SOME PROBLEMS ON VARIATIONAL ITERATION METHOD

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CERTIFICATE

This is to certify that the project report entitled "***SOME PROBLEMS ON VARIATIONAL ITERATION METHOD***" submitted by **Jaysmita Patra** to the National Institute of Technology, Rourkela, Orissa, for the partial fulfillment of requirements for the degree of master of science in Mathematics and the review work is carried out by her under my supervision and guidance .It has fulfilled all the guidelines required for the submission of her research project paper for M.Sc. degree. In my opinion, the contents of this project submitted by her is worthy of consideration for M.Sc. degree and in my insight this work has not been submitted to any other institute or university for the award of any degree.

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ABSTRACT

In this research project paper, I introduce some basic idea of Variational iteration method and its algorithm to solve the equations ODE & PDE, fractional differential equation, fractal differential equation and differential-difference equations. Also, some linear and nonlinear differential equations like Burger's equation, Fisher's equation, Wave equation and Schrodinger equation are solved by using Variational iteration method. Then I compare this method with Adomian decomposition method (ADM) and modified Variational iteration method (MVIM). The advantage of VIM, it does not require a small parameter in an equation as perturbation technique needs. The VIM is used to solve effectively, easily, and accurately a large class of non-linear problems with approximations which converge rapidly to accurate solutions. For linear problems, its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified.

KYE WORDS- Variational iteration method, Adomian decomposition method, Modified variational iteration method, Lagrange multiplier, Stationary condition, Restricted variation, Correction functional, Burger's equation, Fisher's equation, Schrodinger's equation, Wave equation and KDV equation.

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CHAPTER-1

INTRODUCTION

The Variational Iteration Method (VIM) was first developed by Chinese mathematician **Ji-Huan He**, professor at Donghua University. The VIM was initially proposed toward the end of the most recent century and completely grew in 2006 and 2007.

The method VIM is used to solve effectively, easily, and accurately a large class of non-linear problems with approximations, which converge rapidly to the accurate solutions. For linear problems, its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified.

This method VIM introduces a reliable and efficient process for a wide variety of scientific and engineering applications like linear or nonlinear, homogeneous or inhomogeneous, equations and systems of equations etc. Also this method is more powerful than other existing techniques, such as the Adomian method, perturbation method etc. This method gives rapidly convergent successive approximations of the exact solution if such a solution exists, otherwise a few approximations can be used for numerical purpose. The existing numerical techniques suffer from the restrictive assumptions that are used to handle nonlinear terms. The VIM has no specific requirements, such as linearization, small parameters, Adomian polynomials etc. for nonlinear operators. Another important advantage of VIM method is capable of greatly reducing the size of calculation while still maintaining high accuracy of numerical solution. Moreover, the power of the method gives it a wider applicability in handling a huge number of analytical and numerical applications in real life problems.

Recently, fractional-order calculus has been studied as an alternative calculus in mathematics. Numerous problems in physics, chemistry, biology, and engineering can be modelled with fractional derivatives. On the other hand, in control society, fractional-order dynamic systems and controls have gained an increasing attention, and also motion of an elastic column fixed at one end loaded at the other can be formulated in terms of a system of fractional differential equations. Since most fractional differential equations do not have exact analytic solutions, approximate and numerical techniques, therefore, are used extensively.

The variational iteration method (VIM) is relatively new approaches to provide approximate solutions to linear and nonlinear problems. The variational iteration method, (VIM) was successfully applied to find the solutions of several classes of variational problems.

The VIM is an iterative method based on the use of Lagrange multiplier, restricted variations and correction functional which has found a wide application for the solution of nonlinear ordinary and partial differential equations. This method does not require the presence of small parameters in the differential equation, and provides the solution (or an approximation to it) as a sequence of iterates. The method does not require that the nonlinearities be differentiable with respect to the dependent variable and its derivatives. Being different from the other non-linear analytical methods, such as perturbation methods, this method does not depend on small parameters and initial gauss can easily chosen even also with unknown parameters so it gives a wide application in non-linear problems without linearization or small perturbations.

1.1. General Lagrange multiplier

Lagrange multiplier is well known in optimization and calculus of variations. Inokuti et al. suggested a method of General Lagrange multiplier [15]. In order to understand the concept of the general Lagrange multiplier, we consider an algebraic equation

$$f(x = 0), \qquad x \in R \tag{1}$$

If "x_n" is an approximate root of the above equation, it follows

$$f(x_n) \neq 0 \tag{2}$$

Now to improve its accuracy, we can write the following correction equation

$$x_{n+1} = x_n + \lambda f(x_n) \tag{3}$$

Where λ is a general Lagrange multiplier, which can be identified optimally by setting

$$\frac{dx_{n+1}}{dx_n} = 0 \tag{4}$$

Then we have, $1 + \lambda f'(x_n) = 0 \Rightarrow \lambda = 1/f'(x_n)$

This leads to the well-known Newton iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{\acute{f}(x_n)} \tag{5}$$

There are alternative approaches to construction of correction equation. Now we write an another correction for x_n as follows

$$x_{n+1} = x_n + \lambda g(x_n) f(x_n) \tag{6}$$

Here $g(x)$ is an auxiliary function. After identification of the multiplier, we have a general iteration formulation

$$x_{n+1} = x_n - \frac{g(x_n) f(x_n)}{g(x_n) f'(x_n) + g'(x_n) f(x_n)} \tag{7}$$

The value of the auxiliary function should not be zero or small value during the all iteration steps, so$|g(x_n)| > 1$, if we choose $g(x_n) = e^{-\alpha x_n}$ then the iteration formulation, Eq. (7), reduces to

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) + \alpha f(x_n)} \tag{8}$$

This iteration formulation is very effective when $f' x_n$ is small.

To show this we can consider the following example i.e. consider the equation

$$sin\, x = 0 \tag{9}$$

If we begin with $x_0 = 1.6$ the Newton iteration is not valid for cos 1.6 is a small value. The below table shows the iteration procedure, the nearest solution near $x_0 = 1.6$ is $x = \pi$

Iteration	x ($\alpha = 1$)	x ($\alpha = 2$)	x (α) = 0.97
0	1.6	1.6	1.6
1	2.57	2.09	2.60
2	2.96	2.48	2.98
3	3.12	2.78	3.12
4	3.14	2.99	3.14

We also obtained some iteration formulae by General Lagrange multiplier

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)f^2(x_n)}{2f'^3(x_n)} \tag{10}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)f^2(x_n)}{2f'^3(x_n)} + \frac{[f'''(x_n)f'^3(x_n) - 3f''^2(x_n)f'^2(x_n)]f^3(x_n)}{2f^7(x_n)} \tag{11}$$

If Eq. (1) is replaced by differential equation, then a correctional functional similar to Eq. (3) can be established.

1.2. Stationary conditions

The problem of optimization is ubiquitous in nature. The simplest problem of the calculus of variation [7] is to determine a function $y = f(x)$ for which the given functional

$$J = \int_{x_1}^{x_2} F(y, y'; x)dx + g_1(x)y|_{x=x_1} - g_2(x)y|_{x=x_2} \tag{12}$$

is a maximum or a minimum.

The extremum condition (stationary condition) of the functional (12) requires that

$$\delta J = \delta \int_{x_1}^{x_2} F(y, y'; x)dx + g_1(x)y|_{x=x_1} - g_2(x)y|_{x=x_2}$$

$$= \int_{x_1}^{x_2} \delta F(y, y'; x)dx + g_1(x)y|_{x=x_1} - g_2(x)y|_{x=x_2}$$

$$= \int_{x_1}^{x_2} \left\{\frac{dF}{dy}\delta y + \frac{dF}{dy'}\delta y'\right\} dx + g_1\delta y|_{x=x_1} - g_2\delta y|_{x=x_2}$$

$$= \int_{x_1}^{x_2} \left\{\frac{dF}{dy}\delta y + \frac{dF}{dy'}\frac{d}{dx}(\delta y)\right\} dx + g_1\delta y|_{x=x_1} - g_2\delta y|_{x=x_2}$$

$$= \int_{x_1}^{x_2} \left\{\left[\frac{dF}{dy} - \frac{d}{dx}\left(\frac{dF}{dy'}\right)\right]\delta y + \frac{d}{dx}\left(\frac{dF}{dy'}\delta y\right)\right\} dx + g_1\delta y|_{x=x_1} - g_2\delta y|_{x=x_2}$$

$$= \int_{x_1}^{x_2} \left\{\left[\frac{dF}{dy} - \frac{d}{dx}\left(\frac{dF}{dy'}\right)\right]\delta y\right\} dx + \left[\frac{dF}{dy'}\delta y\right]_{x_1}^{x_2} + g_1\delta y|_{x=x_1} - g_2\delta y|_{x=x_2}$$

$$= 0$$

For arbitrary δy, from the above relation, we have

$$\frac{dF}{dy} - \frac{d}{dx}\left(\frac{dF}{dy'}\right) = 0 \tag{13}$$

And the boundary conditions

$$\frac{dF}{dy'}(x_1) - g_1(x_1) = 0 \text{ and } \frac{dF}{dy'}(x_2) - g_2(x_2) = 0 \tag{14}$$

Where Eq. (13) is called Euler–Lagrange's differential equation, or Euler's equation, and Eq. (14) is known as the natural boundary conditions.

