$, where $k \geq 2$ is a positive integers, they are called k^{th} order balancing numbers. In the line of generalization of balancing numbers to k^{th} order balancing numbers, the g_2-balancing numbers can also be generalized resulting in the k^{th} order g_2-balancing numbers. Chapter 5 is completely devoted to the study of second order g_2-balancing numbers.

It is well known that a series of the form $\sum_{n=1}^{\infty} a_n n^{-s}$, where $\{a_n\}_{n=1}^{\infty}$ is a complex sequence and s a complex number is called a Dirichlet series. The particular case $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ with $\mathrm{Re}(s) > 1$ is known as the Riemann zeta function and is extensively studied by many authors [6, 22, 42, 84, 90]. It is also known that $\zeta(s)$ can be analytically continued to the whole complex plane with a simple pole at $s = 1$, and the trivial zeros are $-2, -4, -6, \ldots$ As a variant of this function, Navas [65] defined the Fibonacci zeta function as $\zeta_F(s) = \sum_{n=1}^{\infty} F_n^{-s}$. He proved that $\zeta_F(s)$ can also be analytically continued to the whole complex plane with trivial zeros at $-2, -6, -10, \ldots$ and simple poles at $0, -4, -8, \ldots$ In Chapter 6, we define the balancing zeta function as $\zeta_B(s) = \sum_{n=1}^{\infty} B_n^{-s}$ and study its analytic continuation, zeros and poles.

Several authors [20,32,34] proved that every sequence of integers, obtained by means of a recurrence relation, is periodic modulo any positive integer m. It is possible to find out the period modulo any given m; however, there is no formula to calculate this period for an arbitrary m. Wall [89] was first to study the properties of such periods of Fibonacci sequence. He proved that the period, as a function of the modulus of the congruence m, is a divisibility sequence. Further, if m and n are co-prime positive integers, then the period modulo mn is the least common multiple of periods modulo m and n. Wall however could not find a single prime p for which the period modulo p is equal to the period modulo p^2 and conjectured that there is no such prime.

The period of the balancing sequence modulo n is denoted by $\pi(n)$ and $\{\pi(n) : n = 1,2,\dots\}$ also constitutes a divisibility sequence. The function π remain unaltered for any power of 2 and hence the fixed points of π are $2^n,\ n = 1,2,\dots$ Further, when m and n are coprime, $\pi(mn)$ is also equal to the least common multiple of $\pi(m)$ and $\pi(n)$. However, in contrast to Wall's conjecture, in case of balancing numbers, there are three primes $13, 31$ and 1546463 such that the period modulo any of these three primes is equal to the period modulo its square. An elaborative study of periodicity of balancing numbers is presented in Chapter 7.

Like periodicity, stability is an important aspect of a recurrent sequence. Given a sequence defined by means of a recurrence relation, the frequency distribution of any residue d modulo m is the number of occurrences of d in a period modulo m. For any prime p, the set of frequency distributions modulo p^k, where k is a positive integer, in general, depends on k. If this set remains unchanged for $k = N, N+1, \dots$ then the sequence is said to be stable for the prime p. It has been shown in [44] that the Fibonacci sequence is stable for primes 2 and 5, while, on the contrary the Lucas sequence is not stable for these primes [18]. The balancing sequence, however, is stable for a large class of primes $p \equiv -1, -3 \pmod 8$ and not stable for primes $p \equiv 3 \pmod 8$. Chapter 8 deals with the stability of balancing numbers modulo primes.

This thesis consists of eight chapters. The contents of Chapter 1 are some notations and preliminaries results used in subsequent chapters. Several possible generalizations of balancing numbers are discussed in Chapters 2-5. The remaining chapters are about some important untouched aspects of balancing numbers. The thesis is nicely concluded with an elaborative reference.

Chapter 1

Notations and Preliminaries

1.1 Notations

Throughout this work, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ denote respectively the set of natural numbers, integers, rational numbers, real numbers and complex numbers. For a non square integer D, $\mathbb{Z}[\sqrt{D}]$ denotes a quadratic field while $\mathbb{Z}[\sqrt{D}]^{\times}$ is obtained from $\mathbb{Z}[\sqrt{D}]$ by removing 0. For $z \in \mathbb{C}$, Re z and Im z denote the real and imaginary parts of z respectively. For $m,n \in \mathbb{Z}$, $m \mid n$ stands for m is a divisor of n, (m,n) and $[m,n]$ are respectively the greatest common divisor and the least common multiple of m and n, and $\binom{m}{n}$ is a binomial coefficient.

1.2 Preliminaries

In this chapter, we present some known number theoretic definitions and results. We shall keep on referring back to it as and when required.

1.2.1 Recurrence relation

A recurrence relation is an equation that defines a sequence using a rule to find a term as a function of some specified previous terms [9, 36]. The simplest form of a recurrence relation is of first order in which each term depends only on its previous term. In general, a k^{th} order linear recurrence relation is of the form

$$A_{n+1} = C_1(n)A_{n-k+1} + \cdots + C_k(n)A_n + h(n), \tag{1.2.1}$$

where the coefficients $C_1, C_2, \ldots, C_k$ and h are real function of n and $C_1 \neq 0$. If $C_i(n)$ are constants, i.e., $C_i(n) = c_i$ for each i and $h(n) = 0$, then (1.2.1) is a k^{th} order homogeneous linear recurrence relation with constant coefficients. Observe that $A_n = \alpha^n$ is a solution of the homogeneous recurrence relation if

$$\alpha^k - c_k\alpha^{k-1} - \cdots - c_1 = 0.$$

The last equation is called the characteristic equation of the recurrence relation and its roots are called the characteristic roots. If the characteristic roots $\alpha_1, \alpha_2, \ldots, \alpha_k$ are all distinct then using the initial values, the general solution can be obtained from

$$A_n = a_1\alpha_1^n + a_2\alpha_2^n + \cdots + a_k\alpha_k^n.$$

The characteristic equation of a linear homogeneous recurrence of order two is of the form

$$\alpha^2 - c_2\alpha - c_1 = 0$$

which has only two roots α_1 and α_2. If both α_1 and α_2 are unequal, then the general solution is

$$A_n = a_1\alpha_1^n + a_2\alpha_2^n.$$

However, in case of equal roots, i.e., $\alpha_1 = \alpha_2 = r$, the sequence is given by

$$A_n = (a_1 + a_2 n)r^n.$$

Finally, if the roots are complex say $\alpha_1 = re^{ia}$, $\alpha_2 = re^{-ia}$ then

$$A_n = (a_1 \cos na + a_2 \sin na)r^n.$$

In all these three cases, the two initial values determines the unknowns a_1 and a_2. The solutions so obtained are known as the closed form or Binet form of the recurrence sequence.

The Binet forms for Fibonacci, Lucas, Pell, and associated Pell sequences are given by

$$F_n = \frac{\alpha_1^n - \alpha_2^n}{\sqrt{5}},\ L_n = \alpha_1^n + \alpha_2^n \quad \text{where} \quad \alpha_1 = \frac{1+\sqrt{5}}{2},\ \alpha_2 = \frac{1-\sqrt{5}}{2}$$

and

$$P_n = \frac{\beta_1^n - \beta_2^n}{2\sqrt{2}},\ Q_n = \frac{\beta_1^n + \beta_2^n}{2} \quad \text{where} \quad \beta_1 = 1+\sqrt{2}, \beta_2 = 1-\sqrt{2}.$$

1.2.2 Diophantine equations

An algebraic equation for which only integer solutions are sought is known as a Diophantine equation. Diophantine equations are originally studied by Greek number theorist Diophantus who is known for his book Arithmetica. The general form of a Diophantine equation in n variables is $f(x_1, x_2, \ldots, x_n) = 0$. An n-tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$ satisfying above equation is a solution to the above equation. An equation having one or more solutions is called solvable [62].

The Diophantine equation $x^2 + y^2 = z^2$ is known as Pythagorean equation in literature and the infinitude of its solutions is well-known. The most celebrated Diophantine equation was posed by Fermat in 1637, who always used to carry the book Arithmetica of Diophantus, wrote in the margin of the book "It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers. I have discovered a truly marvellous proof of this, which this margin is too narrow to contain." In other words, Fermat expressed the impossibility of existence of solution of the Diophantine equation $x^n + y^n = z^n$ in positive integers x, y, z when $n \geq 3$. The problem remains unsolved till A. Wiles announced the proof in 1992 and corrected in 1995 [81, 91].

Andrew Beal in 1993 formulated a conjecture by generalizing the Fermat's Last theorem which states that "If $a^x + b^y = c^z$ where a, b, c, x, y, z are positive integer and x, y and z are all greater than 2, then a, b, c must have a common prime factor." This same conjecture was independently formulated by Robert Tijdeman and Don Zagier [23, 88] though this is commonly known as Beal's conjecture.

Another important conjecture in the theory of Diophantine equations was Catalan's conjecture, proved by Preda Mihăilescu [60], which states that the only solution in the natural numbers of the equation $x^a - y^b = 1$ for $x, y, a, b > 1$ is $x = 3, a = 2, y = 2, b = 3$.

1.2.3 Pell's equations

In 1909, Thue [83] proved the following important theorem: "Let

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

be an irreducible polynomial of degree not less than 3 with integer coefficients and

$$F(x,y) = a_n x^n + a_{n-1} x^{n-1} y + \cdots + a_1 x y^{n-1} + a_0 y^n,$$

the corresponding homogeneous polynomial. If k is a non-zero integer, then the equation $F(x,y) = k$ has either no solutions or only a finite number of solutions in integers." When $n = 2$ and $F(x,y) = x^2 - Dy^2$, where D is a non-square positive integer, then for all non-zero integer k, the Diophantine equation $x^2 - Dy^2 = k$ has either no solution or has infinitely many solutions. Any Diophantine equation of the form

$$x^2 - Dy^2 = 1, \tag{1.2.2}$$

where D is a nonsquare, is called a Pell's equation [29] while the equation

$$x^2 - Dy^2 = k \tag{1.2.3}$$

where $k \neq 1$, is called a generalized Pell's equation. To understand this equation thoroughly over $\mathbb{Z}$, one needs to work in the ring extension $\mathbb{Z}[\sqrt{D}]$ of $\mathbb{Z}$

$$\mathbb{Z}[\sqrt{D}] = \{a + b\sqrt{D} : a, b \in \mathbb{Z}\}.$$

In the ring $\mathbb{Z}[\sqrt{D}]$, factorization gives

$$x^2 - Dy^2 = (x + y\sqrt{D})(x - y\sqrt{D}) = 1.$$

If $\alpha = x + y\sqrt{D}$, then the norm of α is given by

$$N(\alpha) = (x + y\sqrt{D})(x - y\sqrt{D}) = x^2 - Dy^2.$$

It is known that the solutions (x, y) of (1.2.2) are in one-to-one correspondence with the units in $\mathbb{Z}[\sqrt{D}]$ of norm 1 [10, 45]. A fundamental unit of $\mathbb{Z}[\sqrt{D}]$ is the smallest unit of norm 1 with positive rational and irrational parts. If $x_0 + y_0\sqrt{D}$ is a fundamental unit of $\mathbb{Z}[\sqrt{D}]$, then all positive integral solutions of (1.2.2) are obtained from

$$x + y\sqrt{D} = (x_0 + y_0\sqrt{D})^m, m \geq 1.$$

Further, for any non-square D and for any integer $B > 1$, there exists $x, y \in \mathbb{Z}$ such that $0 < y < B$ and

$$|x - y\sqrt{D}| < 1/B.$$

Hence x/y is fairly good rational approximation to $\sqrt{D}$.

The generalized Pell's equation (1.2.3) with $k > 1$, does not behave so well like the case $k = 1$. If $x, y \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that

$$N(x + y\sqrt{D}) = x^2 - Dy^2 = k,$$

then $x + y\sqrt{D}$ is called a solution of generalized Pell's equation $x^2 - Dy^2 = k$. Let $x^* + y^*\sqrt{D}$ and $u + v\sqrt{D}$ be solutions of the generalized Pell's equation and the Pell's equation respectively. Then

$$(u + v\sqrt{D})(x^* + y^*\sqrt{D}) = (ux^* + vy^*D) + (uy^* + vx^*)\sqrt{D}$$

is also a solution of the generalized Pell's equation since

$$N[(u + v\sqrt{D})(x^* + y^*\sqrt{D})] = N(u + v\sqrt{D}) \cdot N(x^* + y^*\sqrt{D}) = 1 \cdot k = k.$$

Two solutions $x_m + y_m\sqrt{D}$ and $x_n + y_n\sqrt{D}$ of (1.2.3) are said to be in the same class if

$$x_m + y_m\sqrt{D} = (x + y\sqrt{D})(u + v\sqrt{D})^m$$

and

$$x_n + y_n\sqrt{D} = (x + y\sqrt{D})(u + v\sqrt{D})^n$$

where $u+v\sqrt{D}$ is the fundamental solution of (1.2.2) and $x+y\sqrt{D}$ is a the fundamental solution of (1.2.3). Thus, corresponding to each fundamental solution $x+y\sqrt{D}$ of (1.2.3), there is a class of solutions of (1.2.3) determined by

$$x_n+y_n\sqrt{D}=(x+y\sqrt{D})(u+v\sqrt{D})^n,\ n=0,1,\ldots$$

where $u+v\sqrt{D}$ is the fundamental solution of (1.2.2). Hence, the generalized Pell's equation (1.2.3) may not have any solution or may have one or more classes of solutions. The bounds for the fundamental solutions of (1.2.3) are described in the following theorem.

Theorem 1.2.1. *[61, Theorem 7.1, p. 267] Let $k>1$ and $x+y\sqrt{D}$ be a fundamental solution of* (1.2.3) *in its class. If $u+v\sqrt{D}$ is the fundamental solution of* (1.2.2)*, then*

$$0<|x|\leq\sqrt{\frac{(u+1)k}{2}},\quad 0\leq y\leq\frac{v\sqrt{k}}{\sqrt{2(u+1)}}. \tag{1.2.4}$$

1.2.4 Balancing numbers

As defined by Behera and Panda [13] and Finkelstein [33], a balancing number is a natural number n such that

$$1+2+\cdots+(n-1)=(n+1)+\cdots+(n+r)$$

for some natural number r, which is called a balancer corresponding to n. It is well-known that if n is a balancing number, then n^2 is a square triangular number or equivalently, $8n^2+1$ is a perfect square, $\sqrt{8n^2+1}$ is called a Lucas-balancing number and the balancer corresponding to n is given by

$$r=\frac{-(2n+1)+\sqrt{8n^2+1}}{2}.$$

If B is any balancing number, the relation $8B^2+1=C^2$ leads to the Pell's equation $C^2-8B^2=1$ whose fundamental solution is $(C,B)=(3,1)$. Hence, the totality of balancing and Lucas-balancing numbers are given by their closed form as

$$B_n=\frac{(1+\sqrt{2})^{2n}-(1-\sqrt{2})^{2n}}{4\sqrt{2}}$$

and

$$C_n=\frac{(1+\sqrt{2})^{2n}+(1-\sqrt{2})^{2n}}{2};\ n=1,2,\ldots$$

Using these closed forms, one can easily obtain the binary recurrences for balancing and Lucas-balancing numbers as

$$B_{n+1}=6B_n-B_{n-1},\ \ C_{n+1}=6C_n-C_{n-1}. \tag{1.2.5}$$

By virtue of backward calculations, one can easily obtain $B_0=0$ and $C_0=1$. The balancing and Lucas-balancing numbers share many important properties [13,70].

a. $B_m^2 - B_n^2 = B_{m+n} \cdot B_{m-n}$.

b. $(C_m + \sqrt{8}B_m)^r = C_{mr} + \sqrt{8}B_{mr}$.

c. $B_{m\pm n} = B_m C_n \pm C_m B_n$.

d. $C_{m\pm n} = C_m C_n \pm 8B_m B_n$.

e. $B_{m+n+1} = B_{m+1}B_{n+1} - B_m B_n$.

1.2.5 Cobalancing numbers

According to Panda and Ray [72], a cobalancing number is a natural number n satisfying

$$1+2+\cdots+n = (n+1)+\cdots+(n+r)$$

for some natural number r, which they call the cobalancer corresponding to n. It is known that if n is a cobalancing number, then $8n^2+8n+1$ is a perfect square and implying that pronic number n^2+n is triangular. Thus, corresponding to each pronic triangular number x, there is a cobalancing number b which is equal to the integral part of $\sqrt{x}$. The cobalancer r corresponding to a cobalancing number n is given by

$$r = \frac{-(2n+1)+\sqrt{8n^2+8n+1}}{2}.$$

The recurrence for cobalancing numbers is

$$b_{n+1} = 6b_n - b_{n-1} + 2, b_0 = b_1 = 0$$

which is not homogeneous and the Binet form is given by

$$b_n = \frac{(1+\sqrt{2})^{2n-1} - (1-\sqrt{2})^{2n-1}}{4\sqrt{2}} - \frac{1}{2}.$$

The following are some important properties satisfied by cobalancing numbers [72].

a. $(b_n - 1)^2 = 1 + b_{n-1}b_{n+1}$.

b. $b_{n+1} = 3b_n + \sqrt{8b_n^2 + 8b_n + 1} + 1$.

c. $2(B_1 + B_2 + \cdots + B_{n-1}) = b_n$.

d. $b_{2n} = B_n b_{n+1} - (B_{n-1} - 1)b_n$.

e. $b_{2n+1} = (B_{n+1} + 1)b_{n+1} - B_n b_n$.

Chapter 2

Balancing-Like Numbers

2.1 Introduction

It is well known that the balancing numbers are solutions of the binary recurrence $B_{n+1} = 6B_n - B_{n-1}$ with initial values $B_0 = 0$ and $B_1 = 1$. These numbers can be considered as a variant of natural numbers since natural numbers are solutions of a similar recurrence $x_{n+1} = 2x_n - x_{n-1}$ with initial values $x_0 = 0$ and $x_1 = 1$. Some properties of natural numbers also hold good for balancing numbers. The sum of first n odd natural numbers is equal to n^2 and the sum of first n even numbers is equal to $n(n+1)$. Parallel to the result Panda [70] proved that

$$B_1 + B_3 + \cdots + B_{2n-1} = B_n^2$$

and

$$B_2 + B_4 + \cdots + B_{2n} = B_n B_{n+1}.$$

Each balancing number x is associated with a Lucas-balancing number y and they are related by the identity $8x^2 + 1 = y^2$. The Lucas-balancing numbers also satisfy a recurrence relation identical with that of balancing numbers. More precisely, $C_{n+1} = 6C_n - C_{n-1}$ with $C_0 = 1$ and $C_1 = 3$. The balancing numbers behave like the trigonometric sine functions. The identity $B_{m+n} = B_m C_n + C_m B_n$ looks like $\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y$ while $B_{2n} = 2B_n C_n$ looks like $\sin 2x = 2 \sin x \cos x$ (see p. 18).

The Fibonacci numbers satisfy the strongly divisibility $(F_m, F_n) = F_{(m,n)}$. The balancing numbers also satisfy $(B_m, B_n) = B_{(m,n)}$.

Our main focus in this chapter is to identify a class of binary recurrence sequences enjoying all the above stated properties of balancing numbers.

G. K.Panda and **S. S. Rout**, A class of recurrent sequences exhibiting some exciting properties of balancing numbers, *World Acad. of Sci. Eng. and Tech.*, **6**, 164–166, 2012.

2.2 Some fascinating properties of balancing-like numbers

We consider a class of second order linear recurrence sequences and establish conditions under which these sequences would satisfy the fascinating properties of balancing numbers discussed in the last section.

We start with a second order linear recurrence

$$x_{n+1} = Ax_n - Bx_{n-1}, x_0 = 0, x_1 = 1 \tag{2.2.1}$$

where A and B are natural numbers such that $A^2 - 4B > 0$. The auxiliary equation of this recurrence is given by

$$\alpha^2 - A\alpha + B = 0$$

which, because of the condition $A^2 - 4B > 0$, has the unequal real roots

$$\alpha_1 = \frac{A + \sqrt{A^2 - 4B}}{2}, \quad \alpha_2 = \frac{A - \sqrt{A^2 - 4B}}{2}.$$

The general solution of (2.2.1) is given by

$$x_n = P\alpha_1^n + Q\alpha_2^n,$$

and using the initial conditions, we get the Binet form

$$x_n = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}, \quad n = 0, 1, 2, \ldots$$

To find the conditions under which

$$x_1 + x_3 + \cdots + x_{2n-1} = x_n^2$$

holds, it is enough to find conditions leading to

$$x_{2n+1} = x_{n+1}^2 - x_n^2.$$

Noting that $\alpha_1 + \alpha_2 = A$ and $\alpha_1\alpha_2 = B$,

$$\begin{aligned} x_{n+1}^2 - x_n^2 &= \left[\frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2}\right]^2 - \left[\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}\right]^2 \\ &= \frac{\alpha_1^{2n+2} + \alpha_2^{2n+2} - \alpha_1^{2n} - \alpha_2^{2n} - 2B^{n+1} + 2B^n}{(\alpha_1 - \alpha_2)^2}, \end{aligned}$$

is equivalent to

$$(\alpha_1 - \alpha_2)(\alpha_1^{2n+1} - \alpha_2^{2n+1}) = \alpha_1^{2n+2} + \alpha_2^{2n+2} - \alpha_1^{2n} - \alpha_2^{2n} - 2B^{n+1} + 2B^n.$$

This gives

$$B(\alpha_1^{2n}+\alpha_2^{2n}) = \alpha_1^{2n}+\alpha_2^{2n}+2B^{n+1}-2B^n.$$

Further rearrangement converts this equation to

$$(B-1)(\alpha_1^{2n}+\alpha_2^{2n}-2B^n) = 0$$

and applying $\alpha_1\alpha_2 = B$, we arrive at

$$(B-1)(\alpha_1^n-\alpha_2^n)^2 = 0$$

which is possible if $\alpha_1^n = \alpha_2^n$ or $B = 1$. If $\alpha_1^n = \alpha_2^n$, then $\alpha_1 = \alpha_2$ or $\alpha_1 = -\alpha_2$. But $\alpha_1 = \alpha_2$ corresponds to $A^2-4B = 0$, which is forbidden by our initial assumption and $\alpha_1 = -\alpha_2$ corresponds to a negative B, which is contrary to the choice of B. Thus, the only option left is $B = 1$.

Conversely, if $B = 1$ then $\alpha_1\alpha_2 = 1$ and

$$\begin{aligned} x_{n+1}^2 - x_n^2 &= \left[\frac{\alpha_1^{n+1}-\alpha_2^{n+1}}{\alpha_1-\alpha_2}\right]^2 - \left[\frac{\alpha_1^n-\alpha_2^n}{\alpha_1-\alpha_2}\right]^2 \\ &= \frac{\alpha_1^{2n+2}+\alpha_2^{2n+2}-\alpha_1^{2n}-\alpha_2^{2n}}{(\alpha_1-\alpha_2)^2} \\ &= \frac{\alpha_1^{2n+1}(\alpha_1-\alpha_2)-\alpha_2^{2n+1}(\alpha_1-\alpha_2)}{(\alpha_1-\alpha_2)^2} \\ &= \frac{\alpha_1^{2n+1}-\alpha_2^{2n+1}}{\alpha_1-\alpha_2} = x_{2n+1}. \end{aligned}$$

The above discussion proves the following theorem.

Theorem 2.2.1. *Let* $x_{n+1} = Ax_n - Bx_{n-1}, x_0 = 0, x_1 = 1$ *be a second order linear recurrence such that A and B are natural numbers satisfying* $A^2-4B > 0$. *Then, for each natural number n, a necessary and sufficient condition for*

$$x_1+x_3+\cdots+x_{2n-1} = x_n^2$$

to hold is $B = 1$.

The balancing numbers also satisfy the identity

$$B_2+B_4+\cdots+B_{2n} = B_nB_{n+1}.$$

Next we investigate the conditions leading to

$$x_2+x_4+\cdots+x_{2n} = x_nx_{n+1}.$$

It is enough to find conditions under which

$$x_n x_{n+1} - x_{n-1} x_n = x_{2n}.$$

This is equivalent to

$$\begin{aligned} x_n(x_{n+1} - x_{n-1}) &= \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}\left[\frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2} - \frac{\alpha_1^{n-1} - \alpha_2^{n-1}}{\alpha_1 - \alpha_2}\right] \\ &= \frac{\alpha_1^{2n+1} + \alpha_2^{2n+1} - \alpha_1^{2n-1} - \alpha_2^{2n-1} - B^n(\alpha_1 + \alpha_2) + B^{n-1}(\alpha_1 + \alpha_2)}{(\alpha_1 - \alpha_2)^2} \\ &= \frac{\alpha_1^{2n} - \alpha_2^{2n}}{\alpha_1 - \alpha_2}. \end{aligned}$$

On rearrangement, we get

$$(\alpha_1 - \alpha_2)(\alpha_1^{2n} - \alpha_2^{2n}) = \alpha_1^{2n+1} + \alpha_2^{2n+1} - \alpha_1^{2n-1} - \alpha_2^{2n-1} - B^n(\alpha_1 + \alpha_2) + B^{n-1}(\alpha_1 + \alpha_2),$$

which finally simplifies to

$$(B-1)(\alpha_1^{2n-1} + \alpha_2^{2n-1}) = B^{n-1}(B-1)(\alpha_1 + \alpha_2).$$

This gives $B = 1$.

Conversely, it can be easily seen that if $B = 1$, then $x_n x_{n+1} - x_{n-1} x_n = x_{2n}$.

The above discussion leads to the following theorem.

Theorem 2.2.2. *Let $x_{n+1} = Ax_n - Bx_{n-1}$, $x_0 = 0$, $x_1 = 1$ be a second order linear recurrence such that A and B are natural numbers satisfying $A^2 - 4B > 0$. Then, for each natural number n, a necessary and sufficient condition for*

$$x_2 + x_4 + \cdots + x_{2n} = x_n x_{n+1}$$

to hold is $B = 1$.

It is known that the Binet form for balancing and Lucas-balancing numbers are

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, \quad \text{and} \quad C_n = \frac{\lambda_1^n + \lambda_2^n}{2}$$

where $\lambda_1 = 3 + 2\sqrt{2}$ and $\lambda_2 = 3 - 2\sqrt{2}$. If we define a new sequence

$$y_n = \frac{\alpha_1^n + \alpha_2^n}{2},$$

then it is easy to verify that

$$x_{2n} = 2x_n y_n$$

$-$a property similar to $B_{2n} = 2B_n C_n$. In addition, we observe that $\alpha_1 - \alpha_2 = \sqrt{A^2 - 4B}$, so that

$$(\alpha_1 - \alpha_2)^2 = A^2 - 4B$$

is a natural number. It is easy to see that

$$y_m + \frac{\sqrt{A^2-4B}}{2} x_m = \alpha_1^m,\ m = 1,2,\dots$$

from which it follows that

$$\left[y_m + \frac{\sqrt{A^2-4B}}{2} x_m\right]\left[y_n + \frac{\sqrt{A^2-4B}}{2} x_n\right]$$

$$= \alpha_1^{m+n} = y_{m+n} + \frac{\sqrt{A^2-4B}}{2} x_{m+n}. \tag{2.2.2}$$

If we choose A and B such that A^2-4B is not a perfect square, then $\sqrt{A^2-4B}$ is irrational and comparing rational and irrational parts from both sides of (2.2.2), we get

$$y_{m+n} = y_m y_n + \frac{A^2-4B}{4} x_m x_n, \quad \text{and} \quad x_{m+n} = x_m y_n + y_m x_n.$$

The above discussion leads to the following theorem.

