PRODUCT OF COMPOSITION AND MULTIPLICATION OPERATOR

A Thesis Submitted
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under the supervision
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A thesis presented for the degree of
Master of Science

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May, 2015

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DECLARATION

I hereby declare that the project report entitled *"Product of Multiplication and Composition operator"* submission is of my own work on the concerned topics and to the best of my knowledge and belief. It includes no materials which is previously published or written by another person nor material which to be substantial extent has been excepted for the award of any other degree or other institute of higher learning.

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ACKNOWLEDGEMENT

It is my great pleasure to express my heart-felt gratitude to all those who helped and encourage me at various stage during the running of the project work entitled *"Product of Composition and Multiplication operator"*.

I am indebted to my guide Professor Shesadev Pradhan for his invaluable guidance and constant support with explaining the mistakes with great patience. His encouragement has always comforted me throughout. I would like to thanks my friends in NIT, Rourkela and its outside to whom I am in contact for their love and friendship.
Finally, to my family I will be always feel proud for their care.

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CERTIFICATE

This is to certify that the project report entitled, *"Product of Multiplication and Composition operator"* is the bonafied work carried out by Sharata Charan Gardia, student of M.Sc Mathematics at National Institute of Technology, Rourkela, during the year 2014-15, in partial fulfillment of the requirements for the award of degree of Master of Science under the guidance of Prof. Shesadev Pradhan, Professor, National Institute of Technology, Rourkela and that the project is a review work by the student by the collection of papers by various sources.

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Contents

ABSTRACT

The composition operator C_T and the multiplication operator M_u together multiplied to give a new operator named as Weighted composition operator $W_{u,T} = C_T M_u$ given by $f \mapsto u \circ T.f \circ T$, on the Orlicz space. The thesis discusses various properties of the operator $W_{u,T}$ starting with basic necessary preliminaries.

1 Linear Operators-Definition and Examples

Let X and Y be an arbitrary set then a mapping i.e. $f\colon X \to Y$ is a rule which send each element of to an unique element of Y i.e.

$$\begin{aligned} f &\colon X \to Y \\ &\colon x \mapsto y = f(x) \end{aligned}$$

Now instead of choosing simply a set, take X and Y to be vector spaces. *X and Y both are the algebraic structure with suitable addition(+) and multiplication(.)* Then the corresponding f we denote by T and we call this an operator i.e.

$$T\colon X \to Y$$

Note: operators are basically extension of the function.

Definition 1.1 (**linear operator**). A linear operator T is an operator such that
(i) the domain $D(T)$ of T is a vector space and the range $R(T)$ lies in a vector space over the same field.
(ii) for all $x, y \in D(T)$ and the scalars α, β

$$\begin{aligned} T(x+y) &= Tx + Ty \\ T(\alpha x) &= \alpha Tx \end{aligned}$$

and combining above two, $\quad T(\alpha x + \beta y) = \alpha Tx + \beta Ty.$

• Basically the operator T which preserve the operation addition as well as scalar multiplication is said to be a linear operator where $D(T)$ is always a vector space and $R(T)$ lies in a vector space. In fact $R(T)$ is also a vector space.

Notation	Definition
$D(T)$	Domain of T
$R(T)$	Range of T
$N(T)$	Null space of T

1.1 Bijective property of Linear operator

A linear operator $T\colon\ D(T) \to R(T) \subset Y$

is said to be one-one if

$$\begin{aligned} &T(x1) = T(x2) \\ &\Rightarrow x1 = x2 \qquad x1, x2 \in D(T) \end{aligned}$$

$T\colon\ D(T) \subset X \to R(T) \subset Y$ is said to be onto if

$$\forall\, y \in R(T)\ \exists\, x \in D(T) \text{ s.t. } y = Tx$$

1.2 Examples

1.2.1 Identity operator

$$\begin{gathered} I_x\colon X \to X \ s.t\ I_x(x) = x, \forall\, x \in X \\ I_x(\alpha x + \beta y) = \alpha x + \beta y = \alpha I_x(x) + \beta I_x(y) \end{gathered}$$

so it is a linear operator.

1.2.2 Zero Operator

$$\begin{gathered} O\colon X \to Y \\ : x \mapsto 0 \\ Ox = 0 \end{gathered}$$

So, zero operator is a linear operator.