1.3. Restricted variation

To show how restricted variation works in Variational iteration method, let we consider a simple algebraic equation.

$$x^2 - 3x + 2 = 0 \tag{15}$$

Now we rewrite Eq.(15) in the form

$$\tilde{x}.x - 3x + 2 = 0 \tag{16}$$

Where $\tilde{x}$ is called restricted variable, the value of $\tilde{x}$ is assumed to be known i.e initial guess solving x from (16) leads to result

$$x = \frac{2}{3-\tilde{x}} \tag{17}$$

Or an iteration of the form

$$x_{n+1} = \frac{2}{3-x_n} \tag{18}$$

This method is often very efficient for good prediction, shown in the below table.

Iteration	Eq. (18)	Newton iteration formulation
0	0.5	0.5
1	0.8	0.875
2	0.909	0.987
3	0.956	0.999
4	0.978	1.000
5	0.989	1.000
6	0.994	1.000
7	0.997	1.000
8	0.998	1.000
9	0.999	1.000

In variational iteration method, initial guess is always chosen with a possible unknown parameter, one iteration leads to highly accurate solutions. So we can illustrate the effectiveness of free parameter in the above example. Now introducing a free parameter in the initial prediction

$$x_0 = 0.5 + b, \tag{19}$$

Here b is a small parameter which to be determined. Now we substituted (19) into (18),

$$x_1 = \frac{2}{3-0.5-b} = \frac{2}{2.5-b} = \frac{2}{2.5\left(1-\frac{1}{2.5}b\right)} = 0.8\left(1+\frac{1}{2.5}b\right) + O(b^2) = 0.8 + 0.32b + O(b^2) \tag{20}$$

Now to identified the value of b we set $x_0 = x_1$ (21)

$$\text{Or} \quad 0.8 + 0.32b = 0.5 + b \tag{22}$$

From the above relation we can idenified immideately that

$$b = 0.4412 \tag{23}$$

So the updated approximated root is $x_1 = 0.9412$

Now consider the restricted variation in a variational functional. Consider temperature distribution in convective straight fins with temperature-dependent thermal conductivity, the dimensionless governing equation is

$$\frac{d}{dx}\left[(1+\beta\theta)\frac{d\theta}{dx}\right]-\psi^2\theta=0, \quad \theta'(0)=0, \quad \theta(1)=1. \qquad (24)$$

Here θ is dimensionless temperature, β and γ are constants.

So the use of the concept of restricted variation, an approximate variational functional can be established as defined below

$$J(\theta)=\int_0^1\left\{\left(1-\beta\tilde{\theta}\right)\left(\frac{d\theta}{dx}\right)^2+\psi^2\theta^2\right\}dx \qquad (25)$$

Where $\tilde{\theta}$ is a restricted variation, i.e., $\delta\tilde{\theta} = 0$.

So we can rewrite (24) in an iteration form

$$J(\theta_{n+1})=\int_0^1\left\{(1-\beta\theta_n)\left(\frac{d\theta_{n+1}}{dx}\right)^2+\psi^2{\theta_{n+1}}^2\right\}dx \qquad (26)$$

Now we begin with the initial guesses satisfying the boundary condition $\theta'(0) = 0$ and $\theta_0(1) = 1$

$$\theta_0 = 1-a+ax^2, \qquad (27)$$

Where a is the free parameter. Now we apply Ritz method to solving θ_1, the trial-function for θ_1 is assumed to be have the form

$$\theta_1 = 1-b+bx^2 \qquad (28)$$

Where the unknown parameter b to be further determined. Substituting (27) into (25) yields

$$J=\int_0^1\{4\left(1+\beta(1-a-ax^2)\right)b^2x^2+\psi^2(1-b+bx^2)^2\}dx$$
$$=\frac{4}{3}\left(1+\beta(1-a)\right)b^2+\frac{4}{5}ab^2+\psi^2(1-b^2)+\frac{2}{3}b(1-b)\psi^2+\frac{1}{5}b^2\psi^2 \qquad (29)$$

Now minimizing the functional, eq. (25), with respect to θ is approximately equivalent to minimizing the above function, eq. (28), with respect to b we have,

$$\frac{dJ}{db}=\frac{8}{3}\left(1+\beta(1-a)\right)b+\frac{8}{5}ab-2\psi^2(1-b)+\frac{2}{3}(1-2b)\psi^2+\frac{2}{5}b\psi^2=0 \qquad (30)$$

We set $\theta_0 = \theta_1$, then we can readily identified that,

$$a=\frac{-(20+20\beta+8\psi^2)+\sqrt{(20+20\beta+8\psi^2)^2+40\psi^2(12-20\beta)}}{24-40\beta} \qquad (31)$$

CHAPTER-2

VARIATION ITERATION METHOD

2.1. Basic idea of VIM

In 1978, Inokuti et al. [15] proposed a general Lagrange multiplier method to solve non-linear problems, which was first proposed to solve problems in quantum mechanics. The main feature of the method is the solution of a mathematical problem with linearization assumption is used as initial approximation or trial-function, and then a more highly precise approximation at some special point can be obtained.

Let's consider the following general non-linear system

$$Lu + Nu = g(x), \tag{31}$$

Where L is a linear operator and N is a non-linear operator.

Assuming $u_0(x)$ is the solution of $Lu = 0$. Now according to the Ref. [15], we can write down an expression to correct the value of some special point, for example at $x = 1$

$$u_{cor}(1) = u_0(1) + \int_0^1 \lambda(Lu_0 + Nu_0 - g)dx, \tag{32}$$

Where λ is General Lagrange multiplier [15], which can be identified optimally via Variational theory and the second term on the right hand side is called the correction.

The author Ji-Huan He modified the above method into an iteration method in the following ways

$$u_{n+1}(x_0) = u_n(x_0) + \int_0^{x_0} \lambda(Lu_n + N\tilde{u}_n - g)dx, \tag{33}$$

Where $u_0(x)$ is the initial approximation with possible unknowns, and $\tilde{u}_n$ is considered as a restricted variation [3] i.e. $\delta\tilde{u}_n = 0$. For arbitrary of x_0, the Eq. (32) can be written as follows

$$u_{n+1}(x) = u_n(x) + \int_0^{x} \lambda\{Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\}d\xi, \tag{34}$$

Eq. (34) is called a correctional functional. The above modified method is called Variational iteration method or VIM.

2.2. Standard variational iteration algorithms

Here we have three standard variational iteration algorithms for solving differential equations like integro-differential equations, fractional differential equations, fractal differential equations, differential-difference equations and fractional/fractal differential-difference equations are given below.

2.2.1. Variational Iteration Algorithm-I

Let us consider following general nonlinear system

$$L[u(t)] + N[u(t)] = 0 \tag{35}$$

Where L is a linear operator and N is a nonlinear operator.

The basic concept of the method is to construct a correction functional for the system (35), i.e. we have

$$u_{n+1}(t) = u_n(t) + \int_{t_0}^{t} \lambda\{Lu_n(s) + N\tilde{u}_n(s)\}ds, \quad (36)$$

Where λ is a general Lagrange multiplier that can be identified optimally via variational theory, u_n is the n^{th} approximate solution, and $\tilde{u}_\text{n}$ denotes a restricted variation, i.e.

$$\delta\tilde{u}_\text{n} = 0.$$

Then after identifying the Lagrange multiplier in Eq. (36), we have the following iteration algorithm

$$u_{n+1}(t) = u_n(t) + \int_{t_0}^{t} \lambda\{Lu_n(s) + Nu_n(s)\}ds, \quad (37)$$

This can be easily understand by taking the following example

PROBLEM 1.

Let's consider the following nonlinear equation of k-th order

$$u^{(k)} + f\left(u, u', u'', \dots, u^{(k)}\right) = 0, \quad (38)$$

Then variational iteration formulation is constructed as follows

$$u_{n+1}(t) = u_n(t) + \int_{t_0}^{t} \lambda\left({u_n}^{(k)} + \tilde{f}_n\right)ds, \quad (39)$$

Where $\delta\tilde{f}_\text{n} = 0$, $f_n = f\ (u_n, u'_n, u''_n, \dots)$. After identifying the multiplier, we have

$$u_{n+1}(t) = u_n(t) + (-1)^n \int_{t_0}^{t} \frac{1}{(n-1)!}(s-t)^{n-1}\left[{u_n}^{(k)}(s) + f_n\right]ds, \quad (40)$$

The main merit of this iteration formula is that the initial solution $u_0(t)$, can be freely chosen, with even unknown parameters contained. However, some repeated and unnecessary iterations are involved in this iteration algorithm at each step.

For initial value problems, we can begin with

$$u_o(t) = u(0) + tu'(0) + \frac{1}{2!}t^2u''(0) + \cdots \frac{1}{k!}t^k u^{(k)}(0) \quad (41)$$

This leads to a series solution converging to the exact solution.

For boundary value problems, the initial guess can be expressed in the form

$$u_o(t) = a_1 g_1(t) + a_2 g_2(t) + \cdots + a_k g_k(t), \quad (42)$$

Where $g_k(t)$ are known functions, $a_k$are unknowns which are determined by the boundary conditions after a few iterations.

2.2.2 Variational Iteration Algorithm-II

After identifying the Lagrange multiplier λ in Eq. (36), we can construct the iteration formula

$$u_{n+1}(t) = u_0(t) + \int_{t_0}^{t} \lambda N u_n(s)ds, \quad (43)$$

This is the Variational iteration algorithm-II. For example 1 given above, Variational iteration algorithm-II gives

$$u_{n+1}(t) = u_0(t) + (-1)^n \int_{t_0}^{t} \frac{1}{(n-1)!}(s-t)^{n-1} f_n ds, \quad (44)$$

Where u_0 must satisfy the initial or boundary conditions. This is the main shortcoming of this algorithm.