Theorem 2.2.3. *Let $x_{n+1} = Ax_n - Bx_{n-1}, x_0 = 0, x_1 = 1$ be a second order linear recurrence such that A and B are natural numbers and A^2-4B is non-square and positive. If y_n is defined as $y_n = \frac{\alpha_1^n+\alpha_2^n}{2}$, then*

$$y_{m+n} = y_m y_n + \frac{A^2-4B}{4} x_m x_n \quad \textit{and} \quad x_{m+n} = x_m y_n + y_m x_n.$$

A well known connection between balancing and Lucas-balancing numbers is

$$C_n^2 = 8B_n^2 + 1.$$

We can expect a similar relationship between the sequences x_n and y_n. Indeed

$$x_n^2 = \left[\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}\right]^2 = \frac{\alpha_1^{2n} + \alpha_2^{2n} - 2B^n}{A^2-4B}.$$

Thus,

$$\frac{(A^2-4B)x_n^2}{4} + B^n = \frac{\alpha_1^{2n} + \alpha_2^{2n} + 2B^n}{4} = \left[\frac{\alpha_1^n + \alpha_2^n}{2}\right]^2 = y_n^2.$$

Writing $D = \frac{A^2-4B}{4}$, the last equation takes the form

$$y_n^2 = B^n + Dx_n^2.$$

In view of the above discussion we have the following theorem.

Theorem 2.2.4. *Let $x_{n+1} = Ax_n - Bx_{n-1}, x_0 = 0, x_1 = 1$ be a second order linear recurrence such that A and B are natural numbers and $A^2-4B > 0$. If y_n is defined as*

$y_n = \frac{\alpha_1^n+\alpha_2^n}{2}$, *then* $y_n^2 = B^n + Dx_n^2$, *where* $D = \frac{A^2-4B}{4}$.

We now try to find a recurrence relation for y_n. Since α_1 and α_2 are roots of the equation

$$\alpha^2 - A\alpha + B = 0$$

it follows that

$$\alpha_1^2 - A\alpha_1 + B = 0 \quad \text{and} \quad \alpha_2^2 - A\alpha_2 + B = 0.$$

Multiplying the last two equations by α_1^{n-1} and α_2^{n-1} respectively and rearranging, we get

$$\alpha_1^{n+1} = A\alpha_1^n - B\alpha_1^{n-1} \quad \text{and} \quad \alpha_2^{n+1} = A\alpha_2^n - B\alpha_2^{n-1}.$$

Adding the last two equation and dividing by 2 we arrive at

$$y_{n+1} = Ay_n - By_{n-1}.$$

It is clear that $y_0 = 1$ and $y_1 = \frac{A}{2}$. This shows that the sequence $\{y_n\}$ satisfies the recurrence relation identical with $\{x_n\}$. Further, if A is even then y_n is an integer sequence.

The above discussion proves the following theorem.

Theorem 2.2.5. *Let* $x_{n+1} = Ax_n - Bx_{n-1}, x_0 = 0, x_1 = 1$ *be a second order linear recurrence such that A and B are natural numbers and* $A^2 - 4B > 0$. *If* y_n *is defined as* $y_n = \frac{\alpha_1^n+\alpha_2^n}{2}$, *the sequence* $\{y_n\}_{n=1}^{\infty}$ *satisfy the recurrence relation* $y_{n+1} = Ay_n - By_{n-1}$. *Further,* y_n *is an integer sequence if A is even.*

We now suppose that A is even, hence $\{y_n\}_{n=1}^{\infty}$ is an integer sequence and choose $B = 1$ so that the greatest common divisor of x_n and y_n is 1 for each n. Let k and n be two natural numbers such that $n > 1$. Then

$$(x_k, x_{nk}) = (x_k, x_k y_{(n-1)k} + y_k x_{(n-1)k}) = (x_k, x_{(n-1)k}).$$

Iterating recursively, we arrive at

$$(x_k, x_{nk}) = (x_k, x_k) = x_k.$$

This proves

Theorem 2.2.6. *Let* $x_{n+1} = Ax_n - x_{n-1}, x_0 = 0, x_1 = 1$ *be a second order linear recurrence such that A is an even natural number and* $A^2 - 4$ *is positive. If m and n are natural numbers and m divides n then* x_m *divides* x_n.

We now look at the converse of this theorem. Assume that m and n are natural numbers such that x_m divides x_n. Then definitely, $m < n$ and by Euclid's division algorithm, there exist natural numbers k and r such that

$$n = mk + r, k \geq 1, 0 \leq r < m.$$

By Theorem 2.2.3

$$x_m = (x_m, x_n) = (x_m, x_{mk+r}) = (x_m, x_{mk}y_r + y_{mk}x_r).$$

Since m divides mk, by Theorem 2.2.6, x_m divides x_{mk} and hence the last equation yields

$$x_m = (x_m, y_{mk}x_r).$$

Further by Theorem 2.2.5, $(x_{mk}, y_{mk}) = 1$ and since x_m divides x_{mk}, using Theorem 2.2.6, we arrive at the conclusion that $(x_m, y_{mk}) = 1$. Thus the last equation results in

$$x_m = (x_m, x_r).$$

Since $r < m$, this is impossible unless $r = 0$. Thus $n = mk$ showing that m divides n.

The above discussion proves the following theorem.

Theorem 2.2.7. *Let $x_{n+1} = Ax_n - x_{n-1}, x_0 = 0, x_1 = 1$ be a second order linear recurrence such that A is an even natural number and $A^2 - 4$ is positive. If x_m divides x_n, then m divides n.*

Let m and n be two natural numbers such that $k = (m,n)$. Thus k divides both m and n. In view of Theorem 2.2.6, x_k divides both x_m and x_n and hence x_k divides (x_m, x_n). Further if $s > k$ and x_s divides x_m and x_n, then by Theorem 2.2.7, s divides both m and n and consequently, s divides k which is a contradiction. Hence if $k = (m,n)$, then k is the largest number such that x_k divides both x_m and x_n. The discussion of this paragraph may be summarized as follows.

Theorem 2.2.8. *Let $x_{n+1} = Ax_n - x_{n-1}, x_0 = 0, x_1 = 1$ be a second order linear recurrence such that A is an even natural number and $A^2 - 4$ is positive. If m and n are natural numbers then $(x_m, x_n) = x_{(m,n)}$.*

From the above theorems, we observe that the class of recurrent sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ behave like sequences of balancing and Lucas-balancing numbers. Therefore, we prefer to call the class of sequences determined by

$$x_{n+1} = Ax_n - x_{n-1}, x_0 = 0, x_1 = 1,$$

balancing-like sequences and

$$y_{n+1} = Ay_n - y_{n-1}, y_0 = 1, y_1 = A/2,$$

Lucas-balancing-like sequences. Observe that Lucas-balancing-like numbers are associated with balancing-like numbers in the same way as the Lucas-balancing numbers are associated with balancing numbers and thereby justifying the names.

Chapter 3

Gap Balancing Numbers-I

3.1 Introduction

As it is already known, a natural number n is a balancing number if

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+m$$

for some natural number m (see p. 17), while n is a cobalancing number if

$$1+2+\cdots+n=(n+1)+(n+2)+\cdots+s$$

for some s (see p. 18). Observe that while defining balancing numbers, we delete a number (and hence maintain a gap) from the list of first m natural numbers so that, the sum of numbers to the left of it is equal to the sum to the right. In the definition of cobalancing numbers, there is no such gap. As a generalization, we may also consider deleting two consecutive numbers from 1 through m, but not from either end, so that the sum to the left of these two numbers is equal to the sum to the right. If two such numbers exist for a given m, we call their sum a gap balancing number or more precisely, a 2-gap balancing number. The actual balancing center is the mean of these two deleted numbers which is not a whole number; hence we prefer to take their sum to define 2-gap balancing numbers.

3.2 2-gap balancing numbers

In this section, we define 2-gap balancing numbers and provide some examples.

Definition 3.2.1. *We call $2n+1$ a 2-gap balancing number (or g_2-balancing number) if*

$$1+2+\cdots+(n-1)=(n+2)+(n+3)+\cdots+(n+r)$$

for some natural number r. We call r the g_2-balancer corresponding to the g_2-balancing

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number $2n+1$.

Example 3.2.2. *Since* $1+2+3=6$, 9 *is a* g_2*-balancing number with* g_2*-balancer* 2. *Similarly, since* $1+2+\cdots+8=11+12+13$, 19 *is a* g_2*-balancing number with* g_2*-balancer* 4.

Remark 3.2.3. *The defining equation for* g_2*-balancing numbers suggests that if* $x=2n+1$ *is a* g_2*-balancing number then*

$$r=\frac{-x+\sqrt{2x^2+7}}{2}.$$

Thus, if x *is a* g_2*-balancing number then* $2x^2+7$ *is a perfect square. It is easy to see that* 9 *is the first* g_2*-balancing number. Since*

$$2\cdot 1^2+7=9=3^2 \quad \textit{and} \quad 2\cdot 3^2+7=25=5^2,$$

we accept 1 and 3 as g_2*-balancing numbers (though these numbers do not satisfy the definition of* g_2*-balancing numbers), just like Behera and Panda [13] accepted* 1 *as the first balancing number and Panda and Ray [72] accepted* 0 *as the first cobalancing number. After adding* 1 *and* 3 *to* g_2*-balancing numbers' list, we can claim that a natural number* x *is a* g_2*-balancing number if and only if* $2x^2+7$ *is a perfect square.*

3.3 Functions generating g_2-balancing numbers

In this section, we present some functions that generate g_2-balancing numbers. The following theorems explore these functions.

Theorem 3.3.1. *If* x *is a* g_2*-balancing number then* $f(x)=3x+2\sqrt{2x^2+7}$ *is also a* g_2*-balancing number. Furthermore,* $f(x)\equiv 1$ *or* -1 $(mod\ 4)$ *according as* $x\equiv 1$ *or* -1 $(mod\ 4)$.

Proof. The identity

$$2f^2(x)+7=\left(4x+3\sqrt{2x^2+7}\right)^2$$

together with Remark 3.2.3 proves that $f(x)$ is a g_2-balancing number. We observe that $2x^2+7\equiv 1 (\text{mod } 4)$ if $x\equiv \pm 1 (\text{mod } 4)$ and hence $\sqrt{2x^2+7}\equiv \pm 1 \ (\text{mod } 4)$. If $x\equiv 1 \ (\text{mod } 4)$, then

$$3x+2\sqrt{2x^2+7}\equiv 3\cdot 1\pm 2 \ (\text{mod } 4)\equiv 1 \ (\text{mod } 4)$$

and if $x\equiv -1 \ (\text{mod } 4)$, then

$$3x+2\sqrt{2x^2+7}\equiv 3\cdot(-1)\pm 2 \ (\text{mod } 4)\equiv -1 \ (\text{mod } 4).$$

□

In the next theorem we consider a function which transforms a g_2-balancing number congruent to $-1 \pmod 4$ to one congruent to $1 \pmod 4$.

Theorem 3.3.2. *If x is a g_2-balancing number and $x \equiv -1 \ (mod\, 4)$, then $g(x) = \frac{11x+6\sqrt{2x^2+7}}{7}$ is also a g_2-balancing number and $g(x) \equiv 1 \ (mod\, 4)$.*

Proof. We first show that if x is a g_2-balancing number and $x \equiv -1 \pmod 4$, then $g(x)$ is a natural number, i.e.,

$$11x+6\sqrt{2x^2+7} \equiv 0 \pmod 7.$$

Since

$$2x^2+7 \equiv 9x^2 \pmod 7,$$

it follows that

$$\sqrt{2x^2+7} \equiv \pm 3x \pmod 7.$$

This gives

$$11x+6\sqrt{2x^2+7} \equiv 11x \pm 18x \pmod 7,$$

implying that

$$11x+6\sqrt{2x^2+7} \equiv -7x \pmod 7 \quad \text{or} \quad 11x+6\sqrt{2x^2+7} \equiv 29x \pmod 7.$$

Thus,

$$11x+6\sqrt{2x^2+7} \equiv 0 \pmod 7$$

or

$$10x+6\sqrt{2x^2+7} \equiv 0 \pmod 7. \tag{3.3.1}$$

Observe that the substitution $x = 3$ (and hence $x \equiv -1 \pmod 4$) in (3.3.1) yields $4 \equiv 0 \pmod 7$ which is false. Thus, the only option left is $11x+6\sqrt{2x^2+7} \equiv 0 \pmod 7$, proving that $g(x)$ is a natural number if x is a g_2-balancing number and $x \equiv -1 \pmod 4$. Now, by virtue of Remark 3.2.3 and the identity

$$2g^2(x)+7 = \Big(\frac{12x+11\sqrt{2x^2+7}}{7}\Big)^2,$$

$g(x)$ is a g_2-balancing number. Finally, we have to show that $g(x) \equiv 1 \pmod 4$. Observe that

$$g(x) \equiv (-1)\cdot(11x+6\sqrt{2x^2+7}) \pmod 4$$

since $7^{-1} \equiv -1 \pmod 4$. Thus, if $x \equiv -1 \pmod 4$ then $g(x) \equiv -1 \pm 6 \equiv 1 \pmod 4$. This ends the proof. □

In contrast to the previous theorem, in the following theorem, we identify a function transforming a g_2-balancing number congruent to $1 \pmod 4$ to one congruent to

$-1 \pmod 4$.

Theorem 3.3.3. *If x is a g_2-balancing number and $x \equiv 1 \ (mod\, 4)$, then $h(x) = \frac{9x+4\sqrt{2x^2+7}}{7}$ is also a g_2-balancing number and $h(x) \equiv -1 \ (mod\, 4)$.*

Proof. First of all we claim that $h(x)$ is a natural number. For this, we have to show that if x is a g_2-balancing number and $x \equiv 1 \pmod 4$, then

$$9x+4\sqrt{2x^2+7} \equiv 0 \pmod 7.$$

Since

$$2x^2+7 \equiv 9x^2 \pmod 7,$$

it follows that

$$\sqrt{2x^2+7} \equiv \pm 3x \pmod 7.$$

Hence

$$9x+4\sqrt{2x^2+7} \equiv 9x \pm 12x \pmod 7,$$

which gives

$$9x+4\sqrt{2x^2+7} \equiv 21x \pmod 7 \quad \text{or} \quad 9x+4\sqrt{2x^2+7} \equiv -3x \pmod 7.$$

Thus, either

$$9x+4\sqrt{2x^2+7} \equiv 0 \pmod 7$$

or

$$12x+4\sqrt{2x^2+7} \equiv 0 \pmod 7. \tag{3.3.2}$$

But the substitution $x = 1$ (and hence $x \equiv 1 \pmod 4$) in (3.3.2) gives $3 \equiv 0 \pmod 7$ which is not true. Hence, the only option left is

$$9x+4\sqrt{2x^2+7} \equiv 0 \pmod 7,$$

proving that $h(x)$ is a natural number if x is a g_2-balancing number and $x \equiv 1 \pmod 4$. We next claim that $h(x)$ is a g_2-balancing number. This easily follows from the identity

$$2h^2(x)+7 = \Big(\frac{8x+9\sqrt{2x^2+7}}{7}\Big)^2$$

and Remark 3.2.3. Lastly, it remains to show that $h(x) \equiv -1 \pmod 4$. Since

$$h(x) \equiv (-1)\cdot(9x+4\sqrt{2x^2+7}) \pmod 4$$

and $x \equiv 1 \pmod 4$ it follows that $h(x) \equiv -1 \pmod 4$. This ends the proof. □

3.4 Listing all g_2-balancing numbers

In the last section, we presented some functions that generate g_2-balancing numbers from the given ones. Indeed, we have seen in Remark 3.2.3 that x is a g_2-balancing number if and only if $2x^2+7$ is a perfect square. In this section, we solve the Diophantine equation $2x^2+7=y^2$ and get the list of all g_2-balancing numbers. Of course, the method of solving $2x^2+7=y^2$ is not direct, rather we convert $2x^2+7=y^2$ to a Pell's equation of the form $8z^2+1=w^2$ and apply certain balancing numbers' treatment (see [13, p. 98]).

Let x be any g_2-balancing number so that $2x^2+7$ is a perfect square. As 7 is a prime, the congruence

$$9x^2 \equiv 2x^2+7 \pmod 7$$

yields

$$3x \equiv \pm\sqrt{2x^2+7} \pmod 7.$$

Since both x and $2x^2+7$ are odd, we also have

$$3x \equiv \pm\sqrt{2x^2+7} \pmod 2.$$

Thus $3x \pm \sqrt{2x^2+7}$ is congruent to 0 modulo 2 and modulo 7. As 2 and 7 are co-primes,

$$3x \pm \sqrt{2x^2+7} \equiv 0 (\mathrm{mod}\ 14),$$

yielding that either

$$\frac{3x+\sqrt{2x^2+7}}{14} \quad \text{or} \quad \frac{3x-\sqrt{2x^2+7}}{14}$$

is a natural number. Since

$$8 \cdot \left[\frac{3x \pm \sqrt{2x^2+7}}{14}\right]^2 + 1 = \left[\frac{3\sqrt{2x^2+7} \pm 2x}{7}\right]^2,$$

it follows that either

$$\frac{3x+\sqrt{2x^2+7}}{14} \quad \text{or} \quad \frac{3x-\sqrt{2x^2+7}}{14}$$

is a balancing number [13, p.98]. Letting

$$B = \frac{3x \pm \sqrt{2x^2+7}}{14},$$

we obtain

$$(14B-3x)^2 = 2x^2+7.$$

This leads to the quadratic equation

$$x^2 - 12Bx + 28B^2 - 1 = 0.$$

The solutions of this equation are

$$x = 6B \pm \sqrt{8B^2+1} = 6B \pm C$$

where C is the Lucas-balancing number associated with B. We further observe that

$$2\cdot(6B \pm C)^2 + 7 = (3C \pm 4B)^2.$$

Thus all the g_2-balancing numbers are of the form $6B \pm C$. Hence

$$\{6B_n - C_n,\ 6B_n + C_n : n = 1,2,\dots\}$$

is the exhaustive list of g_2-balancing numbers. We next show that for each natural number n,

$$6B_n - C_n < 6B_n + C_n < 6B_{n+1} - C_{n+1}.$$

The first part of this inequality is obvious. To prove the second part, we observe that in view of $B_{n-1} = 3B_n - C_n$, $B_{n+1} = 3B_n + C_n$ (see p. 18) and $B_n > 0$ if $n \geq 1$, it follows that $C_n < 3B_n$ for $n > 1$. Also, we know that for each natural number n, $B_{n-1} < B_n$. Hence,

$$\begin{aligned}6B_n + C_n &= 3B_n + 3B_n + C_n = 3B_n + B_{n+1}\\ &< 11B_n + B_{n+1} + 2(B_n - B_{n-1})\\ &= 2(6B_n - B_{n-1}) + B_{n+1} + B_n\\ &= 2B_{n+1} + B_{n+1} + 3B_{n+1} - C_{n+1} = 6B_{n+1} - C_{n+1}.\end{aligned}$$

We shall denote the n^{th} g_2-balancing number by x_n. Thus, the first g_2-balancing number is $x_1 = 6B_1 - C_1 = 6\cdot 1 - 3 = 3$, the second one is $x_2 = 6B_1 + C_1 = 9$, the third one $x_3 = 6B_2 - C_2 = 6\cdot 6 - 17 = 19$, the fourth one is $x_4 = 6B_2 + C_2 = 53$ and so on. In general

$$x_{2n-1} = 6B_n - C_n \quad \text{and} \quad x_{2n} = 6B_n + C_n,\ n = 1,2,\dots.$$

Further, we may write

$$x_0 = 6B_0 + C_0 = 6B_0 + \sqrt{8B_0^2+1} = 6\cdot 0 + \sqrt{8\cdot 0^2 + 1} = 1.$$

The above discussion proves the following theorem.

Theorem 3.4.1. *If x is a g_2-balancing number then $x = 6B_n - C_n$ or $x = 6B_n + C_n$ for some natural number n. In particular, if we denote the $n^{th}g_2$-balancing number by x_n, then $x_{2n-1} = 6B_n - C_n$ and $x_{2n} = 6B_n + C_n, n = 1,2,\dots$*

The next theorem classifies g_2-balancing numbers congruent to 1 and -1 modulo 4.

Theorem 3.4.2. *For $n = 1,2,\dots, x_{2n-1} \equiv -1\ (mod\ 4)$ and $x_{2n} \equiv 1\ (mod\ 4)$.*

To prove this theorem, we need the following lemma.

Lemma 3.4.3. *If n is even, then $6B_n \equiv 0 \ (mod\ 4)$ and $C_n \equiv 1 \ (mod\ 4)$; if n is odd, then $6B_n \equiv 2 \ (mod\ 4)$ and $C_n \equiv -1 \ (mod\ 4)$.*

Proof. We know that B_n is even or odd according as n is even or odd. Therefore, if n is even then $6B_n \equiv 0 \pmod 4$. Further, if n is odd then B_n is odd and $B_n \equiv \pm 1 \pmod 4$ implies $6B_n \equiv \pm 6 \equiv 2 \pmod 4$. Further $C_1 = 3 \equiv -1 \pmod 4$ and $C_2 = 17 \equiv 1 \pmod 4$. Assume that $C_{2n-1} \equiv -1 \pmod 4$ and $C_{2n} \equiv 1 \pmod 4$ for $n = 1, 2, \ldots, k$. Then

$$C_{2k+1} = 6C_{2k} - C_{2k-1} \equiv 6 \cdot 1 - (-1) \equiv -1 \pmod 4$$

and

$$C_{2k+2} = 6C_{2k+1} - C_{2k} \equiv 6 \cdot (-1) - 1 \equiv 1 \pmod 4. \qquad \square$$

Proof of Theorem 3.4.2. We infer from Lemma 3.4.3 that if n is even then $6B_n \equiv 0 \pmod 4$ and $C_n \equiv 1 \pmod 4$. Hence $6B_n + C_n \equiv 0 + 1 = 1 \pmod 4$ and $6B_n - C_n \equiv 0 - 1 = -1 \pmod 4$. Similarly, if n is odd $6B_n + C_n \equiv 2 + (-1) = 1 \pmod 4$ and $6B_n - C_n \equiv 2 + 1 \equiv -1 \pmod 4$. Thus, $x_{2n-1} \equiv -1 \pmod 4$ and $x_{2n} \equiv 1 \pmod 4$, $n = 1, 2, \ldots$ $\square$

3.5 Recurrence relations for g_2-balancing numbers

In the previous section, we have seen that the g_2-balancing numbers are given by

$$x_{2n-1} = 6B_n - C_n \ \text{ and } \ x_{2n} = 6B_n + C_n, \ n = 1, 2, \ldots$$

Since both balancing as well as Lucas-balancing numbers satisfy the recurrence relation $y_{n+1} = 6y_n - y_{n-1}$ (see p. 17), it follows that the g_2-balancing numbers satisfy the recurrence relation $x_{n+2} = 6x_n - x_{n-2}$; $n = 3, 4, \ldots$

In Section 3.3, we developed some non-linear functions for finding a specified type of g_2-balancing numbers from given ones. Here we prove that two of these functions are nothing but shift functions to next g_2-balancing numbers. In this context, we have the following theorems:

Theorem 3.5.1. *Let $g(x) = \frac{11x+6\sqrt{2x^2+7}}{7}$ and $h(x) = \frac{9x+4\sqrt{2x^2+7}}{7}$ be two arithmetic functions. Then $g(x_{2n-1}) = x_{2n}$ and $h(x_{2n}) = x_{2n+1}$.*

Proof. If $x = 6B_n \pm C_n$, then $2x^2 + 7 = (3C_n \pm 4B_n)^2$. Thus, if $x = x_{2n-1}$ then

$$g(x_{2n-1}) = \frac{11(6B_n - C_n) + 6\sqrt{2(6B_n - C_n)^2 + 7}}{7}$$

$$= \frac{11(6B_n - C_n) + 6(3C_n - 4B_n)}{7} = 6B_n + C_n = x_{2n}.$$

If $x = x_{2n}$, then

$$\begin{aligned}
h(x_{2n}) &= \frac{9(6B_n + C_n) + 4\sqrt{2(6B_n + C_n)^2 + 7}}{7} \\
&= \frac{9(6B_n + C_n) + 4(3C_n + 4B_n)}{7} \\
&= 10B_n + 3C_n = 3(3B_n + C_n) + B_n \\
&= 3B_{n+1} + (3B_{n+1} - C_{n+1}) \\
&= 6B_{n+1} - C_{n+1} = x_{2n+1}.
\end{aligned}$$

□

It is important to observe that

$$g(x) = \frac{11x + 6\sqrt{2x^2 + 7}}{7} \quad \text{and} \quad h(x) = \frac{9x + 4\sqrt{2x^2 + 7}}{7}$$

are strictly increasing functions for $x > 0$. So the functions are invertible. It is easy to see that

$$g^{-1}(y) = \frac{11y - 6\sqrt{2y^2 + 7}}{7} \quad \text{and} \quad h^{-1}(y) = \frac{9y - 4\sqrt{2y^2 + 7}}{7}.$$

Thus, we can definitely expect $g^{-1}(x_{2n}) = x_{2n-1}$ and $h^{-1}(x_{2n+1}) = x_{2n}$. The following corollary demonstrates this result.

Corollary 3.5.2. *Let* $\tilde{g}(x) = \frac{11x - 6\sqrt{2x^2+7}}{7}$ *and* $\tilde{h}(x) = \frac{9x - 4\sqrt{2x^2+7}}{7}$ *be two arithmetic functions. Then* $\tilde{g}(x_{2n}) = x_{2n-1}$ *and* $\tilde{h}(x_{2n+1}) = x_{2n}$.

Proof. It is known that if $x = 6B_n \pm C_n$, then $2x^2 + 7 = (3C_n \pm 4B_n)^2$. Thus if $x = x_{2n}$, then

$$\begin{aligned}
\tilde{g}(x_{2n}) &= \frac{11(6B_n + C_n) - 6\sqrt{2(6B_n + C_n)^2 + 7}}{7} \\
&= \frac{11(6B_n + C_n) - 6(3C_n + 4B_n)}{7} \\
&= 6B_n - C_n = x_{2n-1}.
\end{aligned}$$

Further, if $x = x_{2n+1}$, then

$$\begin{aligned}
\tilde{h}(x_{2n+1}) &= \frac{9(6B_{n+1} - C_{n+1}) - 4\sqrt{2(6B_{n+1} - C_{n+1})^2 + 7}}{7} \\
&= \frac{9(6B_{n+1} - C_{n+1}) - 4(3C_{n+1} - 4B_{n+1})}{7} \\
&= 10B_{n+1} - 3C_{n+1} = 3(3B_{n+1} - C_{n+1}) + B_{n+1}
\end{aligned}$$

$$= 3B_n + (3B_n + C_n)$$

$$= 6B_n + C_n = x_{2n}. \qquad \square$$

Theorem 3.5.1 gives functions shifting g_2-balancing numbers to the next ones, while Corollary 3.5.2 provide functions taking g_2-balancing numbers to previous ones. In the following theorem, we consider a function which transforms a g_2-balancing number by two steps.

