A Study on Zografos-Balakrishnan G-family of Distributions

Thesis submitted in partial fulfillment of the requirements for the degree of

Integrated Master of Science

by

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May 2015

Certificate

This is to certify that the thesis entitled "A Study on Zografos-Balakrishnan G-family of Distributions " which is being submitted by **Prashant Tiwari** in the Department of Mathematics, National Institute of Technology, Rourkela, in partial fulfilment for the award of the degree of **Integrated Master of Science**, is a record of bonafide review work carried out by him in the Department of Mathematics under my guidance. He has worked as a project student in this Institute for one year. In my opinion the work has reached the standard, fulfilling the requirements of the regulations related to the Integrated Master of Science degree. The results contained in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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Acknowledgement

It is a great pleasure and proud privilege to express my deep sense of gratitude to my guide Dr. M. R. Tripathy. I am grateful to him for his continuous encouragement and guidance throughout the period of my project work. Without his active guidance it would not have been possible for me to complete the work.

I acknowledge my deep sense of gratitude to the Head of the Department, all faculty members, all non-teaching staff members of the department of mathematics and the authorities of NIT Rourkela.

I also thank to all my friends for their support, co-operation and sincere help in various ways. I express my sincere thanks with gratitude to my parents and other family members, for their support, blessings and encouragement. Finally, I bow down before the almighty who has made everything possible.

Place: NIT Rourkela

Date: May 2015

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Abstract

For any continuous baseline G distribution, Zografos and Balakrishnan (2009) proposed generalized gamma generated G-family of Distributions with an extra positive parameter. Nadarajah and Cordiero (2015) proposed various mathematical properties of such distributions. Lee and Famaye (2014) proposed the T-X family of distributions. As a part of these two families we propose two new gamma generated distributions such as gamma-Gumbel and gamma-normal distribution. Applications are stated as to why we need these distributions and some properties of these distributions have been derived. We have found out various properties of these distributions such as probability density function (pdf), Cumulative distribution function (cdf), survival function and hazard rate function. Expansions of pdf and cdf, asymptotes, quantile function, moment, extreme values, reliability, mean, median, mode, moment generating function, order statistics, Renyi and Shannon Entropy, bivariate generalizations and Maximum likilihood estimator (MLE). We also propose some theorems pertaining to moment of these distributions.

Contents

Chapter 1

Introduction

1.1 Introduction and Literature Review

Statistical distributions are used to describe many real world phenomenon and have various applications. It varies from studying basic things like environmental pollution to as complex as strength of carbon fibers. Even from study of cancer data to diesel engines and geyser eruption. Everywhere we find a glimpse of "How useful statistical distributions can be ". Due to such vivid usage of statistical distributions, it's theory is studied and attempts are made to develop new distributions. But, why we need to develop such distributions? The answer is simple, we want our distributions to be as flexible as possible.

A lot of attempts have been made to develop new distributions. As a consequence, many generalized classes of distributions have been developed such as Weibull, gamma, beta, exponentiated etc. These distributions are then applied to describe various phenomenon in nature. These applications have been discussed by various authors in the past. The common feature of these distributions is that they have more parameters (Normal has 2-parameters, then gamma-normal distribution can have 3 or 4 parameters depending on the nature of gamma Distribution used to generate them that is one parameter or two parameter). Johnson et al. (1994) proposed that use of four-parameter distributions should be sufficient to account for most of the practical problems. According to them, at least three parameters are needed. Any noticeable improvement arising after including a fifth or sixth parameter was discarded. Improvement is not commensurate with the extra labor involved.

There are some well-known methods for generating continuous univariate distributions. These methods include: method of differential equation by Pearson (1895), method of translation by Johnson (1949) and method of quantile functions by Tukey (1960). Pearson (1895) designed a system in which every member the probability density function $p(x)$ satisfies a differential equation. In Pearson system of continuous distributions (1895) every probability density function (p.d.f.) $f(x)$ satisfies

$$\frac{1}{f(x)} \frac{df(x)}{dx} = \frac{a+x}{a_0 + a_1 x + a_2 x^2}, \tag{1.1}$$

where a, a_0, a_1, and a_2 are parameters, refer Johnson et al. (1994). The shape of $f(x)$ is dependent on the parameter values. Pearson classified distributions into a number of types depending on their shapes. Here different types correspond to different solutions of equation (1.1). The form of solution of (1.1) is dependent on the nature of roots of the equation. $a_0 + a_1 x + a_2 x^2 = 0$. For example, when $a_1 = a_2 = 0$, then it corresponds the normal distribution which is not assigned to any particular type. In fact it is a limiting distribution of all type of distributions. Detailed discussion of various type of distribution is provided by the authors, refer Johnson et al. (1994).