PROBLEM 2.

Let's consider the following simple equation

$$u' + u^2 = 0, \qquad u(0) = 1. \tag{45}$$

Using a Lagrange multiplier λ, we first construct an iteration formulation

$$u_{n+1}(t) = u_0(t) + \int_0^t \lambda \left(u'_n(s) + \tilde{u}_n{}^2(s)\right) ds, \tag{46}$$

The Lagrange multiplier can easily identified i.e. $\lambda = -1$ and Variational Iteration Algorithm-I gives the following iteration formula

$$u_{n+1} = u_n - \int_0^t \left(u'_n(s) + u_n{}^2(s)\right) ds, \tag{47}$$

Now Variational Iteration Algorithm-II gives

$$u_{n+1} = u_0 - \int_0^t u_n{}^2(s) ds, \tag{48}$$

If we begin with $u_0(t) = u(0) = 1$, we obtain a convergent series

$$\begin{aligned} u_0(t) &= 1 \\ u_1 &= 1 - t + O(t^2) \\ u_1 &= 1 - t + t^2 + O(t^3), \end{aligned} \tag{49}$$

2.2.3. Variational Iteration Algorithm-III

From Variational Iteration Algorithm-II, Eq. (43), we have

$$u_{n+2}(t) = u_0(t) + \int_{t_0}^t \lambda N u_{n+1}(s) ds, \tag{50}$$

Subtracting Eq.(50) from Eq.(43), we obtain the following Variational Iteration Algorithm-III

$$u_{n+2}(t) = u_{n+1}(t) + \int_{t_0}^t \lambda \{N u_{n+1}(s) - N u_n(s)\} ds, \tag{51}$$

One common property of both Variational Iteration Algorithm-I and Variational Iteration Algorithm-III is the allowed dependence of the initial guess on unknown parameters whose values could be identified by using the initial/boundary conditions after a few iterations. Variational Iteration Algorithm-III, in particular, is highly suitable for boundary value problems of high orders.

PROBLEM 3.

Let's consider the following 4th-order differential equation

$$u^{(4)} + f\left(u, u', u'', u''', u^{(4)}\right) = 0, \tag{52}$$

Then we have the following algorithms are given below

Variational Iteration Algorithm-I

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{6}(s-t)^3 \{u_n{}''(s) + f_n\} ds, \tag{53}$$

Variational Iteration Algorithm-II

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{6}(s-t)^3 f_n ds, \tag{54}$$

Variational Iteration Algorithm-III

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{6}(s-t)^3 (f_n - f_{n+1}) ds, \tag{55}$$

Here the initial guesses for Algorithm-I and Algorithm-III can be chosen to contain unknown parameters such as

$$u_0(t) = A + Bt + Ct^2 + Dt^3, \quad (56)$$

Where *A, B, C,* and *D* are unknowns to be determined. Algorithm-II is suitable for initial value problems, and the initial solution is always chosen to be

$$u_o(t) = u(0) + tu'(0) + \frac{1}{2}t^2u''(0) + \frac{1}{6}t^3u'''(0), \quad (57)$$

PROBLEM 4.

We consider a fourth-order integro-differential equation [12]

$$y^{(4)}(x) = x(1 + e^x) + 3e^x + y(x) - \int_0^x y(x)dx, \quad (58)$$

Where the boundary conditions and initial conditions are given by

$$y(0) = 1, \quad y(1) = 1 + e, \quad y''(0) = 2, \quad y''(1) = 3e. \quad (59)$$

Now we introduce $f(x)$ as

$$f(x) = x(1 + e^x) + 3e^x + y(x) - \int_0^x y(x)dx, \quad (60)$$

Variational Iteration Algorithm-III for Eq. (58) is

$$u_{n+2}(t) = u_{n+1}(t) + \int_0^t \frac{1}{6}(s - t)^3(f_n - f_{n+1})ds, \quad (61)$$

Where

$$f_n(x) = x(1 + e^x) + 3e^x + y_n(x) - \int_0^x y_n(x)dx, \quad (62)$$

Now we begin with

$$y_0(x) = 1 + e^x(a + bx + cx^2 + dx^3) \quad (63)$$

And

$$y_1(t) = y_0(t) + \int_0^t \frac{1}{6}(s - t)^3 f_0 ds, \quad (64)$$

The unknowns *a, b, c* and *d* can be identified after a few iterations by incorporating the initial/boundary conditions. If *N* iterations are sufficient for instance, then from Eq. (59), we have

$$y_N(0) = 1 \quad (65)$$

$$y_N(1) = 1 + e \quad (66)$$

$$y''_N(0) = 2, \quad (67)$$

$$y''_N(1) = 3e, \quad (68)$$

The four unknowns can be determined from equations (65)-(68)

2.2.4 Variational Iteration Algorithms for Ordinary Differential Equations and Partial Differential Equations

Here we summarized the variational iteration algorithms for some frequently used differential equations. So we have

$$\begin{cases} u' + f(u, u') = 0 \\ u_{n+1}(t) = u_n(t) - \int_0^t (u'_n + f_n)\,ds \\ u_{n+1}(t) = u_0(t) - \int_0^t f_n\,ds \\ u_{n+1}(t) = u_n(t) - \int_0^t (f_n - f_{n-1})\,ds \end{cases} \quad (69)$$

$$\begin{cases} u' + \alpha u + f(u,u') = 0 \\ u_{n+1}(t) = u_n(t) - \int_0^t e^{\alpha(s-t)}(u_n' + \alpha u_n + f_n)\,\mathrm{d}s \\ u_{n+1}(t) = u_0(t) - \int_0^t e^{\alpha(s-t)} f_n\,\mathrm{d}s \\ u_{n+1}(t) = u_n(t) - \int_0^t e^{\alpha(s-t)}(f_n - f_{n-1})\,\mathrm{d}s \end{cases} \tag{70}$$

$$\begin{cases} u'' + f(u,u',u'') = 0 \\ u_{n+1}(t) = u_n(t) + \int_0^t (s-t)(u_n'' + f_n)\,\mathrm{d}s \\ u_{n+1}(t) = u_0(t) + \int_0^t (s-t) f_n\,\mathrm{d}s \\ u_{n+1}(t) = u_n(t) + \int_0^t (s-t)(f_n - f_{n-1})\,\mathrm{d}s \end{cases} \tag{71}$$

$$\begin{cases} u'' + \omega^2 u + f(u,u',u'') = 0 \\ u_{n+1}(t) = u_n(t) + \frac{1}{\omega}\int_0^t (u_n'' + \omega^2 u_n + f_n)\sin\omega(s-t)\,\mathrm{d}s \\ u_{n+1}(t) = u_0(t) + \frac{1}{\omega}\int_0^t f_n \sin\omega(s-t)\,\mathrm{d}s \\ u_{n+1}(t) = u_n(t) + \frac{1}{\omega}\int_0^t (f_n - f_{n-1})\sin\omega(s-t)\,\mathrm{d}s \end{cases} \tag{72}$$

$$\begin{cases} u'' - \alpha^2 u + f(u,u',u'') = 0 \\ u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{2\alpha}(e^{\alpha(s-t)} - e^{\alpha(t-s)})(u_n'' - \alpha^2 u_n + f_n)\,\mathrm{d}s \\ u_{n+1}(t) = u_0(t) + \int_0^t \frac{1}{2\alpha}(e^{\alpha(s-t)} - e^{\alpha(t-s)}) f_n\,\mathrm{d}s \\ u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{2\alpha}(e^{\alpha(s-t)} - e^{\alpha(t-s)})(f_n - f_{n-1})\,\mathrm{d}s \end{cases} \tag{73}$$

$$\begin{cases} u''' + f(u,u',u'',u''') = 0 \\ u_{n+1}(t) = u_n(t) - \int_0^t \frac{1}{2}(s-t)^2(u_n''' + f_n)\,\mathrm{d}s \\ u_{n+1}(t) = u_0(t) - \int_0^t \frac{1}{2}(s-t)^2 f_n\,\mathrm{d}s \\ u_{n+1}(t) = u_n(t) - \int_0^t \frac{1}{2}(s-t)^2(f_n - f_{n-1})\,\mathrm{d}s \end{cases} \tag{74}$$

$$\begin{cases} u^{(4)} + f(u,u',u'',u''',u^{(4)}) = 0 \\ u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{6}(s-t)^3(u_n^{(4)} + f_n)\,\mathrm{d}s \\ u_{n+1}(t) = u_0(t) + \int_0^t \frac{1}{6}(s-t)^3 f_n\,\mathrm{d}s \\ u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{6}(s-t)^3(f_n - f_{n-1})\,\mathrm{d}s \end{cases} \tag{75}$$

$$
\begin{cases}
u^{(k)} + f(u, u', u'', \cdots, u^{(k)}) = 0 \\
u_{n+1}(t) = u_n(t) + (-1)^k \int_0^t \frac{1}{(k-1)!}(s-t)^{k-1}(u_n^{(k)} + f_n)\,ds \\
u_{n+1}(t) = u_0(t) + (-1)^k \int_0^t \frac{1}{(k-1)!}(s-t)^{k-1} f_n\,ds \\
u_{n+1}(t) = u_n(t) + (-1)^k \int_0^t \frac{1}{(k-1)!}(s-t)^{k-1}(f_n - f_{n-1})\,ds
\end{cases} \tag{76}
$$

The initial guess can be chosen in the algorithms Eq. (71) are only used for nonlinear oscillators, are

$$u_0(t) = u(0)cos\omega t + \frac{1}{\omega}u'(0)sin\omega t \tag{77}$$

Where ω is the frequency of the oscillator.