Theorem 3.5.3. *Let $f(x) = 3x + 2\sqrt{2x^2+7}$ be an arithmetic function. Then $f(x_n) = x_{n+2}$.*

Proof. Here also we shall use the fact that if

$$x = 6B_n \pm C_n, \text{ then } 2x^2 + 7 = (3C_n \pm 4B_n)^2.$$

Now,

$$f(x_{2n-1}) = 3x_{2n-1} + 2\sqrt{2x_{2n-1}^2 + 7}$$

$$= 3(6B_n - C_n) + 2(3C_n - 4B_n)$$

$$= 10B_n + 3C_n.$$

In the proof of Theorem 3.5.1, it has been shown that $10B_n + 3C_n = x_{2n+1}$. Further,

$$f(x_{2n}) = 3x_{2n} + 2\sqrt{2x_{2n}^2 + 7}$$

$$= 3(6B_n + C_n) + 2(3C_n + 4B_n)$$

$$= 26B_n + 9C_n = 9(3B_n + C_n) - B_n$$

$$= 9B_{n+1} - B_n = 6B_{n+1} + 3B_{n+1} - B_n$$

$$= 6B_{n+1} + C_{n+1} = x_{2n+2}. \qquad \square$$

It is important to note that

$$f(x) = 3x + 2\sqrt{2x^2+7}$$

is strictly increasing for $x > 0$. So the inverse exists and it is easy to see that

$$f^{-1}(y) = 3y - 2\sqrt{2y^2+7}.$$

Thus, we can definitely expect $f^{-1}(x_{2n}) = x_{2n-2}$ and $f^{-1}(x_{2n+1}) = x_{2n-1}$. The following corollary ascertains these claims.

Corollary 3.5.4. *Let* $\tilde{f}(x) = 3x - 2\sqrt{2x^2+7}$ *be an arithmetic function. Then* $\tilde{f}(x_n) = x_{n-2}$.

The proof is similar to that of Theorem 3.5.3 and is omitted.

3.6 Binet form for g_2-balancing numbers

In Section 3.5, we obtained the recurrence relation $x_{n+2} = 6x_n - x_{n-2}$ for g_2-balancing numbers, which is linear, homogeneous and is of fourth order. Using this recurrence relation, we can find the Binet form for g_2-balancing numbers.

Putting $x_n = \alpha^n$ as a trial solution in $x_{n+2} = 6x_n - x_{n-2}$ we get the auxiliary equation $\alpha^4 - 6\alpha^2 + 1 = 0$. The solutions of this biquadratic equation are

$$\alpha_1 = 1+\sqrt{2}, \quad \alpha_2 = 1-\sqrt{2}, \quad \alpha_3 = -\alpha_1, \text{ and } \alpha_4 = -\alpha_2.$$

Hence, the general solution of $x_{n+2} = 6x_n - x_{n-2}$ is given by

$$x_n = A\alpha_1^n + B\alpha_2^n + C\alpha_3^n + D\alpha_4^n$$

and the initial conditions are $x_0 = 1$, $x_1 = 3$, $x_2 = 9$ and $x_3 = 19$. Since $\alpha_3 = -\alpha_1$ and $\alpha_4 = -\alpha_2$, it follows that

$$x_n = (A+(-1)^nC)\alpha_1^n + (B+(-1)^nD)\alpha_2^n.$$

Substitution of initial conditions yields

$$x_n = \begin{cases} \frac{\alpha_1^{n+2}-\alpha_2^{n+2}}{2\sqrt{2}} - \frac{\alpha_1^n+\alpha_2^n}{2} & \text{if } n \text{ is even,} \\ \frac{\alpha_1^{n+2}-\alpha_2^{n+2}}{2\sqrt{2}} - \frac{\alpha_1^n-\alpha_2^n}{\sqrt{2}} & \text{if } n \text{ is odd.} \end{cases}$$

Using this result and the Binet form we get,

$$\begin{aligned} x_{2k} &= \frac{\alpha_1^{2k+2}-\alpha_2^{2k+2}}{2\sqrt{2}} - \frac{\alpha_1^{2k}+\alpha_2^{2k}}{2} \\ &= 2\cdot\frac{\lambda_1^{k+1}-\lambda_2^{k+1}}{4\sqrt{2}} - \frac{\lambda_1^k+\lambda_2^k}{2} \\ &= 2B_{k+1} - C_k = 2(3B_k + C_k) - C_k \\ &= 6B_k + C_k, \end{aligned}$$

and

$$x_{2k-1} = \frac{\alpha_1^{2k+1}-\alpha_2^{2k+1}}{2\sqrt{2}} - \frac{\alpha_1^{2k-1}-\alpha_2^{2k-1}}{\sqrt{2}}$$

$$= \frac{(1+\sqrt{2})\alpha_1^{2k} - (1-\sqrt{2})\alpha_2^{2k}}{2\sqrt{2}} + \frac{(1-\sqrt{2})\alpha_1^{2k} - (1+\sqrt{2})\alpha_2^{2k}}{\sqrt{2}}$$

$$= \frac{\alpha_1^{2k} - \alpha_2^{2k}}{2\sqrt{2}} + \frac{\alpha_1^{2k} + \alpha_2^{2k}}{2} + \frac{\alpha_1^{2k} - \alpha_2^{2k}}{\sqrt{2}} - (\alpha_1^{2k} + \alpha_2^{2k})$$

$$= 2B_k + C_k + 4B_k - 2C_k$$

$$= 6B_k - C_k$$

which are already obtained in Section 3.4.

3.7 Functions transforming g_2-balancing numbers to balancing and related numbers

In this section, we present some functions of g_2-balancing numbers that generate balancing and related numbers.

In the following theorem, we identified two functions transforming g_2-balancing numbers to balancing numbers.

Theorem 3.7.1. *If x is an odd ordered g_2-balancing number then $F(x) = \frac{3x+\sqrt{2x^2+7}}{14}$ is a balancing number. Further, if x is an even ordered g_2-balancing number then $\tilde{F}(x) = \frac{3x-\sqrt{2x^2+7}}{14}$ is a balancing number. In particular, $F(x_{2n-1}) = \tilde{F}(x_{2n}) = B_n$.*

Proof. Since $x_{2n-1} = 6B_n - C_n$ and $x_{2n} = 6B_n + C_n$, we have

$$F(x_{2n-1}) = \frac{3(6B_n - C_n) + 3C_n - 4B_n}{14} = B_n$$

and

$$\tilde{F}(x_{2n}) = \frac{3(6B_n + C_n) - (3C_n + 4B_n)}{14} = B_n.$$ □

The next theorem relates functions of g_2-balancing numbers to Lucas-balancing numbers.

Theorem 3.7.2. *If x is an odd ordered g_2-balancing number then $G(x) = \frac{2x+3\sqrt{2x^2+7}}{7}$ is a Lucas-balancing number. Further, if x is an even ordered g_2-balancing number then $\tilde{G}(x) = \frac{-2x+3\sqrt{2x^2+7}}{7}$ is a Lucas-balancing number. In particular, $G(x_{2n-1}) = \tilde{G}(x_{2n}) = C_n$.*

Proof. Since $x_{2n-1} = 6B_n - C_n$ and $x_{2n} = 6B_n + C_n$, we have

$$G(x_{2n-1}) = \frac{2(6B_n - C_n) + 3(3C_n - 4B_n)}{7} = C_n,$$

and

$$\tilde{G}(x_{2n}) = \frac{-2(6B_n + C_n) + 3(3C_n + 4B_n)}{7} = C_n. \qquad \square$$

3.8 An application of g_2-balancing numbers to an almost Pythagorean equation

The association of balancing and cobalancing numbers with the solutions of Pythagorean and Pythagorean-like equations is well known (see [13, p. 104], [72, p. 1199] and [73, p.69]). In [37], Haggard developed certain links of solutions of the Pythagorean equation $x^2 + y^2 = z^2$ with the solutions of the almost Pythagorean equation $x^2 + y^2 = z^2 + 1$. In this section, we completely solve the Diophantine equation

$$x^2 + (x+4)^2 = y^2 + 1. \tag{3.8.1}$$

We observe that if (3.8.1) holds, then x must be odd. Setting $z = x + 2$, we convert this equation to

$$(z-2)^2 + (z+2)^2 = y^2 + 1,$$

which on simplification gives $2z^2 + 7 = y^2$, implying that z is a g_2-balancing number, so that $z = x_n$ for some n. Now we can list the solutions of the (3.8.1) as

$$x = x_n - 2,\ y = \sqrt{2x_n^2 + 7}, n = 1, 2, \ldots$$

Observe that almost Pythagorean equations corresponding to the g_2-balancing number 9 and 19 are respectively $7^2 + 11^2 = 13^2 + 1$ and $17^2 + 21^2 = 27^2 + 1$.

Chapter 4

Gap Balancing Numbers-II

4.1 Introduction

In the last chapter, as a generalization of balancing and cobalancing numbers, we introduced 2-gap balancing numbers by deleting two consecutive numbers from 1 through m, not from either end, so that the sum to the left of these two numbers is equal to the sum to the right. More precisely, we call $2n+1$ a 2-gap balancing number if

$$1+2+\cdots+(n-1)=(n+2)+\cdots+m$$

holds for some natural number m. We may consider deleting $k<m-2$ consecutive numbers from the list of first m natural numbers, not from either end, so that the sum to the left of these k deleted numbers is equal to the sum to the right, leading to the introduction of k-gap balancing numbers. In this chapter, our objective is to explore 3-gap, 4-gap and 5-gap balancing numbers completely. For arbitrary k, finding all k-gap balancing numbers is a tedious task as the process involves solving a parametrized generalized Pell's equation; however, in such cases, we managed to provide two classes of solutions for each k.

We now give two definitions of the k-gap balancing numbers corresponding to odd and even k.

Definition 4.1.1. *Let k be an odd natural number. We call a natural number n a k-gap balancing number (or g_k-balancing number) if*

$$1+2+\cdots+\left(n-\frac{k+1}{2}\right)=\left(n+\frac{k+1}{2}\right)+\left(n+\frac{k+3}{2}\right)+\cdots+(n+r)$$

for some natural number r, which we call a k-gap balancer (or a g_k-balancer) corresponding to n.

S. S. Rout and G. K.Panda, k-Gap Balancing Numbers, *Period. Math. Hungar.*, **70**(1), 109–121, 2015.

Definition 4.1.2. *Let k be even. If*

$$1+2+\cdots+\left(n-\frac{k}{2}\right)=\left(n+\frac{k}{2}+1\right)+\left(n+\frac{k}{2}+2\right)+\cdots+(n+r)$$

for some natural numbers n and r then we call $2n+1$ *a k-gap balancing number (or* g_k*-balancing number) and r a k-gap balancer (or a* g_k*-balancer) corresponding to this k-gap balancing number.*

For some natural number m, if after deleting k consecutive numbers from $1,2,\ldots,m$, not from either end, the sum of numbers to the left of these deleted numbers is equal to the sum to the right, then the median of these k deleted numbers is the center point of balance. However, when k is even, this median is a fractional number, equal to the average of the two middle deleted numbers. Hence, to keep the k-gap balancing numbers integral, we prefer to define these numbers equal to the sum of two middle terms of the deleted numbers.

It is known from Theorem 3.4.1 that the g_2-balancing numbers partition in exactly two classes. As we will see, the g_k-balancing numbers for $k>2$ also partition into two or more classes. Throughout this chapter, whenever necessary, we will denote the $n^{\text{th}}g_k$-balancing number of Class-I by $U_k(n)$ and that of Class-II by $V_k(n)$. In some cases a third class can also emerges and the n^{th} number of this class will be denoted by $W_k(n)$.

4.2 3-gap balancing numbers

In view of Definition (4.1.1), a natural number x is a 3-gap balancing number (or a g_3-balancing number) if

$$1+2+\cdots+(x-2)=(x+2)+(x+3)+\cdots+(x+r) \tag{4.2.1}$$

for some natural number r, which is the 3-gap balancer (or a g_3-balancer) corresponding to x.

Example 4.2.1. *Since* $1+2+3+4+5+6=10+11$, 8 *is a* g_3*-balancing number with* g_3*-balancer* 3*. Similarly, since* $1+2+\cdots+11=15+16+17+18$, 13 *is a* g_3*-balancing number with* g_3*-balancer* 5*.*

If x is a g_3-balancing number, then in view of (4.2.1), the corresponding g_3-balancer is given by

$$r=\frac{-(2x+1)+\sqrt{8x^2+17}}{2}.$$

Thus, if x is a g_3-balancing number then $8x^2+17$ is a perfect square. Since

$$8\cdot 1^2+17=5^2 \quad \text{and} \quad 8\cdot 2^2+17=7^2,$$

we accept 1 and 2 as g_3-balancing numbers (though these numbers do not satisfy the defining equation for g_3-balancing numbers). After including 1 and 2 in the g_3-balancing numbers' list, we can claim that a natural number x is a g_3-balancing number if and only if $8x^2+17$ is a perfect square.

4.2.1 Computation of g_3-balancing numbers

From the above discussion, we notice that x is a g_3-balancing number if and only if $8x^2+17$ is a perfect square. Thus, the computation of g_3-balancing numbers reduces to solving the Diophantine equation $8x^2+17=y^2$, which is equivalent to the generalized Pell's equation

$$y^2-8x^2=17. \tag{4.2.2}$$

In this section, applying modular arithmetic, we will explore two classes of g_3-balancing numbers.

Let x be any g_3-balancing number so that $8x^2+17$ is a perfect square. Since 17 is a prime, the congruence

$$25x^2 \equiv 8x^2+17 \pmod{17}$$

gives

$$5x \equiv \pm\sqrt{8x^2+17} \pmod{17},$$

or, equivalently,

$$5x \pm \sqrt{8x^2+17} \equiv 0 \pmod{17}.$$

Thus, either

$$\frac{5x+\sqrt{8x^2+17}}{17} \quad \text{or} \quad \frac{5x-\sqrt{8x^2+17}}{17}$$

is a natural number. Since

$$8\left[\frac{5x\pm\sqrt{8x^2+17}}{17}\right]^2+1=\left[\frac{5\sqrt{8x^2+17}\pm 8x}{17}\right]^2,$$

by virtue of [13, p.98] it follows that either

$$\frac{5x+\sqrt{8x^2+17}}{17} \quad \text{or} \quad \frac{5x-\sqrt{8x^2+17}}{17}$$

is a balancing number. Letting

$$B=\frac{5x\pm\sqrt{8x^2+17}}{17},$$

in either case we obtain

$$(17B-5x)^2=8x^2+17,$$

which leads to the quadratic equation

$$x^2 - 10xB + 17B^2 - 1 = 0.$$

The solutions of this equation are $x = 5B \pm C$, where C is the Lucas-balancing number associated with B, i.e., $C^2 = 8B^2 + 1$. We further observe that $8 \cdot (5B \pm C)^2 + 17 = (8B \pm 5C)^2$. Thus the numbers of the form $5B \pm C$ are g_3-balancing numbers and the set

$$\{5B_n + C_n, 5B_{n+1} - C_{n+1} : n = 0, 1, \ldots\} \tag{4.2.3}$$

provides two classes of g_3-balancing numbers. Now a natural question is whether the set in (4.2.3) exhausts all the g_3-balancing numbers. The answer is affirmative and is verified in the following section.

4.2.2 Solutions of $8x^2 + 17 = y^2$ as a generalized Pell's equation

The best way to find all the g_3-balancing numbers is to solve the generalized Pell's equation $y^2 - 8x^2 = 17$. The fundamental solution of the equation $y^2 - 8x^2 = 1$ is $3 + \sqrt{8}$. By the Theorem 1.2.1 (see p. 17), the bounds for the fundamental solutions of (4.2.2) are given by

$$|y| \le \sqrt{34} < 6, \;\; 0 \le x \le \sqrt{17/8} < 2.$$

We search for integers x in the interval $[0, 2)$ such that $8x^2 + 17$ is a perfect square. This happens for $x = 1$ and then, of course, $y = \pm 5$. Also the ratio

$$\frac{5 + \sqrt{8}}{-5 + \sqrt{8}} = -\frac{33}{17} - \frac{10}{17}\sqrt{8} \notin \mathbb{Z}[\sqrt{2}]^{\times},$$

a clear indication that $5 + \sqrt{8}$ and $-5 + \sqrt{8}$ are two fundamental solutions of (4.2.2). Since we are interested in non-negative values of x and y, we need to find out the least non-negative member of the class corresponding to $-5 + \sqrt{8}$, and it is easy to see that this member is $7 + 2\sqrt{8}$. Hence the two classes of solutions are given by

$$y_n + x_n\sqrt{8} = (5 + \sqrt{8})(3 + \sqrt{8})^n \tag{4.2.4}$$

and

$$y_n' + x_n'\sqrt{8} = (7 + 2\sqrt{8})(3 + \sqrt{8})^n, \tag{4.2.5}$$

$n = 0, 1, \ldots$ Solving (4.2.4) and (4.2.5) for x_n and x_n', we get the Binet forms for x_n and x_n' as

$$x_n = \frac{(5 + \sqrt{8})(3 + \sqrt{8})^n - (5 - \sqrt{8})(3 - \sqrt{8})^n}{2\sqrt{8}}$$

and

$$x_n' = \frac{(7 + 2\sqrt{8})(3 + \sqrt{8})^n - (7 - 2\sqrt{8})(3 - \sqrt{8})^n}{2\sqrt{8}}.$$

Thus,

$$x_n = 5 \cdot \frac{(3+\sqrt{8})^n - (3-\sqrt{8})^n}{2\sqrt{8}} + \frac{(3+\sqrt{8})^n + (3-\sqrt{8})^n}{2}$$

$$= 5B_n + C_n.$$

Similarly,

$$x_n^{'} = \frac{(3+\sqrt{8})^n - (3-\sqrt{8})^n}{2\sqrt{8}} + 2 \cdot \frac{(3+\sqrt{8})^{n+1} - (3-\sqrt{8})^{n+1}}{2\sqrt{8}}$$

$$= 2B_{n+1} + B_n = 2B_{n+1} + 3B_{n+1} - C_{n+1} = 5B_{n+1} - C_{n+1}.$$

The above discussion confirms that the set in (4.2.3) is the exhaustive list of g_3-balancing numbers and hence we have the following theorem:

Theorem 4.2.2. *All g_3-balancing numbers of Class I and II are of the form $x = 5B_n \pm C_n$. In particular, $U_3(n) = 5B_n + C_n$ and $V_3(n) = 5B_{n+1} - C_{n+1}$ for $n = 0, 1, \ldots$*

4.3 4-gap balancing numbers

By virtue of Definition (4.1.2), if

$$1 + 2 + \cdots + (n-2) = (n+3) + (n+4) + \cdots + (n+r) \tag{4.3.1}$$

for some natural numbers n and r, then $2n+1$ is called a 4-gap balancing number (or g_4-balancing number) and r a 4-gap balancer (or a g_4-balancer) corresponding to the 4-gap balancing number $2n+1$.

Example 4.3.1. *Since* $1 + 2 + \cdots + 9 = 14 + 15 + 16$, 23 *is a g_4-balancing number with g_4-balancer* 5. *Similarly, since* $1 + 2 + \cdots + 14 = 19 + 20 + \cdots + 23$, 33 *is a g_4-balancing number with g_4-balancer* 7.

Equation (4.3.1) confirms that if $x = 2n+1$ is a g_4-balancing number then the corresponding g_4-balancer is given by

$$r = \frac{-x + \sqrt{2x^2 + 31}}{2}.$$

Thus, if x is a g_4-balancing number then $2x^2 + 31$ is a perfect square. Since $2 \cdot 3^2 + 31 = 7^2$ and $2 \cdot 5^2 + 31 = 9^2$, as usual, we accept 3 and 5 as g_4-balancing numbers. Once we add 3 and 5 to g_4-balancing numbers' list, we can claim that a natural number x is a g_4-balancing number if and only if $2x^2 + 31$ is a perfect square.

4.3.1 Computation of g_4-balancing numbers

Here, using modular arithmetic, we will solve the Diophantine equation $2x^2+31=y^2$ and explore all the g_4-balancing numbers.

If x is a g_4-balancing number then $2x^2+31$ is a perfect square. Since 31 is a prime, the congruence

$$49x^2 \equiv 9(2x^2+31) \pmod{31}$$

gives

$$7x \equiv \pm 3\sqrt{2x^2+31} \pmod{31}.$$

Both x and $2x^2+31$ are odd, so

$$7x \equiv \pm 3\sqrt{2x^2+31} \pmod{2}$$

which implies $7x \pm 3\sqrt{2x^2+31}$ is congruent to 0 modulo 2 and modulo 31. As 2 and 31 are co-primes

$$7x \pm 3\sqrt{2x^2+31} \equiv 0 \pmod{62}$$

yielding that either

$$\frac{7x+3\sqrt{2x^2+31}}{62} \quad \text{or} \quad \frac{7x-3\sqrt{2x^2+31}}{62}$$

is a natural number. Since

$$8\left[\frac{7x\pm 3\sqrt{2x^2+31}}{62}\right]^2+1=\left[\frac{7\sqrt{2x^2+31}\pm 6x}{62}\right]^2,$$

by virtue of [13, p.89], it follows that either

$$\frac{7x+3\sqrt{2x^2+31}}{62} \quad \text{or} \quad \frac{7x-3\sqrt{2x^2+31}}{62}$$

is a balancing number. Letting

$$B=\frac{7x\pm 3\sqrt{2x^2+31}}{62},$$

we obtain

$$(62B-7x)^2=9(2x^2+31),$$

which is equivalent to

$$31x^2-868xB+3844B^2-279=0.$$

The solutions of this equation are $x=14B\pm 3C$. We further observe that $2\cdot(14B\pm 3C)^2+31=(12B\pm 7C)^2$. Thus, numbers of the form $14B\pm 3C$ are g_4-balancing numbers. Hence the set

$$\{14B_n+3C_n, 14B_{n+1}-3C_{n+1} : n=0,1,\dots\} \tag{4.3.2}$$

gives two classes of g_4-balancing numbers and this is the complete list of g_4-balancing numbers which is verified in the section below.

4.3.2 Solutions of $2x^2+31=y^2$ as a generalized Pell's equation

The Diophantine equation $2x^2+31=y^2$ reduces to the generalized Pell's equation

$$y^2-2x^2=31. \tag{4.3.3}$$

The fundamental solution of the equation $y^2-2x^2=1$ is $3+2\sqrt{2}$. From Equation (1.2.4) of Theorem 1.2.1 (see p. 17), the bounds for the fundamental solutions of the (4.3.3) are given by

$$0<|y|\leq\sqrt{62}<8,\ \ 0\leq x\leq\sqrt{31/2}<4.$$

The only integers x in the interval $[0,4)$ such that $2x^2+31$ is a perfect square is 3 and substitution in (4.3.3) gives $y=\pm 7$. It is easy to see that $7+3\sqrt{2}$ and $-7+3\sqrt{2}$ are not in the same class and hence these are two fundamental solutions of (4.3.3). Further, the least non-negative member of the class corresponding to $-7+3\sqrt{2}$ is $9+5\sqrt{2}$. Therefore two classes of solutions of (4.3.3) are given by

$$y_n+x_n\sqrt{2}=(7+3\sqrt{2})(3+2\sqrt{2})^n, \tag{4.3.4}$$

and

$$y_n^{'}+x_n^{'}\sqrt{2}=(9+5\sqrt{2})(3+2\sqrt{2})^n,\quad n=0,1,\ldots \tag{4.3.5}$$

Solving (4.3.4) and (4.3.5) for x_n and $x_n^{'}$, we get the Binet forms for x_n and $x_n^{'}$ as

$$x_n=\frac{(7+3\sqrt{2})(3+2\sqrt{2})^n-(7-3\sqrt{2})(3-2\sqrt{2})^n}{2\sqrt{2}}$$

and

$$x_n^{'}=\frac{(9+5\sqrt{2})(3+2\sqrt{2})^n-(9-5\sqrt{2})(3-2\sqrt{2})^n}{2\sqrt{2}}.$$

Thus

$$\begin{aligned}x_n&=7\cdot\frac{(3+2\sqrt{2})^n-(3-2\sqrt{2})^n}{2\sqrt{2}}+3\cdot\frac{(3+2\sqrt{2})^n+(3-2\sqrt{2})^n}{2}\\&=14B_n+3C_n.\end{aligned}$$

Similarly,

$$\begin{aligned}x_n^{'}&=3\cdot\frac{(3+\sqrt{8})^{n+1}-(3-\sqrt{8})^{n+1}}{2\sqrt{2}}-\frac{(3+\sqrt{8})^n+(3-\sqrt{8})^n}{2}\\&=6B_{n+1}-C_n=6B_{n+1}-(B_{n+1}-3B_n)=5B_{n+1}+3B_n\\&=5B_{n+1}+3(3B_{n+1}-C_{n+1})=14B_{n+1}-3C_{n+1}.\end{aligned}$$

The above discussion confirms that the set in (4.3.2) is the exhaustive list of g_4-balancing numbers and hence we have the following theorem:

Theorem 4.3.2. *The g_4-balancing numbers partition into two classes and are of the form $14B_n \pm 3C_n$. In particular,*

$$U_4(n) = 14B_n + 3C_n \text{ and } V_4(n) = 14B_{n+1} - 3C_{n+1} \text{ for } n = 0, 1, \ldots$$

4.4 5-gap balancing numbers

By Definition (4.1.1), we call a natural number x a 5-gap balancing numbers (or g_5-balancing number) if

$$1 + 2 + \cdots + (x-3) = (x+3) + (x+4) + \cdots + (x+r) \tag{4.4.1}$$

for some natural number r, which we call a 5-gap balancer (or a g_5-balancer) corresponding to x.

Example 4.4.1. *Since* $1+2+3+4 = 10$, 7 *is a g_5-balancing number with g_5-balancer* 3. *Similarly, since* $1+2+\cdots+12 = 18+19+20+21$, 15 *is a g_5-balancing number with g_5-balancer* 6.

Equation (4.4.1) when solved for r gives

$$r = \frac{-(2x+1) + \sqrt{8x^2+49}}{2}.$$

Thus, if x is a g_5-balancing number then $8x^2+49$ is a perfect square. Since $8 \cdot 2^2 + 49 = 9^2$ and $8 \cdot 3^2 + 49 = 11^2$, we accept 2 and 3 as g_5-balancing numbers. Once we accept 2 and 3 as g_5-balancing numbers, we can claim that a natural number x is a g_5-balancing number if and only if $8x^2+49$ is a perfect square.

4.4.1 Computation of g_5-balancing numbers

From the above observation we have noticed that x is a g_5-balancing number if and only if $8x^2+49$ is a perfect square. Here we will solve the Diophantine equation $8x^2 + 49 = z^2$ and provide the two classes of g_5-balancing numbers.

Let x be any g_5-balancing number so that $8x^2+49$ is a perfect square, say $8x^2+49 = y^2$. We distinguish two cases.

Case-I: $7 \mid x$

If $7 \mid x$ then $7 \mid y$. Letting $x = 7u$ and $y = 7v$, the equation $8x^2 + 49 = y^2$ reduces to $8u^2 + 1 = v^2$ implying that u is a balancing number (see p. 17). Hence one class of solutions can be written as $x = 7B_n$, $n = 1, 2, \ldots$.

