A review on root system of Lie superalgebras and some partial results on splints of A(m,n)

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CERTIFICATE

Certified that the project work entitled, "**A review on root system of Lie superalgebras and some partial results on splints of A(m,n)**"submitted by '*SUSHREE SANGEETA PRADHAN*', Roll no. *413ma2068* in partial fulfillment of the degree of **Master of Science (Research) in Mathematics** offered by **National Institute of Technology, Rourkela** during the academic year $2014-2015$ is a review work carried out by the student under my superivision.

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ABSTRACT

Lie superalgebras are important in *theoretical physics* where they are used to describe the mathematics of supersymmetry. This dissertation deals with the splints of the root systems of **Classical Lie superalgebra** which can be seen as a generalisation of a Lie algebra to include a $\mathbb{Z}_2 - grading$.

The term '*Splints*' is first coined by **David A Richter** which play an important role in determining the branching rules of a module over a complex semisimple Lie algebra. These results have been extended to classical Lie superalgebras which gave interesting results with regards to the graded algebras.

Contents

INTRODUCTION

Lie Superalgebras are the graded form of the basic Lie algebras and can appear as Lie algebras of certain generalized groups, nowadays called Lie supergroups, whose function algebras are algebras with commuting and anticommuting variables. Briefing the main features of **finite-dimensional Lie Superalgebras** we come across the concept of *superderivation* and it's other implications like *solvability* and the *derived series*.

Let A be a $\mathbb{Z}_2-$graded algebra over K. A $K-$linear map $\partial : A \longrightarrow A$ of degree α^2 is a $left\ superderivation$ provided

$$\partial(bc) = \partial(b)c + (-1)^{\alpha b} b\partial(c)$$

for all $b, c \in A$. Similarly ∂ is a $right - derivation\ of\ degree\ \alpha$ if

$$\partial(bc) = b\partial(c) + (-1)^{\alpha c}\partial(b)c$$

for all $b, c \in A$.

Let $\mathfrak{g}$ ba a Lie Superalgebra. If V and W are subspaces of $\mathfrak{g}$, we write $[V, W]$ for the subspace spanned by all $[v, w]$ with $v \in V,\ w \in W$. A $\mathbb{Z}_2-$graded subspace $\mathfrak{a}$ of $\mathfrak{g}$ is an *ideal* if $[\mathfrak{a},\ \mathfrak{g}] \subset \mathfrak{a}$. The *derived series* $\mathfrak{g}^{(i)}$ of $\mathfrak{g}$ is defined by setting

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)},\ \mathfrak{g}^{(i)}] \ \text{ if } i \geq 0.$$

The *lower central series* $\mathfrak{g}^{[i]}$ of $\mathfrak{g}$ is defined by

$$\mathfrak{g}^{[0]} = \mathfrak{g}, \quad \mathfrak{g}^{[i+1]} = [\mathfrak{g},\ \mathfrak{g}^{[i]}] \ \text{ if } i \geq 0.$$

We say $\mathfrak{g}$ is *solvable* (resp. *nilpotent*) if $\mathfrak{g}^{(n)} = 0$ (resp. $\mathfrak{g}^{[0]} = 0$) for large n and that $\mathfrak{g}$ is *abelian* if $[\mathfrak{g}, \mathfrak{g}] = 0$.

David. Richter introduced the concept of *Splints in root systems* in the case of *semisimple Lie algebras*. It later played an significant role in stating the concepts of *branching rule* for Lie algebras. Now, these concepts have been generalised for

Lie superalgebras, and also the idea of *splints* has been extended. The *stems* of the *splints* include both the even and odd parts of the root system for *Lie superalgebras*. This exhibits many interesting properties to be seen in this paper.

1. PRELIMINARY

1.1. Graded Algebraic Structures

Let Γ be one of the rings $\mathbb{Z}$ (ring of integers) or $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ (ring of integers modulo 2). Considering gradations with values in Γ, the two elements of $\mathbb{Z}_2$ will be denoted by $\overline{0}$ (residue class of even integers) and $\overline{1}$ (residue class of odd integers). If $a \in \mathbb{Z}$ then the integer $(-1)^a$ depends only on the residue class modulo 2 of a. Hence for all $\alpha \in \mathbb{Z}$ the integer $(-1)^\alpha$ is well defined.