For example, we consider an oscillator of the form is

$$u'' + u^3 = 0, \tag{78}$$

This can be rewritten in the form,

$$u'' + \omega^2 u + f = 0, \tag{79}$$

Where $f = -\omega^2 u + u^3$.

So the similar iteration algorithms can be constracted for partial differential equation also. For example the partial differential equation is of the form

$$L_1 u + L_2 u + L_3 u + Nu = 0, \tag{80}$$

Where with respect to *x, y* and *z*, L_1, L_2 and L_2 are the linear operators respectively and N is a nonlinear operator. Then according to VIM the correction functionals can be constructed as follows

$$u_{n+1/3} = u_n + \int_0^x \lambda_1\{L_1 u_n(s,y,z) + L_2\tilde{u}_n(s,y,z) + L_3\tilde{u}_n(s,y,z) + N\tilde{u}_n(s,y,z)\}ds$$

$$u_{n+2/3} = u_{n+1/3} + \int_0^y \lambda_1\{L_1\tilde{u}_{n+1/3}(s,y,z) + L_2 u_{n+1/3}(s,y,z) + L_3\tilde{u}_{n+1/3}(s,y,z) + N\tilde{u}_{n+1/3}(s,y,z)\}ds$$

$$u_{n+1} = u_{n+2/3} + \int_0^z \lambda_1\{L_1\tilde{u}_{n+2/3}(s,y,z) + L_2\tilde{u}_{n+2/3}(s,y,z) + L_3 u_{n+2/3}(s,y,z) + N\tilde{u}_{n+2/3}(s,y,z)\}ds$$

(81)

After the identification of Lagrange multipliers λ_1, λ_2 and λ_3 the variational iteration algorithms (I, II and III) can be easily constructed

PROBLEM 5.

Consider the following partial differential equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} - u^3(x,y) = 0. \tag{82}$$

Now we begin with

$$u_0(x,y) = u(0,0) + xu_x(0,y) + yu_y(0,y) + yu_y(x,0), \tag{83}$$

Then the variational iteration algorithm can obtained as follows

$$\begin{cases} u_{n+1}(x,y) = u_n(x,y) + \int_0^x (s-x)\left\{\frac{\partial^2 u_n(s,y)}{\partial s^2} + \frac{\partial^2 u_n(s,y)}{\partial y^2} - u_n^3(s,y)\right\} \mathrm{d}s, \\ u_{n+1}(x,y) = u_0(x,y) + \int_0^x (s-t)\left\{\frac{\partial^2 u_n(s,y)}{\partial y^2} - u_n^3(s,y)\right\} \mathrm{d}s, \\ u_{n+1}(x,y) = u_n(x,y) + \int_0^x (s-t)\left\{\frac{\partial^2 u_n(s,y)}{\partial y^2} - u_n^3(s,y) - \frac{\partial^2 u_{n-1}(s,y)}{\partial y^2} + u_n^3(s,y)\right\} \mathrm{d}s. \end{cases} \quad (84)$$

We can also convert the partial differential equations into ordinary differential equations if only traveling wave solutions are sought. Applying the transformation $\eta = x + \omega y$, we get the ODE,

$$(1+\omega^2)\frac{d^2u}{d\eta^2} - u^3 = 0, \quad (85)$$

$$\Rightarrow \frac{d^2u}{d\eta^2} - \frac{1}{(1+\omega^2)}u^3 = 0, \quad (86)$$

With the initial conditions are $u(\eta) = \mathrm{A}$ and $\dot{u}(\eta) = \mathrm{B}$

Now we can construct the following iteration algorithms

$$\begin{cases} u_{n+1}(\eta) = u_n(\eta) + \int_0^\eta (s-\eta)(\frac{d^2u_n(s)}{ds^2} - \frac{1}{1+\omega^2}u_n^3(s))\mathrm{d}s, \\ u_{n+1}(\eta) = u_0(\eta) - \frac{1}{1+\omega^2}\int_0^\eta (s-\eta)u_n^3(s)\mathrm{d}s, \\ u_{n+1}(\eta) = u_n(\eta) - \frac{1}{1+\omega^2}\int_0^\eta (s-\eta)(u_n^3(s) - u_{n-1}^3(s))\mathrm{d}s, \end{cases} \quad (87)$$

So we can begin with

$$u_0(\eta) = u(0) + \eta u'(0) = A + B\eta, \quad (88)$$

Then we have a series of solution, which can be easily obtained

2.2.5 Variational Iteration Algorithms for Fractional Differential Equation

In recent decades there are a rapid development in the theory of the fractional calculus and its applications, with fractional synchronization attracting particular interest. Because of their exact description of many nonlinear phenomena Fractional differential equations have received recently much more attention. The variational iteration method is a greatly successful technique or method that was first proposed in 1998 to solve fractional differential equations whose effectiveness and accuracy was clearly discussed by Draganescu[6], Odibat and Momani[14] by applying it in to some complex differential equations of fractional order.

Now let's consider the fractional differential equation

$$\frac{D^\alpha u}{Dt^\alpha} + f = 0, \quad (89)$$

Here $\frac{D^\alpha u}{Dt^\alpha}$ is called the Caputo's fractional derivative that is widely used in the literature and is defined as

$$\frac{D^p u(t)}{Dt^p} = \frac{1}{\Gamma(m+1-p)}\int_a^t \frac{u^{(m+1)}(\tau)}{(t-\tau)^{p-m}}d\tau, \qquad \mathrm{m} < \mathrm{p} < \mathrm{m+1}, \quad (90)$$

p called the order of the fractional derivative where p is some real number and Γ denotes the gamma function.

Now for this case, $0 < \alpha < 1$, we can rewrite Eq. (89) in the following form

$$\frac{du}{dt} + \frac{D^{\alpha}u}{Dt^{\alpha}} - \frac{du}{dt} + f = 0, \tag{91}$$

And then the variational iteration algorithms are given as follows

$$\begin{cases} u_{n+1}(t) = u_n(t) - \int_0^t (\frac{D^{\alpha}u_n}{Dt^{\alpha}} + f_n)\mathrm{d}s \\ u_{n+1}(t) = u_0(t) - \int_0^t (\frac{D^{\alpha}u_n}{Dt^{\alpha}} - \frac{du_n}{dt} + f_n)\mathrm{d}s \\ u_{n+1}(t) = u_0(t) - \int_0^t \left\{ (\frac{D^{\alpha}u_n}{Dt^{\alpha}} - \frac{du_n}{dt} + f_n) - (\frac{D^{\alpha}u_{n-1}}{Dt^{\alpha}} - \frac{du_{n-1}}{dt} + f_{n-1}) \right\}\mathrm{d}s. \end{cases} \tag{92}$$

And for the case $1 < \alpha < 2$, the above iteration formulas also valid. Now we can rewrite Eq. (89) in the form

$$\frac{d^2u}{dt^2} + \frac{D^{\alpha}u}{Dt^{\alpha}} - \frac{d^2u}{dt^2} + f = 0, \tag{93}$$

Then we have the following iteration formulae

$$\begin{cases} u_{n+1}(t) = u_n(t) + \int_0^t (s-t)(\frac{D^{\alpha}u_n}{Dt^{\alpha}} + f_n)\mathrm{d}s \\ u_{n+1}(t) = u_0(t) + \int_0^t (s-t)(\frac{D^{\alpha}u_n}{Dt^{\alpha}} - \frac{d^2u_n}{dt^2} + f_n)\mathrm{d}s \\ u_{n+1}(t) = u_n(t) + \int_0^t (s-t)\left\{ (\frac{D^{\alpha}u_n}{Dt^{\alpha}} - \frac{d^2u_n}{dt^2} + f_n) - (\frac{D^{\alpha}u_{n-1}}{Dt^{\alpha}} - \frac{d^2u_{n-1}}{dt^2} + f_{n-1}) \right\}\mathrm{d}s. \end{cases} \tag{94}$$

When α is close to 1 then Eq. (92) batter and when α approaches to 2 then Eq. (94) is batter. If N is natural number then we have general case, $N < \alpha <$ N+1 the iteration formulas are as follows

$$\begin{cases} u_{n+1}(t) = u_n(t) + (-1)^N \int_0^t \frac{1}{(N-1)!}(s-t)^{N-1}(\frac{D^{\alpha}u_n}{Dt^{\alpha}} + f_n)\mathrm{d}s \\ u_{n+1}(t) = u_0(t) + (-1)^N \int_0^t \frac{1}{(N-1)!}(s-t)^{N-1}(\frac{D^{\alpha}u_n}{Dt^{\alpha}} - \frac{d^Nu_n}{dt^N} + f_n)\mathrm{d}s \\ u_{n+1}(t) = u_n(t) + (-1)^N \int_0^t \frac{1}{(N-1)!}(s-t)^{N-1}\left\{ (\frac{D^{\alpha}u_n}{Dt^{\alpha}} - \frac{d^Nu_n}{dt^N} + f_n) - (\frac{D^{\alpha}u_{n-1}}{Dt^{\alpha}} - \frac{d^Nu_{n-1}}{dt^N} + f_{n-1}) \right\}\mathrm{d}s \end{cases} \tag{95}$$

Or

$$\begin{cases} u_{n+1}(t) = u_n(t) + (-1)^{N+1} \int_0^t \frac{1}{N!}(s-t)^N (\frac{D^{\alpha}u_n}{Dt^{\alpha}} + f_n)\mathrm{d}s \\ u_{n+1}(t) = u_0(t) + (-1)^{N+1} \int_0^t \frac{1}{N!}(s-t)^N (\frac{D^{\alpha}u_n}{Dt^{\alpha}} - \frac{d^{N+1}u_n}{dt^{N+1}} + f_n)\mathrm{d}s \\ u_{n+1}(t) = u_n(t) + (-1)^{N+1} \int_0^t \frac{1}{N!}(s-t)^N \left\{ (\frac{D^{\alpha}u_n}{Dt^{\alpha}} - \frac{d^{N+1}u_n}{dt^{N+1}} + f_n) - (\frac{D^{\alpha}u_{n-1}}{Dt^{\alpha}} - \frac{d^{N+1}u_{n-1}}{dt^{N+1}} + f_{n-1}) \right\}\mathrm{d}s \end{cases} \tag{96}$$

Here when α is close to N, then Eq. (95) is batter and when α is close to N+1 then Eq. (96) is more effective.