Case-II: $7 \nmid x$

In this case, $49 \nmid 81x^2$, $49 \nmid 8x^2+49$ and the congruence

$$81x^2 \equiv 4(8x^2+49) \pmod{49}$$

gives

$$9x \equiv \pm 2\sqrt{8x^2+49} \pmod{49}.$$

Thus,

$$9x \pm 2\sqrt{8x^2+49} \equiv 0 \pmod{49}$$

which confirms that either

$$\frac{9x+2\sqrt{8x^2+49}}{49} \quad \text{or} \quad \frac{9x-2\sqrt{8x^2+49}}{49}$$

is a natural number. Since

$$8\left[\frac{9x \pm 2\sqrt{8x^2+49}}{49}\right]^2 + 1 = \left[\frac{9\sqrt{8x^2+49} \pm 16x}{49}\right]^2,$$

it follows that either

$$\frac{9x+2\sqrt{8x^2+49}}{49} \quad \text{or} \quad \frac{9x-2\sqrt{8x^2+49}}{49}$$

is a balancing number. Letting

$$B = \frac{9x \pm 2\sqrt{8x^2+49}}{49},$$

we obtain

$$(49B-9x)^2 = 4(8x^2+49),$$

which is equivalent to

$$x^2 - 18Bx + 49B^2 - 4 = 0. \tag{4.4.2}$$

The solutions of this equation are $x = 9B \pm 2C$. We further observe that $8(9B \pm 2C)^2 + 49 = (16B \pm 9C)^2$. Hence the set

$$\{9B_n + 2C_n, 9B_{n+1} - 2C_{n+1} : n = 0, 1, \dots\} \tag{4.4.3}$$

gives two classes of g_5-balancing numbers.

4.4.2 Solutions of $8x^2 + 49 = y^2$ as a generalized Pell's equation

We can write the equation $8x^2 + 49 = y^2$ as the generalized Pell's equation

$$y^2 - 8x^2 = 49. \tag{4.4.4}$$

The fundamental solution of the equation $y^2-8x^2=1$ is $3+\sqrt{8}$. By the Theorem 1.2.1 (see p. 17), the bounds for the fundamental solutions of the (4.4.4) are given by

$$0<|y|\leq\sqrt{98}<10,\ \ 0\leq x\leq 7/\sqrt{8}<3.$$

The integers x in the interval $[0,3)$ for which $8x^2+49$ is a perfect square are $x=0,2$ and the corresponding values of y are $\pm 7,\pm 9$ respectively. Thus, the possible fundamental solutions of $y^2-8x^2=49$ are $\pm 7+0\sqrt{8}$, $\pm 9+2\sqrt{8}$. Since the ratio of 7 and -7 lies in $\mathbb{Z}[\sqrt{2}]^{\times}$, there are three fundamental solutions of (4.4.4) and they are $7, 9+2\sqrt{8}, -9+2\sqrt{8}$ and the smallest non-negative fundamental solution corresponding to $-9+2\sqrt{8}$ is $11+3\sqrt{8}$. Hence three classes of solutions are given by

$$y_n+x_n\sqrt{8}=(7+0\sqrt{8})(3+\sqrt{8})^n, \tag{4.4.5}$$

$$y_n^{'}+x_n^{'}\sqrt{8}=(9+2\sqrt{8})(3+\sqrt{8})^n, \tag{4.4.6}$$

and

$$y_n^{''}+x_n^{''}\sqrt{8}=(11+3\sqrt{8})(3+\sqrt{8})^n,\quad n=0,1,\ldots \tag{4.4.7}$$

Solving (4.4.5), (4.4.6) and (4.4.7) for x_n, $x_n^{'}$ and $x_n^{''}$ we get the Binet forms for x_n, $x_n^{'}$ and $x_n^{''}$ as

$$x_n=7\cdot\frac{(3+\sqrt{8})^n-(3-\sqrt{8})^n}{2\sqrt{8}}=7B_n,$$

$$x_n^{'}=\frac{(9+2\sqrt{8})(3+\sqrt{8})^n-(9-2\sqrt{8})(3-\sqrt{8})^n}{2\sqrt{8}}$$

and

$$x_n^{''}=\frac{(11+3\sqrt{8})(3+\sqrt{8})^n-(11-3\sqrt{8})(3-\sqrt{8})^n}{2\sqrt{8}}.$$

But

$$x_n^{'}=9\cdot\frac{(3+\sqrt{8})^n-(3-\sqrt{8})^n}{2\sqrt{8}}+2\cdot\frac{(3+\sqrt{8})^n+(3-\sqrt{8})^n}{2}$$

$$=9B_n+2C_n,$$

and

$$x_n^{''}=3\cdot\frac{(3+\sqrt{8})^{n+1}-(3-\sqrt{8})^{n+1}}{2\sqrt{8}}+2\cdot\frac{(3+\sqrt{8})^n-(3-\sqrt{8})^n}{2\sqrt{8}}$$

$$=3B_{n+1}+2B_n=3B_{n+1}+2(3B_{n+1}-C_{n+1})=9B_{n+1}-2C_{n+1}.$$

Thus, the set

$$\{9B_n+2C_n, 9B_{n+1}-2C_{n+1}, 7B_{n+1}: n=0,1,\ldots\}$$

provides the complete list of g_5-balancing numbers.

Theorem 4.4.2. *The g_5-balancing numbers partition into three classes. These numbers in Class I and II are of the form $9B_n \pm 2C_n$ and those in Class III are of the form $7B_n$. In particular,*

$$U_5(n) = 9B_n + 2C_n, \ V_5(n) = 9B_{n+1} - 2C_{n+1} \ \textit{and} \ W_5(n) = 7B_{n+1} \ n = 0, 1, \ldots$$

4.5 k-gap balancing numbers

In this section, we try to explore certain classes of k-gap balancing numbers for an arbitrary positive integer k. Let k be odd. By virtue of Definition 4.1.1, the k-gap balancer r corresponding to a k-gap balancing number n is given by

$$r = \frac{-(2n+1) + \sqrt{8n^2 + 2k^2 - 1}}{2}.$$

Thus, if k is odd and n is a g_k-balancing number then $8n^2 + 2k^2 - 1$ is a perfect square. Since

$$8\left(\frac{k-1}{2}\right)^2 + 2k^2 - 1 = (2k-1)^2 \quad \text{and} \quad 8\left(\frac{k+1}{2}\right)^2 + 2k^2 - 1 = (2k+1)^2,$$

we accept $\frac{k-1}{2}$ and $\frac{k+1}{2}$ as the first and second g_k-balancing numbers and thereafter we can claim that a natural number n is a g_k-balancing number if and only if $8n^2 + 2k^2 - 1$ is a perfect square.

Also, if k is even, by virtue of Definition 4.1.2, the k-gap balancer r corresponding to a k-gap balancing number $x = 2n + 1$ is given by

$$r = \frac{-x + \sqrt{2x^2 + 2k^2 - 1}}{2}.$$

Consequently, for even k, if x is a g_k-balancing number then $2x^2 + 2k^2 - 1$ is a perfect square. Since

$$2 \cdot (k-1)^2 + 2k^2 - 1 = (2k-1)^2 \quad \text{and} \quad 2 \cdot (k+1)^2 + 2k^2 - 1 = (2k+1)^2,$$

we accept $k - 1$ and $k + 1$ as g_k-balancing numbers and then we can claim that a natural number x is a g_k-balancing number if and only if $2x^2 + 2k^2 - 1$ is a perfect square.

The above discussion proves the following theorem:

Theorem 4.5.1. *The requirement for a natural number x to be a g_k-balancing number is that $8x^2 + 2k^2 - 1$ be a perfect square if k is odd and $2x^2 + 2k^2 - 1$ be a perfect square if k is even.*

4.5.1 Computation of g_k-balancing numbers

We will list the g_k-balancing numbers by assuming k even and k odd separately.

Case-I: Let k be even and x be any g_k-balancing number. Then $2x^2+2k^2-1$ is a perfect square. From the congruence

$$(k-1)^2(2x^2+2k^2-1) \equiv (2k-1)^2x^2 \pmod{2k^2-1},$$

it follows that

$$(k-1)\sqrt{2x^2+2k^2-1} \equiv \pm(2k-1)x \pmod{2k^2-1}.$$

Since both x and $2x^2+2k^2-1$ are odd, we also have

$$(2k-1)x \pm (k-1)\sqrt{2x^2+2k^2-1} \equiv 0 \pmod{2}.$$

Thus,

$$(2k-1)x \pm (k-1)\sqrt{2x^2+2k^2-1}$$

is congruent to 0 modulo 2 and $2k^2-1$. As 2 and $2k^2-1$ are coprimes, it follows that either

$$\frac{(2k-1)x+(k-1)\sqrt{2x^2+2k^2-1}}{2(2k^2-1)}$$

or

$$\frac{(2k-1)x-(k-1)\sqrt{2x^2+2k^2-1}}{2(2k^2-1)}$$

is a natural number. Since

$$8\left[\frac{(2k-1)x \pm (k-1)\sqrt{2x^2+2k^2-1}}{2(2k^2-1)}\right]^2+1 = \left[\frac{(2k-1)\sqrt{2x^2+2k^2-1} \pm 2(k-1)x}{2k^2-1}\right]^2,$$

by virtue of [13, p.98], it follows that either

$$\frac{(2k-1)x+(k-1)\sqrt{2x^2+2k^2-1}}{2(2k^2-1)}$$

or

$$\frac{(2k-1)x-(k-1)\sqrt{2x^2+2k^2-1}}{2(2k^2-1)}$$

is a balancing number. Letting

$$B = \frac{(2k-1)x \pm (k-1)\sqrt{2x^2+2k^2-1}}{2(2k^2-1)}$$

we arrive at the quadratic equation

$$[(2k-1)^2-2(k-1)^2]x^2-4(2k-1)(2k^2-1)Bx+4B^2(2k^2-1)^2-(k-1)^2(2k^2-1)=0.$$

The solutions are $x = (4k-2)B \pm (k-1)C$. Thus two classes of g_k-balancing numbers are given by

$$\{(4k-2)B_n + (k-1)C_n, (4k-2)B_{n+1} - (k-1)C_{n+1} : n = 0, 1, \dots\}.$$

Case-II: Let k be odd and x is any g_k-balancing number. Then $8x^2 + 2k^2 - 1$ is a perfect square. From the congruence

$$\left(\frac{k-1}{2}\right)^2 (8x^2 + 2k^2 - 1) \equiv (2k-1)^2 x^2 (\text{mod } 2k^2 - 1),$$

it follows that

$$\frac{k-1}{2}\sqrt{8x^2 + 2k^2 - 1} \equiv \pm(2k-1)x \ (\text{mod } 2k^2 - 1).$$

Thus,

$$(2k-1)x \pm \left(\frac{k-1}{2}\right)\sqrt{8x^2 + 2k^2 - 1} \equiv 0 \ (\text{mod } 2k^2 - 1)$$

and hence either

$$\frac{2(2k-1)x \pm (k-1)\sqrt{8x^2 + 2k^2 - 1}}{2(2k^2 - 1)}$$

or

$$\frac{2(2k-1)x \pm (k-1)\sqrt{8x^2 + 2k^2 - 1}}{2(2k^2 - 1)}$$

is a natural number. Since

$$8\left[\frac{2(2k-1)x \pm (k-1)\sqrt{8x^2 + 2k^2 - 1}}{2(2k^2 - 1)}\right]^2 + 1 = \left[\frac{(2k-1)\sqrt{8x^2 + 2k^2 - 1} \pm 4(k-1)x}{2k^2 - 1}\right]^2,$$

by virtue of [13] it follows that either

$$\frac{2(2k-1)x \pm (k-1)\sqrt{8x^2 + 2k^2 - 1}}{2(2k^2 - 1)}$$

or

$$\frac{2(2k-1)x \pm (k-1)\sqrt{8x^2 + 2k^2 - 1}}{2(2k^2 - 1)}$$

is a balancing number. Letting

$$B = \frac{2(2k-1)x \pm (k-1)\sqrt{8x^2 + 2k^2 - 1}}{2(2k^2 - 1)}$$

we arrive at the quadratic equation

$$[4(2k-1)^2 - 8(k-1)^2]x^2 - 8(2k-1)(2k^2-1)Bx + 4B^2(2k^2-1)^2 - (k-1)^2(2k^2-1) = 0$$

and the solutions are $x = (2k-1)B \pm \left(\frac{k-1}{2}\right)C$. Thus, two classes of g_k-balancing numbers are given by

$$\{(2k-1)B_n + \left(\frac{k-1}{2}\right)C_n, (2k-1)B_{n+1} - \left(\frac{k-1}{2}\right)C_{n+1} : n = 0, 1, \dots\}.$$

The above discussion proves the following theorem.

Theorem 4.5.2. *It is always possible to explore two classes of g_k-balancing numbers for any arbitrary positive integer $k \geq 2$. For $n = 1,2,\dots$ the n^{th} members of Class-I and Class-II are given by*

$$U_k(n) = \begin{cases} (4k-2)B_n + (k-1)C_n & \text{if } k \text{ is even,} \\ (2k-1)B_n + \left(\frac{k-1}{2}\right)C_n & \text{if } k \text{ is odd} \end{cases}$$

and

$$V_k(n) = \begin{cases} (4k-2)B_{n+1} - (k-1)C_{n+1} & \text{if } k \text{ is even,} \\ (2k-1)B_{n+1} - \left(\frac{k-1}{2}\right)C_{n+1} & \text{if } k \text{ is odd,} \end{cases}$$

respectively.

4.5.2 Some more classes of g_k-balancing numbers

One can explore three or more classes of g_k-balancing numbers for certain values of k. This class corresponds to those k for which $2k^2 - 1$ is a perfect square, and more precisely, corresponds to cases when k is an odd Pell number. If k is odd, as we know, the requirement for a natural number x to be a g_k-balancing number is that $8x^2 + 2k^2 - 1$ be a perfect square, i.e.,

$$8x^2 + 2k^2 - 1 = y^2 \tag{4.5.1}$$

for some natural number y. Letting $2k^2 - 1 = l^2$, Equation (4.5.1) reduces to generalized Pell's equation

$$8x^2 + l^2 = y^2.$$

We calculate a particular class of solutions of this equation corresponding to the case $l \mid x$. Of course, then $l \mid y$ and writing $x = lu$ and $y = lv$, (4.5.2) reduces to

$$8u^2 + 1 = v^2$$

which is a Pell's equation and the complete solution is given by $u = B_n$ and $v = C_n$; $n = 1,2,\dots$ (see p. 17). Thus $x = lB_n$ and $y = lC_n$; $n = 1,2,\dots$ Hence our third class of solution is

$$\{lB_n : n = 1,2,\dots\}.$$

Since $2k^2 - 1$ is a perfect square $k = P_{2n+1}$ and $2k^2 - 1 = 2P_{2n+1}^2 - 1 = Q_{2n+1}^2$. Thus $l = Q_{2n+1}$ and the third class of solution is given by

$$W_k(n) = \{Q_{2n+1}B_n : n = 1,2,\dots\}.$$

Our discussion will not be complete if we do not investigate the case where $2k^2 - 1$ contains a non-trivial square factor. Writing $2k^2 - 1 = lm^2$ and considering the particular case $m \mid x$ (and hence $m \mid y$), the Equation (4.5.1) and $2x^2 + 2k^2 - 1^2 = y^2$ reduce to the generalized Pell's equations

$$8u^2 + l = v^2 \quad \text{and} \quad 2u^2 + l = v^2$$

which are simpler than the original ones. By solving these equations, some other classes of g_k-balancing numbers can be obtained.

Chapter 5

Higher Order Gap Balancing Numbers

5.1 Introduction

The concept of higher order balancing and cobalancing numbers was introduced by Panda in [69]. Accordingly, for a given positive integer m, the number n^m is called an m^{th} order balancing number if

$$1^m + 2^m + \cdots + (n-1)^m = (n+1)^m + \cdots + (n+r)^m$$

holds for some positive integer r, while n^m is called a m^{th} order cobalancing number if

$$1^m + 2^m + \cdots + n^m = (n+1)^m + \cdots + (n+r)^m.$$

Panda in [69] proved that no third order balancing or cobalancing number exists. He further conjectured that no m^{th} order balancing or cobalancing numbers exists if $k \geq 2$.

Using the concepts of both higher order balancing numbers and gap balancing numbers, we dedicate this chapter for the study of higher order gap balancing numbers. Generalizing the definitions of gap balancing numbers, we define higher order gap balancing numbers as follows.

Definition 5.1.1. *Let k be the fixed odd positive integer. We call the positive integer x^m a m^{th} order k-gap balancing number if*

$$1^m + 2^m + \cdots + \left(x - \frac{k+1}{2}\right)^m = \left(x + \frac{k+1}{2}\right)^m + \left(x + \frac{k+3}{2}\right)^m + \cdots + y^m,$$

for some positive integer y.

S. S. Rout, Second order gap balancing numbers, *Journal of Numbers*, Article ID 216738, 5 pages, 2014.

Definition 5.1.2. *Let k be the fixed even positive integer. We call the positive integer $x^m+(x+1)^m$ a m^{th} order k-gap balancing number if*

$$1^m+2^m+\cdots+\left(x-\frac{k}{2}\right)^m=\left(x+\frac{k}{2}+1\right)^m+\left(x+\frac{k}{2}+2\right)^m+\cdots+y^m, \tag{5.1.1}$$

for some positive integer y.

In this chapter, our study is limited to second order gap balancing numbers only. More specifically, we prove the following theorem.

Theorem 5.1.3. *There does not exist any non trivial second order* 2*-gap balancing number.*

5.2 Prerequisites

To prove Theorem 5.1.3, we have to deal with Diophantine equations of degree three. Therefore, we need to discuss real cubic field $\mathbb{Q}(\theta)$, where $\theta^3=2$ (see [51]). The necessary informations for our problem are as follows:

i. The integers of $\mathbb{Q}(\theta)$ are of the form $\alpha=A+B\theta+C\theta^2$, where A,B and C are rational integers where $(1,\theta,\theta^2)$ is an integral basis for $\mathbb{Q}(\theta)$.

ii. The ring of integers of $\mathbb{Q}(\theta)$ is a unique factorization domain.

iii. By virtue of Dirichlet's theorem on units, there is only one fundamental unit of the field $\mathbb{Q}(\theta)$, which we designate by ε_0, with $0<\varepsilon_0<1$, is given by

$$\varepsilon_0=-1+\theta.$$

All the units of the field are given by $\pm\varepsilon_0^m$, where m is any rational integer. Any such power of ε_0 is of the form $a+b\theta+c\theta^2$, where a,b and c are rational integers. Norm of $\alpha=A+B\theta+C\theta^2$ is given by $N(\alpha)=A^3+2B^3+4C^3-6ABC$. All units of norm 1 in $\mathbb{Q}(\theta)$ are of the form ε_0^m.

First, we find the number of equivalence classes of associated primes of norm $3,5$ and 71.

Since $x^3-2\equiv(x-2)^3 \pmod 3$, 3 is a perfect cube in $\mathbb{Q}(\theta)$ apart from unit factors. So

$$3=(\theta+1)^3(\theta-1),$$

and 3 is the cube of a prime of norm 3 times an unit factor. Hence, there is only one equivalence class of associated primes of norm 3 in $\mathbb{Q}(\theta)$, as any integer of norm 3 in $\mathbb{Q}(\theta)$ must divide 3, apart from unit factors and there is only one such integer.

5 is a rational prime of the form $3r+2$. So 5 splits into two primes in $\mathbb{Q}(\theta)$, one of first degree and other of second degree because $5 \nmid (-108)$, which is the discriminant of the field. That is

$$5 = (-3+2\theta^2)(9+8\theta+6\theta^2)$$

where the first factor is a prime in $\mathbb{Q}(\theta)$ of norm 5 and second factor is also prime in $\mathbb{Q}(\theta)$ of norm 25. Hence there is only one equivalence class of associated primes of norm 5 in $\mathbb{Q}(\theta)$, as any integer in $\mathbb{Q}(\theta)$ with norm 5 must divide 5, and apart from unit factors, there is only one such integer.

Lastly, since 71 is a rational prime of the form $3r+2$, it splits into two primes in $\mathbb{Q}(\theta)$

$$71 = (5-3\theta)(25+15\theta+9\theta^2).$$

Thus, the norm of first factor of 71 is 71 and the norm of second factor is 5041. Hence there is only one equivalence class of associated primes of norm 71 in $\mathbb{Q}(\theta)$.

To solve (5.1.1) for $m=2$ and $k=2$, we need the following results.

Theorem 5.2.1. *(See [76, p.152]) Let a,b and c be non zero integers. Then the equation $ax^3+by^3=c$ has only finitely many solutions in integers (x,y).*

Theorem 5.2.2. *(See [40, p.97]) Let $K(\rho)$ be a cubic field over the field of rational numbers, and let $\alpha = A\rho^2+B\rho+C$ be an integer in the ring $(1,\rho,\rho^2)$. Suppose $A \equiv B \equiv 0 \pmod{p^k}$, where p is an odd rational prime, and $(\alpha, p)=1$. Further, suppose that $tA+sB \not\equiv 0 \pmod{p^{2k}}$, where t,s and k are rational integers, and $k>0$. Then, if $\alpha^n = A_n\rho^2+B_n\rho+C_n$, tA_n+sB_n is never zero for any $n \neq 0$.*

Theorem 5.2.3. *(See [64]) The Diophantine equation $Ax^3+By^3=C$ $(C=1$ or 3; $3 \nmid AB$ if $C=3; A>B$; A,B positive integers) has at most one solution in non zero integers (x,y). There is the unique exception for the equation $2x^3+y^3=3$ which has exactly two integral solutions $(x,y)=(1,1)$ and $(x,y)=(4,-5)$.*

5.3 Proof of Theorem 5.1.3

Let x be a second order 2-gap balancing number. Equation (5.1.1) with $m=2$ and $k=2$ is equivalent to

$$\frac{(x-1)x(2x-1)}{6} = \frac{y(y+1)(2y+1)}{6} - \frac{(x+1)(x+2)(2x+3)}{6}. \tag{5.3.1}$$

Simplification of the above equation gives

$$2(2x+1)^3+22(2x+1) = (2y+1)^3-(2y+1).$$

Setting $A = -(2x+1)$ and $B = (2y+1)$, we get

$$-2[A^3 + 11A] = B^3 - B. \tag{5.3.2}$$

We shall now prove several lemmas which together imply that the only solution of (5.3.2) subject to the conditions

$$A \text{ is negative and odd, } B \text{ is positive and odd} \tag{5.3.3}$$

is $(A,B) = (-3,5)$.

Lemma 5.3.1. *All the integral solutions of* (5.3.2) *satisfying the condition* (5.3.3) *correspond to the integral solutions of the equations*

$$2u^3 + v^3 = 5, 15, 25, 71, 75, 213, 355, 1065, 1775 \text{ and } 5325. \tag{5.3.4}$$

Proof. Let (A,B) be any integral solution of (5.3.2) subject to the conditions (5.3.3). Let $(A,B) = d$. Substituting $A = du$ and $B = dv$ in (5.3.2), we get

$$d^2(v^3 + 2u^3) = v - 22u, \tag{5.3.5}$$

where

$$(u,v) = 1, \ u \text{ is negative and odd, } v \text{ is positive and odd.} \tag{5.3.6}$$

Let

$$2u^3 + v^3 = c. \tag{5.3.7}$$

Then $v - 22u = cd^2$, or equivalently,

$$v = 22u + cd^2. \tag{5.3.8}$$

Substituting (5.3.8) in (5.3.7), we get

$$d^6c^3 + 66uc^2d^4 + 1452u^2d^2c + 10650u^3 = c \tag{5.3.9}$$

and from (5.3.7) and (5.3.9), we obtain

$$d^6c^3 + 66uc^2d^4 + 1452u^2d^2c + 5324c = 5325v^3.$$

Therefore $c|5325v^3$, hence $c|5325$ as $(c,v) = 1$. So the possible values of c are

$$1, 3, 5, 15, 25, 71, 75, 213, 355, 1065, 1775 \text{ and } 5325.$$

Thus, solving (5.3.2) subject to the Condition (5.3.3) can be reduced to solving

$$2u^3 + v^3 = 1, 3, 5, 15, 25, 71, 75, 213, 355, 1065, 1775 \text{ and } 5325. \tag{5.3.10}$$

However, by Theorem 5.2.3 the equations $2u^3 + v^3 = 1, 3$ has solutions $(u,v) = (0,1), (1,-1), (1,1), (4,-$
which violates the condition in (5.3.6). □

Lemma 5.3.2. *The equation* $2u^3+v^3=5$ *has no solution in rational integers* (u,v) *satisfying* (5.3.6).

Proof. The norm of the integer $-3+2\theta^2$ in $\mathbb{Q}(\theta)$ is 5. Any other integer of norm 5 in $\mathbb{Q}(\theta)$ must be of the form $(-3+2\theta^2)\varepsilon_0^m$ as all primes of norm 5 in $\mathbb{Q}(\theta)$ are associated. Let $\varepsilon_0^m = a_m + b_m\theta + c_m\theta^2$ and

$$X_m + Y_m\theta + Z_m\theta^2 = (-3+2\theta^2)(a_m+b_m\theta+c_m\theta^2).$$

Hence

$$Z_m = -3c_m + 2a_m. \tag{5.3.11}$$

We seek all integers of the form $a+b\theta$ in $\mathbb{Q}(\theta)$ of norm 5 . Hence Z_m must be zero, and thus, the congruence $Z_m \equiv 0 \pmod{t}$ is solvable for every modulus t. We shall show that

$$Z_m \not\equiv 0 \pmod{3}.$$

First, we will show that Z_m is periodic modulo 3. Since

$$\varepsilon_0^3 + 3\varepsilon_0^2 + 3\varepsilon_0 - 1 = 0,$$

we have

$$\varepsilon_0^{m+3} + 3\varepsilon_0^{m+2} + 3\varepsilon_0^{m+1} - \varepsilon_0^m = 0, \tag{5.3.12}$$

from which it follows that

$$Z_{m+3} + 3Z_{m+2} + 3Z_{m+1} - Z_m = 0. \tag{5.3.13}$$

Therefore,

$$Z_{m+3} \equiv Z_m \pmod{3}. \tag{5.3.14}$$

Therefore, we only need to check $Z_i \not\equiv 0 \pmod{3}$ for $i = 0,1$ and 2. Using (5.3.11), we have $Z_0 \equiv Z_2 \equiv 2 \pmod{3}$ and $Z_1 \equiv 1 \pmod{3}$ and none of these is zero. This completes the proof of Lemma 5.3.2. □

Lemma 5.3.3. *The only integral solution of the equation* $2u^3+v^3=71$ *satisfying* (5.3.6) *is* $(u,v)=(-3,5)$.

Proof. We seek all the integers of $\mathbb{Q}(\theta)$ which are of the form $a+b\theta$. Since all primes of norm 71 in $\mathbb{Q}(\theta)$ are associated, any such prime must be an associate of $5-3\theta$. Let $\varepsilon_0^m = a_m + b_m\theta + c_m\theta^2$ be a unit of $\mathbb{Q}(\theta)$. Our requirement is that the coefficient of θ^2 in

$$(5-3\theta)(a_m+b_m\theta+c_m\theta^2)$$

be zero. This gives

$$5c_m - 3b_m = 0. \tag{5.3.15}$$

We claim that (5.3.15) is impossible for $m \neq 0$. Now let

$$x_m + y_m\theta + z_m\theta^2 = (5-3\theta)\varepsilon_0^m$$

we have $z_m = 5c_m - 3b_m = 0$ by (5.3.15). Also from (5.3.12)

$$z_{m+3} \equiv z_m \pmod 3. \tag{5.3.16}$$

Since $b_0 = 0,\ c_0 = 0$, we have $z_0 = 0$. Further $b_1 = 1,\ c_1 = 0$ and $b_2 = -2,\ c_2 = 1$ yield

$$z_1 = 0 \pmod 3 \ \text{ and } \ z_2 = 2 \pmod 3.$$

Therefore, by (5.3.16) $z_m = 0$ only when $m \equiv 0, 1 \pmod 3$. Now $\varepsilon_0^3 = 1 + 3\theta - 3\theta^2$ results

$$b_3 \equiv c_3 \equiv 0 \pmod 3 \quad \text{and} \quad z_3 \equiv 5c_3 - 3b_3 \equiv 3 \pmod 9.$$

By Theorem 5.2.2, z_m is never zero for any $m \neq 0$, which completes the proof of the lemma. □

Lemma 5.3.4. *The only integral solution of the equation* $2u^3 + v^3 = 25$ *satisfying* (5.3.6) *is* $(u, v) = (-1, 3)$.