1. Let V be avector space over the field K. A $\Gamma-$*gradation of the vector space* V is a family $(V_\gamma)_{\gamma\in\Gamma}$ of subspaces of V such that

$$V = \bigoplus_{\gamma\in\Gamma} V_\gamma$$

The *vector space* V is said to be $\Gamma-$*graded* if it is equipped with a $\Gamma-$*gradation*.

An element of V is called *homogeneous* of degree γ, $\gamma \in \Gamma$, if it is an element of V_γ. In the case $\Gamma = \mathbb{Z}_2$ the elements of $V_{\overline{0}}$ (resp. $V_{\overline{1}}$) are also called *even* (resp. *odd*).

Every element $y \in V$ has a unique decomposition of the form

$$y = \sum_{\gamma\in\Gamma} y_\gamma \quad ; \quad y_\gamma \in V_\gamma, \quad \gamma \in \Gamma$$

(where, of element y_γ is called the *homogeneous component* of y of degree γ. A subspace U of V is called $\Gamma - graded$ (or simply *graded*) if the homogeneous components of all of its elements, i.e. if

$$U = \bigoplus_{\gamma\in\Gamma} (U \cap V_\gamma).$$

On any $\mathbb{Z}-$*graded* vector space $V = \bigoplus_{j\in\mathbb{Z}} V_j$ there exists a natural $\mathbb{Z}_2-$*gradation* which is said to be induced by the $\mathbb{Z} - gradation$ and which is defined by

$$V_{\overline{0}} = \bigoplus_{j\in\mathbb{Z}} V_{2j} \quad ; \quad V_{\overline{1}} = \bigoplus_{j\in\mathbb{Z}} V_{2j+1}$$

2. Let

$$W = \bigoplus_{\gamma \in \mathit{\Gamma}}$$

be a second $\mathit{\Gamma}-$graded vector space. A 1inear mapping

$$\mathfrak{g} \; : V \longrightarrow W$$

is said to be *homogeneous* of degree $\gamma,\ \gamma \in \mathit{\Gamma}$, if

$$\mathfrak{g}(V_\alpha) \subset W_{\alpha+\gamma} \quad \text{for all } \alpha \in \mathit{\Gamma}.$$

The mapping $\mathfrak{g}$ is ca11ed a homomorphism of the $\mathit{\Gamma}-$graded vector space V into the $\mathit{\Gamma}-$graded vector space W if $\mathfrak{g}$ is homogeneous of degree 0. It is now evident how we define an isomorphism or an automorphism of $\mathit{\Gamma}-$graded vector spaces.

3. Let A be an algebra over K. The *algebra* A is said to be $\mathit{\Gamma} -$ *graded* if the underlying vector space of A is $\mathit{\Gamma}-$graded,

$$A = \bigoplus_{\gamma \in \mathit{\Gamma}} A_\gamma,$$

and if, furthermore,

$$A_\alpha A_\beta \subset A_{\alpha+\beta} \quad \text{for all } \alpha,\ \beta \in \mathit{\Gamma}$$

Evidently, A_0 is a subalgebra of A. If A has a unit element then this element 1ies in A_0.

A homomorphism of $\mathit{\Gamma}-$graded algebras is by definition a homomorphism of the underlying algebras as well as of the underlying $\mathit{\Gamma}-$graded vector spaces; in particular, a homomorphism is homogeneous of degree 0. Similar remarks apply for isomorphisms and automorphisms.

A graded subalgebra (resp. ideal) of a $\mathit{\Gamma}-$graded algebra A is a subalgebra (resp. ideal) of the algebra A which is, in addition, a graded subspace of the $\mathit{\Gamma}-$graded vector space A. The quotient algebra of a $\mathit{\Gamma}-$graded algebra modulo a (two-sided) graded ideal is again a $\mathit{\Gamma}-$graded algebra.

1.2. Representations in Groups

Let G be a finite group, let F be a field and let V be a vector space over F.

1. A *linearrepresentation* of G is any homomorphism from G into $GL(V)$. The *degree* of the representation is the dimension of V.

2. Let $n \in \mathbb{Z}^+$. A *matrixrepresentation* of G is any homomorphism from G into $GL_n(F)$.

3. A linear or matrix representation is *faithful* if it is injective.

4. The *groupring* of G over F is the set of all formal sums of the form

$$\sum_{g\in G} \alpha_g g \qquad \alpha \in F$$

with componentwise addition and multiplication $(\alpha g)(\beta h) = (\alpha\beta)(gh)$(where α and β are multiplied in F and gh is the product in G) extended to sums via the distributive law.