1.2.3 Differentiation or Differential Operator

Let X be the vector space of all polynomial defined on [a,b].
$x(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n, n \geq 1$
Define $T\colon X \to X$ by

$$Tx(t) = \frac{d}{dt} x(t)$$

Also we can define

$$\begin{aligned} &T\colon C^1[a,b] \to C[a,b] \quad by \\ &T : f \mapsto f' \end{aligned}$$

Note: As polynomials are well differentiable, so the operator T is well defined. But if we will replace X by $C[a,b]$, then the operator T is not well defined as

there are function which are continuous but not differentiable. That is why we took the domain as the vector space of all polynomial defined on [a,b].
So,

$$Tx(t) = \frac{d}{dt}x(t) = x'(t)$$

is again a polynomial.
Again

$$T(\alpha x + \beta y) = (\alpha x(t) + \beta y(t))' = \alpha(x'(t)) + \beta(y'(t)) = \alpha Tx(t) + \beta Ty(t)$$

So, T becomes linear.

1.2.4 Integral operator

Let $X = C[a, b]$ be set of all continuous function defined on the interval $[a, b]$. Then the operator

$$T\colon X \to X$$

defined by

$$T(x(t)) = \int_a^b \mathrm{x}(\tau)\mathrm{d}\tau$$

is well defined.
considering

$$T(x(t)) = tx(t),\ t \in [a, b]$$

T is linear as

$$T(\alpha x + \beta y) = t[\alpha x(t) + \beta y(t)] = \alpha tx(t) + \beta ty(t) = \alpha Tx + \beta Ty$$

Hence, T is linear.

2 Bounded Linear Operator

Definition 2.1 (**Norm**)**.** A norm on a field i.e. real or complex vector space X is a real valued function on X whose value at an $x \in X$ is denoted by $\|x\|$ which satisfy the following properties:

1. $\|x\| \geq 0$
2. $\|x\| = 0 \Leftrightarrow x = 0$
3. $\|\alpha x\| = |\alpha|\|x\|$

4. $\|x+y\| \leq \|x\| + \|y\|$

Then $(X, \|.\|)$ is a normed space.

Definition 2.2. (**Normed space**)
A normed space is a vector space with a norm defined on it.

Let $(X, \|.\|_x)$ and $(Y, \|.\|_y)$ be normed spaces and $T : D(T) \subset X \to Y$ be a linear operator. Then we say; the operator T is bounded, if there is a real number $c > 0$ such that $\forall x \in D(T)$,

$$\|Tx\| \leq c.\|x\| \tag{1}$$

Let $B(X, Y)$ be the set of all bounded linear operator from X to Y.
For $T \in B(X, Y)$ norm of T is defined by

$$\|T\| = sup\{\|T(x)\| \colon \|x\| \leq 1\}$$

$\|Tx\| \leq c\|x\|$ gives the relation between the norm and the operator also.
What should be the minimum value of c here so that (1) holds ?

$$\frac{\|Tx\|}{\|x\|} < c, \text{ true } \forall x \neq 0 \in D(T)$$
$$\text{so, sup } \frac{\|Tx\|}{\|x\|} \text{ is the minimum value for } c.$$

so let $\|T\| = \sup \frac{\|Tx\|}{\|x\|}$
This minimum value for c, we call it as ***Norm*** of the bounded linear operator T. So $\|Tx\| \leq \|T\|\|x\|$ and If $T = 0$, then $\|T\| = 0$.

2.1 Examples

1. Identity operator I : $X \to X$.

2. Zero operator 0: $X \to X$.

3. Differential operator

4. Integral operator

$$T \colon C[0,1] \to C[0,1]$$
$$: x \mapsto y = Tx$$

where $y(t) = \int_a^b k(t,\tau)x(\tau)d\tau$.where k is a given function which is called the kernel of T. Assuming k is continuous on the closed region $J \times J$ in $t - \tau plane$.

