A Review on Some G-family Distributions

Thesis submitted in partial fulfillment of the requirements for the degree of

Integrated Master of Science

by

Mr. Gopal Krishna Dila

Under the guidance of

Dr. Manas Ranjan Tripathy

Department of Mathematics
National Institute of Technology
Rourkela-769008
India

May 2015

Certificate

This is to certify that the thesis entitled **"A Review on Some G-family Distributions"**, which is being submitted by **Mr. Gopal Krishna Dila** in the Department of Mathematics, National Institute of Technology, Rourkela, in partial fulfilment for the award of the degree of **Integrated Master of Science**, is a record of bonafide review work carried out by him under my guidance. He has worked as a project student in this Institute for one year. In my opinion the work has reached the standard, fulfilling the requirements of the regulations related to the Master of Science degree. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

Place: NIT Rourkela
Date: May 2015

Dr. M. R. Tripathy
Assistant Professor
Department of Mathematics
NIT Rourkela-769008
India

Acknowledgement

It is a great pleasure and proud privilege to express my deep sense of gratitude to my guide, Prof. M. R. Tripathy. I am grateful to him for his, continuous encouragement and guidance throughout the period of my project work. Without his active guidance it would not have been possible for me to complete the work.

I acknowledge my deep sense of gratitude to the Head of the Department, all faculty members, all nonteaching staff members of the department of mathematics and the authorities of NIT Rourkela.

I also thank to all my friends for their support, co-operation and sincere help in various ways. I express my sincere thank with gratitude to my parents and other family members, for their support, blessings and encouragement. Finally, I bow down before the almighty who has made everything possible.

Place: NIT Rourkela

Date: May 2015

(Gopal Krishna Dila)

Roll No-410MA5087

Abstract

In this thesis we study the properties of some G-family of distributions, such as, Kumaraswamy-G family, Zografos-Balakrishnan-G family and Ristic-Balakrishnan-G family for any continuous baseline G distribution. Here, we provide a thorough study of general mathematical properties of these family of distributions. We are trying to find out some new distributions by making use of the families demonstrated. We discuss the properties of Zografos-Balakrishnan-generalized exponential distribution, such as, probability density function (pdf), cumulative distribution function (cdf), hazard rate function, moments, quantile function, entropy and estimation of the parameters by maximum likelihood method.

Contents

Chapter 1

Introduction

In statistics continuous distributions are generally amenable to more elegant mathematical treatment than discrete distributions. This makes them especially useful as approximations to discrete distributions. Continuous distributions are used in both construction of models and in applying statistical techniques. The fact that most uses of continuous distributions in model building are as approximations to discrete distributions may be less widely appreciated but is no less true.

An essential property of a continuous random variable is that there is zero probability that it takes any specified numerical value, but in general a non-zero probability, calculable as a definite integral of a probability density function that it takes a value in specified intervals. When an observed value is represented by a continuous random variable, the recorded value is, of necessity "discretized". For example, if measurements are made to the nearest 0.01 units, then all values actually in the interval (8.665,8.675) will be recorded as 8.67.

In this chapter a review of Kumaraswamy-G family and Zografos-Balakrishnan-G family is discussed which are very much essential for the development of the entire project work. Later, we discuss the properties of Zografos-Balakrishnan-generalized Exponential distribution, such as, probability density function (pdf), cumulative distribution function (cdf), hazard rate function, moments, quantile function, entropy and estimation of the parameters by maximum likelihood method.

1.1 A Review of Kumaraswamy-G family

Kumaraswamy (1980) introduced a distribution for double bounded random processes with hydrological applications. Based on this, he described a new family of generalized distributions (denoted as Kw) to extend several well-known distributions. In econometrics, commonly the data are modeled by finite range distributions. Summed up generalized beta distributions have been generally mulled over in insights and various authors have created different classes of these distribution. Eugene et al. (2002) proposed a general class of distributions for a random variable defined from the logit of the beta random variable by employing two parameters whose role is to introduce skewness and to vary tail weight. Following the work of Eugene et al. (2002), who defined the beta normal distribution, Nadarajah and Kotz (2004) introduced the beta Gumbel distribution, Nadarajah and Gupta (2004) proposed the beta Frechet distribution and Nadarajah and Kotz (2006) worked with the beta exponential distribution. However, all these works lead to some mathematical difficulties because the beta distribution is not fairly tractable and, in particular, it's cumulative distribution function (cdf) involves the incomplete beta function ratio.