Now, we can say V to be an $FG-$module if V is a vector space over F and $\phi : G \to GL(V)$ is a representation. So, if V is an $FG-$module affording the representation ϕ, then a subspace U of V is called $G - invariant$ or $G - stable$ if $g.u \in U$ for all $g \in G$ and all $u \in U$ (i.e., if $\phi(g)(u) \in U$ for all $g \in G$ and all $u \in U$). It follows easily that the $FG - submodules$ of V are precisely the $G - stable$ subspaces of V.

2. LIE SUPERALGEBRAS

Introducing the basic notions and definitions in the theory of Lie superalgebras, we have

$$\mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$$

denoting the group of two elements and let $\mathcal{C}_n$ denote the symmetric group in n letters.

Definition: A Lie superalgebra is a superalgebra $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ with bilinear multiplication $[.,.]$ satisfying the following two axioms: for homogeneous elements $a, b, c \in \mathfrak{g}$,

1. Skew-supersymmetry: $[a, b] = -(-1)^{|a|.|b|}[b, a]$.
2. Super Jacobi identity: $[a, [b, c]] = [[a, b], c] + (-1)^{|a|.|b|}[b, [a, c]]$.

For a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$, the even part $\mathfrak{g}_{\overline{0}}$ is a Lie algebra. Hence, if $\mathfrak{g}_{\overline{1}} = 0$, then $\mathfrak{g}$ is just a usual Lie algebra. A Lie superalgebra $\mathfrak{g}$ with purely odd part, i.e., $\mathfrak{g}_{\overline{0}} = 0$, has to be abelian, i.e., $[\mathfrak{g}, \mathfrak{g}] = 0$.

A superalgebra $A = A_{\overline{0}} \oplus A_{\overline{1}}$ can be made into a Lie superalgebra by letting

$$[a, b] := ab - (-1)^{|a|.|b|}ba,$$

for $a, b \in A$ and extending $[.,.]$ by bilinearity.

2.1. Representations of Lie Superalgebras

Let L be a Lie supera1gebra and 1et V be a $\mathbb{Z}_2-$graded vector space. As $Hom(V)$ has a natural $\mathbb{Z}_2-$gradation which converts it into an associative superalgebra.

A *graded representation* ρ of L in V is an even linear mapping

$$\rho \;:\; L \longrightarrow \text{Hom}V$$

such that

$$\rho(< A, B >) = \rho(A)\rho(B) - (-1)^{\alpha\beta}\rho(B)\rho(A)$$

$$\text{for all } A \in L_\alpha,\; B \in L_\beta;\; \alpha,\; \beta \in \mathbb{Z}_2$$

A $\mathbb{Z}_2-$graded vector space equipped with a graded representation of L is called a (left) *graded* $L-$*module.*

2.2. <u>Classical Lie Superalgebras</u>

The finite dimensional Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is *simple* if it does not contain any non-trivial ideal and the necessary condition is that the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is faithful and $[\mathfrak{g}_{\bar{1}},\ \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{0}}$. $\mathfrak{g}$ is *classical* if it is simple and the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely *reducible*.

2.3. <u>The general and special linear Lie superalgebras.</u>

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace so that $\mathrm{End}(V)$ is a superalgebra. The $\mathrm{End}(V)$ equipped with the supercommutator, forms a Lie superalgebra called the general linear Lie superalgebra and denoted by $\mathfrak{gl}(v)$.

2.4. <u>The ortho-symplectic Lie superalgebras.</u>

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace. A bilinear form

$$B(.,.) : V \times V \longrightarrow V$$

is called even (respectively, odd), if $B(V_i, V_j) = 0$ unless $i + j = \bar{0}$ (respectively, $i + j = \bar{1}$). An even bilinear form B is said to be supersymmetric if $B|_{V_{\bar{0}} \times V_{\bar{0}}}$ is symmetric and $B|_{V_{\bar{1}} \times V_{\bar{1}}}$ is skew-symmetric, and it is called skew-supersymmetric if $B|_{V_{\bar{0}} \times V_{\bar{0}}}$ is skew-symmetric and $B|_{V_{\bar{1}} \times V_{\bar{1}}}$ is symmetric.