Burr (1942) proposed a system of continuous distributions which can take different type of shapes. These distributions from the system satisfy the differential equation given by

$$dF = F(1-F)g(x)dx \tag{1.2}$$

where $0 \leq$ F ≤ 1 and $g(x)$ is a non-negative function over x. Now corresponding to the ways of solving the above equation, Burr (1942) gave 12 solutions to the equation in (1.2) and these solutions then correspond to different choices of $g(x)$. Further refer Fry (1993) and Johnson et al. (1994) for detailed discussion about Burr Types I-XII distributions.

Johnson (1949) used method of translation and proposed a system to generate distributions by use of normalization transformation. It has the general form given by

$$Z = \gamma + \delta g\left(\frac{x-\xi}{\lambda}\right). \tag{1.3}$$

Here $g(.)$ is the transformation function used, and Z is a standardized normal random variable, where γ and δ are shape parameters, λ is a scale parameter and ξ is the location parameter of the distribution. Then without loss of generality it can be assumed that δ and λ are positive. By use of above, Johnson proposed three different transformation functions. He proposed a different family corresponding to each of the transformation. As a consequence he then defined the lognormal family of distributions, the family of bounded system of distributions and the family of unbounded system of distributions. Most of the basic distributions known to us such as normal, log-normal, gamma, beta, exponential distributions, and many others are a part of one of these families of distributions. For further queries refer Johnson et al. (1994).

Tukey (1960) proposed the lambda distribution, which was later generalized by Ramberg and Schmeiser (1972,1974) and Ramberg (1979) as class of generalized lambda distributions (GLD). Now this generalized lambda distributions (GLD) is defined in terms of quantile function as

$$Q(y) = Q(y;\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \lambda_1 + \frac{y^{\lambda_3}-(1-y)^{\lambda_4}}{\lambda_2}, \quad \text{where} \quad 0 \le y \le 1. \tag{1.4}$$

Here we have parameters as λ_1 and λ_2, which are location and scale parameters respectively. We also have λ_3 and λ_4 which determine the skewness and kurtosis of the distribution. Then the pdf corresponding to that is given by

$$f(x) = \frac{\lambda_2}{\lambda_3 y^{\lambda_3-1} + \lambda_4(1-y)^{\lambda_4-1}}, \quad \text{with} \quad x = Q(y). \tag{1.5}$$

For existence of a valid pdf we need to have the sign of $\lambda_3 y^{\lambda_3-1} + \lambda_4(1-y)^{\lambda_4-1}$ to be same for all y in [0, 1] and that λ_2 takes the same sign throughout. Freimer (1988) has made the comparison between the two systems, that is he noted the similarity and differences between the Pearsons system and generalized lambda distributions. During the comparison it was pointed out that Pearsons family does not include logistic distribution family. Also generalized lambda distributions does not include all the skewness and kurtosis values. To overcome these shortcomings an extended generalized lambda distributions was proposed by Karian and Dudewicz (2000) which contains both generalized lambda distributions and generalized beta distribution and it is defined as

$$f(x) = \begin{cases} \frac{(x-\beta_1)_3^{\beta}(\beta_1+\beta_2-x)^{\beta_4}}{B(\beta_3+1,\beta_4+1)\beta_2^{(\beta_3+\beta_4+1)}}, & \text{for} \quad \beta_1 \leq x \leq \beta_1+\beta_2 \\ 0 \quad \text{otherwise}, \end{cases} \tag{1.6}$$

where B(.,.) is the complete beta function. A detailed discussion about the generalized lambda distributions and extended generalized lambda distributions is provided in, Karian and Dudewicz (2000)

Azzalini (1985) proposed the skew normal family of distributions. If X and Y are independent random variables, where it's p.d.f. is symmetric about zero. Then, for any λ we get

$$0.5 = P(X - \lambda Y < 0) = \int_{-\infty}^{\infty} \mathrm{g_y(Y)G_x(\lambda y)}\, \mathrm{d}x, \tag{1.7}$$

where $2g_y(Y)G_x(\lambda y)$ is a probability density function. Now, let us assume, X and Y are standard normal that is then pdf of the skew-normal family of distributions is given as

$$2\varphi(x)\Phi(\lambda x). \tag{1.8}$$

Here $\phi(x)$ and $\Phi(x)$ are pdf and cdf of standard normal distribution respectively. The distribution is characterized by a single parameter λ. We can also add Location and scale parameters by using the translation $Y = \mu + \sigma X$. Detailed description of the above distribution can be found in Johnson et al. (1994).