Now

$$T\colon (C[0,1], \|.\|_\infty)$$

where $C[0,1]$ is a normed space with respect to the normed $\|.\|_\infty$, where

$$\|x\| = \sup_{0\leq t\leq 1} |x(t)|$$

T is bounded as

$$\begin{aligned}
\|Tx\| = \|y\| &= max \mid y(t)|, t \in J \\
&= max \mid \int_0^1 k(t,\tau)x(\tau)d\tau| \\
&\leq max \int_0^1 |k(t,\tau)|.|x(\tau)|d\tau
\end{aligned}$$

Since k is a continuous on closed region $J \times J$, so k is bounded. So,

$$\mid k(t,\tau)| \leq M \ \forall (t,\tau) \in J \times J.$$
$$\text{Hence, } \|Tx\| \leq M\|x\|$$

$\Rightarrow$ T is bounded operator.

3 Banach Space

Definition 3.1 (**Norm**). A norm on a field i.e. real or complex vector space X is a real valued function on X whose value at an $x \in X$ is denoted by $\|x\|$ which satisfy the following properties:

1. $\|x\| \geq 0$
2. $\|x\| = 0 \Leftrightarrow x = 0$
3. $\|\alpha x\| = |\alpha|\|x\|$
4. $\|x + y\| \leq \|x\| + \|y\|$

Then $(X, \|.\|)$ is a normed space.

Definition 3.2. (**Normed space**)
A normed space is a vector space with a norm defined on it.

A **Banach space** is a complete normed space(complete in the norm defined by norm).

Theorem 3.1. Let Y be a subspace of a Banach space X. Then Y is closed if and only if Y is complete(i.e the concept of converging Cauchy sequence).

Proof. Let Y be closed subspace in X. Let $\{x_n\}$ be a Cauchy sequence in Y, then $\{x_n\} \in X$.
Since X is complete,

$$\exists \text{ some } x \in X \text{ s.t. } \{x_n\} \to x$$

Now, every neighborhood of x will contain some points in Y.
Take $\{x_n\} \neq x$ with n very large which implies that x is an accumulation point of Y and since Y is closed(assumption), $x \in Y$.
Hence Y is complete.
Conversely,
Let Y be complete and x be a limit point of Y.

$$\Rightarrow \exists \text{ a sequence } \{x_n\} \subset Y,$$
$$\text{s.t. } \{x_n\} \to x \text{ for } n \to \infty$$

And we know that every convergent sequence is a Cauchy sequence.
Now Y is complete. So $\{x_n\}$ converges in Y and hence $x \in Y$.
Thus Y is closed. □

4 Product of composition and multiplication operators

Definition 4.1. (**Lebesgue outer measure**)
Lebesgue outer measure or the simply outer measure of a set is given by $m^\star(A) = inf \sum l(I_n)$,where the infimum is taken over all finite or countable collection of intervals $[I_n]$ such that $A \subseteq \cup I_n$.
Some well-known properties are as follows.

1. $m^\star(A) \geq 0$,
2. $m^\star(\phi) = 0$,
3. $m^\star(A) \leq m^\star(B)$ if $A \subseteq B$

Definition 4.2. (measurable sets)
A set X is said to be measurable if for each set A we have

$$m^\star(A) = m^\star(A \cap X) + m^\star(A \cap X^c).$$

Definition 4.3. (σ**-algebra**)
A collection of subsets of any arbitrary space X is said to be a $\sigma - algebra$ if X belongs to the collection and the collection is closed under formation of countable unions and of complements.

Mathematically,
A collection B of subsets of the space X is called a $\sigma - algebra$ if

1. $\phi \in B$
2. if $E \in B$, then it's compliment $X - E \in B$.
3. if $E_n \in B$, n=1,2,3,... is a countable sequence of sets in B then $\bigcup_{n=1}^{\infty} E_n \in B$.

Examples:

1. $B = \{\phi, X\}$ (trivial σ-algebra)
2. $B =$ power set of X (full σ-algebra)

Definition 4.4. (Measure)
A function μ defined by $\mu \colon B \to R^+ \cup \{\infty\}$ is said to be a measure if

1. $\mu(\phi) = 0$.

2. if E_n is countable collection of sets and are pairwise disjoint in B i.e $E_n \cap E_n = \phi$ for $n \neq m$, then $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$.

Definition 4.5. (Measurable space)
Let X be any subspace and B is a σ-algebra of X. Then the pair (X, B) is called a measurable space.

Definition 4.6. (Measure space)
Let X be any subspace and B is a σ-algebra of X. Then (X, B, μ) is called a measure space, where μ is a measure on measurable space (X, B).