Let B be a non-degenerate even supersymmetric bilinear form on a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$. For $s \in \mathbb{Z}_2$, let

$$\mathfrak{osp}(V)_s := \{g \in \mathfrak{gl}(V)_s | B(g(x), y) = -(-1)^{s.|x|} B(x, g(y)), \forall x, y \in V\},$$

$$\mathfrak{osp}(V) := \mathfrak{osp}(V)_{\bar{0}} \oplus \mathfrak{osp}(V)_{\bar{1}}.$$

The lie superalgebra $\mathfrak{osp}(V)$ is the subalgebra of $\mathfrak{gl}(V)$ that preserves a non-degenerate supersymmetric bilinear form. Its even subalgebra is isomorphic to $\mathfrak{so}(V_{\bar{0}}) \oplus \mathfrak{sp}(V_{\bar{1}})$, a direct sum of the orthogonal lie algebra on $V_{\bar{0}}$ and the symplectic lie algebra on $V_{\bar{1}}$.

3. ROOT SYSTEM OS LIE SUPERALGEBRAS

3.1. Cartan Matrix

Definition (*Cartan Subalgebra*): Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a classical Lie superalgebra. A Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$ is defined as the maximal nilpotent subalgebra of $\mathcal{G}$ coinciding with its own normalizer, that is

$$\mathcal{H} \text{ nilpotent and } \{X \in \mathcal{G} | [X, \mathcal{H}] \subseteq \mathcal{H}\} = \mathcal{H}$$

Definition: Let $\mathcal{G}$ be a basic Lie superalgebra with Cartan subalgebra $\mathcal{H}$ and simple root system $\Delta^0 = (\alpha_1, \ldots \alpha_r)$. Then we define the *Cartan matrix* of $\mathcal{G}$ as:

$$A_{ij} = 2\frac{\alpha_i.\alpha_j}{\alpha_i.\alpha_i} \qquad \text{if } \alpha_i.\alpha_i \neq 0$$

$$A_{ij} = \frac{\alpha_i.\alpha_j}{\alpha_i.\alpha_i} \qquad \text{if } \alpha_i.\alpha_i = 0$$

If $\alpha_i.\alpha_i \neq 0$, then $A_{ii} = 2$. If $\alpha_i.\alpha_i = 0$, then $A_{ii} = 0$.

A cartan subalgebra of a classical Lie superalgebra is diagonalizable. Therefore, we have the root decomposition

$$\mathcal{G} = \oplus_{\alpha \in \mathcal{H}} \qquad \text{where} \quad \mathcal{G}_\alpha = \{a \in \mathcal{G} | \ [h, a] = \alpha(h)a \text{ for } h \in \mathcal{H}$$

The set $\Delta = \{\alpha \in \mathcal{H} | \ \mathcal{G}_\alpha \neq 0\}$ is called the root system. So, $\Delta = \Delta_0 \cup \Delta_1$, where Δ_0 is called the *system of even* and Δ_1 that of *odd roots*.

3.2. Dynkin Diagram

Let $\mathfrak{g}$ be a basic Lie superalgebras of rank r and dimension n. Let $\Delta^0 = (\alpha_1, \ldots, \alpha_r)$ be a simple root system of $\mathfrak{g}$, A be the associated Cartan matrix. One can associate to Δ^0 a Dynkin diagram according to the following rules.
Using the Cartan matrix A :

1. One associates to each simple even root a white dot, to each simple odd root of zero length ($A_{ii} = 0$) a grey dot.

2. The $i-$th and $j-$th dots will be joined by η_{ij} lines where

$$\eta_{ij} = \max(|A_{ij}|, |A_{ji}|)$$

3. We add an arrow on the lines connecting the $i-$th and $j-$th dots when $\eta_{ij} > 1$ and $|A_{ij}| \neq |A_{ji}|$, pointing from j to i if $|A_{ij}| > 1$.

3.3. Classification of Lie Superalgebras:

1. THE BASIC LIE SUPERALGEBRA $A(m-1, n-1) = sl(m, n)$.

   ....