1.2 A Review of Zografos-Balakrishnan-G family and Ristic-Balakrishnan-G family

Zografos and Balakrishnan (2009) and Ristic and Balakrishnan (2012) proposed two generalized gamma-generated distributions with an extra positive parameter, for any continuous baseline G distribution. They studied the mathematical properties of these family and put forward some models. The number of parameters of The Zografos-Balakrishnan-G and Ristic-Balakrishnan-G distributions is equal to that of the G distribution plus an additional shape parameter $a > 0$. For $a = 1$, the G distribution is a basic exemplar of the Zografos-Balakrishnan-G and Ristic-Balakrishnan-G distributions with a continuous crossover towards cases with different shapes (for example, a particular combination of skewness and kurtosis).

Well known distributions can be extended in many ways. The earliest of the extended distributions is the class of distributions generated by a standard beta random variable introduced by Eugene et al. (2002). The more recent ones are: the class of distributions generated by Kumaraswamy (1980), random variable introduced by Cordeiro and de Castro (2011); the class of distributions generated by McDonald (1984), generalized beta random variable introduced by Alexander et al. (2012); the class of distributions generated by Ng and Kotz (1995), Kummer beta random variable introduced by Pescim et al. (2012).

Chapter 2

Kumaraswamy-G family of distribution

2.1 Introduction

The pdf and cdf of the Kumaraswamy's distribution (2009) (Kw), with shape parameters $a < 0$ and $b < 0$ are defined by

$$f(x) = abx^{a-1}(1 - x^a)^{b-1} \text{ and } F(x) = 1 - (1 - x^a)^b.$$

Let us start from a parent continuous distribution function $G(x)$. A way of generating families of distributions is starting from a parent distribution with pdf $g(x) = \frac{dG(x)}{dx}$ to apply the quantile function to a family of distributions on the interval $(0, 1)$.From an arbitrary parent cdf $G(x)$, the cdf $F(x)$ of the Kw-G distribution is defined by:

$$F(x) = 1 - (1 - G(x)^a)^b. \tag{2.1}$$

where $a > 0$ and $b > 0$ are two additional parameters which introduce skewness and vary tail weights. Because of its tractable distribution function, the Kw-G distribution can be used quite effectively even if the data are censored. The probability density function (pdf) of this family of distribution is

$$f(x) = abg(x)G(x)^{a-1}(1 - G(x)^a)^{b-1} \tag{2.2}$$

The Kumaraswamy-G density function has an upper hand over the class of generalized beta distributions as it does not involve any type of special function. When $a = 1$, the Kw-G distribution coincides with the beta-G distribution generated by the $beta(1, b)$ distribution. One of the significant advantages of the Kw family of generalized distributions is its capacity of fitting skewed data that can not be properly fitted by any existing distributions.

2.2 Some special Kw generalized distributions

Some special Kw generalized distributions as discussed by Cordeiro and Castro (2011) are mentioned below.

a. Kumaraswamy-normal: The Kumaraswamy-normal density function can be obtained by taking $G(.)$ and $g(.)$ to be the cdf and pdf of the normal $N(\mu; \sigma^2)$ distribution, so that

$$f(x) = \frac{ab}{\sigma}\phi(\frac{x-\mu}{\sigma})[\Phi(\frac{x-\mu}{\sigma})]^{a-1}(1 - \Phi(\frac{x-\mu}{\sigma})^a)^{b-1},$$

where $x \in R$,$\mu \in R$ is the location parameter, $\sigma > 0$ is the scale parameter, $a, b > 0$ are the shape parameters, and $\phi(.)$ and $\Phi(.)$ are the probability density function and cummulative distribution function of the standard normal distribution, respectively.