2.2.6 Variational Iteration Algorithms for Fractal Differential Equations

By the transformation of the standard integer dimensional space-time (x, t) into a fractal space-time [4, 5], the fractal derivative is defined as follows

$$\frac{du(t)}{dt^D} = \lim_{s \to t} \frac{u(t)-u(s)}{t^D - s^D} \tag{97}$$

Here D is the order of the fractal derivative. For example, consider the fractal derivative relaxation equation [4]

$$\frac{du}{dt^D} + Bu = 0, \qquad 0 < D < 1, \quad u(0) = 1, \tag{98}$$

The analytical solution of Eq. (98) is

$$u(t) = exp(-Bt^D), \tag{99}$$

Now we write down a general fractal differential equation of the form

$$\frac{du}{dt^D} + f = 0 \tag{100}$$

and to convert Eq.(100) into an ordinary differential equation , we use the transformation $t^D = x$, we have

$$\frac{du}{dx} + f = 0 \tag{101}$$

So the iteration algorithms proposed above can be directly applied is given by

$$\begin{cases} u_{n+1}(x) = u_n(x) - \int_0^x \left(\frac{du_n}{dx} + f_n\right) \mathrm{d}x \\ u_{n+1}(x) = u_0(x) - \int_0^x f_n \, \mathrm{d}x \\ u_{n+1}(x) = u_0(x) - \int_0^x \left\{ f_n - f_{n-1} \right\} \mathrm{d}x \end{cases} \tag{102}$$

Or

$$\begin{cases} u_{n+1}(t^D) = u_n(t^D) - \int_0^{t^D} \left(\frac{du_n}{dt^D} + f_n\right) Dt^{D-1} \, \mathrm{d}t \\ u_{n+1}(t^D) = u_0(t^D) - \int_0^{t^D} f_n Dt^{D-1} \, \mathrm{d}t \\ u_{n+1}(t^D) = u_0(t^D) - \int_0^{t^D} \left\{ f_n - f_{n-1} \right\} Dt^{D-1} \, \mathrm{d}x. \end{cases} \tag{103}$$

Let's taking an another example i.e. consider the n^{th} order fractal differential equation is

$$\frac{d^N u}{dt^D} + f = 0 \tag{104}$$

That can be converted by the transformation $\quad t^D = x^N \quad$ (105)

in to the ordinary differential equation

$$\frac{d^N u}{dt^N} + f = 0 \tag{106}$$

Then the Variational iteration algorithms are

$$\begin{cases} u_{n+1}(x) = u_n(x) + (-1)^N \int_0^x \frac{1}{(N-1)!}(s-x)^{N-1}(u_n^{(N)} + f_n)\mathrm{d}s \\ u_{n+1}(x) = u_0(x) + (-1)^N \int_0^t \frac{1}{(N-1)!}(s-x)^{N-1} f_n \,\mathrm{d}s \\ u_{n+1}(x) = u_n(x) + (-1)^N \int_0^t \frac{1}{(N-1)!}(s-t)^{N-1}(f_n - f_{n-1})\mathrm{d}s. \end{cases} \quad (107)$$

2.2.7 Variational Iteration Algorithms for Differential-difference Equations

The another approach to the formulation of discontinuous problems is differential-difference model which also attracted much attention due to its ability to exactly describe many real life problems in engineering, nanotechnology etc. that can be written in the general form

$$\frac{d^k u_i}{dt^k} + f(\dots, u_{i-1}, u_i, u_{i+1}, \dots) = 0 \quad (108)$$

Then the Variational iteration algorithms are

$$\begin{cases} u_{i,n+1}(t) = u_{i,n}(t) + (-1)^k \int_0^t \frac{1}{(k-1)!}(s-t)^{k-1}(\frac{d^k u_{i,n}}{dt^k} + f_n)\mathrm{d}s \\ u_{i,n+1}(t) = u_{i,0}(t) + (-1)^k \int_0^t \frac{1}{(k-1)!}(s-t)^{k-1} f_n \,\mathrm{d}s \\ u_{i,n+1}(t) = u_{i,n}(t) + (-1)^k \int_0^t \frac{1}{(k-1)!}(s-t)^{k-1}(f_{i,n} - f_{i,n-1})\mathrm{d}s. \end{cases} \quad (109)$$

2.3 Physical understanding

Here physical understanding of fractional, fractal and differential-difference equation are given.

2.3.1 For Fractional Calculus

Although the fractional calculus was first invented by Newton and Leibnitz over three centuries ago, but it became a hot topic recently owing to the development of the computer and also it gives the exact description of many real-life problems. To understand the physical interpretation of the fractional calculus, let we begin with a simple function

$$x = x^2 \quad (110)$$

When $x = x_1$ and $x = x_2$, then we have,

$$y_1 = x_1^2 \quad \text{and} \quad y_2 = x_2^2 \quad (111)$$

Therefore, we have the difference

$$\Delta y = y_1 - y_2 = x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2) = (x_1 + x_2)\Delta x \quad (112)$$

Now in case $x_1 \to x_2$ or $\Delta x \to 0$, we have the differential,

$$dy = 2xdx \quad (113)$$

The above derivation, however, is only valid for continuous functions. To show its applications, we consider the action in String Theory,

$$S = mc\int ds \tag{114}$$

Where m is the mass of a particle, c is the speed of light, and ds is the relativistic metric that can be expressed in the form

$$ds = \sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2} = c\sqrt{1 - \frac{u^2}{c^2}dt} \tag{115}$$

Substitution of Eq. (101) in Eq. (100) gives

$$S = \int mc^2\sqrt{1 - \frac{u^2}{c^2}dt} \tag{116}$$

From which equation of free motion can be obtained in the usual way

$$\frac{d}{dt}\left(\frac{mu}{\sqrt{1-\frac{u^2}{c^2}dt}}\right) = 0 \tag{117}$$

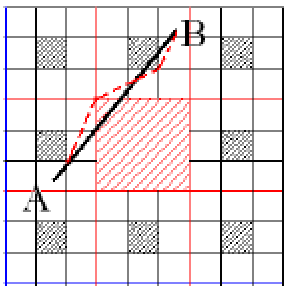
So the above derivation assumes that space and time are both continuous. Now the distance between two points, for example, cannot be expressed in the form of Eq. (115) in a discontinuous world (e.g., in the Julia set). Now consider a plane with fractal structure (see Fig.1). The shortest path between two points is not a line and we have

$$ds_E = kds^D \tag{118}$$

Here ds_E is the actual distance between two points (i.e. discontinuous line in Fig.1), ds is the line distance between two points (continuous line in Fig.1), D is the fractal dimension and k is a constant. Now the action in a discontinuous space-time can therefore be written in the following form by using fractional calculus

$$S = mc\int ds_E = \int mc^{1+D}k\,(1 - \frac{u^2}{c^2})^{D/2}dt^D \tag{119}$$

Therefore the Fractional calculus is valid for discontinuous problems.



. Fig.1 The distance between two points in a discontinuous space time.

Let we consider a well-known predator-prey model (i.e. the Lotka–Volterra equation).

$$\frac{dx}{dt} = x(a - by), \tag{120}$$

$$\frac{dy}{dt} = -y(c - dy), \tag{121}$$

Here y is the number of predators (for example, wolves), x the number of its prey (for example, rabbits) and a, b, c, and d are the parameters representing the interaction of the two species. In general, the growth of the two populations is discontinuous and a simple modification of the predator-prey model is to replace dy/dt and dx/dt by fractional derivatives

$$\frac{D^{\alpha}x}{Dt^{\alpha}} = x(a - by) \tag{122}$$
$$\frac{D^{\beta}x}{Dt^{\beta}} = -y(c - dy) \tag{123}$$
Where the populations of the predator and prey may be greatly affected by the fractional orders, α and β.