Proof. In this case we seek all integers of the form $a + b\theta$ of norm 25. Here we employ the same method of proof as we did for Lemma 5.3.3. Observe that

$$25 = (3-\theta)(9 + 3\theta + \theta^2).$$

The coefficient of θ^2 in

$$(3-\theta)(a_m + b_m\theta + c_m\theta^2)$$

is $3c_m - b_m$. We will show that $3c_m - b_m \neq 0$ if $m \neq 0$. Using the techniques applied in the previous lemma one can easily see that

$$b_3 \equiv c_3 \equiv 0 \pmod 3 \quad \text{and} \quad z_3 \equiv 3c_3 - b_3 \equiv -3 \pmod 9.$$

By Theorem 5.2.2, $z_m \neq 0$ if $m \neq 0$. □

Lemma 5.3.5. *The equation* $2u^3 + v^3 = 15$ *has no rational solutions in u and v satisfying* (5.3.6).

Proof. In $\mathbb{Q}(\theta)$ the integer $1 - 3\theta + 2\theta^2$ is of norm 15. Any other integer of norm 15 must be of the form $(1 - 3\theta + 2\theta^2)\varepsilon_0^m$ as all primes of norm 5 in $\mathbb{Q}(\theta)$ are associated and so also all primes of norm 3. Let $\varepsilon_0^m = a_m + b_m\theta + c_m\theta^2$ and

$$X_m + Y_m\theta + Z_m\theta^2 = (1 - 3\theta + 2\theta^2)(a_m + b_m\theta + c_m\theta^2).$$

Hence

$$Z_m = c_m - 3b_m + 2a_m. \tag{5.3.17}$$

We seek all integers of the form $a+b\theta$ in $\mathbb{Q}(\theta)$ of norm 15. Hence, Z_m must be zero, and the congruence $Z_m \equiv 0 \pmod{t}$ is solvable for every modulus t. We next show that

$$Z_m \not\equiv 0 \pmod{31}.$$

Also, by manual verification, we have $\varepsilon_0^{10} \equiv -6 \pmod{31}$, $\varepsilon_0^{20} \equiv 5 \pmod{31}$ and $\varepsilon_0^{30} \equiv 1 \pmod{31}$. Therefore Z_m satisfies the conditions

$$Z_{m+10} \equiv -6Z_m \pmod{31}, \quad Z_{m+20} \equiv 5Z_m \pmod{31}, \quad Z_{m+30} \equiv Z_m \pmod{31}, \tag{5.3.18}$$

and we only need to check $Z_i \not\equiv 0 \pmod{31}$ for $i = 0, 1, \ldots, 9$. From (5.3.17), the values of Z_0 to Z_9 modulo 31 are $2, -5, 9, -10, -2, 14, 16, 1, -6, 10$ and none of these is zero. This completes the proof of Lemma 5.3.5. □

Lemma 5.3.6. *The equation* $2u^3 + v^3 = 75$ *is not solvable in rational integers* u *and* v *satisfying* (5.3.6).

Proof. Here, our focus is on integers of the form $a+b\theta$ from $\mathbb{Q}(\theta)$ having norm 75. The integer $11-6\theta-2\theta^2$ has norm 75. Any other integer of norm 75 must be of this form $(11-6\theta-2\theta^2)\varepsilon_0^m$. Therefore

$$X_m + Y_m\theta + Z_m\theta^2 = (11-6\theta-2\theta^2)(a_m + b_m\theta + c_m\theta^2).$$

Hence

$$Z_m = 11c_m - 6b_m - 2a_m. \tag{5.3.19}$$

We will show that $Z_m \not\equiv 0 \pmod{31}$ also Z_m satisfies the (5.3.18) therefore we only check $Z_i \not\equiv 0 \pmod{31}$ for $i = 0, 1, \ldots, 9$. Using (5.3.19) the values of Z_0 to Z_9 modulo 31 are $-2, -4, 21, 9, -1, -3, -10, -19, 25, 12$ and none of these is zero. This completes the proof of Lemma 5.3.6. □

Lemma 5.3.7. *The equations* $2u^3 + v^3 = 213, 355, 1065, 1775, 5325$ *are impossible in rational integers* (u, v) *satisfying* (5.3.6).

Proof. Using the norm of 3, 5 and 71 and multiplicative property of the norm, the integers having norms $213, 355, 1065, 1775, 5325$ are $-11+3\theta+5\theta^2, -27+9\theta+10\theta^2, 45+11\theta-37\theta^2, 15-14\theta+3\theta^2$ and $-43+20\theta+12\theta^2$ respectively. Also, we know that all primes of norm 3, 5 and 71 in $\mathbb{Q}(\theta)$ are associated. Therefore, the integers whose norms are $213, 355, 1065, 1775, 5325$ respectively can be represented by

$$X_m^1 + Y_m^1\theta + Z_m^1\theta^2 = (-11+3\theta+5\theta^2)(a_m^1 + b_m^1\theta + c_m^1\theta^2),$$

$$X_m^2 + Y_m^2\theta + Z_m^2\theta^2 = (-27+9\theta+10\theta^2)(a_m^2 + b_m^2\theta + c_m^2\theta^2),$$

$$X_m^3 + Y_m^3\theta + Z_m^3\theta^2 = (45+11\theta-37\theta^2)(a_m^3 + b_m^3\theta + c_m^3\theta^2),$$

$$X_m^4 + Y_m^4\theta + Z_m^4\theta^2 = (15 - 14\theta + 3\theta^2)(a_m^4 + b_m^4\theta + c_m^4\theta^2),$$

$$X_m^5 + Y_m^5\theta + Z_m^5\theta^2 = (-43 + 20\theta + 12\theta^2)(a_m^5 + b_m^5\theta + c_m^5\theta^2).$$

Hence,

$$Z_m^1 = -11c_m^1 + 3b_m^1 + 5a_m^1,\ Z_m^2 = -27c_m^2 + 9b_m^2 + 10a_m^2,\ Z_m^3 = 45c_m^3 + 11b_m^3 - 37a_m^3$$

$$Z_m^4 = 15c_m^4 - 14b_m^4 + 3a_m^4,\ \ Z_m^5 = -43c_m^5 + 20b_m^5 + 12a_m^5.$$

But we seek all integers of the form $a+b\theta$. That means $Z_m^i \equiv 0 \pmod t$ for any modulus t and for $i = 1,2,\dots,5$. We want to show that for all $i = 1,2,\dots,5$ $Z_m^i \not\equiv 0 \pmod t$ for some t. Also Z_m^i satisfy the congruences $Z_{m+10}^i \equiv -6Z_m^i \pmod{31}$, $Z_{m+20}^i \equiv 5Z_m^i \pmod{31}$ and $Z_{m+30}^i \equiv Z_m^i \pmod{31}$. Using these congruences the values of Z_m^i modulo 31 are as follows. The values of Z_0^1 to Z_9^1 modulo 31 are $5,-2,-12,16,-14,13,19,2,15,-13$, the values of Z_0^2 to Z_9^2 modulo 31 are $10,-1,-4,21,-2,20,2,-6,1,-14$, the values of Z_0^3 to Z_9^3 modulo 31 are $6,17,-14,-15,11,-2,20,-12,5,10$, the values of Z_0^4 to Z_9^4 modulo 31 are $3,-17,-10,9,4,7,7,-7,7,7$ and the values of Z_0^5 to Z_9^5 modulo 31 are $12,8,-3,5,-2,3,2,-15,17,2$ and none of these is zero. This completes the proof of Lemma 5.3.7 □

Till now we have got the solutions (u,v) of (5.3.4) satisfying (5.3.6). We need to find the solutions (A,B) of (5.3.2) for which the exact value of d should be calculated. It follows from Lemma 5.3.2 to Lemma 5.3.7 that the only integral solution of (5.3.4) are $(u,v) = (-1,3),(-3,5)$. In both the cases, both u and v are relatively prime and odd, u is negative and v is positive. Therefore from (5.3.5), we have

$$25d^2 = 25 \quad \text{and} \quad 71d^2 = 71.$$

In either case, $d = 1$

Thus, the only integral solution of (5.3.2) satisfying conditions (5.3.3) are $(A,B) = (-1,3),(-3,5)$. Hence the only integral solution of (5.3.1) is $(x,y) = (1,2)$. But $x = 1$ does not satisfy the defining equation (5.1.1) for $k = m = 2$. This completes the proof of the Theorem 5.1.3.

Chapter 6

Balancing Dirichlet Series

6.1 Introduction

A series of the form $\sum_{n=1}^{\infty} a_n n^{-s}$ where $a_n, n \in \mathbb{N}$ is a complex sequence and $s = \sigma + it \in \mathbb{C}$ is a complex number is called a Dirichlet series. Functions defined by this series very often relate algebraic properties in analytic terms. This mostly happens when a_n is a multiplicative function such as number of divisors of n, sum of divisors of n or the Möbius function and so on. When $a_n = 1$ and $\mathrm{Re}(s) > 1$, one gets $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ which is the Riemann zeta function, extensively available in literature [6,22,42,84,90]. $\zeta(s)$ can be analytically continued to the whole complex plane, with only one simple pole at $s = 1$. Also, $\zeta(s)$ has an important symmetry around the line $\mathrm{Re}(s) = \sigma = 1/2$ in the form of a functional equation. The trivial zeros of $\zeta(s)$ are located at $-2, -4, -6, \ldots$ and its values at negative odd integers are rational, and in fact, given by the Bernoulli numbers [74].

The series $\zeta_F(s) = \sum_{n=1}^{\infty} F_n^{-s}$ where F_n is the n^{th} Fibonacci number is a variant of the Riemann zeta function is known as the Fibonacci zeta function and its analytic continuation was studied by Navas [65]. Unlike $\zeta(s)$, this series has trivial zeros at $-2, -6, -10, \ldots$ and simple poles $0, -4, -8, \ldots$ Also $\zeta_F(s)$ takes rational numbers at negative odd integers. This motivates us to consider the analytic continuation of the series

$$\zeta_B(s) = \sum_{n=1}^{\infty} B_n^{-s}, \tag{6.1.1}$$

where B_n is the n^{th} balancing number (see p. 17). We call the series, the balancing zeta function. We will show that $\zeta_B(s)$ has simple poles at $0, -2, -4, \ldots$ and can be extended to a meromorphic function on $\mathbb{C}$. However as we will see, the balancing zeta function has no trivial zeros unlike the Riemann zeta function.

S. S. Rout and G. K.Panda, Balancing Dirichlet series and related L-functions, *Indian J. Pure Appl. Math*, **45**(6), 943–952, 2014.

6.2 Analytic continuation of balancing zeta function

Analytic continuation of a series consists of extending its domain of analyticity to the whole complex plane. The balancing zeta function is analytic in the half plane $\mathrm{Re}(s) > 1$. In this section, we will show that it can be analytically continued to the whole complex plane except at poles.

It is known that the balancing numbers satisfy the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$, $n \geq 1$ with $B_0 = 0, B_1 = 1$ and the Binet form is given by

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \quad \text{where} \quad \lambda_1 = 3+2\sqrt{2} \quad \text{and} \quad \lambda_2 = 3-2\sqrt{2}$$

For any complex number z

$$\begin{aligned} B_n^z =& \Big(\frac{\lambda_1^n - \lambda_2^n}{4\sqrt{2}}\Big)^z = 2^{-5z/2}\Big(\lambda_1^n - \lambda_2^n\Big)^z \\ =& 2^{-5z/2}\lambda_1^{nz}\Big(1-\Big(\frac{\lambda_2}{\lambda_1}\Big)^n\Big)^z \\ =& 2^{-5z/2}\lambda_1^{nz}\Big(1-\Big(\frac{1}{\lambda_1^{2n}}\Big)\Big)^z \\ =& 2^{-5z/2}\lambda_1^{nz}\sum_{k=0}^{\infty}(-1)^k\binom{z}{k}\lambda_1^{-2nk} \\ =& 2^{-5z/2}\sum_{k=0}^{\infty}(-1)^k\binom{z}{k}\lambda_1^{n(z-2k)}. \end{aligned}$$

This expression is valid for any $z \in \mathbb{C}$ and this binomial series converges since $\lambda_1 > 1$. Substituting $z = -s$ in the final expression for B_n^z in (6.1.1) we get,

$$\sum_{n=1}^{\infty} B_n^{-s} = 2^{5s/2}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(-1)^k\binom{-s}{k}\lambda_1^{n(-s-2k)}. \tag{6.2.1}$$

Thus,

$$\begin{aligned} \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\Big|(-1)^k\binom{-s}{k}\lambda_1^{n(-s-2k)}\Big| =& \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\Big|(-1)^k\Big|\Big|\binom{-s}{k}\Big|\Big|\lambda_1^{n(-s-2k)}\Big| \\ \leqslant& \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(-1)^k\binom{-|s|}{k}\lambda_1^{-n(\sigma+2k)} \\ =& \sum_{n=1}^{\infty}\lambda_1^{-n\sigma}(1-\lambda_1^{-2n})^{-|s|} \\ \leqslant& (1-\lambda_1^{-2})^{-|s|}\sum_{n=1}^{\infty}\lambda_1^{-n\sigma} < \infty. \end{aligned}$$

Thus, the series in (6.2.1) is absolutely convergent and we can interchange the order of summation, i.e.,

$$\begin{aligned}\zeta_B(s) = \sum_{n=1}^{\infty} B_n^{-s} = 2^{5s/2} \sum_{k=0}^{\infty} (-1)^k \binom{-s}{k} \sum_{n=1}^{\infty} \left(\lambda_1^{-(s+2k)}\right)^n \\ =& 2^{5s/2} \sum_{k=0}^{\infty} (-1)^k \binom{-s}{k} \left(\frac{1}{1-\lambda_1^{-(s+2k)}} - 1\right) \\ =& 2^{5s/2} \sum_{k=0}^{\infty} (-1)^k \binom{-s}{k} \left(\frac{\lambda_1^{-(s+2k)}}{1-\lambda_1^{-(s+2k)}}\right) \\ =& 2^{5s/2} \sum_{k=0}^{\infty} (-1)^k \binom{-s}{k} \left(\frac{1}{\lambda_1^{(s+2k)} - 1}\right).\end{aligned}$$

This infinite series determines a function holomorphic on $\mathbb{C}$ except at the poles derived from $\lambda_1^{(s+2k)} - 1 = 0$. Let $k_0 = \max\{1, -\sigma\}$. For each $s \in \mathbb{C}$ and $k > k_0$, we have

$$|\lambda_1^{(s+2k)} - 1| \geq \lambda_1^{(\sigma+2k)} - 1 > \lambda_1^{(\sigma+k)}.$$

Hence

$$\Big|\sum_{k>k_0}^{\infty} (-1)^k \binom{-s}{k} \Big(\frac{1}{\lambda_1^{(s+2k)} - 1}\Big)\Big| \leq \lambda_1^{-\sigma} \sum_{k=0}^{\infty} \binom{-|s|}{k} \lambda_1^{-k} = \lambda_1^{-\sigma}(1-\lambda_1^{-1})^{-|s|} < \infty,$$

proving that the series in (6.1.1) converges uniformly and absolutely on compact subsets of $\mathbb{C}$ which does not contain any poles of the function

$$f_k(s) = \binom{-s}{k} \frac{1}{\lambda_1^{(s+2k)} - 1}.$$

The poles of $f_k(s)$ are given by $s = -2k + \frac{2\pi i n}{\log \lambda_1}$ for $k \geq 0$ and $n \in \mathbb{Z}$. Thus, the poles of the series $\zeta_B(s)$ lie on the line $\sigma = -2k$ and spaced in an interval of $\frac{2\pi i}{\log \lambda_1}$. Hence, the series $\zeta_B(s)$ can be analytically continued to the whole complex plane and its simple poles are located at $s_{k,n} = -2k + \frac{2\pi i n}{\log \lambda_1}$. The residue of $\zeta_B(s)$ at $s_{k,n}$ is

$$\mathrm{Res}_{s=s_{k,n}} \zeta_B(s) = (-1)^k 2^{5s_{k,n}/2} \binom{-s_{k,n}}{k} \lim_{s \to s_{k,n}} \frac{s - s_{k,n}}{\lambda_1^{(s+2k)} - 1}.$$

By L'Hôpital's rule,

$$\lim_{s \to s_{k,n}} \frac{s - s_{k,n}}{\lambda_1^{(s+2k)} - 1} = \frac{1}{\log \lambda_1}.$$

The above discussion proves the following theorem.

Theorem 6.2.1. *The function $\zeta_B(s)$ can be analytically continued to the whole complex plane and can be expressed as*

$$\zeta_B(s) = 2^{5s/2}\sum_{k=0}^{\infty}(-1)^k\binom{-s}{k}\Big(\frac{1}{\lambda_1^{(s+2k)}-1}\Big)$$

which is holomorphic except for simple poles at $s = s_{k,n} = -2k + \frac{2\pi in}{\log\lambda_1}$, $n \in \mathbb{Z}$ and the residue at $s = s_{k,n}$ is given by $\frac{(-1)^k 2^{5s_{k,n}/2}\binom{-s_{k,n}}{k}}{\log\lambda_1}$.

6.3 Values of $\zeta_B(s)$ at integral arguments

6.3.1 Values at negative integers

In this section, we discuss the values of $\zeta_B(s)$ at negative integers. We have already verified that $0, -2, -4, -6, \ldots$ are simple poles for $\zeta_B(s)$. The following theorem shows the value of $\zeta_B(s)$ at odd negative integer.

Theorem 6.3.1. *If m is an odd natural number, then*

$$\zeta_B(-m) = -2^{(-5m+3)/2}\sum_{k=0}^{\frac{m-1}{2}}(-1)^k\binom{m}{k}\frac{B_{2k-m}}{1-C_{2k-m}}.$$

Proof. Let $m \geq 0$ be an integer which is not a multiple of 2. Then

$$\zeta_B(-m) = 2^{-5m/2}\sum_{k=0}^{\infty}(-1)^k\binom{m}{k}\Big(\frac{1}{\lambda_1^{-m+2k}-1}\Big)$$

and since all terms with $k > m$ are zero, it is a finite sum belonging to $\mathbb{Q}(\sqrt{2})$. Let $\sigma_k = (-1)^k\binom{m}{k}(\lambda_1^{-m+2k}-1)^{-1}$ and $\alpha_k = \sigma_k + \sigma_{m-k}$ so that $\alpha_k = \alpha_{m-k}$ and

$$\zeta_B(-m) = 2^{-5m/2}\sum_{k=0}^{\frac{m-1}{2}}\alpha_k.$$

Now

$$\begin{aligned}
\alpha_k &= (-1)^k\binom{m}{k}\frac{1}{\lambda_1^{-m+2k}-1} + (-1)^{m-k}\binom{m}{m-k}\frac{1}{\lambda_1^{m-2k}-1}\\
&= (-1)^k\binom{m}{k}\left[\frac{1}{\lambda_1^{-m+2k}-1} + \frac{(-1)^m}{\lambda_1^{m-2k}-1}\right]\\
&= (-1)^k\binom{m}{k}\left[\frac{1}{\lambda_1^{-m+2k}-1} + \frac{(-1)^m}{\lambda_2^{2k-m}-1}\right]\\
&= (-1)^k\binom{m}{k}\left[\frac{(\lambda_1^{-m+2k}-1)(-1)^m + \lambda_2^{-m+2k}-1}{(\lambda_1^{-m+2k}-1)(\lambda_2^{2k-m}-1)}\right].
\end{aligned}$$

So, if $m \not\equiv 0 \pmod 2$, then

$$\zeta_B(-m) = 2^{-5m/2} \sum_{k=0}^{\frac{m-1}{2}} (-1)^k \binom{m}{k} \frac{\lambda_2^{-m+2k} - \lambda_1^{-m+2k}}{2 - (\lambda_2^{-m+2k} + \lambda_1^{-m+2k})}$$

$$= -2^{(-5m+3)/2} \sum_{k=0}^{\frac{m-1}{2}} (-1)^k \binom{m}{k} \frac{B_{2k-m}}{1 - C_{2k-m}}. \qquad \square$$

6.3.2 Values at positive integers

We know that $B_n^z = 2^{-5z/2} \sum_{k=0}^{\infty} (-1)^k \binom{z}{k} \lambda_1^{n(z-2k)}$ and we also have

$$(-1)^k \binom{-s}{k} = \binom{s+k-1}{k}.$$

For $m \in \mathbb{N}$,

$$B_n^{-m} = 2^{5m/2} \sum_{k=0}^{\infty} (-1)^k \binom{-m}{k} \lambda_1^{n(-m-2k)}$$

$$= 2^{5m/2} \sum_{k=0}^{\infty} \binom{m+k-1}{k} \lambda_1^{-n(m+2k)}$$

$$= 2^{5m/2} \sum_{k=0}^{\infty} \binom{m+k-1}{m-1} \lambda_1^{-n(m+2k)}.$$

Taking $d = m + 2k$ and

$$S_m = \{d \geq m : d \equiv m \pmod 2\},$$

we get

$$B_n^{-m} = 2^{5m/2} \sum_{d \in S_m} \binom{\frac{d+m-2}{2}}{m-1} \lambda_1^{-nd}.$$

Let

$$S_m^+ = \{d \geq m : d \equiv m \pmod 4\} \text{ and } S_m^- = \{d \geq m+2 : d \equiv m \pmod 4\}.$$

Then

$$B_n^{-m} = 2^{5m/2} \left(\sum_{d \in S_m^+} \binom{\frac{d+m-2}{2}}{m-1} \lambda_1^{-nd} + \sum_{d \in S_m^-} \binom{\frac{d+m-2}{2}}{m-1} \lambda_1^{-nd} \right).$$

To take sum over n, we need to collect like powers $l = nd$, so that l runs over all natural numbers and we restrict to $d \mid l$. We thus have

$$\sum_{n=1}^{\infty} B_n^{-m} = 2^{5m/2} \sum_{l=1}^{\infty} \left(\sum_{\substack{d \mid l \\ d \in S_m^+}} \binom{\frac{d+m-2}{2}}{m-1} + \sum_{\substack{d \mid l \\ d \in S_m^-}} \binom{\frac{d+m-2}{2}}{m-1} \right) \lambda_1^{-l}.$$

The above discussion proves the following theorem.

Theorem 6.3.2. *$\zeta_B(m) = 2^{5m/2}\sum_{l=1}^{\infty} a_l \lambda_1^{-l}$ for $m \in \mathbb{N}$, where the coefficients a_l are combinations of sums of the powers of divisors of l.*

When $m = 1$, we have $S_1^+ = \{d \geq 1 : d \equiv 1 \pmod 4\} = \{1,5,9,\dots\}$ and $S_1^- = \{d \geq 1 : d \equiv 3 \pmod 4\} = \{3,7,11,\dots\}$. Also all the binomial coefficients reduce to 1 and then

$$\sum_{n=1}^{\infty} B_n^{-1} = 2^{5/2}\sum_{l=1}^{\infty}\left(\sum_{\substack{d|l \\ d\in S_1^+}} 1 + \sum_{\substack{d|l \\ d\in S_1^-}} 1\right)\lambda_1^{-l}$$

$$= 2^{5/2}\sum_{l=1}^{\infty}(d_1(l)+d_3(l))\lambda_1^{-l}$$

where

$$d_i(n) = \sum_{\substack{d|n \\ d\equiv i \pmod 4}} 1.$$

Hence we have the following identities:

$$\sum_{n=1}^{\infty} B_{2n}^{-1} = 2^{5/2}\sum_{l\equiv 0 \pmod 2}(d_1(l)+d_3(l))\lambda_1^{-l},$$

$$\sum_{n=1}^{\infty} B_{2n+1}^{-1} = 2^{5/2}\sum_{l\equiv 1 \pmod 2}(d_1(l)+d_3(l))\lambda_1^{-l}.$$

It is interesting to observe the special values of $\zeta_B(s)$ when s is a natural number. André-Jeannin [4] proved that the Fibonacci Dirichlet series $\sum_{n=1}^{\infty}\frac{1}{F_n}$ is an irrational number. Also, Duverney et al. [28] proved that $\sum_{n=1}^{\infty}\frac{1}{F_n^{2k}}$ is transcendental for $k = 1,2,\dots$ Similarly one can prove that $\zeta_B(2k)$ is transcendental for $k = 1,2,\dots$, which is, indeed, a particular case of transcendence of binary linear recurrence sequences proved in [31]. Ram Murty [63] also proved the same results for the Fibonacci Dirichlet series using q-series, where he used q-exponential for irrationality of $\sum_{n=1}^{\infty}\frac{1}{F_n^s}$ at $s = 1$ and q-logarithm for transcendence of the series for $s = 2k$ and establish a connection to Ramanujan's mock-theta function.

6.4 Balancing *L*-function

Let χ be a Dirichlet character with modulo p. The Dirichlet L-function is defined as $L(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$. We define the balancing L function as

$$\mathcal{L}_{\beta}(s,\chi) := \sum_{n=1}^{\infty}\frac{\chi(n)}{B_n^s}, \quad Re(s) = \sigma > 1. \tag{6.4.1}$$

We know from [6] that the $\zeta(s)$ and $L(s,\chi)$ can be unified using the Hurwitz zeta function $\zeta(s,a)=\sum_{n=1}^{\infty}\frac{1}{(n+a)^s}$. Similarly, we can write the balancing L-function in terms of the balancing zeta function using the balancing zeta function in arithmetic progression which we define as

$$\zeta_B(s,(r,p)) := \sum_{\substack{n\geq 1 \\ n\equiv r \pmod{p}}} \frac{1}{B_n^s}. \tag{6.4.2}$$

Thus, $\zeta_B(s,(1,1))=\zeta_B(s)$. Further,

$$\begin{aligned}
\zeta_B(s,(r,p)) &= 4^s 2^{s/2}\sum_{n=0}^{\infty}\left(\lambda_1^{pn+r}-\lambda_2^{pn+r}\right)^{-s} \\
&= 2^{5s/2}\sum_{n=0}^{\infty}\lambda_1^{-(pn+r)s}\left(1-\left(\frac{\lambda_2}{\lambda_1}\right)^{pn+r}\right)^{-s} \\
&= 2^{5s/2}\sum_{n=0}^{\infty}\lambda_1^{-(pn+r)s}\sum_{k=0}^{\infty}(-1)^k\binom{-s}{k}\left(\frac{\lambda_2}{\lambda_1}\right)^{(pn+r)k} \\
&= 2^{5s/2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\binom{-s}{k}\lambda_1^{-(pn+r)(s+2k)} \\
&= 2^{5s/2}\sum_{k=0}^{\infty}(-1)^k\binom{-s}{k}\frac{\lambda_1^{-(s+2k)r}}{1-\lambda_1^{-(s+2k)p}}.
\end{aligned} \tag{6.4.3}$$

This shows that the function $\zeta_B(s,(r,p))$ can be analytically continued to the whole complex plane except for the simple poles at $s=-2k+\frac{2n\pi i}{p\log\lambda_1}$.