b. Kumaraswamy-Weibull: The cdf of the weibull distribution is $G(x) = 1 - \exp[-(\beta x)^c]$, with parameters $\beta > 0$ and $c > 0$ and for $x > 0$. So, the density of the Kw-Weibull distribution, say $KwW(a, b, c, \beta)$, reduces to:

$$f(x) = abc\beta^c(x)^{c-1}\exp[-(\beta x)^c][1-\exp[-(\beta x)^c]]^{a-1}[1-(1-\exp[-(\beta x)^c])^a]^{b-1}, x, a, c, \beta > 0$$

c. Kumaraswamy-gamma: $G(y) = \frac{\Gamma_{\beta y}(\alpha)}{\Gamma(\alpha)}, for y, \alpha, \beta > 0$, where $\Gamma(.)$ is the gamma function for a gamma random variable Y and

$$\Gamma_z(\alpha) = \int_0^z t^{\alpha-1}\exp(-t)dt$$

is the incomplete gamma function. The pdf of a random variable X following a $KwGa$ distribution, say $X \sim KwGa(a, b, \beta, \alpha)$ is

$$f(x) = \frac{ab\beta^{\alpha}x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)^{ab}}\Gamma_{\beta x}(\alpha)^{a-1}\{\Gamma(\alpha)^{a} - \Gamma_{\beta x}(\alpha)^{a}\}^{b-1}$$

2.3 A general expansion for the density function

The expansion of the density function (f(x)) expressed by Cordeiro and Castro(2011) is

$$f(x) = g\,(x) \sum_{i,j=0}^{\infty} \sum_{r=0}^{j} w_{i,j,r} G(x)^{r},$$

where the coefficients

$$w_{i,j,r} = w_{i,j,r}(a, b) = (-1)^{i+j+r} ab \binom{a(i+1)-1}{j} \binom{b-1}{i} \binom{j}{r}$$

are constants satisfying

$$\sum_{i,j=0}^{\infty} \sum_{r=0}^{j} w_{i,j,r} = 1.$$

2.4 General formulae for the moments

Let Y and X follow the baseline G and Kw-G distribution, respectively. The s-th moment of X, say μ'_s expressed in terms of the (s, r)-th PWMs $\tau_{s,r} = E\{Y^s G(Y)^r\}$ of Y for $r = 0, 1, \cdots$, as defined by Greenwood et al. (1979). For a integer

$$\mu'_s = \sum_{r=0}^{\infty} w_r \tau_{s,a(r+1)-1},$$

whereas for a real non-integer

$$\mu'_s = \sum_{i,j=0}^{\infty} \sum_{r=0}^{j} w_{i,j,r} \tau_{s,r}.$$

Some power series expansions are preferred to calculate the moments of any Kw-G distribution than computing the moments directly by numerical integration of the expression

$$\mu'_s = ab\int x^s g(x)G(x)^{a-1}(1-G(x)^a)^{b-1}dx.$$

2.5 Maximum likelihood estimation

Let us consider independent random variables $X_1, \cdots, X_n$, each X_i following a Kw-G distribution with parameter vector $\theta = (a, b, \gamma)$. So, the log-likelihood function $l = l(\theta)$ for the model parameters obtained by Cordeiro (2011) is:

$$l(\theta) = n\{\log(a) + \log(b)\} + \sum_{i=1}^{n} \log\{g(x_i;\gamma)\} + (a-1)\sum_{i=1}^{n} \log\{G(x_i;\gamma)\} + (b-1)\sum_{i=1}^{n} \log\{1 - G(x_i;\gamma)^a\}$$

On differentiating with respect to the parameters:

$$\tfrac{\partial l(\theta)}{\partial a} = \tfrac{n}{a} + \sum_{i=1}^{n} \log\{G(x_i;\gamma)\}\{1 - \tfrac{(b-1)G(x_i;\gamma)^a}{1-G(x_i;\gamma)^a}\}, \tfrac{\partial l(\theta)}{\partial b} = \tfrac{n}{b} + \sum_{i=1}^{n} \log\{1 - G(x_i;\gamma)^a\}$$

and

$$\tfrac{\partial l(\theta)}{\partial \gamma_j} = \sum_{i=1}^{n}\{\tfrac{1}{g(x_i,\gamma)}\tfrac{\partial g(x_i,\gamma)}{\partial \gamma_j} + \tfrac{1}{G(x_i,\gamma)}\tfrac{\partial G(x_i,\gamma)}{\partial \gamma_j}\{1 - \tfrac{a(b-1)}{G(x_i;\gamma)^{-a}-1}\}\},$$