2.3.2 For Fractal differential equation

In many applications the fractal derivative is simpler than fractional counterpart and it is also valid for discontinuous cases. Now we can re-write Eq. (97) is in the form
$$\frac{du(t)}{dt^{D}} = \frac{1}{k}\lim_{A\to B}\frac{u(A)-u(B)}{\tilde{x}_A-\tilde{x}_B}, \tag{124}$$
Here k is a constant and A and B are arbitrary points in discontinuous space or space-time (as shown in Fig.2). $(\tilde{x}_A\ ,\ \tilde{x}_B)$ are called the fractal coordinates and are defined by
$$\tilde{x}_A = k(\tilde{x}_A - 0)^D = k(\tilde{x}_A)^D \tag{125}$$
$$\tilde{x}_B = k(\tilde{x}_B - 0)^D = k(\tilde{x}_B)^D \tag{126}$$
Here $(x_A, x_B)$the coordinates and D are is the fractional dimension in *x*-direction. Substituting Eq. (125) and (126) into (124), we obtain
$$\frac{du(t)}{dt^{D}} = \lim_{A\to B}\frac{u(A)-u(B)}{(x_A)^D-(x_B)^D}, \tag{127}$$

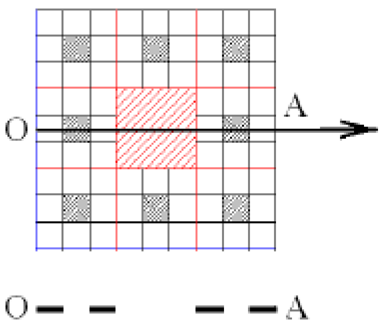


Fig. 2 A schematic diagram of distance between O and A (i.e. the fractal coordinate of A) in a fractal space, *D*=ln2/ln3, while the fractional dimensions for the plane are ln8/ln3.

The fractal differential model is particularly suitable for describing the discontinuous matter and which is the preferred model for describing flow or heat conduction through porous media. For example, the principle of mass conservation can be written in the form
$$\frac{\partial\rho}{\partial t} + k_1\frac{\partial(\rho u)}{\partial x^{D_1}} + k_2\frac{\partial(\rho v)}{\partial y^{D_2}} + k_3\frac{\partial(\rho w)}{\partial z^{D_3}} = 0, \tag{128}$$
In the *x, y,* and *z* directions D_1 , D_2 and D_3 are the fractal dimensions of porosity respectively and $k_i(i = 1,2,3)$ are constants which are related to the fractal dimensions. In particular, we can have $k_i = 1$ when $D_i = 1$. Similarly, for the momentum equation in one-dimensional porous flow, can be written in the form
$$\frac{\partial u}{\partial t} + ku\frac{\partial u}{\partial x^{D}} = -\frac{k}{\rho}\frac{\partial p}{\partial x^{D}} + k\frac{\partial}{\partial x^{D}}\left(\mu k\frac{\partial u}{\partial x^{D}}\right), \tag{129}$$
Here $D\ =\ 1$, and $k\ =\ 1$, then Eq. (126) turns out to be the classical one.

Now in porous media the one-dimensional heat conduction equation can be expressed as

$$\frac{\partial T}{\partial t} + k\frac{\partial}{\partial x^D}\left(\mu k\frac{\partial T}{\partial x^D}\right) = 0 \quad (130)$$

Where μ is the conduction coefficient and when $D = 1$ we have $k = 1$.

Also an oscillator is swinging in a porous medium can be described by fractal differential equations. For example, the Duffing equation with fractal damp we have the following equation

$$\frac{d^2u}{dt^2} + u + k\mu\frac{\partial u}{\partial t^D} + \varepsilon u^3 = 0 \quad (131)$$

By replacing $\partial / \partial x$ in the classical approach by $k\partial/\partial x^n$, it is easy to establish fractal differential equations for discontinuous media.

2.3.3 Physical understanding of Differential-difference equations

Recently the differential-difference model has achieve much attention because of its ability to describe exactly many real-life problems in textile engineering, nanotechnology, and stratified hydrostatic flows. This can batter understood by considering the flow through a lattice where the conservation of mass requires

$$\frac{d\rho_i}{dt} + \rho_{i+1}u_{i+1} - \rho_{i-1}u_{i-1} = 0 \quad (132)$$

Where ρ_i and u_i respectively are the gas density and velocity at the *i*-th lattice point.

(i-1)th lattice point i-th lattice point (i+1)th lattice point

Fig. 3 Conservation of mass

CHAPTER-3

APPLICATIONS

This method is vastly applicable on higher order nonlinear equations. It has many applications like Boundary Value Problems of various-orders, Boussinesq Equations, Thomas-Fermi Model, Unsteady Flow of Gas through Porous Medium, Boundary Layer Flows, Blasius Problem, Goursat Problems, Laplace ProblemsHeat and Wave Like Models, Burger Equations, Couple Burger equations, Parabolic Equations, KdVs of Third, Fourth and Seventh-orders, Evolution Equations, Higher-dimensional IBVPS,Helmholtz Equations, Fisher's Equations, Schrödinger Equations, Sine-Gordon Equations, Telegraph Equations, Flierl Petviashivili Equations, Lane-Emden Equations, Emden-Fowler Equations etc. Let's discusses the some of following examples to understand this method in a batter way.

3.1 Burger's equation

Consider that the one-dimensional Burger's equation, which has the form

$$u_t + uu_x - vu_{xx} = 0 \tag{133}$$

With initial condition $u(x,0) = \frac{\alpha+\beta+(\beta-\alpha)exp(\gamma)}{1+exp(\gamma)}$, $t \geq 0.$ (134)

Where $\gamma = (\alpha/v)(x-\lambda)$ and the parameters α, β, γ and v are arbitrary constants

Now to solve Eq.(133) by means of VIM, let's construct a correction functional as follows

$$u_{n+1}(x,t) = u_n(x,0) + \int_0^t \lambda\{u_t + u\tilde{u}_x - v\tilde{u}_{xx}\}d\tau, \tag{135}$$

Where $\delta\tilde{u}_n$ is considered as a restricted variation Now making above correction functional stationary, and noticing that $\delta y(0) = 0$, so its stationary conditions are obtained as follows

$$\lambda'^{(\tau)} = 0. \tag{136}$$

$$1 + \lambda(\tau)|_{\tau=t} = 0. \tag{137}$$

Where Eq. (136) is called Lagrange–Euler equation, and Eq. (136) natural boundary condition. The Lagrange multiplier, therefore, can be identified as $\lambda = -1$ and the following Variational iteration formula can be obtained

$$u_{n+1}(x,0) = u_n(x,0) - \int_0^t [(u_t)_n + u_n u_{nx} - vu_{nxx}]d\tau \tag{138}$$

We start with an initial approximation $u_0 = u(x,0)$ given by Eq. (134), by the Variation iteration formula we can obtain directly the other components as

$$u_1(x,t) = \frac{\alpha+\beta+(\beta-\alpha)exp(\gamma)}{1+exp(\gamma)} + \frac{2\alpha\beta^2 exp(\gamma)}{v[1+exp(\gamma)]^2}t, \tag{139}$$

$$u_2(x,t) = \frac{\alpha+\beta+(\beta-\alpha)exp(\gamma)}{1+exp(\gamma)} + \frac{2\alpha\beta^2 exp(\gamma)}{v[1+exp(\gamma)]^2}t + \frac{\alpha^3\beta^2 exp(\gamma)[-1+exp(\gamma)]}{v^2[1+exp(\gamma)]^3}t^2 \tag{140}$$

$$u_3(x,t) = u_2 + \frac{\alpha^4\beta^2 exp(\gamma)[1-4exp(\gamma)+exp(\gamma)^2]}{3v^3[1+exp(\gamma)]^4}t^2 \tag{141}$$

and so on.

And in the same manner the rest of components of the iteration formula (138) were obtained using the Maple Package. The solution of $u(x,t)$ in a closed form is

$$u(x,t) = \frac{\alpha+\beta+(\beta-\alpha)exp(\zeta)}{1+exp(\zeta)} \tag{142}$$

Where $\zeta = (\alpha/\nu)(x - \beta t - \lambda)$ is the required solution.

3.2. Fisher's equation

Let's consider the Fisher's equation

$$u_t = u_{xx} + u(1-u) \tag{143}$$

subject to the initial condition $u(x,0) = \alpha$.

To solve this equation by Variational iteration method, let substitute $Lu = u_{n_\tau}$, $Ru_n = -u_{n_{xx}}$, $Nu_n = -u_n(1-u_n)$, $g = 0$, where $L \equiv \partial/\partial t$, R is a linear operator which has partial derivatives with respect to x, $Nu(x,t)$ is a nonlinear term and $g(x,t)$ is an inhomogeneous term and we obtain the following iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \{u_{n_\tau} - u_{n_{xx}} - u_n(1-u_n)\} d\tau \tag{144}$$

Where $\lambda = -1$ and with the initial approximation $u_0(x,t) = \alpha$. Then we obtain the solution by several approximation as follows

$$u_1(x,t) = \alpha + \alpha(1-\alpha)t$$

$$u_2(x,t) = \alpha + \alpha(1-\alpha)t + \alpha(1-\alpha)(1-2\alpha)\frac{t^2}{2!} + \left[-\alpha^2(1-\alpha)^2\frac{t^3}{3!}\right],$$

$$u_3(x,t) = \alpha + \alpha(1-\alpha)t + \alpha(1-\alpha)(1-2\alpha)\frac{t^2}{2!} + \alpha(1-\alpha)(1-6\alpha+6\alpha^2)\frac{t^3}{3!}$$
$$+ \left[-\alpha^2(1-2\alpha)(1-\alpha)^2\frac{t^4}{3} - \alpha^2(1-\alpha)^2(3-20\alpha+20\alpha^2)\frac{t^5}{60}\right.$$
$$\left. + \alpha^3(1-2\alpha)(1-\alpha)^3\frac{t^6}{18} - \alpha^4(1-\alpha)^4\frac{t^7}{63}\right]$$

and so on. Then the solution is

$$u(x,t) = \alpha + \alpha(1-\alpha)t + \alpha(1-\alpha)(1-2\alpha)\frac{t^2}{2!} + \alpha(1-\alpha)(1-6\alpha+6\alpha^2)\frac{t^3}{3!} + \cdots$$
$$= \frac{\alpha e^t}{1-\alpha+\alpha e^t} \tag{145}$$

Which is the required solution of equation (133).