The following theorem, which uses the notion of Gauss sum [6], establishes the analytic continuation of the balancing L-function. For any Dirichlet character χ modulo p, the sum

$$G(n,\chi)=\sum_{m=1}^{p}\chi(m)\exp\left(\frac{2\pi imn}{p}\right) \tag{6.4.4}$$

is called the Gauss sum associated with χ.

Theorem 6.4.1. *The function $\mathcal{L}_B(s,\chi)$ can be analytically continued to the whole complex plane which is holomorphic except for simple poles at $s=s_{k,n}=-2k+\frac{2\pi in}{p\log\lambda_1}$ and the residue at $s=s_{k,n}$ is given by*

$$Res_{s=s_{k,n}}\mathcal{L}_B(s,\chi)=2^{5s_{k,n}/2}\binom{-s_{k,n}}{k}\frac{(-1)^k}{p\log\lambda_1}\chi(-1)G(n,\chi).$$

Proof. Observe that

$$\mathcal{L}_B(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{B_n^s}$$

$$= \sum_{r=1}^{p} \chi(r)\zeta_B(s,(r,p)). \tag{6.4.5}$$

It is clear from (6.4.5) that the poles of $\zeta_B(s,(r,p))$ are the poles of the balancing L-function $\mathcal{L}_B(s,\chi)$. Therefore the balancing L-function can be analytically continued to the whole complex plane. For residue of the balancing L-function at the poles, we first find out the residue of $\zeta_B(s,(r,p))$.

$$\begin{aligned} Res_{s=s_{k,n}}\zeta_B(s,(r,p)) &= 2^{5s_{k,n}/2}(-1)^k\binom{-s_{k,n}}{k}\lim_{s\to s_{k,n}}(s-s_{k,n})\frac{\lambda_1^{(s+2k)r}}{1-\lambda_1^{(s+2k)p}} \\ &= 2^{5s_{k,n}/2}(-1)^k\binom{-s_{k,n}}{k}\exp\left(-\frac{2n\pi i}{p}r\right)\lim_{s\to s_{k,n}}\frac{(s-s_{k,n})}{1-\lambda_1^{(s+2k)p}} \\ &= 2^{5s_{k,n}/2}(-1)^k\binom{-s_{k,n}}{k}\exp\left(-\frac{2\pi i}{p}\right)^{nr}\frac{1}{p\log\lambda_1}. \end{aligned}$$

Hence the residue of balancing L-function at the requisite pole is

$$\begin{aligned} Res_{s=s_{k,n}}\mathcal{L}_B(s,\chi) &= \sum_{r=1}^{p}\chi(r)Res_{s=s_{k,n}}\zeta_B(s,(r,p)) \\ &= 2^{5s_{k,n}/2}\binom{-s_{k,n}}{k}\frac{(-1)^k}{p\log\lambda_1}\sum_{r=1}^{p}\chi(r)\exp\left(-\frac{2\pi i}{p}\right)^{nr}. \end{aligned}$$

Using (6.4.4), we get

$$\begin{aligned} Res_{s=s_{k,n}}\mathcal{L}_B(s,\chi) &= 2^{5s_{k,n}/2}\binom{-s_{k,n}}{k}\frac{(-1)^k}{p\log\lambda_1}G(-n,\chi) \\ &= 2^{5s_{k,n}}\binom{-s_{k,n}/2}{k}\frac{(-1)^k}{p\log\lambda_1}\chi(-1)G(n,\chi). \qquad \square \end{aligned}$$

The following result gives the zeros of the balancing L-function at odd negative integers.

Theorem 6.4.2. *Let χ be any non-principal character modulo p and $\chi(-1) = -1$. Then $\mathcal{L}_B(-m,\chi) = 0$ for $m \equiv 1 \pmod 2$.*

Proof. Using (6.4.3) and (6.4.5) we get

$$\mathcal{L}_B(s,\chi) = \sum_{r=1}^{p}\chi(r)2^{5s/2}\sum_{k=0}^{\infty}(-1)^k\binom{-s}{k}\frac{\lambda_1^{-(s+2k)r}}{1-\lambda_1^{-(s+2k)p}}.$$

Hence for $s = -m$, we have

$$\mathcal{L}_B(-m,\chi) = \sum_{r=1}^{p}\chi(r)2^{-5m/2}\sum_{k=0}^{\infty}(-1)^k\binom{m}{k}\frac{\lambda_1^{(m-2k)r}}{1-\lambda_1^{(m-2k)p}}.$$

Proceeding like the proof of Theorem 6.2.1, we get

$$
\begin{aligned}
&\sum_{k=0}^{\infty}(-1)^k\binom{m}{k}\frac{\lambda_1^{(m-2k)r}}{1-\lambda_1^{(m-2k)p}}\\
&=\frac{1}{2}\sum_{k=0}^{m}\left[(-1)^k\binom{m}{k}\frac{\lambda_1^{(m-2k)r}}{1-\lambda_1^{(m-2k)p}}+(-1)^{m-k}\binom{m}{m-k}\frac{\lambda_1^{(-m+2k)r}}{1-\lambda_1^{(-m+2k)p}}\right]\\
&=\frac{1}{2}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\left[\frac{\lambda_1^{(m-2k)r}}{1-\lambda_1^{(m-2k)p}}+\frac{(-1)^m\lambda_1^{(-m+2k)r}}{1-\lambda_1^{(-m+2k)p}}\right]\\
&=\frac{1}{2}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\left[\frac{\lambda_1^{(m-2k)r}}{1-\lambda_1^{(m-2k)p}}+\frac{(-1)^m\lambda_2^{(m-2k)r}}{1-\lambda_2^{(m-2k)p}}\right]\\
&=\frac{1}{2}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{1}{1-\lambda_1^{(m-2k)p}}\left[\lambda_1^{(m-2k)r}+\frac{(-1)^m\lambda_2^{(m-2k)r}}{-\lambda_1^{-(m-2k)p}}\right]\\
&=\frac{1}{2}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{1}{1-\lambda_1^{(m-2k)p}}\left(\lambda_1^{(m-2k)r}-(-1)^m\lambda_1^{(m-2k)p-r}\right).
\end{aligned}
$$

Since, $m\equiv 1 \pmod 2$ we have

$$
\begin{aligned}
&\mathcal{L}_B(-m,\chi)\\
&=2^{\frac{-5m}{2}-1}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{1}{1-\lambda_1^{(m-2k)p}}\sum_{r=1}^{p}\chi(r)\left(\lambda_1^{(m-2k)r}+\lambda_1^{(m-2k)p-r}\right).
\end{aligned}
$$

As $\chi(-1)=-1$,

$$\sum_{r=1}^{p}\chi(r)\lambda_1^{(m-2k)r}=-\sum_{r=1}^{p}\chi(r)\lambda_1^{(m-2k)p-r}.$$

Thus,

$$\mathcal{L}_B(-m,\chi)=0.$$

This completes the proof of the theorem. □

Chapter 7

Periodicity of Balancing Numbers

7.1 Introduction

In the previous chapters, we discussed a lot about the recurrence relation $B_{n+1} = 6B_n - B_{n-1}, B_0 = 0, B_1 = 1$ satisfied by balancing numbers. Using this recurrence, we can easily prepare a small list of balancing numbers: $B_0 = 0, B_1 = 1, B_2 = 6, B_3 = 35, B_4 = 204, B_5 = 1189, B_6 = 6930, B_7 = 40391, B_8 = 235416, B_9 = 1372105, B_{10} = 7997214$. Notice that the repetition of the unit place digit starts from B_6. In the language of periodicity, the period of the balancing sequence modulo 10 is 6, i.e., after every six consecutive balancing numbers, the pattern of the unit place digits repeats. We describe this situation by saying that the period of the balancing sequence modulo 10 is 6.

The modular periods of a recurrence sequence have many interesting properties. In the year 1960, while studying the periodicity of Fibonacci numbers, Wall [89] conjectured that the period of the Fibonacci sequence modulo some prime p might be same with the period of the sequence modulo p^2, although he could not find any such prime in the first $10,000$ natural numbers. Recently, Elsenhans and Jahnel [30] extended this search for primes up to 10^{14} but could not find any such prime. As we will subsequently observe, the balancing numbers, in this regard, behave quite nicely; the period of the balancing sequence modulo $13, 31$ and 1546463 are same with the period of the sequence modulo $13^2, 31^2$ and 1546463^2 respectively. Another deficiency with the periods of Fibonacci sequence is that, no formula is available to calculate the modular period; however, the periods of the balancing sequence is computable for certain class of moduli.

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7.2 Definitions and properties

In this section, we first show that the balancing sequence modulo any natural number is periodic and then study certain divisibility property of its period. Throughout this paper, $\pi(n)$ denotes the least period (subsequently we simply call period) of the balancing sequence modulo n.

It is well known that $B_0 = 0$; a natural question is: "Given any natural number m, is there any natural number n such that $B_n \equiv 0 \pmod{m}$?" The following theorem answers this question in affirmative.

Theorem 7.2.1. *For any natural number m, there exists a natural number n such that* $B_n \equiv 0$ $(mod\ m)$.

Proof. It is well known that modulo m, there are m distinct least residues $0, \ldots, m-1$ and each balancing number is congruent to one of these residues. By pigeon hole principle, there exists two balancing numbers B_r and B_s $(r > s)$ both congruent to some i, $0 \leq i \leq m-1$ modulo m i.e., $B_r \equiv B_s \pmod{m}$. Hence, $B_r^2 \equiv B_s^2 \pmod{m}$ and by virtue of the identity $B_r^2 - B_s^2 = B_{r+s} \cdot B_{r-s}$ (see p. 18), it follows that

$$B_{r+s}B_{r-s} \equiv 0 \pmod{m}.$$

Therefore, either B_{r+s} or B_{r-s} is congruent to zero modulo m. □

It is known that for any integer k, B_n divides B_{nk} [70]. Thus, if $B_n \equiv 0 \pmod{m}$ then $B_{nk} \equiv 0 (\mathrm{mod}\ m)$. By virtue of Theorem 7.2.1, the set $S = \{n > 1 : B_n \equiv 0 (\mathrm{mod}\ m)\}$ is non-empty. By well ordering principle, the set S has a least element say $\tau(m)$. It has the following properties:
$B_{\tau(m)} \equiv 0 (\mathrm{mod}\ m)$ and if $n > 1$, $B_n \equiv 0 (\mathrm{mod}\ m)$ then $\tau(m)$ divides n.

Observe that if $B_n \equiv 0 \pmod{m}$ then using the identity $B_{n-1}B_{n+1} = B_n^2 - 1$ (see p. 18), we infer that if $B_{n-1} \equiv -1 \pmod{m}$, then $B_{n+1} \equiv 1 \pmod{m}$ and if $B_{n-1} \equiv 1 \pmod{m}$, then $B_{n+1} \equiv -1 \pmod{m}$. If $B_n \equiv 0$ and $B_{n+1} \equiv 1 \pmod{m}$ then $B_{kn} \equiv 0 \pmod{m}$ and by a simple mathematical induction on k, it can be easily verified that $B_{kn+1} \equiv 1 \pmod{m}$ for each natural number k. Further, if $B_n \equiv 0$ and $B_{n+1} \equiv -1 \pmod{m}$, then $B_{2n} = 2B_nC_n \equiv 0 \pmod{m}$ and $B_{2n+1} = B_{n+1}^2 - B_n^2 \equiv 1 \pmod{m}$ and thus $B_{2kn} \equiv 0 \pmod{m}$. Again by means of mathematical induction, it is easy to see that $B_{2kn+1} \equiv 1 \pmod{m}$ for each natural number k. Since the smallest value of n for which $B_n \equiv 0 \pmod{m}$ is $\tau(m)$, it is clear that, the minimum value of n for which $B_n \equiv 0, B_{n+1} \equiv 1 \pmod{m}$ is $\tau(m)$ if $B_{\tau(m)+1} \equiv 1 \pmod{m}$ and is $2\tau(m)$ if $B_{\tau(m)+1} \equiv -1 \pmod{m}$.

The above discussion proves the following theorem:

Theorem 7.2.2. *The sequence of balancing numbers modulo any natural number m is periodic.*

We are now in a position to define the period of the balancing sequence modulo any natural number m.

Definition 7.2.3. *A natural number t is called the period of the balancing sequence modulo m if* $B_t \equiv 0, B_{t+1} \equiv 1 \pmod{m}$ *and if for some natural number n,* $B_n \equiv 0, B_{n+1} \equiv 1 \pmod{m}$ *then t divides n.*

In view of Definition 7.2.3, we observe that for each natural number m, $\pi(m)$ is the smallest natural number to satisfy $B_{\pi(m)} \equiv 0$ and $B_{\pi(m)+1} \equiv 1 \pmod{m}$.

It is known that if m divides n then B_m divides B_n [70, Lemma 2.10], i.e., the balancing sequence is a divisibility sequence. The periods of balancing sequence modulo natural numbers also enjoy a similar property.

Theorem 7.2.4. *If m divides n, then* $\pi(m)$ *divides* $\pi(n)$.

Proof. Let m and n be natural numbers such that m divides n and let q be the quotient. Then $n = mq$ and we have $B_{\pi(mq)} \equiv 0$ and $B_{\pi(mq)+1} \equiv 1 \pmod{mq}$. But then $B_{\pi(mq)} \equiv 0$ and $B_{\pi(mq)+1} \equiv 1 \pmod{m}$ and by virtue of the definition of period, $\pi(m)$ divides $\pi(mq)$. □

The above discussion confirms that like the balancing sequence, the sequence of periods $\{\pi(n) : n = 1, 2, \dots\}$ is also a divisibility sequence. A similar property also holds for the Fibonacci sequence [89].

The following theorem provides a connection among periods of balancing sequence modulo m, n and mn. Similar result also exists for Fibonacci sequence [89].

Theorem 7.2.5. *If* $(m,n) = 1$ *then* $\pi(mn) = [\pi(m), \pi(n)]$.

Proof. Since both m and n divide mn, by virtue of the Theorem 7.2.4, both $\pi(m)$ and $\pi(n)$ divide $\pi(mn)$ and hence $[\pi(m), \pi(n)]$ divides $\pi(mn)$. Conversely, since $\pi(m)$ and $\pi(n)$ divide $[\pi(m), \pi(n)]$,

$$B_{[\pi(m),\pi(n)]} \equiv 0, B_{[\pi(m),\pi(n)]+1} \equiv 1 \pmod{m}$$

and

$$B_{[\pi(m),\pi(n)]} \equiv 0, B_{[\pi(m),\pi(n)]+1} \equiv 1 \pmod{n}.$$

Since $(m,n) = 1$,

$$B_{[\pi(m),\pi(n)]} \equiv 0, B_{[\pi(m),\pi(n)]+1} \equiv 1 \pmod{mn}$$

and hence $\pi(mn)$ divides $[\pi(m),\pi(n)]$. □

In [70], Panda has identified many interesting properties in which the balancing numbers behave like natural numbers. An important and interesting question concerning the periodicity of balancing sequence is that, like natural numbers, whether prime numbers can be the period of the balancing sequence modulo itself. The following theorem answers this question almost in negative.

Theorem 7.2.6. *If p is a prime, then p divides B_p if and only if $p = 2$.*

Proof. If $p = 2$ then $B_p = 6$ and hence p divides B_p. Conversely we will show that if p is an odd prime then p does not divide B_p. The use of Binet form for balancing numbers gives

$$B_p = \frac{(3+\sqrt{8})^p - (3-\sqrt{8})^p}{2\sqrt{8}} = \frac{2[\binom{p}{1}3^{p-1}\sqrt{8} + \binom{p}{3}3^{p-3}8\sqrt{8} + \cdots + (\sqrt{8})^p]}{2\sqrt{8}}.$$

Since, p divides $\binom{p}{r}$ for $r = 1,2,\ldots,p-1$, it follows that

$$B_p \equiv (\sqrt{8})^{p-1} = 8^{\frac{p-1}{2}} \pmod{p}$$

and the only prime that divides $8^{(p-1)/2}$ is 2. Hence, if p is an odd prime then p does not divide B_p. □

7.3 Periods of balancing sequence modulo primes

The next important aspect of periodicity of the balancing sequence is to link the period and the modulus of congruence. The following lemma will play a vital role in exploring periods of the balancing sequence modulo primes.

Lemma 7.3.1. *The number 8 is a quadratic residue modulo primes of the form $8x \pm 1$ and quadratic non-residue modulo primes of the form $8x \pm 3$.*

Proof. Let p be an odd prime. Then the Legendre symbol

$$\left(\frac{8}{p}\right) = \left(\frac{2^2\cdot 2}{p}\right) = \left(\frac{2^2}{p}\right)\left(\frac{2}{p}\right) = \left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod 8 \\ -1 & \text{if } p \equiv \pm 3 \pmod 8. \end{cases}$$

Hence, 8 is a quadratic residue modulo primes of the form $8x \pm 1$ and quadratic non-residue modulo primes of the form $8x \pm 3$. □

One of the most important work relating to periodicity of balancing numbers is to find explicit formulas for periods modulo primes. The following theorem, though does not give a formula to calculate periods modulo odd primes, provides adequate idea about numbers which are multiples of periods.

Theorem 7.3.2. *If p is a prime of the form $8x \pm 1$, then $B_{p-1} \equiv 0 \ (mod\ p)$ and $B_p \equiv 1 \ (mod\ p)$; further if the prime p is of the form $8x \pm 3$, then $B_p \equiv -1 \ (mod\ p)$ and $B_{p+1} \equiv 0 \ (mod\ p)$.*

Proof. We start with the Binet form of balancing numbers

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{4\sqrt{2}}; \ \lambda_1 = 3+2\sqrt{2}, \ \lambda_2 = 3-2\sqrt{2}.$$

Expanding λ_1^n and λ_2^n, we get

$$B_n = \frac{\binom{n}{1}3^{n-1}\cdot 2\sqrt{2} + \binom{n}{3}3^{n-3}\cdot 16\sqrt{2} + \cdots + \binom{n}{k}3^{n-k}\cdot 2^{3k/2}}{2\sqrt{2}} \tag{7.3.1}$$

where $k = 2\left[\frac{n-1}{2}\right]+1$, i.e., k is the largest odd number less than or equal to n. Since, for any odd prime p, $\binom{p}{k} \equiv 0$ (mod p) if $1 \le k < p$, it follows from (7.3.1) that

$$B_p \equiv 2^{3(p-1)/2} = 8^{(p-1)/2} \ (\text{mod } p).$$

By Euler's criterion [6, p. 180],

$$\left(\frac{8}{p}\right) \equiv 8^{(p-1)/2} \ (\text{mod } p).$$

Thus, by virtue of Lemma 7.3.1, the last two identities yield

$$B_p \equiv \begin{cases} 1 & \text{if } p \equiv \pm 1 \ (\text{mod } 8) \\ -1 & \text{if } p \equiv \pm 3 \ (\text{mod } 8) \end{cases}.$$

Further, we note that $\binom{p+1}{k} \equiv 0$ (mod p) if k is not a member of the set $\{0,1,p,p+1\}$. Now (7.3.1) yields,

$$B_{p+1} \equiv 3^p + 3\cdot 8^{\frac{p-1}{2}} \ (\text{mod } p)$$

and by Fermat's little theorem $3^p \equiv 3$ (mod p), for $p \ne 3$ we have

$$\frac{1}{3}B_{p+1} \equiv 1 + 8^{\frac{p-1}{2}} \ (\text{mod } p) \equiv 1 + \left(\frac{8}{p}\right) \ (\text{mod } p)$$

and by virtue of Lemma 7.3.1, $\left(\frac{8}{p}\right) = -1$ if $p \equiv \pm 3$ (mod 8). Thus, $B_{p+1} \equiv 0$ (mod p) if $p \ne 3$ and $p \equiv \pm 3$ (mod 8). If $p = 3$, then $B_{p+1} = 204 \equiv 0$ (mod p). Lastly, we observe that, $\binom{p-1}{k} \equiv (-1)^k$ (mod p) if $k = 0,1,\ldots,p-1$. Hence, in view of (7.3.1), we have

$$B_{p-1} \equiv -\left(3^{p-2} + 3^{p-4}\cdot 8 + \cdots + 3\cdot 8^{(p-3)/2}\right)$$

$$\equiv 3(8^{(p-1)/2} - 1) \equiv 3\left[\left(\frac{8}{p}\right) - 1\right] \pmod{p}.$$

Since, by Lemma 7.3.1, $p \equiv \pm 1 \pmod 8$ implies $\left(\frac{8}{p}\right) = 1$, $B_{p-1} \equiv 0 \pmod p$ follows. □

Theorem 7.3.2 has some important implications. Using this theorem, we can find natural numbers that are multiples of periods of the balancing sequence modulo primes. The following corollary identifies these multiples.

Corollary 7.3.3. *If p is a prime then $\pi(p)$ divides $p-1$ if $p \equiv \pm 1 \pmod 8$ and $\pi(p)$ divides $p+1$ if $p \equiv \pm 3 \pmod 8$. In other words, if p is an odd prime then $\pi(p)$ divides $p^2 - 1$.*

Proof. If p is a prime and $p \equiv \pm 1 \pmod 8$, then by Theorem 7.3.2, $B_{p-1} \equiv 0 \pmod p$, $B_p \equiv 1 \pmod p$ and by definition of period of the balancing sequence, $\pi(p)$ divides $p-1$. If $p \equiv \pm 3 \pmod 8$, then by Theorem 7.3.2, $B_p \equiv -1 \pmod p, B_{p+1} \equiv 0 \pmod p$, and by virtue of the recurrence relation $B_{n+1} = 6B_n - B_{n-1}, B_{p+2} \equiv 1 \pmod p$. Hence in this case, $\pi(p)$ divides $p+1$. Therefore, if p is an odd prime then $\pi(p)$ divides $p-1$ or $p+1$ and hence divides $p^2 - 1$. □

Corollary 7.3.3 provides an upper bound for period of the balancing sequence modulo primes. Since $\pi(2) = 2$, it is clear that for any prime p, $\pi(p) \le p+1$.

The converse of Corollary 7.3.3 is also true. The following theorem asserts that if for any prime p, $\pi(p)$ divides $p-1$ then $p \equiv \pm 1 \pmod 8$ and if $\pi(p)$ divides $p+1$ then $p \equiv \pm 3 \pmod 8$.

Theorem 7.3.4. *If p is a prime and $\pi(p)$ divides $p-1$, then $p \equiv \pm 1 \pmod 8$ and if $\pi(p)$ divides $p+1$ then $p \equiv \pm 3 \pmod 8$.*

Proof. Let $\pi(p)$ divide $p-1$ and assume to the contrary that $p \equiv \pm 3 \pmod 8$. Then $B_{p-1} \equiv 0 \pmod p$ and $B_p \equiv 1 \pmod p$. Since $p \equiv \pm 3 \pmod 8$, by virtue of Theorem 7.3.2, $B_p \equiv -1 \pmod p$ is a contradiction to $B_p \equiv 1 \pmod p$. Similarly, assume to the contrary that $\pi(p)$ divides $p+1$ and $p \equiv \pm 1 \pmod 8$. By definition of period, $B_{p+1} \equiv 0 \pmod p$ and $B_{p+2} \equiv 1 \pmod p$ and by virtue of the recurrence $B_{p-1} = 6B_p - B_{p+1}$, it follows that $B_{p-1} \equiv -1 \pmod p$ and if $p \equiv \pm 1 \pmod 8$, then by Theorem 7.3.2, $B_{p-1} \equiv 0 \pmod p$, which is a contradiction. □

In Theorems 7.3.2-7.3.4, we discussed the behaviour of $\pi(m)$ when m is a prime. The following theorem, which deals with the behaviour of $\pi(m)$ when m is a prime power, explores the resemblance of π with Euler totient function. Similar result also exists for the Fibonacci numbers [89].

Theorem 7.3.5. *If $\pi(p^2) \neq \pi(p)$, then $\pi(p^l) = p^{l-1}\pi(p)$. Further, if k is the largest integer such that $\pi(p^k) = \pi(p)$ and $l > k$, then $\pi(p^l) = p^{l-k}\pi(p)$.*

Proof. The congruence $B_{\pi(p^l)} \equiv 0(\text{mod } p^l)$ gives $B_{\pi(p^l)} = kp^l$ for some natural number k. By De-Moivre's Theorem for balancing numbers

$$C_{p\pi(p^l)} + \sqrt{8}B_{p\pi(p^l)} = (C_{\pi(p^l)} + \sqrt{8}B_{\pi(p^l)})^p.$$

Hence, for $l > 1$

$$B_{p\pi(p^l)} = k\binom{p}{1}C^{p-1}_{\pi(p^l)}p^l + 8k^3\binom{p}{3}C^{p-3}_{\pi(p^l)}p^{3l} + \cdots + 8^{\frac{p-1}{2}}k^p p^{pl} \equiv 0 \ (\text{mod } p^{l+1}). \quad (7.3.2)$$

We next show that for any natural number s,

$$B_{s\pi(p^l)+1} \equiv B^s_{\pi(p^l)+1} \ (\text{mod } p^{l+1}). \quad (7.3.3)$$

We prove this by induction. This result is obviously true for $s = 1$. Assume that (7.3.3) is true for $s \leq n$. Notice that

$$B_{(n+1)\pi(p^l)+1} = B_{n\pi(p^l)+1+\pi(p^l)} = B_{n\pi(p^l)+1}B_{\pi(p^l)+1} - B_{n\pi(p^l)}B_{\pi(p^l)}.$$

Since, by definition of period, p^l divides $B_{n\pi(p^l)}$ for each natural number n, $B_{n\pi(p^l)}B\pi(p^l)$ is divisible by p^{2l} and hence by p^{l+1} for each natural number l. Now, by the inductive hypothesis,

$$B_{(n+1)\pi(p^l)+1} \equiv B^n_{\pi(p^l)+1}B_{\pi(p^l)+1} = B^{n+1}_{\pi(p^l)+1}(\text{mod } p^{l+1}),$$

indicates that the hypothesis is also true for $s = n+1$. For $s = p$, we have

$$B_{p\pi(p^l)+1} \equiv B^p_{\pi(p^l)+1}(\text{mod } p^{l+1}).$$

But, by the definition of period, $B_{\pi(p^l)+1} \equiv 1(\text{mod } p^l)$ and hence, $B_{\pi(p^l)+1} = 1 + rp^l$ for some natural number r and

$$B_{p\pi(p^l)+1} \equiv (1+rp^l)^p = 1 + \sum_{i=1}^{p}\binom{p}{i}(rp^l)^i \equiv 1(\text{mod } p^{l+1}). \quad (7.3.4)$$

It is clear from (7.3.2) and (7.3.4) that $\pi(p^{l+1})$ divides $p\pi(p^l)$. Since $\pi(p^l)$ divides $\pi(p^{l+1})$, it follows that $\pi(p^{l+1}) = \pi(p^l)$ or $\pi(p^{l+1}) = p\pi(p^l)$. For $l = 1$, the conclusion is that $\pi(p^2) = \pi(p)$ or $\pi(p^2) = p\pi(p)$; so if $\pi(p^2) \neq \pi(p)$, then $\pi(p^2) = p\pi(p)$. Further, if k is the largest integer such that $\pi(p^k) = \pi(p)$, then $\pi(p^{k+t}) = p\pi(p^{k+t-1}) = \cdots = p^t\pi(p^k) = p^t\pi(p)$ for each natural number t. □

Example 7.3.6. *By manual calculation, it can be checked that $\pi(5) = 6$ and $\pi(25) = 30$*

a clear indication that $\pi(5) \neq \pi(5^2)$ *and hence* $\pi(5^2) = 5\pi(5) = 30$. *In contrast to this,* $\pi(13) = \pi(13^2) = 14$ *and* $\pi(13^3) = 182 = 13\pi(13)$.