   Root system ($1 \leq i \neq j \leq m$ and $1 \leq k \neq l \leq n$):

$$\Delta = \{e_i - e_j, \delta_k - \delta_l, e_i - \delta_k, \delta_k - e_i\}$$

$$\Delta_{\overline{0}} = \{e_i - e_j, \delta_k - \delta_l\}, \quad \Delta_{\overline{1}} = \{e_i - \delta_k, \delta_k - e_i\}$$

   Simple root system:

$$\alpha_1 = \delta_1 - \delta_2, \ldots, \ \alpha_{n-1} = \delta_{n-1} - \delta_n, \ \alpha_n = \delta_n - e_1,$$

$$\alpha_{n+1} = e_1 - e_2, \ldots, \ \alpha_{m+n-1} = e_{m-1} - e_m.$$

   where

$$(e_i, e_j) = 1 \text{ for } i = j, \text{ otherwise } 0,$$

$$(\delta_i, \ \delta_j) = -1 \text{ for } i = j, \text{ otherwise } 0$$

   and

$$(\delta, \ e_i) = 0$$

   The positive roots ($1 \leq i < j \leq m$ and $1 \leq k < l \leq n$):

$$\delta_k - \delta_l = \alpha_k + \cdots + \alpha_{l-1}$$

$$e_i - e_j = \alpha_{n+i} + \cdots + \alpha_{n+j-1}$$

$$\delta_k - e_i = \alpha_k + \cdots + \alpha_{n+i-1}$$

   The Dynkin diagram:

$$\bigcirc \cdots \bigcirc \longrightarrow \bigotimes \longrightarrow \bigcirc \cdots \bigcirc$$

The Cartan matrix:

$$
\begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & \cdots & \cdots & & & \cdots & 0 \\
-1 & \ddots & \ddots & \ddots & & & & & & & \vdots \\
0 & \ddots & & \ddots & 0 & & & & & & \\
\vdots & \ddots & \ddots & \ddots & -1 & \ddots & & & & & \\
0 & & 0 & -1 & 2 & -1 & \ddots & & & & \vdots \\
\vdots & & & \ddots & -1 & 0 & 1 & \ddots & & & \vdots \\
\vdots & & & & \ddots & -1 & 2 & -1 & 0 & & 0 \\
 & & & & & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\
 & & & & & & 0 & \ddots & & \ddots & 0 \\
 & & & & & & & \ddots & \ddots & \ddots & -1 \\
0 & \cdots & & & \cdots & \cdots & 0 & \cdots & 0 & -1 & 2
\end{pmatrix}
$$

2. THE BASIC LIE SUPERALGEBRA $B(m,n) = osp(2m+1|2n)$.

. . . .

Root system ($1 \le i \ne j \le m$ and $1 \le k \ne l \le n$):

$$\Delta = \{\pm e_i \pm e_j, \ \pm \delta_k \pm \delta_l, \ \pm 2\delta_k, \ \pm e_i \pm \delta_k, \ \pm \delta_k\}$$

$$\Delta_{\overline{0}} = \{\pm e_i \pm e_j, \ \pm \delta_k \pm \delta_l, \ \pm 2\delta_k\}, \quad \Delta_{\overline{1}} = \{\pm e_i \pm \delta_k, \ \pm \delta_k\}$$

Simple root system:

$$\alpha_1 = \delta_1 - \delta_2, \ldots, \ \alpha_{n-1} = \delta_{n-1} - \delta_n, \ \alpha_n = \delta_n - e_1,$$

$$\alpha_{n+1} = e_1 - e_2, \ldots, \ \alpha_{n+m-1} = e_{m-1} - e_m.$$

The positive roots ($1 \leq i < j \leq m$ and $1 \leq k < l \leq n$):

$$\begin{aligned}
\delta_k - \delta_l &= \alpha_k + \cdots + \alpha_{l-1} \\
\delta_k + \delta_l &= \alpha_k + \cdots + \alpha_{l-1} + 2\alpha_l + \cdots + 2\alpha_{n+m} \\
2\delta_k &= 2\alpha_k + \cdots + 2\alpha_{n+m} \\
e_i - e_j &= \alpha_{n+i} + \cdots + \alpha_{n+j-1} \\
e_i + e_j &= \alpha_{n+i} + \cdots + \alpha_{n+j-1} + 2\alpha_{n+j} + \cdots + 2\alpha_{n+m} \\
e_i &= \alpha_{n+i} + \cdots + \alpha_{n+m} \\
\delta_k - e_i &= \alpha_k + \cdots + \alpha_{n+i-1} \\
\delta_k + e_i &= \alpha_k + \cdots + \alpha_{n+i-1} + 2\alpha_{n+i} + \cdots + 2\alpha_{n+m} \\
\delta_k &= \alpha_k + \cdots + \alpha_{n+m}.
\end{aligned}$$