3.3 Schrodinger equation

Let's consider the linear Schrodinger equation

$$u_t + iu_{xx} = 0, \qquad u(x,0) = 1 + cosh \tag{146}$$

Then the correction functional is given by

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial u_n}{\partial \xi} + i\frac{\partial^2(\tilde{u}_n)(x,\xi)}{\partial x^2}\right) d\xi \tag{147}$$

The stationary conditions are given by

$$1+\lambda=0,$$
$$\lambda'=0 \quad (148)$$

From which we have $\lambda=-1.$ (149)

Substituting this value of the Lagrange multiplier $\lambda=-1$ into the functional (147) gives the iteration formula

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left(\frac{\partial u_n}{\partial\xi}+i\frac{\partial^2(\tilde{u}_n)(x,\xi)}{\partial x^2}\right)d\xi, \qquad n\geq 0. \quad (150)$$

Now we can start with initial approximation $u(x,0)=1+cosh(2x)$ and using the iteration formula (150), we obtained following successive approximations

$u_0(x,t)=1+cosh(2x)$

$u_1(x,t)=1+cosh(2x)+4itcosh(2x)$

$u_2(x,t)=1+cosh(2x)+4itcosh(2x)+\frac{(4it)^2}{2!}cosh(2x)$

$u_3(x,t)=1+cosh(2x)+4itcosh(2x)+\frac{(4it)^2}{2!}cosh(2x)+\frac{(4it)^3}{3!}cosh(2x)$

And so on. Then,

$$u_n(x,t)=1+cosh(2x)\left(1+(4it)+\frac{(4it)^2}{2!}+\frac{(4it)^3}{3!}+\frac{(4it)^4}{4!}+\cdots\right) \quad (151)$$

The VIM introduces the use of $u(x,t)=\lim_{n\to\infty}u_n(x,t)$, which gives the exact solution by

$$u(x,t)=1+cosh(2x)e^{4it}, \quad (152)$$

obtained upon using the Taylor expansion of e^{4it} which is the required solution.

3.4 Wave equation

Let's consider the first-order wave equation in one-dimension

$$u_t+cu_{xx}=0, c>0 \quad (153)$$

With initial and boundary conditions

$$u(0,t)=sin\left(\frac{-c\pi t}{l}\right), \qquad u_x(0,t)=\frac{\pi}{l}cos\left(\frac{-c\pi t}{l}\right) \quad (154)$$

$$u(x,o)=sin\left(\frac{\pi x}{l}\right), \qquad u_t(x,0)=-\frac{c\pi}{l}cos\left(\frac{\pi x}{l}\right) \quad (155)$$

Then the correction functional for Eq. (153) in the t-direction takes the form

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t\lambda\left[\frac{\partial u_n(x,t)}{\partial\tau}+c\frac{\partial\tilde{u}_n(x,t)}{\partial x}\right]d\tau \quad (156)$$

Now stationary conditions of above equations are $\lambda'(\tau)=0,\ 1+\lambda(\tau)|_{\tau=t}=0$ which gives the solution $\lambda=-1$. Now substituting the value of λ in the equation (155), we have

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left[\frac{\partial u_n(x,t)}{\partial\tau}+c\frac{\partial\tilde{u}_n(x,t)}{\partial x}\right]d\tau \quad (157)$$

Considering the given conditions (154) and (155), it is clear that the solution contains $sin\left(\frac{\pi x}{l}\right)$. So we can choose $u_0(x,t)=u(x,t)=sin\left(\frac{\pi x}{l}\right)$

Now using this chosen value in the equation (157) we obtained the following successive approximation

$u_0(x,t) = sin\left(\frac{\pi x}{l}\right)$

$u_1(x,t) = sin\left(\frac{\pi x}{l}\right) - cos\left(\frac{\pi x}{l}\right)\left[\frac{c\pi t}{l}\right]$

$u_2(x,t) = sin\left(\frac{\pi x}{l}\right)\left[1 - \frac{1}{2!}\left(\frac{c\pi t}{l}\right)^2\right] - cos\left(\frac{\pi x}{l}\right)\left[\frac{c\pi t}{l}\right]$

$u_3(x,t) = sin\left(\frac{\pi x}{l}\right)\left[1 - \frac{1}{2!}\left(\frac{c\pi t}{l}\right)^2\right] - cos\left(\frac{\pi x}{l}\right)\left[\frac{c\pi t}{l} - \frac{1}{3!}\left(\frac{c\pi t}{l}\right)^3\right],$

Then the n^{th} approximation is

$$u_n(x,t) = sin\left(\frac{\pi x}{l}\right)\left[1 - \frac{1}{2!}\left(\frac{c\pi t}{l}\right)^2 + \frac{1}{4!}\left(\frac{c\pi t}{l}\right)^4 - \frac{1}{6!}\left(\frac{c\pi t}{l}\right)^6 + \cdots\right] - cos\left(\frac{\pi x}{l}\right)\left[\frac{c\pi t}{l} - \frac{1}{3!}\left(\frac{c\pi t}{l}\right)^3 + \frac{1}{5!}\left(\frac{c\pi t}{l}\right)^5 - \frac{1}{7!}\left(\frac{c\pi t}{l}\right)^7 + \cdots\right] \quad (158)$$

Then the solution is

$$u(x,t) = sin\left(\frac{\pi x}{l}\right) cos\left(\frac{c\pi t}{l}\right) - cos\left(\frac{\pi x}{l}\right) sin\left(\frac{c\pi t}{l}\right) = sin\left[\pi\left(\frac{x-ct}{l}\right)\right] \quad (159)$$

Which is required exact solution for Eq. (152), with given conditions (154) and (155).

CHAPTER-4

COMPARISIONS WITH VARIATIONAL ITERATION METHOD

4.1. With Adomian Decomposition method

4.1.1. Analysis of Adomian Decomposition method

Adomian decomposition method defines the unknown function $u(x)$ by an infinite series

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{160}$$

Where the components $u_n(x)$ are usually determined by recurrently. Now the nonlinear operator $F(u)$ can be decomposed into an infinite series of polynomials, which is given by

$$F(u) = \sum_{n=0}^{\infty} A_n, \tag{161}$$

Where An are the so-called Adomian polynomials of $u_0, u_1, \ldots, u_n$ defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}, \qquad n = 0,1,2,3, \ldots, \tag{162}$$

Or equivalently

$$A_0 = F'(u_0)$$

$$A_1 = u_1 F'(u_0)$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2}u^2{}_1 F''(u_0)$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3}u^3{}_1 F'''(u_0)$$

$$A_4 = u_4 F'(u_0) + \left(u_1 u_3 + \frac{1}{2}u^2{}_2\right) F''^{(u_0)} + \frac{1}{2}u^2{}_1 u_2 F'''(u_0) + \frac{1}{24}u^4{}_1 F^{iv}(u_0)$$

and so on. (163)

Now these polynomials can be generated for all classes of nonlinearity according to specific algorithms defined by (161) and Adomian decomposition method provides the components of the exact solution, where these components should follow the summation given in (160).

Now we can show comparison between Variational iteration methods and Adomian decomposition method by taking an example

PROBLEM 6.

Let's consider the homogeneous advection problem

$$u_t + u u_x = 0 \tag{164}$$

With initial condition $u(x,0) = -x$

4.1.2 Implementation of Variational iteration method

Now according to the method the correctional functional is given by

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \tilde{u}_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x}\right)d\xi \tag{165}$$

This yields the stationary conditions $\lambda'(\xi) = 0,\ 1 + \lambda(\xi) = 0$ and this in turns gives $\lambda = -1$, where λ is the Lagrange multiplier. Now substitute the value of λ in the equation (164), we have

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \tilde{u}_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x}\right)d\xi, \qquad n \geq 0 \tag{166}$$

Now according to the method, using initial condition in equation (165), we have successive approximations

$u_0(x,t) = -x$

$u_1(x,t) = -x - xt$

$u_2(x,t) = -x - xt - xt^2 - \frac{1}{3}xt^3$

$u_3(x,t) = -x - xt - xt^2 - xt^3 - \frac{2}{3}xt^4 - small\ terms.$

$u_4(x,t) = -x - xt - xt^2 - xt^3 - xt^4 - \frac{14}{15}xt^5 - small\ terms.$

$u_5(x,t) = -x - xt - xt^2 - xt^3 - xt^4 - xt^5 - small\ terms.$

Then n^{th} approximation is given by,

$$u_n(x,t) = -x - xt - xt^2 - xt^3 - xt^4 - xt^5 - \cdots - xt^n - small\ terms. \tag{167}$$