Definition 4.7. (Convex function)
A continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be a convex function if for any two arbitrary points x_1 and x_2 in $\mathbb{R}$ (on the graph of $f(x)$),

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

Examples:

1. $f(x) = x^2$
2. $f(x) = e^x$

Definition 4.8. (Measurable function)
Let φ be a function defined by $\varphi\colon X \to Y$, where (X, E) and (Y, F) are the measurable space. Then φ is said to be measurable if $\varphi^{-1}(F) \subset E$.

Now we write Ω for the space of consideration.

Definition 4.9. (Orlicz function)
Let (Ω, B, μ) be a finite measure space and let φ be a continuous convex function defined by $\varphi\colon [0, \infty) \to [0, \infty)$ such that

1. $\varphi(x) = 0$ if and only if $x = 0$
2. $\varphi(x) \to \infty$ as $x \to \infty$

The function φ is known as an Orlicz function.

Definition 4.10. (Orlicz space)
The Orlicz space denoted by $L_{\varphi,\Omega,B,\mu}$ or L_φ consist of all complex-valued measurable function f on Ω such that for some $k > 0$,

$$\int_\Omega \varphi(k|f(\omega)|)d\mu < \infty, \quad \omega > 0$$

We simply denote the Orlicz space by L_φ. An atom of a measure space (Ω, B, μ) is an element $A \in B$ with $\mu(A) > 0$ such that for each $S \in B$, if $S \subset A$, then either $\mu(S) = 0$ or $\mu(S) = \mu(A)$.And the measure space having no atom is called a non-atomic measure space.

Definition 4.11. (Measurable Transformation)
Let (Ω_1, B_1) and (Ω_2, B_2) be two measurable space. A mapping or transformation T defined by

$$T\colon \Omega_1 \to \Omega_2$$

is said to be measurable if for any measurable set $A \in B_2$, the inverse image will be in B_2.

$$\text{i.e. } T^{-1}(A) = \{\omega_1 \colon T(\omega_1) \in A\} \subset B_1$$

Definition 4.12. (Non-singular measurable transformation)
Suppose $T\colon \Omega_1 \to \Omega_2$ is a measurable transformation which satisfies whenever $\mu(A) = 0 \Rightarrow \mu(T{-}1(A)) = 0$ for $A \in B$. Then T is called a non-singular measurable transformation.

If T is non-singular, then the measure μT^{-1} which is given by $(\mu T^{-1})(B) = \mu(T^{-1}(B))$ for $A \in B$, is absolutely continuous with respect to the measure μ.

4.1 Radon-Nikodym Theorem

The following theorem is important which we will use frequently in composition and multiplication operator.
Theorem
Let (X, Σ) be a measurable space. Now if a σ-finite measure ν on the measurable space (X, Σ) is absolutely continuous with respect to another σ-finite measure μ on (X, Σ), then there exits a measurable function $f\colon X \to [0, \infty)$ such that for any measurable set A in X

$$\nu(A) = \int_A f d\mu$$

and the function f is called the Radon-Nicodym derivative of ν with respect to μ.It is denoted by $\frac{d\nu}{d\mu}$.
So by using Radon-Nikodym Theorem, there exists a non-negative measurable function f_T such that

$$(\mu T^{-1})(A) = \int_A f_T d\mu, \text{ for every } A \in B.$$

The function f_T is known as Radon-Nikodym derivative of the measure μT^{-1} with respect to the measure μ and hence it is written as $\frac{d\mu T^{-1}}{d\mu}$.

4.2 Measurable function

Let f be a extended real-valued function defined on a measurable set X. Then f is called measurable function if for each $\alpha \in R$, the set $\{x \colon f(x) > \alpha\}$ is measurable.

4.3 Multiplication and Composition operator

Definition 4.13. (Multiplication operator)
Let T be a non-singular transformation defined by

$$T \colon \Omega \to \Omega$$

and u be a complex valued measurable function defined on Ω.The bounded linear transformation $M_u \colon F(\Omega) \to F(\Omega)$ defined by $M_u f = u.f$ for every $f \in F(\Omega)$ is called a **multiplication operator** induced by u. In fact a continuous multiplication linear transformation is known as multiplication operator.
(Here $F(\Omega)$ is the topological vector space of complex valued function defined on Ω.)