Chapter 3

Zografos-Balakrishnan-G family

3.1 Introduction

For any baseline cumulative distribution function (cdf) G(x), and $x \in R$, Zografos and Balakrishnan (2009) defined a distribution with probability density function (pdf) f(x) and cdf F(x) given by:

$$f(x) = \frac{1}{\Gamma(a)}\{-\log[1 - G(x)]\}^{a-1}g(x) \tag{3.1}$$

and

$$F(x) = \frac{\gamma(a, \log[1 - G(x)])}{\Gamma(a)} = \frac{1}{\Gamma(a)}\int_0^{-\log[1-G(x)]} t^{a-1}\exp(-t)dt, \tag{3.2}$$

respectively, for $a > 0$, where $g(x) = \frac{dG(x)}{dx}$,

$$\Gamma(a) = \int_0^{\infty} t^{a-1}\exp(-t)dt$$

is the gamma function, and

$$\gamma(a, z) = \int_0^{z} t^{a-1}\exp(-t)dt$$

is the incomplete gamma function. The distribution given by (3.1) and (3.2) is known as the Zografos-Balakrishnan-G distribution. The corresponding hazard rate function (hrf) is

$$h(x) = \frac{\{-\log[1-G(x)]\}^{a-1} g(x)}{\Gamma(a, -\log[1-G(x)])}. \tag{3.3}$$

3.2 Some special Zografos-Balakrishnan-G distributions

Here, we present and study some special cases of this family because it extends several widely-known distributions in the literature.

a. Zografos-Balakrishnan-normal distribution The Zografos-Balakrishnan-normal (GN) distribution [Nadarajah (2013)] obtained by taking the cdf and pdf of the normal $N(\mu;\sigma^2)$ distribution in Zografos-Balakrishnan-G family is:

$$f_{GN}(x) = \frac{1}{\sigma\Gamma(a)}\{-\log[1-\Phi(\frac{x-\mu}{\sigma})]\}^{a-1}[\phi(\frac{x-\mu}{\sigma})],$$

where $x \in R$,$\mu \in R$ is the location parameter, $\sigma > 0$ is the scale parameter, $a > 0$ is the shape parameter, and $\phi(.)$ and $\Phi(.)$ are the probability density function and cumulative distribution function of the standard normal distribution, respectively.

b. Zografos-Balakrishnan-weibull: The cdf of the weibull distribution is $G(x) = 1 - \exp[-(\beta x)^\alpha]$, with parameters $\beta > 0$ and $\alpha > 0$. So, the density of the Zografos-Balakrishnan-weibull distribution [Nadarajah (2013)], say $GW(a, \alpha, \beta)$, reduces to:

$$f_{GW}(x) = \frac{\alpha\beta^{\alpha a}}{\Gamma(a)}(x)^{a\alpha-1}\exp[-(\beta x)^\alpha], x > 0$$

The above pdf is very important as it extends many distributions that are previously considered in the literature. It is identical to the generalized gamma distribution.

3.3 Expansions for density function and distribution function

Some useful expansions for the pdf and cdf were derived by Nadarajah et al. (2013) using the concept of exponentiated distributions. For an arbitrary baseline distribution function G(x), an rv say $X \sim \exp -G(a)$ is said to have the exponentiated-G distribution with parameter $a > 0$, if its pdf and cdf are

$$f_a^*(x) = aG^{a-1}(x)g(x) \tag{3.4}$$

and

$$F_a^*(x) = G^a(x) \tag{3.5}$$

respectively.

Note: For $a < 1$ and $a > 1$ and for larger values of x, the multiplicative factor $aG(x)^{a-1}$ is smaller and greater than one, respectively. For smaller values of x, the reverse assertion also hold.