7.4 Periods of balancing sequences modulo balancing, Pell and associated Pell numbers

It is well known that every balancing number is product of a Pell number and an associated Pell number. More precisely, $B_n = P_n Q_n$, where P_n and Q_n are respectively, the n^{th} Pell and n^{th} associated Pell numbers [73, Theorem 3.1]. Thus, it is meaningful to study periodicity of the balancing sequence modulo Pell and associated Pell numbers. Even the study of periodicity of the balancing sequence modulo balancing numbers and Lucas-balancing numbers would be interesting.

The following theorem asserts that the period of the balancing sequence modulo a balancing number or Lucas-balancing number is a multiple of the index of that number.

Theorem 7.4.1. *For any natural number* $k > 1$, $\pi(B_k) = 2k$ *and* $\pi(C_k) = 4k$.

Proof. Since

$$B_{2k} = 2B_k C_k \equiv 0 \pmod{B_k}$$

and

$$B_{2k+1} = 6B_{2k} - B_{2k-1} \equiv -B_{2k-1} = B_{k-1}^2 - B_k^2 \equiv B_k B_{k-2} + 1 \equiv 1 \pmod{B_k},$$

$\pi(B_k)$ divides $2k$. Also, modulo B_k, B_k is the first balancing number to vanish, while $B_{k+1} \equiv 6B_k - B_{k-1} \equiv B_k - B_{k-1} \pmod{B_k}$ which is not congruent to 1 modulo B_k unless $k = 1$. Hence $\pi(B_k) > k$. Since $\pi(B_k)$ divides $2k$, we have $k < \pi(B_k) \leq 2k$. Since, between k and $2k$ there is no proper divisor of $2k$, it follows that $\pi(B_k) = 2k$.

Similarly,

$$B_{4k} = 2B_{2k} C_{2k} = 4B_k C_k C_{2k} \equiv 0 \pmod{C_k}$$

and

$$\begin{aligned} B_{4k+1} &= 6B_{4k} - B_{4k-1} \\ &\equiv -B_{4k-1} = B_{2k-1}^2 - B_{2k}^2 \\ &\equiv B_{2k-1}^2 = B_{2k} B_{2k-2} + 1 \pmod{C_k} \end{aligned}$$

and since $B_{2k} = 2B_k Ck \equiv 0 \pmod{C_k}$, it follows that $B_{4k+1} \equiv 1 \pmod{C_k}$. Thus $\pi(C_k)$ divides $4k$. Further, modulo C_k, the first balancing number that vanishes is greater than B_k since $B_k < C_k$. Thus $\pi(C_k) > k$. We next show that for $k > 1$, B_{2k} is the first balancing number which is congruent to 0 modulo C_k. Since m divides n implies B_m divides B_n [70], it follows that if for some natural number x,

$$B_x \equiv 0 \pmod{m} \quad \text{then} \quad B_{xy} \equiv 0 \pmod{m}$$

for each natural number y. Let

$$S = \{n > 1 : B_n \equiv 0 \pmod{C_k}\}.$$

Since $B_{2k} \equiv 0 \pmod{C_k}$, it follows that the set S is non-empty. By well ordering principle, the set S has a least element l and hence, $2k = ul$ for some natural number u. If $u = 1$, then $l = 2k$. If $u \geq 2$, then $l = 2k/u \leq k$ which is not possible because $B_k < C_k$. Thus $l = 2k$ and since,

$$B_{2k+1} = 6B_{2k} - B_{2k-1} \equiv C_k - B_{k-1} \pmod{C_k}$$

and

$$C_k - B_{k-1} = 6C_{k-1} - C_{k-2} - B_{k-1} > 4C_{k-1} \geq 12 > 1,$$

it follows that B_{2k+1} is not congruent to 1 modulo C_k. Hence $\pi(C_k) > 2k$, implying that $2k < \pi(C_k) \leq 4k$ and since $\pi(C_k)$ divides $4k$, we must have $\pi(C_k) = 4k$. □

Since Pell and associated Pell numbers are factors of balancing numbers, it is important and interesting to study the periods of balancing sequence modulo Pell and associated Pell numbers. The following theorem provides formulas for the periods of the balancing sequence modulo Pell numbers.

Theorem 7.4.2. *For any natural number* k, $\pi(P_{2k}) = 2k$ *and* $\pi(P_{2k+1}) = 2(2k+1)$.

Proof. Since $P_{2k} = 2B_k$ [73, p. 46], by virtue of Theorems 7.2.4 and 7.4.1, $\pi(B_k) = 2k$ divides $\pi(P_{2k})$. We next show that $\pi(P_{2k})$ divides $2k$. For this, it is sufficient to show that

$$B_{2k} \equiv 0 \text{ and } B_{2k+1} \equiv 1 \pmod{P_{2k}}.$$

Since $B_{2k} = P_{2k}Q_{2k}$ [73, Theorem 3.1], it is obvious that $B_{2k} \equiv 0 \pmod{P_{2k}}$. The identities

$$\begin{aligned} B_{2k+1} &= 6B_{2k} - B_{2k-1} \\ &\equiv -B_{2k-1} = -P_{2k-1}Q_{2k-1} \pmod{P_{2k}} \end{aligned}$$

and

$$\begin{aligned} P_{2k-1}Q_{2k-1} &= (Q_{2k} - P_{2k})(P_{2k} - P_{2k-1}) \equiv -Q_{2k}P_{2k-1} \\ &= -(P_{2k+1} - P_{2k})P_{2k-1} \end{aligned}$$

$$\equiv -P_{2k+1}P_{2k-1} = -(P_{2k}^2 + 1) \equiv -1 \pmod{P_{2k}}$$

confirms that $B_{2k+1} \equiv 1 \pmod{P_{2k}}$. Thus, $\pi(P_{2k})$ divides $2k$ and combined with $2k$ divides $\pi(P_{2k})$, we get $\pi(P_{2k}) = 2k$.

To complete the proof of the theorem, we must show that $\pi(P_{2k+1}) = 2(2k+1)$. We first observe that

$$\begin{aligned} B_{2(2k+1)} &= P_{2(2k+1)}Q_{2(2k+1)} = 2B_{2k+1}Q_{2(2k+1)} \\ &= 2P_{2k+1}Q_{2k+1}Q_{2(2k+1)} \equiv 0 \pmod{P_{2k+1}} \end{aligned}$$

and

$$\begin{aligned} B_{2(2k+1)+1} &= 6B_{2(2k+1)} - B_{4k+1} \\ &\equiv -B_{4k+1} = -(B_{2k+1}^2 - B_{2k}^2) \\ &\equiv B_{2k}^2 = B_{2k-1}B_{2k+1} + 1 \\ &\equiv 1 \pmod{P_{2k+1}} \end{aligned}$$

and hence, $\pi(P_{2k+1})$ divides $2(2k+1)$. Since $B_{2k+1} \equiv 0 \pmod{P_{2k+1}}$ but

$$\begin{aligned} B_{2k+2} &\equiv -B_{2k} = -P_{2k}Q_{2k} \\ &= -P_{2k}(P_{2k+1} - P_{2k}) \\ &\equiv P_{2k}^2 = P_{2k+1}P_{2k-1} - 1 \equiv -1 \pmod{P_{2k+1}}, \end{aligned}$$

it follows that $\pi(P_{2k+1}) > 2k+1$. Thus,

$$2k+1 < \pi(P_{2k+1}) \leq 2(2k+1)$$

and since $\pi(P_{2k+1})$ divides $2(2k+1)$, we conclude that $\pi(P_{2k+1}) = 2(2k+1)$. □

Theorem 7.4.2 asserts that the period of the balancing sequence modulo an even Pell number is equal to its index, while the period modulo an odd Pell number is twice its index. In contrast to these observations, the reverse is true for the balancing sequence modulo associated Pell numbers.

Theorem 7.4.3. *For any natural number k, $\pi(Q_{2k}) = 4k$ and $\pi(Q_{2k+1}) = 2k+1$.*

Proof. Since $Q_{2k} = C_k$, by virtue of Theorems 7.4.1 $\pi(Q_{2k}) = 4k$. Further, $B_{2k+1} =$

$P_{2k+1}Q_{2k+1} \equiv 0 \pmod{Q_{2k+1}}$ and

$$\begin{aligned} B_{2k+2} &= 3B_{2k+1} + C_{2k+1} \equiv C_{2k+1} = Q_{2(2k+1)} \\ &= Q_{2k+1}^2 + 2P_{2k+1}^2 \equiv 2P_{2k+1}^2 \\ &= 1 + Q_{2k+1}^2 \equiv 1 (\bmod Q_{2k+1}) \end{aligned}$$

clearly indicate that $\pi(Q_{2k+1})$ divides $2k+1$. Also $B_k < C_k = Q_{2k} < Q_{2k+1}$ which shows that modulo Q_{2k+1}, none of the first k balancing numbers vanishes and hence, $\pi(Q_{2k+1}) > k$. Since there is no proper divisor of $2k+1$ greater than k, it follows that $\pi(Q_{2k+1}) = 2k+1$. □

7.5 A fixed point theorem for π

In the previous section, we have seen that if p is an odd prime then $\pi(p)$ divides $p-1$ or $p+1$ according as $p \equiv \pm 1 (\bmod 8)$ or $p \equiv \pm 3 \pmod 8$ and if $\pi(p) \neq \pi(p^2)$ then $\pi(p^n) = p^{n-1}\pi(p)$ for each natural number n. Thus, for some odd prime p, possibly $\pi(p)$ is equal to $p-1$ or $p+1$ and

$$\pi(p^n) = (p-1)p^{n-1} \text{ or } \pi(p^n) = (p+1)p^{n-1}$$

and hence $\pi(p^n) = p^n$ never occurs. But it is easy to see that $\pi(1) = 1, \pi(2) = 2, \pi(4) = 4$ and so on. The following theorem deals with the fixed points of the arithmetic function π.

Theorem 7.5.1. *For any natural number $n > 1$, $\pi(n) = n$ if and only if $n = 2^k$ for some natural number k.*

Proof. We first assume that $n = 2^k$ for some natural number k. Since $\pi(2) \neq \pi(2^2)$, by virtue of Theorem 7.3.5, $\pi(n) = 2^{k-1}\pi(2)$. Since $\pi(2) = 2$, we have $\pi(n) = 2^{k-1} \cdot 2 = 2^k = n$.

Conversely assume that n is a natural number with at least one odd prime factor. We will show that $\pi(n) \neq n$. Using the fundamental theorem of arithmetic, we can factorize n as $n = 2^r p_1^{e_1} p_2^{e_2} \ldots p_l^{e_l}$ where $r \geq 0, e_i > 0$ for $i = 1, 2, \ldots, l$. We may assume without loss of generality that $2 < p_1 < \cdots < p_l$. By virtue of Theorems 7.2.5 and 7.3.5

$$\pi(n) = [\pi(2^r), \pi(p_1^{e_1}), \ldots, \pi(p_l^{e_l})] = [2^r, p_1^{f_1}\pi(p_1), \ldots, p_l^{f_l}\pi(p_l)]$$

where $f_1 < e_1, f_2 < e_2, \ldots, f_l < e_l$ are suitable non-negative integers. Since for any prime p, the maximum value of $\pi(p)$ is $p+1$ and $\pi(p)$ divides $p+1$ or $p-1$, it follows that p does not divide $\pi(p)$. If p and q are odd primes and $q < p$, then $\pi(q) \leq q+1 < p$ and hence p does not divide $\pi(q)$. In the present case, $p_1, p_2, \ldots, p_l$ are odd primes, $p_1 < p_2 < \cdots < p_l$ and $p_l^{e_l}$ divides n; but $p_l^{e_l}$ does not divide any of $2^r, p_1^{f_1}\pi(p_1), \ldots, p_l^{f_l}\pi(p_l)$ and thus $p_l^{e_l}$ does not divide $\pi(n)$. Hence $\pi(n)$ can never be equal to n, if n has an odd prime factor. □

Chapter 8

Stability of the Balancing Sequence

8.1 Introduction

In the last chapter, we discussed about the periodicity of the balancing sequence with respect to various class of moduli. What we observed is that the periods of the balancing sequence are computable for certain class of moduli, while, for some other classes, only a multiple of the period is available. Given any positive integer m, we denoted the period of the balancing sequence by $\pi(m)$. By definition, $\pi(m)$ is the smallest natural number to satisfy $B_{\pi(m)} \equiv 0$ and $B_{\pi(m)+1} \equiv 1 \pmod{m})$. The rank of apparition or simply rank of the balancing sequence modulo m is the least positive integer r such that $B_r \equiv 0 \pmod{m})$ and is denoted by $\alpha(m)$, i.e., $\alpha(m)$ is the index of the first non-zero balancing number B_n which is divisible by m. Niven [66] introduced the notion of uniform distribution of an integer sequence as follows. He called a sequence $\mathcal{A} = \{a_n : n = 0, 1, \dots\}$ uniformly distributed modulo $m \geq 2$ if

$$\lim_{N\to\infty} \frac{1}{N} \#\{n < N : a_n \equiv b \pmod{m})\} = \frac{1}{m}$$

for any $b \in \{0, 1, \dots, m-1\}$.

For a fixed modulus m and residue r, we denote the number of occurrences of the residue r in a period of the balancing sequence by $\nu_B(m,r)$. This function is the frequency distribution function of balancing sequence modulo m. In the early 1970's, interest in the distribution functions of binary recurrence sequences centered on the characterization of those sequences that have constant frequency distribution function, i.e., sequences that are uniformly distributed. Denoting by

$$\Omega_B(m) = \{\nu_B(m,r) : r \in \{0, 1, \dots, m-1\}\}, \qquad (8.1.1)$$

the set of all frequencies of residues modulo m in a period, the balancing sequence is uniformly distributed whenever $\#\{\Omega_B(m)\} = 1$. Stability of the balancing sequence comes into picture when $\#\{\Omega_B(m)\}$ is not constant and this generalize the concept of uniform distribution and also the notion of f-uniform distribution modulo prime powers [85]. The concrete and precise definition of stability was due to Carlip and Jacobson [19]. We prefer to state this definition for the balancing sequence, though it can be stated for any arbitrary sequence.

Definition 8.1.1. *The balancing sequence is said to be stable modulo a prime p if there is a positive integer N such that $\Omega_B(p^k) = \Omega_B(p^N)$ for all $k \geq N$.*

Bundschuh [18] studied the stability of Lucas sequence modulo 2 and 5. He showed that Lucas sequence is not stable modulo 2 and 5. Several authors studied the stability of binary recurrence sequences modulo certain numbers. As we will see, the balancing sequence is stable for two particular class of primes. In this connection, we will completely describe the function $\nu_B(p^k,\cdot)$ and show that $\nu_B(2^k,\cdot) = \{1\}$ and hence $\Omega_B(2^k) = \{1\}$, $\nu_B(p^k,\cdot) = \{1\}$ when $p \equiv -1 \pmod 8$ and $\nu_B(p^k,\cdot) = \{2\}$ when $p \equiv -3 \pmod 8$. These results would confirm the stability of the balancing sequence modulo p when $p \equiv -1,-3 \pmod 8$. Finally we have shown that balancing sequence is not stable modulo primes $p \equiv 3 \pmod 8$. However, for some primes (not all) $p \equiv 1 \pmod 8$ the balancing sequence is stable.

Throughout this chapter, p represents an odd prime. For any non-zero integer a, $\text{ord}_p a = m$ if $p^m \mid a$ but $p^{m+1} \nmid a$. Important properties of ord_p are

$$\text{ord}_p(ab) = \text{ord}_p(a) + \text{ord}_p(b),\ \text{ord}_p(a+b) \geq \min(\text{ord}_p(a), \text{ord}_p(b)).$$

Thus $a \equiv b (\text{mod } p^k)$ is equivalent to $\text{ord}_p(a-b) \geq k$ [47]. We also need the following results concerning the balancing sequence and its rank of apparition. These results will be used in the subsequent sections.

Lemma 8.1.2. *If the integers m and n are of the same parity, then*

$$B_m - B_n = 2B_{\frac{m-n}{2}} C_{\frac{m+n}{2}}, \tag{8.1.2}$$

$$C_m - C_n = 16 B_{\frac{m-n}{2}} B_{\frac{m+n}{2}}. \tag{8.1.3}$$

Proof. It is well known that $B_{x\pm y} = B_x C_y \pm C_x B_y$ (see p. 18). Thus

$$B_{x+y} - B_{x-y} = 2B_y C_x;$$

and taking $x+y = m$, $x-y = n$, we get

$$B_m - B_n = 2B_{\frac{m-n}{2}} C_{\frac{m+n}{2}}.$$

Similarly by virtue of the formula

$$C_{x\pm y} = C_x C_y \pm 8B_x B_y,$$

$C_m - C_n = 16B_{\frac{m-n}{2}}B_{\frac{m+n}{2}}$ follows. □

Lemma 8.1.3. *If $B_n \equiv 0 \pmod p$ then $B_{2n} \equiv 0 \pmod p$ and $B_{2n+1} \equiv 1 \pmod p$.*

Proof. Since $B_n \equiv 0 \pmod p$,

$$B_{2n} = 2B_n C_n \equiv 0 \pmod p$$

and

$$B_{2n+1} = B_n B_{n+2} - B_{n-1}B_{n+1} \equiv -(B_n^2 - 1) \equiv 1 \pmod p$$

since $B_{n+1}B_{n-1} = B_n^2 - 1$ (see p. 17). □

Lemma 8.1.4. *For any prime p, $\pi(p) = \alpha(p)$ or $\pi(p) = 2\alpha(p)$.*

Proof. Since $B_{\alpha(p)} \equiv 0 \pmod p$ by Lemma 8.1.3, we get $B_{2\alpha(p)} \equiv 0 \pmod p$ and $B_{2\alpha(p)+1} \equiv 1 \pmod p$. Thus, $\pi(p) \mid 2\alpha(p)$ and hence $\pi(p) = \alpha(p)$ or $\pi(p) = 2\alpha(p)$. □

Lemma 8.1.5. *If $\alpha(p^2) \neq \alpha(p)$, then $\alpha(p^{l+1}) = p^l\alpha(p)$ for $l \geq 1$.*

Proof. The congruence $B_{\alpha(p^l)} \equiv 0 (\text{mod } p^l)$ gives $B_{\alpha(p^l)} = kp^l$ for some natural number k. By De-Moivre's Theorem for balancing numbers [70]

$$C_{p\alpha(p^l)} + \sqrt{8}B_{p\alpha(p^l)} = (C_{\alpha(p^l)} + \sqrt{8}B_{\alpha(p^l)})^p. \tag{8.1.4}$$

Using binomial theorem in (8.1.4), we get

$$B_{p\alpha(p^l)} = k\binom{p}{1}C_{\alpha(p^l)}^{p-1}p^l + 8k^3\binom{p}{3}C_{\alpha(p^l)}^{p-3}p^{3l} + \cdots + 8^{\frac{p-1}{2}}k^p p^{pl} \equiv 0 (\text{mod } p^{l+1}).$$

It is clear from above equation that $\alpha(p^{l+1})$ divides $p\alpha(p^l)$. Since $\alpha(p^l)$ divides $\alpha(p^{l+1})$, it follows that

$$\alpha(p^{l+1}) = \alpha(p^l) \text{ or } \alpha(p^{l+1}) = p\alpha(p^l).$$

For $l = 1$, the conclusion is that

$$\alpha(p^2) = \alpha(p) \text{ or } \alpha(p^2) = p\alpha(p);$$

so if

$$\alpha(p^2) \neq \alpha(p), \text{ then } \alpha(p^2) = p\alpha(p).$$

Continuing in this process we will arrive at $\alpha(p^{l+1}) = p^l\alpha(p)$. □

The following lemma, which relates the order of B_n with order of n, will play a crucial role.

Lemma 8.1.6. *If $n \in \mathbb{Z}$ and p is any arbitrary prime then $\alpha(p) \mid n$ if and only if $p \mid B_n$. Furthermore, if $\alpha(p) \mid n$, then*

$$\text{ord}_p B_n = 1 + \text{ord}_p n.$$

Proof. The proof of the first part follows directly from the definition of $\alpha(p)$. To prove the second part, let us suppose that

$$\text{ord}_p B_n = t \text{ and } \text{ord}_p n = s. \tag{8.1.5}$$

From second part of (8.1.5), $n = kp^s$ where $(k, p) = 1$, and $\alpha(p) \mid n$ implies $\alpha(p) \mid kp^s$. Since $\alpha(p) \mid p^2 - 1$, by Corollary 7.3.3, $(\alpha(p), p) = 1$. Thus $\alpha(p) \mid k$ gives $k = a\alpha(p)$ for some integer a. Putting the value of k in n, we get

$$n = a\alpha(p)p^s. \tag{8.1.6}$$

By definition, $p^t \| B_n$ if and only if $\alpha(p^t) \| n$. By Lemma 8.1.5, $p^{t-1}\alpha(p) \| n$. Using the value of n in (8.1.6), we get

$$p^{t-1}\alpha(p) \| a\alpha(p)p^s$$

which implies $p^{t-1} \| ap^s$. Since $(k, p) = 1$ and $k = a\alpha(p)$, we have $(a, p) = 1$. Therefore $t - 1 = s$. □

Similar results hold for Lucas balancing numbers. The proof of the following lemma is similar to that of Lemma 8.1.6 and hence it is omitted.

Lemma 8.1.7. *Let $n \in \mathbb{Z}$ and for any prime $p \equiv 3 \ (mod\ 8)$ define $\beta(p) = \min\{r : C_r \equiv 0 \ (mod\ p)\}$. Then*

$$\beta(p) \mid n \Leftrightarrow p \mid C_n \text{ and } \beta(p) \mid n \Rightarrow \text{ord}_p C_n = 1 + \text{ord}_p n.$$

8.2 Stability of balancing sequence modulo 2

The Fibonacci sequence is stable modulo 2 and 5 [44]. In this section, we will show that the balancing sequence is also stable modulo 2.

Theorem 8.2.1. $\nu_B(2^k, b) = 1$ *for every residue b modulo 2^k and for every $k \in \mathbb{N}$.*

Proof. By virtue of Theorem 7.5.1, $\pi(n) = n$ if and only if $n = 2^k$ for any $k = 0, 1, \dots$ Using this result, we will show that each residue $b \in \{0, 1, \dots, 2^k - 1\}$ occurs only once in a period modulo 2^k. Since B_n is even or odd according as n is even or odd, it follows that the least residue of $B_n, 0 \leq n \leq 2^k - 1$ modulo 2^k is also even or odd according as n is even or odd. To complete the proof, we have to show that no two least residue of $B_n, 0 \leq n \leq 2^k - 1$ are congruent modulo 2^k. Since B_{2m+1} and B_{2n} are incongruent modulo 2^k, it is sufficient to show that

$$B_{2m+1} \not\equiv B_{2n+1} \pmod{2^k} \text{ for } 0 < 2m+1 < 2n+1 < 2^k, \tag{8.2.1}$$

and

$$B_{2i} \not\equiv B_{2j} \pmod{2^k} \text{ for } 0 \leq 2i < 2j \leq 2^k. \tag{8.2.2}$$

Since $\pi(2^k) = 2^k$, it follows that $2^k \mid B_n$ if and only if $2^k \mid n$. Let us assume the contrary of (8.2.1). Then

$$B_{2m+1} \equiv B_{2n+1} \pmod{2^k} \text{ for } 0 < 2m+1 < 2n+1 < 2^k,$$

hence, 2^k divides $B_{2n+1} - B_{2m+1}$. Using (8.1.2), $2^k \mid 2B_{n-m}C_{n+m+1}$ implies

$$2^{k-1} \mid B_{n-m} \text{ as } (2, C_x) = 1$$

for any natural number x. Then $2^{k-1} \mid n-m$ which is a contradiction since $n - m < 2^{k-1}$. Thus (8.2.1) holds and similarly (8.2.2) can be proved. □

From equation (8.1.1), we have $\Omega_B(2^k) = \{1\}$. The following corollary is a consequence of the above theorem which ascertains the stability of balancing sequence modulo 2.

Corollary 8.2.2. *The balancing sequence is stable modulo* 2.

8.3 Stability of balancing sequence modulo primes $p \equiv -1, -3 \pmod 8$

In this section, we will show that balancing sequence is stable for a large class of primes. These primes are precisely those $\equiv -1, -3 \pmod 8$. To establish this claim, we need help of a series of lemmas.

Lemma 8.3.1. *If $A = \{a_1, a_2, \ldots, a_r\}$ are distinct residues modulo p, then $A + mp$, $m = 0, 1, \ldots, p^{k-1} - 1$ are also distinct residues modulo p^k.*

Proof. Suppose that for some integers l, m and $0 \leq i, j \leq p^{k-1} - 1$,

$$a_l + ip \equiv a_m + jp \pmod{p^k}. \tag{8.3.1}$$

Then

$$p^k \mid (i-j)p + (a_l - a_m),$$

which shows that p divides $a_l - a_m$. In other words, $a_l \equiv a_m \pmod p$, which is a contradiction to (8.3.1). □

Lemma 8.3.2. *If $p \equiv -1 \pmod 8$, then $\pi(p) \mid \frac{p-1}{2}$. Furthermore, $\pi(p)$ is odd.*

Proof. If $p \equiv -1 \pmod 8$, then $p = 8x-1$ for some integer x. Then by Corollary 7.3.3, $\pi(p) \mid p-1 = 8x-2$. Thus,

$$B_{8x-2} \equiv 0, \text{ and } B_{8x-3} \equiv -B_1,\ B_{8x-4} \equiv -B_2 \pmod p)$$

and so on. In other words,

$$B_r + B_{8x-2-r} \equiv 0 \pmod p \ \text{ for } \ r = 1,2,\ldots,4x-2.$$

In particular,

$$B_{4x-2} + B_{4x} \equiv 0 \pmod p \ \text{ gives } \ 6B_{4x-1} \equiv 0 \pmod p.$$

Hence

$$B_{4x-1} = B_{\frac{p-1}{2}} \equiv 0 \pmod p \ \text{ as } \ (6,p) = 1.$$

We claim that $B_{\frac{p+1}{2}} \equiv 1 \pmod p$. Observe that

$$\text{ord}_p(B_{\frac{p+1}{2}} - B_1) = \text{ord}_p(2 \cdot B_{\frac{p-1}{2}} \cdot C_{\frac{p+3}{2}}) = 0 + 1 + \text{ord}_p\left(\frac{p-1}{2}\right) + \text{ord}_p(C_{\frac{p+3}{2}}) \geq 1,$$

which shows that $B_{\frac{p+1}{2}} \equiv 1 \pmod p$ and combining with $B_{\frac{p-1}{2}} \equiv 0 \pmod p$ we conclude that

$$\pi(p) \Big| \frac{p-1}{2} = 4x - 1$$

which implies that $\pi(p)$ is odd. □

Lemma 8.3.3. *If $p \equiv -1 \pmod 8$, then $\pi(p) = \alpha(p)$.*

Proof. In view of Lemma 8.1.4, $\pi(p) = \alpha(p)$ or $\pi(p) = 2\alpha(p)$. But by Lemma 8.3.2, $\pi(p)$ is odd, hence the lemma. □

Lemma 8.3.4. *If $p \equiv -3 \pmod 8$, then $B_{\frac{p+1}{2}} \equiv 0 \pmod p$.*

Proof. Let $p = 8x-3$. By Corollary 7.3.3

$$B_p \equiv -1 \pmod p, \ \ B_{p+1} \equiv 0 \pmod p. \tag{8.3.2}$$

Using the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$ (see p. 17) and (8.3.2) it is easy to see that

$$B_{\frac{p-1}{2}} + B_{\frac{p+3}{2}} \equiv 0 \pmod p. \tag{8.3.3}$$

Hence, $6B_{\frac{p+1}{2}} \equiv 0 \pmod p$ and $(6,p) = 1$ implies $B_{\frac{p+1}{2}} \equiv 0 \pmod p$. □

Lemma 8.3.5. *If $p \equiv -3 \pmod 8$, then for every x such that $0 \le x \le \frac{\pi(p)}{2}$,*

$$B_x \equiv B_{\frac{\pi(p)}{2}-x} \pmod p \text{ and } B_x \equiv -B_{\frac{\pi(p)}{2}+x} \pmod p.$$

Furthermore, $\pi(p) = 2\alpha(p)$.