The bounded linear transformation or composition transformation $C_T \colon F(\Omega) \to F(\Omega)$ by $C_T f = f \circ T$ for every $f \in F(\Omega)$ on a Banach function space is called a **composition operator** and is induced by T.
A weighted composition operator denoted by $W_{u,T}$ is the product of a multiplication operator with a composition operator.

$$W_{u,T}(f) = C_T M_u(f)$$

Now we shall derive the equation for the weighted composition operator $W_{u,T}$.

$$\begin{aligned}
W_{u,T} f &= C_T M_u f \\
&= C_T(u.f), && \because M_u f = u.f \\
&= (u.f) \circ T && \because C_T f = f \circ T \\
&= u(T) f(T)
\end{aligned}$$

Hence we got the equation for $W_{u,T}$ i.e

$$W_{u,T} = u(T)f(T)$$

We proceed for the weighted composition operator $W_{u,T}$ given by

$$W_{u,T}f(\omega) = u(T(\omega))f(T(\omega))$$

where u and f are complex valued measurable function and T is a non-singular measurable transformation on the Orlicz space L_φ.

4.4 Boundedness of operator $W_{u,T}$

The class of all bounded operator on a Banach space X is denoted by $\mathfrak{B}(X)$ and the kernel and range of an operator P on X by $ker(P)$ and $R(P)$ respectively.
Now, suppose $W_{u,T}$ is bounded in L_φ i.e it's range is in L_φ. Then this thing is denoted by writing $W_{u,T} \in \mathfrak{B}(L_\varphi)$.
For verifying the boundedness of weighted composition operator $W_{u,T}$, it can be easily seen that C_T and M_u are both bounded linear transformation and therefore the operator $W_{u,T}$ is bounded, writing $W_{u,T} \in \mathfrak{B}(L_\varphi)$.

5 Adjoint of $W_{u,T}$

5.1 Δ_2- condition

Definition 5.1. (Δ_2- condition)
We say the Orlicz function φ satisfy the Δ_2- condition if for some $k > 0$,

$$\varphi(2x) \leq k\varphi(x) \quad \forall x > 0$$

Now we assume φ satisfies Δ_2- condition.
u is a complex-valued measurable mapping.
T is a non-singular measurable transformation.
$L(\mu)$ is the linear space of all complex-valued measurable functions. Let (Ω, B, μ) be a finite measure space . Then we define
$L^\phi(\mu) = \{f \colon X \to C$ is measurable s.t. $\int_X \phi(\epsilon|f|)d\mu < \infty$ for some $\epsilon > 0\}$.

Theorem 5.1. If the mapping $W_{u,T} \colon L_\varphi \to L(\mu)$, is such that $W_{u,T}(L_\varphi) \subseteq L_\varphi$ then $W_{u,T} \in \mathfrak{B}(L_\varphi)$.

Proof. let us consider a sequence f_n and f, g both are in L_φ i.e $f \in L_\varphi, g \in L_\varphi$ such that

$$\|f_n - f\|_\varphi \to 0 \text{ and } \|W_{u,T} f_n - g\|_\varphi \to 0$$

as $n \to \infty$. Then we can find a subsequence f_{n_k} of f_n which will satisfy $\varphi(|f_{n_k} - f|) \to 0$ almost everywhere(a.e) on the space Ω. Now,

$$\varphi(|f_{n_k} - f|) \to 0 \quad a.e$$
$$or, \quad \varphi(|u.f_{n_k} - u.f|) \to 0 \quad a.e$$

Now since T is non-singular,

$$\varphi(|u \circ T.f_{n_k} \circ T - u \circ T.f \circ T|) \to 0 \quad a.e \ on \ \Omega.$$

Again we can find a subsequence $f_{n_{k'}}$ of f_{n_k} such that

$$\varphi(|u \circ T.f_{n_{k'}} \circ T - g|) \to 0 \quad a.e \ on \ \Omega$$

also from above, we have

$$\varphi(|u \circ T.f_{n_{k'}} \circ T - u \circ T.f \circ T|) \to 0 \quad a.e \ on \ \Omega.$$

So, we can write

$$\|W_{u,T} f_{n_{k'}} - g\|_\varphi \to 0 \quad and \tag{2}$$
$$\|W_{u,T} f_{n_{k'}} - W_{u,T} f\|_\varphi \to 0 \tag{3}$$

as $n_{k'} \to \infty$. This implies from (2) and (3), $W_{u,T} f = g$ and by closed graph theorem, $W_{u,T}$ is bounded on L_φ i.e. $W_{u,T} \in \mathfrak{B}(L_\varphi)$. □

Theorem 5.2. Let u and f_T belongs to $L^\infty(\mu)$. Then $W_{u,T} \in \mathfrak{B}(L_\varphi)$.