Later Eugene et al. (2002) used the well-known beta distribution as a generator, and developed family of beta-generated distributions. Then cdf of beta-generated random variable X is given by

$$G(x) = \int_0^{F(x)} \mathrm{b(t)}\, \mathrm{d}t. \tag{1.9}$$

In the above expression b(t) is the pdf of the beta random variable(RV) and F(x) is the cdf of any RV. Then the density function of the beta-generated distribution is given by

$$g(x) = \frac{1}{B(\alpha,\beta)} f(x)F^{\alpha-1}(x)(1-F(x))^{\beta-1}. \tag{1.10}$$

This family is a generalization of distributions of order statistics for RV X with c.d.f. $F(x)$ as in Eugene and Jones (2002). After that many beta-generated distributions have been studied such as beta-Gumbel distribution by Nadarajah and Kotz (2004), beta-exponential distribution by Nadarajah and Kotz (2005), beta-Weibull distribution by Famoye (2005), beta-gamma by Kong (2007),and beta-Pareto (2008) by Akinsete.

Extention of the beta-generated family of distributions was done by using beta distribution in place of the Kumaraswamy distribution. Then we get, pdf of the Kumaraswamy generalized distributions as

$$g(x) = \alpha\beta f(x)F^{\alpha-1}(x)(1 - F^{\alpha}(x))^{\beta-1}. \tag{1.11}$$

Several generalized distributions have been studied in the literature including the Beta generalized distributions by Eugene (2002), Beta and generalized-gamma generated distributions by Cordiero et al. (2009), Ferreira and Steel(2006) introduced a method to generate skewed distributions through inverse probability integral transformations. Properties of gamma family has been studied by Cordiero et al.(2015).

This article proposes two new distributions in statistics such as gamma-Gumbel and Zorafos-Balakrishnan(ZB) Normal Distributions. The thesis is organized as follows : Chapter 2 proposes Gamma-Gumbel distribution and it's properties. Chapter 3 proposes ZB-Normal distribution and it's properties. Finally the thesis ends with conclusion and scope for future work in chapter 4.

1.2 Some basic results

In this section we provide somebasic properties of Zografos-Balakrishnan Family of Distributions. We state the formulae for the density function, distribution function, hazard function and survival function is given in this section. Further we discuss the formula for quantile and moment in this section. The pdf and cdf for ZB-G family is given by

$$\begin{aligned} f(x) &= \frac{1}{\Gamma(a)}\{-\log[1-G(x)]\}^{a-1}g(x) \\ F(x) &= \frac{\gamma(a,-\log[1-G(x)])}{\Gamma(a)} = \int\limits_0^{-\log[1-G(x)]} t^{a-1}\exp(-t)dt \end{aligned} \tag{1.12}$$

The hazard function is given by

$$h(x) = \frac{f_{ggu}(x)}{1-F_{ggu}(x)} = \frac{f_{ggu}(x)}{S_{ggu}(x)} \tag{1.13}$$

Quantile function is then given by

$$F^{-1}(u) = G^{-1}\left\{1-\exp\left(-Q^{-1}(a,1-u)\right)\right\} \qquad for, 0<u<1 \tag{1.14}$$