Proof. Let us assume that $B_x \equiv B_y \pmod p$ for some $0 \le y < x \le \frac{\pi(p)}{2}$. Then

$$C_x \equiv \pm C_y \pmod p, \text{ since } C_n = \sqrt{8B_n^2+1}.$$

Therefore we have $B_{x\pm y} \equiv 0 \pmod p$. By Lemma 8.3.4, for $0 \le x \le \frac{\pi(p)}{2}$,

$$B_x = 0 \text{ iff } x = 0, \frac{\pi(p)}{2}.$$

Hence

$$x \pm y = 0 \text{ or } \frac{\pi(p)}{2}.$$

We observe that $x - y = 0$ gives trivial solution $x = y$ which is not possible since $x > y$. Again, $x + y = 0$ gives $x = -y$ which is also not possible since both x and y are non-negative and $x > y$. $x - y = \frac{\pi(p)}{2}$ gives $x = \frac{\pi(p)}{2} + y$. This is absurd since $0 \le x \le \frac{\pi(p)}{2}$. Thus, we are left with one option $x + y = \frac{\pi(p)}{2}$ or equivalently, $x = \frac{\pi(p)}{2} - y$. Hence $B_y \equiv B_{\frac{\pi(p)}{2}-y} \pmod p$. From (8.3.3) we have $B_{\frac{\pi(p)}{2}-k} \equiv -B_{\frac{\pi(p)}{2}+k} \pmod p$. Thus, $B_x \equiv -B_{\frac{\pi(p)}{2}+x} \pmod p$ and the proof is complete. □

We next prove two important theorem assuring the stability of the balancing sequence modulo p for $p \equiv -1, -3 \pmod 8$.

Theorem 8.3.6. *If $p \equiv -3 \pmod 8$ and $k \in \mathbb{N}$, then $\nu_B(p^k, b) = 2$ for each feasible residue b modulo p^k. Hence the balancing sequence is stable modulo p for $p \equiv -3 \pmod 8$.*

Proof. First, we will show that the number of occurrences of each feasible residue modulo p in a period is 2. In Lemma 8.3.5, we have shown that each feasible residue of the balancing numbers B_x modulo p occurs twice for $x \in \{0, 1, \ldots, \frac{\pi(p)}{2}\}$. Since $B_x \equiv -B_{\frac{\pi(p)}{2}+x} \pmod p$, we have

$$\#\{x : B_x \equiv b \pmod p\} = 2 \text{ for each } x \in \{0, 1, \ldots, \pi(p)-1\}$$

and hence $\nu_B(p, b) = 2$ for each feasible residue b modulo p. Using Lemma 8.3.1 we get $\nu_B(p^k, b) = 2$ for each feasible residue b modulo p^k. □

From (8.1.1), $\Omega_B(p^k) = \{2\}$.

Theorem 8.3.7. *If $p \equiv -1 \pmod 8$ and $k \in \mathbb{N}$, then $\nu_B(p^k, b) = 1$ for each feasible residue b modulo p^k. Hence the balancing sequence is stable modulo p for $p \equiv -1 \pmod 8$.*

Proof. Since $p \equiv -1 \pmod 8$, by Lemma 8.3.3 $\pi(p) = \alpha(p)$. Therefore, each feasible residue b occurs only once such that $B_r \equiv b \pmod p$ for $0 \le r < \pi(p)$; otherwise $\alpha(p) < \pi(p)$. Then Lemma 8.3.1 confirms that $\nu_B(p^k, b) = 1$ for each feasible residue b of the balancing sequence modulo p^k. □

In view of (8.1.1), $\Omega_B(p^k) = \{1\}$.

8.4 Stability of balancing sequence modulo primes $p \equiv 1, 3 \pmod 8$

Modulo 8, there are four classes of primes $p \equiv \pm 1, \pm 3 \pmod 8$. In the last section, we have proved that the balancing sequence is stable modulo p for $p \equiv -1, -3 \pmod 8$. But unfortunately, it is not stable modulo primes $p \equiv 1, 3 \pmod 8$. For certain primes of this class, the balancing sequence is indeed stable.

The following lemmas, describing the structure of the period and behaviour of balancing numbers occurring in a period, will play crucial roles while proving the main results of this section.

Lemma 8.4.1. *If* $p \equiv 3 \pmod 8$*, then* $4 \mid \pi(p)$.

Proof. For $p = 3, \pi(p) = 4$, hence $4 \mid \pi(p)$. For $P > 3$ and $p \equiv 3 \pmod 8$, firstly, we will prove

$$B_{\frac{p-1}{2}} \equiv 1 \pmod p. \tag{8.4.1}$$

Observe that

$$\mathrm{ord}_p(B_{\frac{p-1}{2}} - B_1) = \mathrm{ord}_p(2B_{\frac{p-3}{4}} C_{\frac{p+1}{4}}) = \mathrm{ord}_p(B_{\frac{p-3}{4}}) + \mathrm{ord}_p(C_{\frac{p+1}{4}}). \tag{8.4.2}$$

In view of Corollary 7.3.3, $\pi(p) \mid p+1$. Using this result in (8.4.2), we get

$$\mathrm{ord}_p(B_{\frac{p-1}{2}} - B_1) = 0 + 1 + \mathrm{ord}_p\left(\frac{p+1}{4}\right) \ge 1. \tag{8.4.3}$$

Proceeding as in Lemma 8.3.4 and using (8.3.2), it is easy to see that $B_{\frac{p+1}{2}} \equiv 0 \pmod p$. Using recurrence relation $B_n = 6B_{n-1} - B_{n-2}$ and (8.4.1), we get $B_{\frac{p+3}{2}} \equiv 1 \pmod p$ which confirms that $\pi(p) \nmid (p+1)/2 = 4x+2$; but $\pi(p) \mid p+1 = 8x+4$ which implies that $4 \mid \pi(p)$. □

Lemma 8.4.2. *If* $p \equiv 3 \pmod 8$ *and* $x \in \mathbb{N}$*, then*

$$B_{p^x \frac{\pi(p)}{4}} \not\equiv B_{3p^x \frac{\pi(p)}{4}} \pmod p.$$

Proof. First we will show that for $x \in \mathbb{N}$,

$$B_{p^x \frac{\pi(p)}{4}} \equiv (-1)^x B_{\frac{\pi(p)}{4}} \pmod{p}.$$

Assume that x is even. Then

$$\mathrm{ord}_p(B_{p^x \frac{\pi(p)}{4}} - B_{\frac{\pi(p)}{4}}) = \mathrm{ord}_p\left(2B_{\frac{\pi(p)}{4} \cdot \frac{p^x-1}{2}} C_{\frac{\pi(p)}{4} \cdot \frac{p^x+1}{2}}\right).$$

Since x is even, $\alpha(p) \mid \frac{\pi(p)}{4}\frac{p^x-1}{2}$. Using Lemma 8.1.6 we find

$$\begin{aligned}
&\mathrm{ord}_p(B_{p^x \frac{\pi(p)}{4}} - B_{\frac{\pi(p)}{4}}) \\
&= \mathrm{ord}_p 2 + 1 + \mathrm{ord}_p\left(\frac{\pi(p)}{4}\left(\frac{p^x-1}{2}\right)\right) + \mathrm{ord}_p\left(C_{\frac{\pi(p)}{4}(\frac{p^x+1}{2})}\right) \\
&\geq 1.
\end{aligned}$$

Now let x be odd. Since $B_{-n} = -B_n$, it can be easily proved that

$$B_{p^x \frac{\pi(p)}{4}} \equiv -B_{\frac{\pi(p)}{4}} \pmod{p}.$$

A similar argument as above will lead to

$$B_{3 \cdot p^x \frac{\pi(p)}{4}} \equiv (-1)^x B_{3 \cdot \frac{\pi(p)}{4}} \pmod{p}.$$

To complete the proof, it remains to show that $B_{\frac{\pi(p)}{4}} \not\equiv B_{3 \cdot \frac{\pi(p)}{4}} \pmod{p}$. It is obvious since

$$B_{\frac{\pi(p)}{4}} \equiv -B_{3 \cdot \frac{\pi(p)}{4}} \pmod{p}.$$ □

Lemma 8.4.3. *If* $p \equiv 3 \ (mod\ 8)$ *and* $k \in \mathbb{N}$, *then there are two distinct feasible residues of* B_n *with* $0 \leq n < \pi(p)p^{k-1}$ *occurring at least* $p^{[k/2]}$ *times in a period modulo* p^k.

Proof. Let us assume that $n, j \in \mathbb{Z}$ and $\pi\left(p^{[(k-1)/2]+1}\right) | n$. We claim that

$$B_{n + \frac{\pi(p)}{4}(1+2j)p^{[(k-1)/2]}} \equiv B_{\frac{\pi(p)}{4}(1+2j)p^{[(k-1)/2]}} \pmod{p^k}. \tag{8.4.4}$$

Indeed, if $p \equiv 3 \pmod 8$, by virtue of Lemma 8.4.1, $4 \mid \pi(p)$ and $\pi(p) \mid n$ implies $4 \mid n$. Thus,

$$\begin{aligned}
&\mathrm{ord}_p(B_{n + \frac{\pi(p)}{4}(1+2j)p^{[(k-1)/2]}} - B_{\frac{\pi(p)}{4}(1+2j)p^{[(k-1)/2]}}) \\
=&\mathrm{ord}_p\left(2B_{\frac{n}{2}} C_{\frac{n}{2} + \frac{\pi(p)}{4}(1+2j)p^{[(k-1)/2]}}\right) \\
=&\mathrm{ord}_p(2) + \mathrm{ord}_p(B_{\frac{n}{2}}) + \mathrm{ord}_p\left(C_{\frac{n}{2} + \frac{\pi(p)}{4}(1+2j)p^{[(k-1)/2]}}\right) \\
=&0 + 1 + \mathrm{ord}_p\left(\frac{n}{2}\right) + 1 + \mathrm{ord}_p\left(\frac{n}{2} + \frac{\pi(p)}{4}(1+2j)p^{[(k-1)/2]}\right)
\end{aligned}$$

$$\geq 2(1+[(k-1)/2]) > 2(1+(k-1)/2-1) = k-1,$$

which proves (8.4.4). Since $\pi\big(p^{[(k-1)/2]+1}\big) \mid n$ by assumption, $\pi(p)p^{[(k-1)/2]} \mid n$. Therefore,

$$n = \pi(p)p^{[(k-1)/2]}i \ \text{ with } \ \text{some } \ i < p^{[k/2]}.$$

For every fixed $j = 0,1$

$$\begin{aligned} 0 \leq n + \frac{\pi(p)}{4}(1+2j)p^{[(k-1)/2]} &= \Big(\pi(p)i + \frac{\pi(p)}{4}(1+2j)\Big)p^{[(k-1)/2]} \\ &\leq \Big(\pi(p)p^{[k/2]} - \pi(p) + \frac{3\pi(p)}{4}\Big)p^{[(k-1)/2]} \\ &= \pi(p)p^{k-1} - \frac{\pi(p)}{4}p^{[(k-1)/2]} < \pi(p)p^{k-1}. \end{aligned}$$

Now, it remains to show that $B_{\frac{\pi(p)}{4}p^{[(k-1)/2]}}$ and $B_{\frac{3\pi(p)}{4}p^{[(k-1)/2]}}$ are incongruent modulo p^k for which it is enough to show that they are incongruent modulo p which is established in Lemma 8.4.2. □

Lemma 8.4.4. *If $p \equiv 3 \ (mod\ 8)$ and $k \in \mathbb{N}$, then for every integer x with $1 \leq x \leq [(k-1)/2]$ there exist $(p-1)p^{k-2x-1}$ distinct feasible residue b of B_n with $0 \leq n < \pi(p)p^{k-1}$ occurring at least $2p^x$ times in a period modulo p^k.*

Proof. Suppose $n \in \mathbb{Z}$ and $p^{x-1} \| n$. Then

$$\begin{aligned} \operatorname{ord}_p(B_{n+\pi(p)p^{k-x-1}} - B_n) &= \operatorname{ord}_p\Big(2B_{\frac{\pi(p)}{2}p^{k-x-1}}C_{n+\frac{\pi(p)}{2}p^{k-x-1}}\Big) \\ &= 1 + \operatorname{ord}_p\Big(\frac{\pi(p)}{2}p^{k-x-1}\Big) + 1 + \operatorname{ord}_p\Big(n + \frac{\pi(p)}{2}p^{k-x-1}\Big) \\ &= 1 + (k-x-1) + 1 + x - 1 = k. \end{aligned}$$

(Here we are using the inequality $\operatorname{ord}_p(a+b) \geq \min(\operatorname{ord}_p(a), \operatorname{ord}_p(b))$ as $0 \leq x-1 \leq \frac{k-3}{2}$ and $\frac{k-1}{2} \leq k-x-1 \leq k-2$). Therefore

$$B_{n+\pi(p)p^{k-x-1}} \equiv B_n (\text{mod } p^k)$$

for all n such that $p^{x-1} \| n$. We need to count the number of integers n with $0 \leq n < \pi(p)p^{k-1}$ and $p^{x-1} \| n$ for which a given b occurs as a residue of B_n modulo p^k. This is equivalent to counting the number of integers n with $0 \leq n < \pi(p)p^{k-x-1}$ and $p^{x-1} \| n$ for which a given b occurs as a residue of B_n modulo p^k and then to multiply this number by p^x. Hence we have to check the distribution of the $2(p-1)p^{k-2x-1}$ numbers

$$B_j (\text{mod } p^k) : \Big(1 \leq j < \pi(p)p^{k-x-1}, \ 2 \nmid j, \ \frac{\pi(p)}{4} | j, \ p^{x-1} \| j\Big).$$

We claim that half of them, i.e., $(p-1)p^{k-2x-1}$ of the B_n's are pairwise incongruent modulo p^k and other half are congruent to the first half in some way. We claim that

$$B_{\frac{\pi(p)}{2}\cdot p^{k-x-1}-\frac{\pi(p)}{4}j} \equiv B_{\frac{\pi(p)}{4}j} (\text{mod } p^k), 1 \le j < p^{k-x-1},\ \ 2 \nmid j,\ \ p^{x-1} \| j$$

and

$$B_{\pi(p)p^{k-x-1}-\frac{\pi(p)}{4}j} \equiv B_{\frac{\pi(p)}{2}p^{k-x-1}+\frac{\pi(p)}{4}j} (\text{mod } p^k), 1 \le j < p^{k-x-1},\ \ 2 \nmid j,\ \ p^{x-1} \| j.$$

Observe that

$$\begin{aligned}
&\text{ord}_p(B_{\frac{\pi(p)}{2}p^{k-x-1}-\frac{\pi(p)}{4}j} - B_{\frac{\pi(p)}{4}j}) \\
&= \text{ord}_p\left(2B_{\frac{\pi(p)}{4}(p^{k-x-1}-j)} C_{\frac{\pi(p)}{4}p^{k-x-1}}\right) \\
&= \text{ord}_p(2) + \text{ord}_p(B_{\frac{\pi(p)}{4}(p^{k-x-1}-j)}) + \text{ord}_p(C_{\frac{\pi(p)}{4}p^{k-x-1}}) \\
&= 1 + \text{ord}_p\Big(\frac{\pi(p)}{4}(p^{k-x-1}-j)\Big) + 1 + \text{ord}_p\Big(\frac{\pi(p)}{4}p^{k-x-1}\Big) \\
&= 2 + x - 1 + k - x - 1 = k
\end{aligned}$$

and

$$\begin{aligned}
\text{ord}_p\Big(B_{\pi(p)p^{k-x-1}-\frac{\pi(p)}{4}j} - B_{\frac{\pi(p)}{2}p^{k-x-1}+\frac{\pi(p)}{4}j}\Big) &= \text{ord}_p\Big(2B_{\frac{\pi(p)}{4}p^{k-x-1}-\frac{\pi(p)}{4}j} C_{\frac{3\pi(p)}{4}p^{k-x-1}}\Big) \\
&= 1 + \text{ord}_p\Big(\frac{\pi(p)}{4}(p^{k-x-1}-j)\Big) + 1 + \text{ord}_p\Big(\frac{3\pi(p)}{4}p^{k-x-1}\Big) \\
&\ge 2 + x - 1 + k - x - 1 = k.
\end{aligned}$$

It only remains to show that

$$B_{\pi(p)p^{k-x-1}-\frac{\pi(p)}{4}j} \not\equiv B_{\frac{\pi(p)}{4}j} (\text{mod } p^k),\ 1 \le j < p^{k-x-1},\ \ 2 \nmid j,\ \ p^{x-1} \| j.$$

Since

$$\text{ord}_p(B_{\pi(p)p^{k-x-1}-\frac{\pi(p)}{4}j} - B_{-\frac{\pi(p)}{4}j}) = \text{ord}_p(2B_{\frac{\pi(p)}{2}p^{k-x-1}} C_{\frac{\pi(p)}{2}p^{k-x-1}-\frac{\pi(p)}{4}j}) \ge k,$$

it follows that

$$B_{\pi(p)p^{k-x-1}-\frac{\pi(p)}{4}j} \equiv B_{-\frac{\pi(p)}{4}j} \equiv -B_{\frac{\pi(p)}{4}j} (\text{mod } p^k), 1 \le j < p^{k-x-1},\ \ 2 \nmid j,\ \ p^{x-1} \| j.$$

Hence (8.4) follows and the proof is complete. □

Lemma 8.4.5. *If $p \equiv 3$ (mod 8), then there exist $\frac{\pi(p)}{2} - 1$ distinct feasible residue b of B_n with $0 \le n < \pi(p)$ occurring exactly twice.*

Proof. In view of Lemma 8.4.3 with $k = 1$, there exist two distinct feasible residues b of B_n modulo p for $n = 0, 1, \ldots, \pi(p) - 1$ occurring only once. Hence we need to check the distribution of the remaining $\pi(p)-2$, namely,

$$B_r \pmod p, \text{ for } 0 \le r < \pi(p), r \notin \left\{\frac{\pi(p)}{4}, \frac{3\pi(p)}{4}\right\}.$$

We claim that half of them, i.e, $\frac{\pi(p)}{2}-1$ of the B_n's are pairwise incongruent modulo p and the other half are congruent to the first half in some manner, i.e.,

$$B_i \equiv B_{\frac{\pi(p)}{2}-i} \pmod p, \; B_{\frac{\pi(p)}{2}+i} \equiv B_{\pi(p)-i} \pmod p \text{ for } 0 \le i < \frac{\pi(p)}{4}.$$

But

$$\mathrm{ord}_p(B_{\frac{\pi(p)}{2}-i} - B_i) = \mathrm{ord}_p(2B_{\frac{\pi(p)}{4}-i}C_{\frac{\pi(p)}{4}}) = \mathrm{ord}_p(2) + \mathrm{ord}_p(B_{\frac{\pi(p)}{4}-i}) + \mathrm{ord}_p(C_{\frac{\pi(p)}{4}}) \ge 1,$$

shows that $B_i \equiv B_{\frac{\pi(p)}{2}-i} \pmod p$. Similarly it can be easily shown that

$$B_{\frac{\pi(p)}{2}+i} \equiv B_{\pi(p)-i} \pmod p.$$

To complete the proof, it remains to show

$$B_i \not\equiv B_{\frac{\pi(p)}{2}+i} \pmod p.$$

Since $B_i \equiv -B_{\frac{\pi(p)}{2}+i} \pmod p$ and the case $B_i \equiv 0 \equiv B_{\frac{\pi(p)}{2}+i} \pmod p$ leads to contradiction to the definition of period, it follows that $B_i \not\equiv B_{\frac{\pi(p)}{2}+i} \pmod p$. □

Remark 8.4.6. *If $p \equiv 3 \pmod 8$ and $k \in \mathbb{N}$, then there exist $p^{k-1}(\frac{\pi(p)}{2}-1)$ distinct feasible residues b of B_n modulo p^k with $0 \le n < \pi(p)p^{k-1}$ occurring exactly twice.*

Using the last three lemmas, we can prove the following important theorem.

Theorem 8.4.7. *If $p \equiv 3 \pmod 8$ and $i \in \{0,1\}$, then*

$$\nu_B(p^k,b) = \begin{cases} p^{[k/2]} & \text{if } b \equiv B_{\left(\frac{\pi(p)}{4}+i\cdot\frac{\pi(p)}{2}\right)p^{[(k-1)/2]}} \pmod{p^k}, \\ 2p^x & \text{if } b \equiv B_{\frac{\pi(p)}{4}j+i\cdot\frac{\pi(p)}{2}p^{k-x-1}},\; p^{x-1} \| j,\; 2 \nmid j,\; 1 \le j < p^{k-x-1} \\ & \text{for some } x \in \{1,2,\ldots,[(k-1)/2]\}, \\ 2 & \text{otherwise}. \end{cases}$$

Proof. In view of Lemmas 8.4.3, 8.4.4 and Remark 8.4.6 we have the following results:

$$\nu_B(p^k,b) \ge p^{[k/2]}, \;\; \nu_B(p^k,b) \ge 2p^x \text{ and } \nu_B(p^k,b) = p^{k-1}(\frac{\pi(p)}{2}-1). \tag{8.4.5}$$

Hence

$$\sum_{b=0}^{p^k-1} \nu_B(p^k,b) \ge 2p^{[k/2]} + \sum_{x=1}^{[(k-1)/2]} (p-1)p^{k-2x-1}(2p^x) + p^{k-1}\left(\frac{\pi(p)}{2}-1\right) = \pi(p)p^{k-1}. \tag{8.4.6}$$

By virtue of Theorem 7.3.5 the left hand side of (8.4.6) equals $\pi(p)p^{k-1}$. Thus, equality holds in (8.4.5) for every feasible residue b modulo p^k. □

In the above theorem, the second case occurs if $k \geq 3$ and in this case, there are exactly $(p-1)p^{k-2x-1}$ distinct feasible residues b modulo p^k. In the third case, for each $k \in \mathbb{N}$, exactly $p^{k-1}(\frac{\pi(p)}{2}-1)$ distinct feasible residue b modulo p^k are possible. Using (8.1.1), we get

$$\Omega_B(p^k) = \{2, 2p, 2p^2, \ldots, 2p^{[(k-1)/2]}, p^{[k/2]}\}. \tag{8.4.7}$$

Thus, the following corollary is a direct consequence of Theorem 8.4.7.

Corollary 8.4.8. *If $p \equiv 3 \pmod 8$, then balancing sequence is not stable modulo p.*

We next search for primes $p \equiv 1 \pmod 8$ for whom the balancing sequence is stable. In the following theorem, we limit the search for such primes in the class of associated Pell numbers.

Theorem 8.4.9. *If the prime $p \equiv 1 \pmod 8$ is an odd indexed associated Pell number, then balancing sequence is stable modulo p.*

Proof. Since p is an odd indexed associated Pell number, use of Theorem 7.4.3 confirms that $\pi(p)$ is odd. Using the arguments given in the proof of Lemma 8.3.3, it is easy to see that $\pi(p) = \alpha(p)$. Now proceeding like the proof of Theorem 8.3.7, one can easily verify that the balancing sequences is stable modulo such a prime. $\square$

For some members in the class of primes $p \equiv 1 \pmod 8$, $\pi(p)$ is a multiple of 4. For example 17 is one such prime with $\pi(17) = 8$. The following theorem confirms that the balancing sequence is not stable modulo any such prime.

Theorem 8.4.10. *Let p be a prime such that $p \equiv 1 \pmod 8$ and $4 \mid \pi(p)$. If $i \in \{0,1\}$ then*

$$\nu_B(p^k, b) = \begin{cases} p^{[k/2]} & \text{if } b \equiv B_{\left(\frac{\pi(p)}{4} + i\cdot\frac{\pi(p)}{2}\right)p^{[(k-1)/2]}} \pmod{p^k}, \\ 2p^x & \text{if } b \equiv B_{\frac{\pi(p)}{4}j + i\cdot\frac{\pi(p)}{2}p^{k-x-1}},\ p^{x-1} \| j,\ 2 \nmid j,\ 1 \leq j < p^{k-x-1}, \\ & \text{for some } x \in \{1, 2, \ldots, [(k-1)/2]\} \\ 2 & \text{otherwise} \end{cases}$$

Proof. The proof is similar to the proof of Theorem 8.4.7, hence it is omitted. $\square$

There are some primes $p \equiv 1 \pmod 8$ for which $4 \nmid \pi(p)$. Such type of primes are excluded from Theorem 8.4.10. For example, if $p = 137, \pi(p) = 34$ and one can check that the balancing sequence is stable modulo 137. It is an open problem to identify some more subclass of primes for which the balancing sequence is stable or not stable.

Conclusion

In this thesis, we generalized balancing numbers in two different ways resulting in balancing-like numbers and gap balancing numbers. The gap balancing numbers are further generalized to sequence gap balancing numbers in the line of generalization of balancing numbers to sequence balancing numbers. We only proved that no second order gap balancing number exists. However, proving the existence or non existence of gap balancing numbers of order higher than two is an open problem.

We generalize the Riemann zeta function to balancing zeta function in the line of its generalization to Fibonacci zeta function and studied its analytic continuation, poles and residues.

We also studied the behaviour of the balancing sequence modulo primes and other natural numbers and observed that, the period of the balancing sequence modulo any of the three primes $13, 31$ and 1546463 is equal to the period modulo its square. After exhaustive verification of special cases, we believe that there is no other prime with this property. It is an open problem to prove or disprove our claim.

Lastly, we studied the stability of the balancing sequence and proved that the sequence is stable for primes $p \equiv -1, -3 \pmod 8$ and not stable for primes $p \equiv 3 \pmod 8$. For certain subclass of primes $p \equiv 1 \pmod 8$, we proved that the balancing sequence is not stable; however, there are examples of primes in this class for which the sequence is stable. Thus, for the class of primes $p \equiv 1 \pmod 8$, it is an open problem to identify all primes for which the balancing sequence is stable.

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