Proof. Let u and f_T belongs to $L^\infty(\mu)$ i.e. $u \in L^\infty(\mu)$ and $f_T \in L^\infty(\mu)$.

If $f_t > 1$, then

$$\begin{aligned}
&\int_\Omega \varphi\left(\frac{|W_{u,T}f|}{\|f\|_\varphi \|u\|_\infty \|f_T\|_\infty}\right) d\mu \\
&= \int_\Omega \varphi\left(\frac{|u \circ T.f \circ T|}{\|f\|_\varphi \|u\|_\infty \|f_T\|_\infty}\right) d\mu \\
&= \int_\Omega \varphi\left(\frac{|(u.f) \circ T|}{\|f\|_\varphi \|u\|_\infty \|f_T\|_\infty}\right) d\mu \\
&= \int_\Omega \varphi\left(\frac{|(u.f)|}{\|f\|_\varphi \|u\|_\infty \|f_T\|_\infty}\right) d\mu \circ T^{-1} \\
&\leq \int_\Omega \varphi\left(\frac{|f|}{\|f\|_\varphi \|f_T\|_\infty}\right) f_T d\mu \\
&\leq \int_\Omega \varphi\left(\frac{|f|}{\|f\|_\varphi}\right) \frac{f_T}{\|f_T\|_\infty} d\mu \\
&\leq \int_\Omega \varphi\left(\frac{|f|}{\|f\|_\varphi}\right) d\mu \\
&\leq 1
\end{aligned}$$

$\therefore \|W_{u,T}f\|_\varphi \leq \|f\|_\varphi \|u\|_\infty \|f_T\|_\infty$
and when $f_T \leq 1$,

$$\begin{aligned}
&\int_\Omega \varphi\left(\frac{|W_{u,T}f|}{\|f\|_\varphi \|u\|_\infty}\right) d\mu \\
&\leq \int_\Omega \varphi\left(\frac{|f|}{\|f\|_\infty}\right) f_T d\mu \\
&\leq \int_\Omega \varphi\left(\frac{|f|}{\|f\|_\infty}\right) d\mu \\
&\leq 1
\end{aligned}$$

$\therefore \quad \|W_{u,T}f\|_\varphi \leq \|f\|_\varphi \|u\|_\infty$
So, we take $k = max\{1, \|f_T\|_\infty\}$
which follows to

$$\|W_{u,T}f\|_\varphi \leq k.\|f\|_\varphi \|u\|_\infty$$

$\therefore W_{u,T} \in \mathfrak{B}(L_\varphi)$ □

Using the above results and observation, we extend the results for the weighted composition operator on the Orlicz space L_φ as follows.

Theorem 5.3. The linear transformation $W_{u,T}\colon L \mapsto L(\mu)$, is a bounded operator on L_φ if and only if $u \in L^\infty(\mu T^{-1})$.

Proof. Suppose the linear transformation is $W_{u,T}\colon L \mapsto L(\mu)$ is a bounded operator on L_φ.
Assume the contrary that, u is not bounded with respect to the measure μT^{-1} i.e. $u \notin L^\infty(\mu T^{-1})$,which is equivalently saying $u \circ T$ is not bounded with respect to the measure μ.Then from theorem 5.1 , for each natural number $n, \chi E_n \in L_\varphi$ and

$$\begin{aligned}
&\|W_{u,T\chi E_n}\|_\varphi \\
&= \|M_{u\chi E_n}\|_{\varphi,\mu T^{-1}} \\
&= \|M_{u\circ T\chi E_n}\|_\varphi \\
&\geq n\|\chi E_n\|_\varphi,
\end{aligned}$$

where $E_n = \{x \in \Omega\colon |u\,(T(x))\,| > n\}$. This means that the weighted composition operator $W_{u,T}$ has no bound and hence it contradicts the hypothesis. $\therefore u \in L^\infty(\mu T^{-1})$.
Conversely,
Let $u \in L^\infty(\mu T^{-1})$ i.e. $u \circ T \in L^\infty$. Then by using theorem 5.1, $f \mapsto u \circ T.f$ is a bounded operator on L_φ. Also since $f_T \in L^\infty$, $f \mapsto f \circ T$ is bounded. Hence $W_{u,T}$ is bounded on L_φ. □