The Dynkin diagram:

$$\bigcirc \cdots \bigcirc \longrightarrow \bigotimes \longrightarrow \bigcirc \cdots \bigcirc \Longrightarrow \bigcirc$$

The Cartan matrix:

$$\begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & \cdots & \cdots & & & \cdots & 0 \\
-1 & \ddots & \ddots & \ddots & & & & & & & \vdots \\
0 & \ddots & & \ddots & 0 & & & & & & \\
\vdots & \ddots & \ddots & \ddots & -1 & \ddots & & & & & \\
0 & & 0 & -1 & 2 & -1 & \ddots & & & & \vdots \\
\vdots & & & \ddots & -1 & 0 & 1 & \ddots & & & \vdots \\
\vdots & & & & \ddots & -1 & 2 & -1 & 0 & & 0 \\
 & & & & & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\
 & & & & & & 0 & \ddots & & -1 & 0 \\
 & & & & & & & \ddots & -1 & 2 & -1 \\
0 & \cdots & & & \cdots & \cdots & 0 & \cdots & 0 & -2 & 2
\end{pmatrix}$$

3. THE BASIC LIE SUPERALGEBRA $C(n+1) = osp(2|2n)$

. . . .

Root system ($1 \leq k \neq l \leq n$):

$$\Delta = \{\pm\delta_k \pm \delta_l, \ \ \pm \delta_k, \ \ \pm e \pm \delta_k\}$$

$$\Delta_{\bar{0}} = \{\pm\delta_k \pm \delta_l\}, \quad \Delta_{\bar{1}} = \{\pm e \pm \delta_k\}$$

Simple root system:

$$\alpha_1 = e - \delta_1,\ \alpha_2 = \delta_1 - \delta_2, \ldots,\ \alpha_n = \delta_{n-1} - \delta_n,\ \alpha_{n+1} = 2\delta_n$$

All the positive roots $(1 \leq k < l \leq n)$:

$$\begin{aligned}
\delta_k - \delta_l &= \alpha_{k+1} + \cdots + \alpha_l \\
\delta_k + \delta_l &= \alpha_{k+1} + \cdots + \alpha_l + 2\alpha_{l+1} + \cdots + 2\alpha_n + \alpha_{n+1} \qquad (l < n) \\
\delta_k + \delta_n &= \alpha_{k+1} + \cdots + \alpha_{n+1} \\
2\delta_k &= 2\alpha_{k+1} + \cdots + 2\alpha_n + \alpha_{n+1} \qquad (k < n) \\
2\delta_n &= \alpha_{n+1} \\
e - \delta_k &= \alpha_1 + \cdots + \alpha_k \\
e - \delta_k &= \alpha_1 + \cdots + \alpha_k + 2\alpha_{k+1} + 2\alpha_n + \alpha_{n+1} \qquad (k < n) \\
e - \delta_n &= \alpha_1 + \cdots + \alpha_{n+1}.
\end{aligned}$$

The Dynkin diagram:

$$\bigotimes \longrightarrow \bigcirc \cdots \bigcirc \Longleftarrow \bigcirc$$

The Cartan matrix:

$$\begin{pmatrix}
0 & 1 & 0 & \cdots & \cdots & 0 \\
-1 & 2 & -1 & 0 & & 0 \\
0 & -1 & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ddots & \ddots & -1 & 0 \\
\vdots & & \ddots & -1 & 2 & -2 \\
0 & \cdots & \cdots & 0 & -1 & 2
\end{pmatrix}$$

4. THE BASIC LIE SUPERALGEBRA $D(m, n) = osp(2m|2n)$

. . . .

Root System $(1 \leq i \neq j \leq m$ and $1 \leq k \neq l \leq n)$:

$$\Delta = \{\pm e_i \pm e_j,\ \ \pm\delta_k \pm \delta_l,\ \ \pm 2\delta_k,\ \ \pm e_i \pm \delta_k\}$$

$$\Delta_{\bar{0}} = \{\pm e_i \pm e_j,\ \ \pm \delta_k \pm \delta_l,\ \ \pm 2\delta_k\}, \qquad \Delta_{\bar{1}} = \{\pm e_i \pm \delta_k\}$$

Simple Root System:

$$\alpha_1 = \delta_1 - \delta_2, \ldots\ \alpha_{n-1} = \delta_{n-1} - \delta_n,\ \alpha_n = \delta_n - e_1,$$

$$\alpha_{n+1} = e_1 - e_2, \ldots,\ \alpha_{n+m-1} = e_{m-1} - e_m,\ \alpha_{n+m} = e_{m-1} + e_m$$