Using $u = \lim_{n\to\infty} u_n$ we have from Eq. (167)

$$u(x,t) = -x(1 + t + t^2 + t^3 + t^4 + t^5 + \cdots), \tag{168}$$

This leads to the solution

$$u(x,t) = \frac{x}{t-1} \tag{169}$$

4.1.3 Implementation of Adomian Decomposition method

Here we first rewrite Eq. (164) in an operator form,

$$Lu = -uu_x,$$
$$u(x,0) = -x \tag{170}$$

Where the differential operator L is

$$L = \frac{\partial}{\partial t} \tag{171}$$

The inverse L^{-1} is assumed as an integral operator given by

$$L^{-1}(*) = \int_0^t (*)\,dt \tag{172}$$

Applying the inverse operator L^{-1} on both sides of (170) and using the initial condition we find

$$u(x,t) = -x - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \tag{173}$$

According to the Adomian Decomposition method, we have the functional equation

$$\sum_{n=0}^{\infty} u_n(x,t) = -x - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right), \tag{174}$$

where u_n are the so-called Adomian polynomials. Identifying the zeroth component $u_0(x,t)$ by $-x$, the remaining components $u_n(x,t\ ), n \geq 1$, can be determined by using the recurrence relation

$$u_0(x,t) = -x,$$
$$u_{k+1}(x) = -L^{-1}(A_k) \qquad k \geq 0, \tag{175}$$

Where A_k are Adomian polynomials that represent the nonlinear term uu_x and given by

$$A_0 = u_0 u_{0_x}$$
$$A_1 = u_0 u_{1_x} + u_1 u_{0_x}$$
$$A_2 = u_0 u_{2_x} + u_1 u_{1_x} + u_2 u_{0_x}$$

and so on. (176)

Other polynomials can be generated in a similar way to enhance the accuracy of approximation. Combining (174) and (175) yields

$u_0(x,t) = -x,$
$u_1(x,t) = -xt,$
$u_2(x,t) = -xt^2,$
$u_3(x,t) = -xt^3$
$u_4(x,t) = -xt^4$

and so on. (177)

In view of (176), the solution $u(x,t)$ is readily obtained in a series form by

$$u(x) = -x(1 + t + t^2 + t^3 + t^4 + \cdots), \tag{178}$$

Or in closed form by $u(x,t) = \frac{x}{t-1}$ (179)

4.1.4 Basic difference between Adomian Decomposition method and Variational iteration method

The two methods are powerful and efficient methods that both give approximations of higher accuracy and closed form solutions if existing. He's variational iteration method gives several successive approximations through using the iteration of the correction functional. However, Adomian decomposition method provides the components of the exact solution, where these components should follow the summation given in (159). Moreover, the VIM requires the evaluation of the Lagrangian multiplier λ whereas ADM requires the evaluation of the Adomian polynomials that mostly require tedious algebraic calculations. It is interesting to point out that unlike the successive approximations obtained by the VIM; the ADM provides the solution in successive components that will be added to get the series solution.

More importantly, the VIM reduces the volume of calculations by not requiring the Adomian polynomials; hence the iteration is direct and straightforward. However, ADM requires the use of Adomian polynomials for nonlinear terms, and this needs more work. For x nonlinear equations that arise frequently to express nonlinear phenomenon, He's variational iteration method facilitates the computational work and gives the solution rapidly if compared with Adomian method.

4.2. WITH MODIFIED VARIATIONAL ITERATION METHOD

4.2.1 Modified Variational Iteration method

Modified Variational Iteration method is a little modification of Variational Iteration method. To illustrate the basic concept of the modified variational iteration method (MVIM), we consider the following partial differential equation

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = g(x,t),$$
$$u(x,0) = f(x), \qquad (180)$$

Where $L = \frac{\partial}{\partial t}, R$ is a linear operator which has partial derivatives with respect to x, $Nu(x,t)$ is a nonlinear term and $g(x,t)$ is an inhomogeneous term. Partial differential equation (179) covers a large branch of applications such as soliton equations like Burger's, coupled Burger's, Schrödinger, KdV, modified KdV and also compacton equations like $k(n,n)$ and many others important equations.

Now using Variational iteration method, we have $\lambda = -1$ and we have the following iteration formula

$$U_{n+1} = U_n - \int_0^t \{L(U_n) + R(U_n) + NU_n - g\}d\tau. \qquad (181)$$

We can rewrite the Eq. (180) as

$$U_{n+1} = U_n - \int_0^t \{R(U_n - U_{n-1}) + (G_n - G_{n-1})\}d\tau \qquad (182)$$

Where $U_{-1} = 0, U_0 = f(x), U_1 = U_0 - \int_0^t \{R(U_0 - U_{-1}) + (G_0 - G_{-1}) - g\}d\tau$, and $G_0(x,t)$ is obtained from

$$NU_n(x,t) = G_n(x,t) + O(t^{n+1}) \qquad (183)$$

Eq. (182) can be solved iteratively to obtain an approximate solution that takes the form

$$u(x,t) \simeq U_n(x,t), \qquad (184)$$

Where n is the final iteration step. This is the required Modified Variational iteration method (MVID)

Now we can show comparison between Variational iteration methods and Modified Variational iteration methods by taking an example.

4.2.2. KDV equation

Let's consider the KdV equation which takes the form

$$u_t - 6uu_x + u_{xxx} = 0, \qquad x \in R$$
$$u(x,0) = \frac{-k^2}{2} sech^2\left[\frac{k}{2}x\right] \qquad (185)$$

Applying VIM, the following VIM results are obtained

$$U_0(x,t) = \frac{-k^2}{2} sech^2\left[\frac{k}{2}x\right]$$
$$U_1(x,t) = U_0(x,t) - \frac{k^5}{2} sech^2\left[\frac{k}{2}x\right] Tanh\left[\frac{k}{2}x\right]$$

$$U_2(x,t) = U_1(x,t) - \frac{k^8}{8} sech^4\left[\frac{k}{2}x\right](2 - cosh[kx])t^2$$
$$+ 512 sech^6\left[\frac{k}{2}x\right] Tanh\left[\frac{k}{2}x\right](2 - cosh[kx])t^3,$$
$$U_3(x,t) = U_2(x,t) - \frac{k^8}{8} sech^4\left[\frac{k}{2}x\right](2 - cosh[kx])t^2$$
$$+ \frac{k^{11}}{48} sech^5\left[\frac{k}{2}x\right]\left(11 sinh\left[\frac{k}{2}x\right] - sinh\left[\frac{3k}{2}x\right]\right)t^3$$
$$+ 64(970 - 1163 cosh[kx] + 232 cosh[2kx]$$
$$- 11 cosh[3kx]) sech^{10}\left[\frac{k}{2}x\right]t^4 + O(t^5),$$

and so on. (186)

Applying Modified Variational iteration method using formula Eq. (181) with

$U_{-1} = 0,$

And $G_n(x,t)$ is calculated from the relation

$$-6U_n(x,t)\big(U_n(x,t)\big)_x = G_n(x,t) + O(t^{n+1}), \quad (187)$$

The following MVIM results are obtained

$$U_0(x,t) = \frac{-k^2}{2} sech^2\left[\frac{k}{2}x\right]$$
$$U_1(x,t) = U_0(x,t) - \frac{k^5}{2} sech^2\left[\frac{k}{2}x\right] Tanh\left[\frac{k}{2}x\right]$$
$$U_2(x,t) = U_1(x,t) - \frac{k^8}{8} sech^4\left[\frac{k}{2}x\right](2 - cosh[kx])t^2$$
$$U_3(x,t) = U_2(x,t) + \frac{k^{11}}{48} sech^5\left[\frac{k}{2}x\right]\left(11 sinh\left[\frac{k}{2}x\right] - sinh\left[\frac{3k}{2}x\right]\right)t^3$$

and so on. (188)

This solution is convergent to the exact solution

$$u(x,t) = \frac{-k^2}{2} sech^2\left[\frac{k}{2}(x - k^2 t)\right] \quad (189)$$

4.2.3 Basic difference between Modified Variational iteration method and Variational iteration method

MDVI helps to overcome some of dis advantages of VIM. MVIM eliminates all the unneeded terms and the repeated computations in VIM. MVIM is powerful in saving time and calculations. MVIM is often useful to engineering and non-specialists and others to have an approximate closed form solution to describe the nonlinear problems. MVIM can deal with highly nonlinear differential equations with no need to small parameter or linearization. The solution procedure is very simple by means of variational iteration theory, and few iterations lead to high accurate solutions. More precisely Variational Iteration method (VIM) has some repeated computations and the calculation of unneeded terms but by using Modified Variational iteration method (MVIM) we overcome the disadvantages of VIM, which stops the repeats of the old computations and eliminate the unneeded terms.

CONCLUSION

In this paper we discuss the elementary introduction of the method VIM, algorithms etc. also here we have studied few problems with this this method and we conclude that this method does not required small parameters in equation as the perturbation technique do. The main part of the method is the construction of correction functional which can easily constructed by the Lagrange multiplier, and the multiplier can optimally identified by variational theory. The application of restricted variations in correction functional makes it much easier to determine. In this method the initial approximation can be freely selected with unknown constants, which can be determined via various methods which can be easily calculated. The approximations obtained by this method are valid not only for small parameter, but also for very large parameter this is the most important advanture of this method. Furthermore their first-order approximations are of extreme accuracy. Comparison of this method with Adomian's method reveals that the approximations obtained by the proposed method converge to its exact solution faster than those of Adomian's. Hence we finally concluded that this is one of the best method for solving higher order nonlinear problems.

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