5.2 Adjoint of $W_{u,T}$

Definition 5.2. Let $W_{u,T}\colon L_\varphi \mapsto L(\mu)$ is a bounded linear operator on L_φ, then there exists an unique operator $W^\star_{u,T}\colon L_\varphi \mapsto L(\mu)$ such that

$$(x, W^\star_{u,T}y) = (W_{u,T}x, y) \text{ for all } x, y \in L_\varphi$$

The operator $W^\star_{u,T}$ is linear and bounded with the following criterion.

1. $\|W^\star_{u,T}\| = \|W_{u,T}\|$
2. $(W^\star_{u,T})^\star = W_{u,T}$

The operator $W^\star_{u,T}$ is called the adjoint of $W_{u,T}$.

Theorem 5.4. Let $W_{u,T}$ be bounded on L_φ. Then the adjoint $W^\star_{u,T}$ is given by

$$W^\star_{u,T}(g) = f_T.u.E(g) \circ T^{-1} \text{ for each } g \in L_\psi.$$

(Here φ and ψ are two complementary Orlicz function.)

Proof. Let $A \in B$ be such that $\mu(A) < \infty$, where B is a measurable space. Then for $g \in L_\psi$,

$$\begin{aligned}
&(W^\star_{u,T} F_g)(\chi A) \\
&= F_g(W_{u,T}\chi A) \qquad (by\ the\ definition\ of\ adjoint) \\
&= \int (W_{u,T}\chi A).g d\mu \\
&= \int u \circ T.\chi A \circ T g d\mu \\
&= \int E(u \circ T.g).\chi A \circ T d\mu \\
&= \int u \circ T.E(g).\chi A \circ T d\mu \\
&= \int f_T.u.E(g) \circ T^{-1}.\chi A d\mu \\
&= (F_{f_T.u.E(g)\circ T^{-1}})(\chi A)
\end{aligned}$$

Hence, $(W^\star_{u,T} F_g) = F_{f_T.u.E(g)\circ T^{-1}}$ and
$(W^\star_{u,T} g) = f_T.u.E(g) \circ T^{-1}$, for each $g \in L_\psi$ □

6 Bounded Weighted Composition on Orlicz space

We have the transformation T i.e.

$$T \colon \Omega \to \Omega$$

. By using Radon-Nikodym derivative there exists non-negative measurable function f_0 such that $\mu T^{-1}(A) = \int_A f_0 d\mu$. Here f_0 is called the Radon-Nikodym derivative of the measure μT^{-1} with respect to the measure μ and it is denoted by

$$f_0 = \frac{d\mu T^{-1}}{d\mu}$$

On the basis of above we have the following results.

Theorem 6.1. Let $T \colon \Omega \to \Omega$ and $u \colon \Omega \to C$ be two measurable transformation. Then the operator $M_{u,T} \colon L^\phi(\mu) \to L^\phi(\mu)$ is bounded operator if and only if there exists a constant $M > 0$ such that

$$f_0(x)\phi(|u(x)y|) \leq \phi(M|y|)$$

for $\mu-$almost all $x \in \Omega$, $y \in C$.

Proof. Suppose that, $f_0(x)\phi(|u(x)y|) \leq \phi(M|y|)$.
Then for every $f \in L^{\phi}(\mu)$,