All positive roots($1 \le i < j \le m$ and $1 \le k < l \le n$) :

$$\begin{aligned}
\delta_k - \delta_l &= \alpha_k + \cdots + \alpha_{l-1} \\
\delta_k + \delta_l &= \alpha_k + \cdots + \alpha_{l-1} + 2\alpha_l + \cdots + 2\alpha_{n+m-2} + \alpha_{n+m-1} + \alpha_{n+m} \\
2\delta_k &= 2\alpha_k + \cdots + 2\alpha_{n+m-2} + \alpha_{n+m-1} + \alpha_{n+m} \\
e_i - e_j &= \alpha_{n+i} + \cdots + \alpha_{n+j-1} \\
e_i + e_j &= \alpha_{n+i} + \cdots + \alpha_{n+j-1} + 2\alpha_{n+j} + \cdots + 2\alpha_{n+m-2} + \alpha_{n+m-1} + \alpha_{n+m} \\
e_i + e_{m-1} &= \alpha_{n+i} + \cdots + \alpha_{n+m-2} + \alpha_{n+m-1} + \alpha_{n+m} \\
e_i + e_m &= \alpha_{n+i} + \cdots + \alpha_{n+m-2} + \alpha_{n+m} \\
\delta_k - e_i &= \alpha_k + \cdots + \alpha_{n+i-1} \\
\delta_k + e_i &= \alpha_k + \cdots + \alpha_{n+i-1} + 2\alpha_{n+i} + \cdots + 2\alpha_{n+m-2} + \alpha_{n+m-1} + \alpha_{n+m} \\
\delta_k + e_{m-1} &= \alpha_k + \cdots + \alpha_{n+m-2} + \alpha_{n+m-1} + \alpha_{n+m} \\
\delta_k + e_m &= \alpha_k + \cdots + \alpha_{n+m-2} + \alpha_{n+m}
\end{aligned}$$

The Cartan matrix:

$$\begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & \cdots & \cdots & & & \cdots & 0 \\
-1 & \ddots & \ddots & \ddots & & & & & & & \vdots \\
0 & \ddots & & \ddots & 0 & & & & & & \\
\vdots & \ddots & \ddots & \ddots & -1 & \ddots & & & & & \\
0 & & 0 & -1 & 2 & -1 & \ddots & & & & \vdots \\
\vdots & & & \ddots & -1 & 0 & 1 & \ddots & & & \vdots \\
\vdots & & & & \ddots & -1 & 2 & -1 & 0 & & 0 \\
 & & & & & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\
 & & & & & & 0 & \ddots & \ddots & -1 & -1 \\
 & & & & & & & \ddots & -1 & 2 & 0 \\
0 & \cdots & & & \cdots & \cdots & 0 & \cdots & -1 & 0 & 2
\end{pmatrix}$$

4. SPLINTS OF LIE SUPERALGEBRAS

Definition: Suppose Δ_0 and Δ are root systems. Then embedding ι of a root system Δ is a bijective map of roots of Δ_0 to a (proper) subset of Δ, that commutes with vector composition law in Δ_0 and Δ. $\iota(\gamma) = \iota(\alpha) + \iota(\beta)$ for all $\alpha, \beta, \gamma \in \Delta$ such that $\gamma = \alpha + \beta$.

$$\iota(\gamma) \neq \iota(\alpha) + \iota(\beta) \text{ if } \alpha \in \Delta_1 \text{ and } \beta \in \Delta_0$$

Definition: A root system of Lie superalgebra Δ "splinters" as (Δ_1, Δ_2) if there are two embeddings $\iota_1 : \Delta_1 \hookrightarrow \Delta$ and where $\iota_2 : \Delta_2 \hookrightarrow \Delta$ where

1. Δ is the disjoint union of the images of ι_1 and ι_2 and
2. neither the rank of Δ_1 nor the rank of Δ_2 equal to or greater than the rank of Δ.

It is equivalent to say that (Δ_1, Δ_2) is a splint of Δ. Each component Δ_1 and Δ_2 is a stem of the splint (Δ_1, Δ_2).