The binomial coefficient generalized to real arguments is given by $\binom{Z}{y} = \Gamma(x+1)/\{\Gamma(y+1)\Gamma(x-y+1)\}$. For any real parameter $a > 0$, we define

$$b_k = \frac{\binom{k+1-a}{k}}{(a+k)\Gamma(a-1)} \sum_{j=0}^{k} \frac{(-1)^{j+k}\binom{k}{j} p_{j,k}}{(a-1-j)}, \tag{3.6}$$

and

$$p_{j,k} = k_{-1} \sum_{m=1}^{k} \frac{[k-m(j+1)]}{m+j} c_m p_{j,k-m}, \tag{3.7}$$

for $k = 1, 2, 3, \cdots$ with $p_{j,0} = 1$ and $c_k = (-1)^{k+1}(x+1)^{-1}$,$a > 0$. Then the pdf and cdf can be expressed as

$$f(x) = \sum_{k=0}^{\infty} b_k f_{a+k}^*(x) \tag{3.8}$$

and

$$F(x) = \sum_{k=0}^{\infty} b_k F_{a+k}^*(x), \tag{3.9}$$

respectively, where $f_{a+k}^*(x)$ and $F_{a+k}^*(x)$ denote the pdf and cdf of the $exp - G(a+k)$ distribution.

3.4 Asymptotes

The asymptotes of pdf, cdf and hrf as $x \to -\infty, +\infty$ calculated by Nadarajah et al. (2013) are given by:

$$f(x) \sim \frac{1}{\Gamma(a)} G^{a-1}(x) g(x)$$

as $x \to -\infty$,

$$F(x) \sim \frac{1}{\Gamma(a+1)} \{-\log[1 - G(x)]\}^{a}$$

as $x \to -\infty$,

$$1 - F(x) \sim \frac{1}{\Gamma(a)} \{-\log[1 - G(x)]\}^{a-1} [1 - G(x)]$$

as $x \to +\infty$,

$$h(x) \sim \frac{1}{\Gamma(a)} G^{a-1}(x) g(x)$$

as $x \to -\infty$,

$$h(x) \sim \frac{g(x)}{1 - G(x)}$$

as $x \to +\infty$.

3.5 Quantile function

Let $X \sim Zografos - Balakrishnan - G$ random variable. The cdf of X is given by (3.2). Inverting $F(x) = u$, we obtain

$$F^{-1}(u) = G^{-1}\{1 - \exp[-Q^{-1}(a, 1 - u)]\} \tag{3.10}$$

for $0 < u < 1$, where $Q^{-1}(a, 1 - u)$ is the inverse of the function $Q(a, x) = 1 - \frac{\gamma(a,x)}{\Gamma(a)}$.

Chapter 4

Zografos-Balakrishnan-generalized exponential distribution

4.1 Introduction

For any baseline cumulative distribution function (cdf) G(x), and $x \in R$, the pdf and cdf defined by Zografos and Balakrishnan (2009) are

$$f(x) = \frac{1}{\Gamma(a)}\{-\log[1 - G(x)]\}^{a-1}g(x) \tag{4.1}$$

and

$$F(x) = \frac{\gamma(a, \log[1 - G(x)])}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_0^{-\log[1-G(x)]} t^{a-1} \exp(-t)dt, \tag{4.2}$$

respectively, for $a > 0$, where $g(x) = \frac{dG(x)}{dx}$,

$$\Gamma(a) = \int_0^\infty t^{a-1} \exp(-t)dt$$

is the gamma function, and

$$\gamma(a, z) = \int_0^z t^{a-1} \exp(-t)dt$$

is the incomplete gamma function.

Here, we use the method proposed by Zografos and Balakrishnan (2009) to define a new model, called as the Zografos-Balakrishnan-generalized exponential distribution(ZB-GE), which generalizes the generalized exponential(GE) distribution by Gupta and Kundu (1999). We try to find out some of the properties of the new model defined.

4.2 Some properties

Gupta and Kundu (1999) introduced the two-parameter generalized exponential distribution, whose cdf and pdf are given, respectively by $(for x, \alpha, \lambda > 0)$

$$G(x;\alpha,\lambda) = (1-e^{-\lambda x})^{\alpha}, g(x;\alpha,\lambda) = \alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}. \tag{4.3}$$

Inserting this g(x) and G(x) into (4.1) and (4.2), we can define the pdf and cdf of the ZB-GE distribution $(for x > 0)$

$$f(x;a,\alpha,\lambda) = \frac{\alpha\lambda}{\Gamma(a)} e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}\{-\log[1-(1-e^{-\lambda x})^{\alpha}]\}^{a-1} \tag{4.4}$$

and

$$F(x;a,\alpha,\lambda) = 1 - \frac{\Gamma(a, -\log[1-(1-e^{-\lambda x})^{\alpha})]}{\Gamma(a)}. \tag{4.5}$$