$$\begin{aligned}
&\int_{\Omega} \phi\left(\frac{M_{u,T}(f)}{M\|f\|_{\phi}}\right) d\mu \\
&= \int_{\Omega} \phi\left(\frac{(u \circ T).(f \circ T)}{M\|f\|_{\phi}}\right) d\mu \\
&= \int_{\Omega} \phi\left(\frac{u(T(x)).f(T(x))}{M\|f\|_{\phi}}\right) d\mu \\
&= \int_{T^{-1}(\Omega)} \phi\left(\frac{(u \circ T).(f \circ T)}{M\|f\|_{\phi}}\right) d\mu T^{-1}(y) \qquad \textit{taking } T(x) = y \\
&= \int_{\Omega} \phi\left(\frac{|u.f|}{M\|f\|_{\phi}}\right) \frac{d\mu T^{-1}}{d\mu} d\mu \\
&\leq \int_{\Omega} f_0 \phi\left(\frac{|u.f|}{M\|f\|_{\phi}}\right) d\mu \qquad \because \frac{d\mu T^{-1}}{d\mu} = f_0 \\
&\leq \int_{\Omega} \phi\left(\frac{|f|}{\|f\|_{\phi}}\right) d\mu \\
&\leq 1
\end{aligned}$$

Therefore,

$$\|M_{u,T}\|_{\phi} \text{ for every } f \in L^{\phi}(\mu).$$

Conversely, Let $M_{u,T} \colon L^{\phi}(\mu) \to L^{\phi}(\mu)$ is a bounded operator. And Suppose that the condition $f_0(x)\phi(|u(x)y|) \leq \phi(M|y|)$ is not true. Then for every $n \in N$ there exists a measurable set $\{F_n\}$ of Ω and some $y_n \in C$ such that

$$F_n = \{x \in \Omega \colon f_0(x)\phi(|u(x)y_n|) > \phi(2^n n^2 |y_n|)\}$$

is a measurable set having positive measure. Now we know that μ is non-atomic and so we can choose a disjoint sequence of measurable sets $\{E_n\}$ such that

1. $E_n \subset F_n$ and
2. $\mu(E_n) = \frac{\phi(|y_1|)}{2^n \phi(n^2|y_n|)}$

Let f is defined as simple function as follows.

$$f = \sum_{n=1}^{\infty} b_n \chi E_n, \quad \textit{where } b_n = n y_n$$

. Now, consider

$$
\begin{aligned}
&\int_\Omega \phi(\alpha f) d\mu \\
&= \int_\Omega \phi\left(\alpha \sum_{n=1}^{\infty} b_n \chi E_n\right) d\mu \\
&= \sum_{n=1}^{\infty} \int_\Omega \phi(\alpha b_n) \chi E_n d\mu \\
&= \sum_{n=1}^{n_0} \phi(\alpha b_n)\mu(E_n) + \sum_{n=n_0}^{\infty} \phi(\alpha b_n)\mu(E_n) \\
&\leq \sum_{n=1}^{n_0} \phi(\alpha b_n)\mu(E_n) + \sum_{n=n_0}^{\infty} \frac{\phi(\alpha b_n)\phi(|y_1|)}{2^n \phi(n^2|y_n|)} \\
&= \sum_{n=1}^{n_0} \phi(\alpha b_n)\mu(E_n) + \sum_{n=n_0}^{\infty} \frac{\phi(n^2|y_n|)\phi(|y_1|)}{2^n \phi(n^2|y_n|)} \\
&< \infty
\end{aligned}
$$

where $n_0 > \alpha$ But

$$
\begin{aligned}
&\int_\Omega \phi(\alpha M_{u,T} f) d\mu \\
&= \int_\Omega f_0 \phi(\alpha |u.f|) d\mu \\
&\geq \sum_{n=n_0}^{\infty} \int_{E_n} f_0 \phi(\frac{1}{n}|u.b_n|) d\mu \qquad \alpha < n_0 \; and \; \alpha > \frac{1}{n_0} \\
&\geq \sum_{n=n_0}^{\infty} \int_{E_n} f_0 \phi(|u.y_n|) d\mu \qquad \because b_n = n y_n \\
&> \sum_{n=n_0}^{\infty} \phi(2^n n^2 |y_n|)\mu(E_n) \\
&= \sum_{n=n_0}^{\infty} 2^n \phi(n^2|y_n|)\mu(E_n) \\
&\geq \sum_{n=n_0}^{\infty} \phi(|y_1|) \\
&= \infty
\end{aligned}
$$

which is a contradiction to the fact that $M_{u,T}$ is a bounded operator on $L^\phi(\mu)$. Hence the inequality $f_0(x)\phi(|u(x)y|) \leq \phi(M|y|)$ holds. $\square$

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