4.1. Splints for root system of type A_r

For $r = 1$, A_1 does not splint as it has only one element. For $r \geq 3$, there can be atleast two splints of A_r, namely $(rA_1,\ A_{r-1})$ and $(A_1 + A_{r-1},\ (r-1)A_1)$. Splints can be described as

$$\Delta_1 = \{e_i - e_r : 0 \leq i \leq r-1\} \text{ and } \Delta_2 = \{e_i - e_j : 0 \leq i < j \leq r-1\},$$

and second by

$$\Delta_1 = \{e_i - e_j : 0 \leq i < j \leq r-1\} \cup \{e_0 - e_r\}$$

and

$$\Delta_2 = \{e_i - e_r : 11 \leq i \leq r-1\}$$

. These splints coincide when $r = 2$, giving the splint $(A_1, 2A_1)$ of (A_2). Whereas A_3 has an additional splint as $(3A_1,\ 3A_1)$.

4.2. Extension to Lie Superalgebras

In the case of *Lie Superalgebras*, the root system $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$ "splinters" as $(\Delta_{(1)\bar{0}} \cup \Delta_{(1)\bar{1}}\ ,\ \Delta_{(2)\bar{0}} \cup \Delta_{(2)\bar{1}})$ if there are two embeddings $\iota_1 : \Delta_{(1)\bar{0}} \cup \Delta_{(1)\bar{1}} \hookrightarrow \Delta$ and $\iota_2 : \Delta_{(2)\bar{0}} \cup \Delta_{(2)\bar{1}} \hookrightarrow \Delta$ and similarly Δ is the disjoint union of both the 'splints' where neither the rank of the splints exceeds the rank of Δ.

Example:

The splints of $A(m-1, n-1)$

Starting from $A(0,1)$

1st Splint:

$$\Delta_1 = \{\delta_2 - e_1,\ \delta_2 - e_2,\ e_1 - e_2\} = A(0,0)$$
$$\Delta_2 = \{\delta_1 - \delta_2,\ \delta_1 - e_1,\ \delta_2 - e_1\} = A_2$$

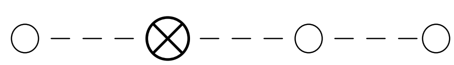
2nd Splint:

$$\Delta_1 = \{\delta_1 - \delta_2,\ \delta_1 - e_1,\ \delta_2 - e_1,\ \delta_1 - e_2\} = A_2 + A_1$$
$$\Delta_2 = \{\delta_2 - e_2,\ e_1 - e_2\} = 2A_1$$

3rd Splint:

$$\Delta_1 = \{\delta_1 - \delta_2,\ \delta_2 - e_1,\ \delta_1 - e_2\} = 3A_1$$
$$\Delta_2 = \{e_1 - e_2,\ \delta_1 - e_1,\ \delta_2 - e_2\} = 3A_1$$

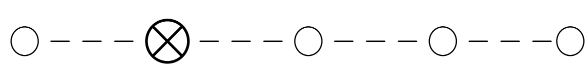
Now for $A(1,1)$



Splint:

$$\Delta_1 = \{\delta_1 - \delta_2,\ \delta_2 - e_1,\ \delta_2 - e_2,\ \delta_2 - e_3,\ \delta_1 - e_3,\ \delta_1 - e_1\} = A_2 + 3A_1$$
$$\Delta_2 = \{e_1 - e_2,\ e_2 - e_3,\ \delta_1 - e_2,\ e_1 - e_3\} = A_2 + A_1$$

For $A(2,1)$



Splint:

$$\Delta_1 = \{\delta_1 - \delta_2,\ \delta_2 - e_1,\ \delta_1 - e_1,\ e_1 - e_2,\ \delta_2 - e_2,\ \delta_1 - e_2,$$

$$e_2 - e_3, \delta_1 - e_4,\ e_1 - e_4,\ \delta_2 - e_4\} = A(0,1) + 4A_1$$

$$\Delta_2 = \{\delta_1 - e_3,\ e_1 - e_3,\ e_3 - e_4,\ \delta_2 - e_3,\ e_1 - e_4\} = A_2 + 2A_1.$$

Thus for generalization the splints of $A(m-1, n-1)$ are:

$$\Delta_1 = A(0, n-1) + (m+n-1)A_1$$
$$\Delta_2 = A_{m-1} + n(m-2)A_1$$

5. Conclusion

Due to time constraint I could not complete the whole work and these results obtained, I could not compare with more general results obtained by one of my Professors PhD. students B.Ransingh. However I want to delve into this problem in future.

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