A random variable X having density function (4.4) is denoted by $X \sim ZB-GE(a,\alpha,\lambda)$. The generalized exponential(GE) distribution is a basic example for $a = 1$. The survival function and the hazard rate functions of ZB-GE distribution are given, respectively, as

$$S(x;a,\alpha,\lambda) = 1 - F(x;a,\alpha,\lambda) = \frac{\Gamma(a, -\log[1-(1-e^{-\lambda x})^{\alpha}])}{\Gamma(a)} \tag{4.6}$$

and

$$h(x;a,\alpha,\lambda) = \frac{\alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}\{-\log[1-(1-e^{-\lambda x})^{\alpha}])\}^{a-1}}{\Gamma(a, -\log[1-(1-e^{-\lambda x})^{\alpha}])} \tag{4.7}$$

4.2.1 Expansions for density function and distribution function

For an arbitrary baseline distribution function G(x), an rv say $X \sim \exp -G(a)$ is said to have the exponentiated-G distribution with parameter $a > 0$, if its pdf and cdf are

$$f_a^*(x) = aG^{a-1}(x)g(x) \tag{4.8}$$

and

$$F_a^*(x) = G^a(x) \tag{4.9}$$

respectively. We can also obtain expansion for the ZB-GE density function given by

$$f(x) = \sum_{k=0}^{\infty} b_k g(x; \alpha_k, \lambda), \tag{4.10}$$

where

$$b_k = \frac{\binom{k+1-a}{k}}{(a+k)\Gamma(a-1)} \sum_{j=0}^{k} \frac{(-1)^{j+k}\binom{k}{j} p_{j,k}}{(a-1-j)}, \alpha_k = \alpha(a+k) and \tag{4.11}$$

$$p_{j,k} = k_{-1} \sum_{m=1}^{k} \frac{[k - m(j+1)]}{m+j} c_m p_{j,k-m}, \tag{4.12}$$

for $k = 1, 2, 3, \cdots$ with $p_{j,0} = 1$ and $c_k = (-1)^{k+1}(x+1)^{-1}$,$a > 0$.

4.2.2 Quantile function

Here, we have calculated the quantile function for the Zografos-Balakrishnan-generalized exponential distribution(ZB-GE).

$$F(x; a, \alpha, \lambda) = \frac{\gamma(a, -\log[1 - (1 - e^{-\lambda x})^{\alpha}])}{\Gamma(a)} = u. \tag{4.13}$$

Inverting F(x)=u, we get $G^{-1}\{1 - exp[-Q^{-1}(a, 1-u)]\}$, for $0 < u < 1$, where $Q^{-1}(a, 1-u)$ is the inverse of the function $Q(a, x) = 1 - \frac{\gamma(a,x)}{\Gamma(a)}$.

Now for Generalized Exponential distribution:

$$
\begin{aligned}
F_{GE} = (1-e^{-\lambda x})^{\alpha} \\
&\Rightarrow u = (1-e^{-\lambda x})^{\alpha} \\
&\Rightarrow \log u = \alpha \log(1-e^{-\lambda x}) \\
&\Rightarrow \frac{\log u}{\alpha} = \log(1-e^{-\lambda x}) \\
&\Rightarrow \log u^{\frac{1}{u}} = \log(1-e^{-\lambda x}) \\
&\Rightarrow 1-u^{\frac{1}{u}} = e^{-\lambda x} \\
&\Rightarrow \log(1-u^{\frac{1}{u}}) = -\lambda x \\
&\Rightarrow x = -\frac{1}{\lambda}\log(1-u^{\frac{1}{u}}) \\
&\Rightarrow x = -\frac{1}{\lambda}\log(1-F_{GE}{}^{\frac{1}{u}})
\end{aligned}
$$

So, now

$$
F^{-1}(u) = -\lambda^{-1}\log(1-[1-exp[-Q^{-1}(a,1-u)]]^{\frac{1}{u}}) \tag{4.14}
$$

4.2.3 Asymptotes

The asymptotes of the density function, distribution function and hazard rate function as $x \to -\infty, +\infty$ are calculated as

$$
f(x) \sim \frac{1}{\Gamma(a)} G^{a-1}(x) g(x)
$$

$$
\Rightarrow f(x) \sim \frac{1}{\Gamma(a)}(1-e^{-\lambda x})^{\alpha(a-1)}\alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}
$$

as $x \to -\infty$,

$$
F(x) \sim \frac{1}{\Gamma(a+1)}\{-\log[1-G(x)]\}^{a}
$$

$$
\Rightarrow F(x) \sim \frac{1}{\Gamma(a+1)}\{-\log[1-(1-e^{-\lambda x})^{\alpha}]\}^{a}
$$

as $x \to -\infty$,

$$1 - F(x) \sim \frac{1}{\Gamma(a)}\{-\log[1 - G(x)]\}^{a-1}[1 - G(x)]$$

$$\Rightarrow F(x) \sim \frac{1}{\Gamma(a)}\{-\log[1 - (1 - e^{-\lambda x})^{\alpha}]\}^{a-1}[1 - (1 - e^{-\lambda x})^{\alpha}]$$

as $x \to +\infty$,

$$h(x) \sim \frac{1}{\Gamma(a)} G^{a-1}(x) g(x)$$

$$\Rightarrow h(x) \sim \frac{1}{\Gamma(a)}(1 - e^{-\lambda x})^{\alpha(a-1)} \alpha\lambda e^{-\lambda x}(1 - e^{-\lambda x})^{\alpha-1}$$

as $x \to -\infty$,

$$h(x) \sim \frac{g(x)}{1 - G(x)}$$

$$\Rightarrow h(x) \sim \frac{\alpha\lambda e^{-\lambda x}(1 - e^{-\lambda x})^{\alpha-1}}{(1 - (1 - e^{-\lambda x})^{\alpha})}$$

as $x \to +\infty$.

4.2.4 Moments

The moment generating function (mgf) of a random variable X having the ZB-GE distribution can be obtained as

$$M(t) = \sum_{k=0}^{\infty} b_k \frac{\Gamma(\alpha + 1)\Gamma(1 - \frac{t}{\lambda})}{\Gamma(\alpha - \frac{t}{\lambda} - 1)}. \tag{4.15}$$

The expression for nth moment of the ZB-GE distribution can be written as

$$E(Y^r) = \alpha\beta \sum_{j=0}^{\infty} t_r \tau_{r,j}, \tag{4.16}$$

where,

$$t^r = \frac{(-1)^j \Gamma(\beta)}{j!} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma[(k+1)\alpha]}{\Gamma(\beta - k)\Gamma[(k+1)\alpha - j]k!}, \tag{4.17}$$

and

$$\tau_{r,j} = r!\lambda_r \sum_{m=0}^{\infty} \frac{(-1)^{m+r}\binom{j}{m}}{(m+1)^{r+1}}. \tag{4.18}$$

4.2.5 Entropy

The Renyi entropy for the Zografos-Balakrishnan-generalized exponential distribution(ZB-GE) is defined as

$$I_R(\gamma) = \frac{\gamma \log \Gamma(a)}{\gamma - 1} + \frac{1}{1-\gamma} \log\{\sum_{k=0}^{\infty} w_k B[\gamma, 1 + (\gamma - 1)(k + a\gamma)]\}, \tag{4.19}$$

where

$$w_k = \alpha_\gamma \lambda^{\gamma-1} \sum_{j=0}^{k} \frac{(-1)^{j+k}\binom{k}{j}\binom{k+\gamma(1-a)}{k}}{\gamma(a-1) - j}.$$

and

$$B(a,b) = \int_0^1 w^{a-1}(1-w)^{b-1} dw$$

is the beta function.

4.3 Maximum likelihood estimation

Let us consider independent random variables $X_1, \cdots, X_n$, each X_i following a $ZB - GE(a, \alpha, \lambda)$ distribution with parameter vector $\theta = (a, \alpha, \lambda)$. So, the log-likelihood function $l = l(\theta)$ for the parameters obtained is

$$l(\theta) = n \log(\frac{\alpha\lambda}{\Gamma(a)}) + \sum_{i=1}^{n} \log\{-\log[1 - (1 - e^{-\lambda x_i})^{\alpha}]\} - \lambda \sum_{i=1}^{n} x_i + (\alpha - 1) \sum_{i=1}^{n} (1 - e^{-\lambda x_i}). \tag{4.20}$$

Chapter 5

Conclusions and Scope of Future Works

We have studied about two G-gamily distributions. We have presented some basic properties of Zografos-Balakrishnan-generalized exponential distribution(ZB-GE). We are now trying to work on the estimation of these distributions. MLE can be studied easily but we do not get that in closed form. Further future work is possible on Bayesian estimation. Lindleys approximation, Gibbs sampling, EM Algorithm can be used. These distributions have many applications as they always tend to give better fit than the parent distribution. We have observed that the pdf is a linear combination of Generalized exponential distributions which elucidates the mathematical expressions related to the properties of ZB-GE. Further, when the shape parameter $a = 1$, it coincides with the generalized exponential distribution and for $\alpha = 1$, it follows one parameter exponential distribution. We have demonstrated that some mathematical properties of the ZB-GE distribution can be readily obtained from those of the GE distribution.

Bibliography

Cordeiro, G. M. and de Castro, M. (2011): A new family of generalized distributions. Journal of Statistical Computation and Simulation, 81, 883-898.

Eugene, N., Lee, C. and Famoye, F. (2002): Beta-normal distribution and its applications. Communications in StatisticsTheory and Methods, 31, 497-512.

Gupta, R. D. and Kundu, D. (1999): Generalized exponential distributions. Australian and New Zealand Journal of Statistics, 41, 173-188.

Johnson, N. L. , Kotz, S. and Balakrishnan, N.(2004): Continuous Univariate Distributions Vol-1, Wiley-interscience, New York.

Johnson, N. L. , Kotz, S. and Balakrishnan, N.(2004): Continuous Univariate Distributions Vol-2, Wiley-interscience, New York.

Kumaraswamy, P. (1980): Generalized probability density-function for double-bounded random-processes. Journal of Hydrology, 46, 79-88.

Nadarajah, S. and Kotz, S. (2004): The beta Gumbel distribution. Mathematical Problems in Engineering 10, 323-332.

Nadarajah, S. and Gupta, A. K. (2004): The beta Frechet distribution. Far East Journal of Theoretical Statistics 14, 15-24, 2004.

Nadarajah, S. and Kotz, S. (2005): The beta exponential distribution. Reliab. Eng. Syst. Saf. 91(6), 689697, 2005.

Nadarajah, S. (2005): The exponentiated Gumbel distribution with climate application. Environmetrics, 17, 13-23.

Nadarajah, S. and Gupta, A. K. (2007): The exponentiated gamma distribution with application to drought data. Calcutta Statistical Association Bulletin, 59, 29-54.

Nadarajah, S. and Kotz, S. (2006): The exponentiated type distributions. Acta Applicandae Mathematicae, 92, 97-111.

Nadarajah, S. and Kotz, S. (2006): The beta exponential distribution. Reliability Engineering and System Safety 91, 689-697.

Nadarajah, S., Cordeiro, G. M., Ortega, E. M. M. (2012): The gamma-G family of distributions:Mathematical properties and applications, Communications in Statistics: Theory and Methods.

Nadarajah, S., Cordeiro, G. M. and Ortega, E. M. M. (2013): The Zografos-Balakrishnan G-family of distributions: Mathematical properties and application.

Ng, K. W. and Kotz, S. (1995): Kummer-gamma and Kummer-beta univariate and multivariate distributions. Research Report, 84, Department of Statistics, The University of Hong Kong, Hong Kong.

Pearson, K. (1895): Contributions to the mathematical theory of evolution. II. Skew variation in homogeneous material. Philos. Trans. Royal Soc. Lond. A 186, 343414, 1895.

Ristic, M.M. and Balakrishnan, N. (2012): The Gamma Exponentiated Exponential Distribution, Journal of Statistical Computation and Simulation, 82, 1191-1206.

Rohatgi, V. K. and Saleh, A. K. Md. E. (2009): An Introduction to Probability and Statistics, Wiley-interscience, New York.

Zografos, K. and Balakrishnan, N. (2009). On families of beta- and generalized gamma generated distributions and associated inference. Statistical Methodology.