where,

$$Q^{-1}(u) = 1-\frac{\gamma(a,x)}{\Gamma(a)}$$

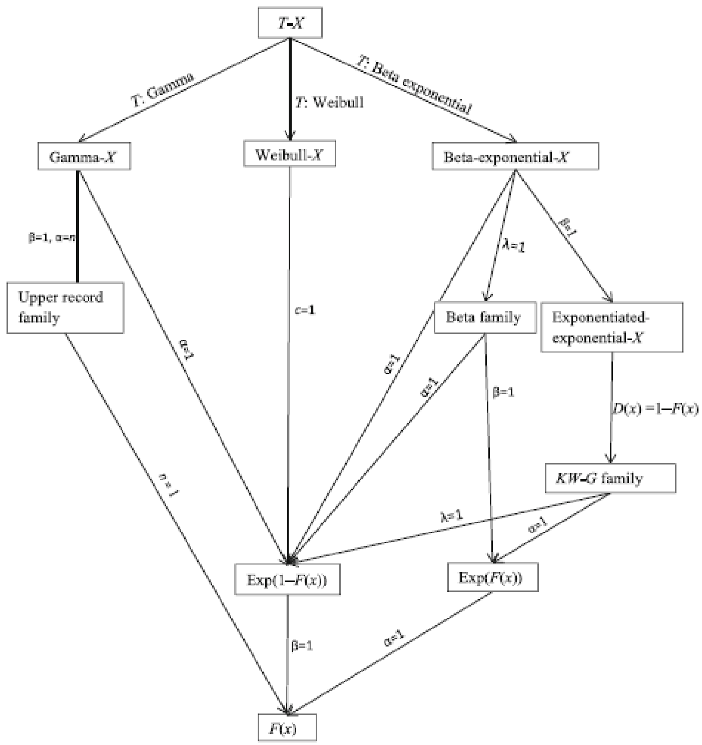
The expectation of Gamma family is given as

$$E\left(X^n\right) = \sum_{k=0}^{\infty}(a+k)\,\Gamma(na+k) \tag{1.15}$$

where,

$$\begin{aligned} \Gamma(n,a) &= \int_{-\infty}^{+\infty} x^n G(x)^n g(x)dx \\ \Gamma(n,a) &= \int_0^1 Q_G(u)^n u^a du \end{aligned} \tag{1.16}$$

The T-X Family of distribution is given by the following tree diagram. We can see the requisite relation between the family of distributions and how they can be converted from one to the other.



Chapter 2

Gamma-Gumbel Distribution

The Gumbel distribution is considered to be most widely applied distribution for data related to climate. Kotz and Nadarajah (2000), described Gumbel distribution to have over 50 applications, ranging from accelerated life testing through to earthquakes, floods, sea currents, wind speedshorse racing, rainfall, queues in supermarkets and track race records. Nadarajah (2006) proposed generalization of the Gumbel distribution with the hope that it will attract wider applicability in climate modeling.In this section we propose Gamma-Gumbel distribution and discuss it's various properties. It should be noted that Gamma-Gumbel will always give a better fit of data than that of corresponding parent distribution.

2.1 Gamma-Gumbel Distribution

The pdf and cdf of Gumbel distribution with location parameter $\mu \in \mathbb{R}$ and scale parameter $\sigma > 0$, is given by

$$g(x) = \frac{1}{\sigma} \exp\left\{ \left(\frac{x-\mu}{\sigma} \right) - \exp\left(\frac{x-\mu}{\sigma} \right) \right\}. \tag{2.1}$$

$$G(x) = 1 - \exp\left\{ - \exp\left(\frac{x-\mu}{\sigma} \right) \right\}. \tag{2.2}$$

Next we discuss the properties of gamma-Gumbel distribution. The pdf, cdf, Hazard and survival function of Gamma-Gumbel will be given as

Probability density function(pdf) :

$$\begin{aligned} f_{GGu}(x) &= \frac{1}{\Gamma(a)}\left\{-log\left[1-G(x)\right]^{a-1}\right\}g(x) \\ &= \frac{1}{\Gamma(a)}\left\{-log\left(exp\left\{-exp\left(\frac{x-\mu}{\sigma}\right)\right\}\right)\right\}^{a-1}g(x) \\ &= \frac{1}{\Gamma(a)}\left(exp\left(\frac{x-\mu}{\sigma}\right)\right)^{a-1}\frac{1}{\sigma}exp\left\{\left(\frac{x-\mu}{\sigma}\right)-exp\left(\frac{x-\mu}{\sigma}\right)\right\} \\ &= \frac{1}{\sigma\Gamma(a)}\left\{exp\left\{a\left(\frac{x-\mu}{\sigma}\right)-exp\left(\frac{x-\mu}{\sigma}\right)\right\}\right\} \end{aligned} \tag{2.3}$$

Cumulative Distribution Function (cdf):

$$\begin{aligned} F_{GGu} &= \frac{1}{\Gamma(a)}\int_0^{\left(exp\left(\frac{x-\mu}{\sigma}\right)\right)^{a-1}} \mathrm{t^{a-1}exp\left(-t\right)d}t \\ &= \frac{\gamma\left(a, exp\left(\frac{x-\mu}{\sigma}\right)\right)^{a-1}}{\Gamma(a)} \\ &= 1-\frac{\Gamma\left(a, exp\left(\frac{x-\mu}{\sigma}\right)\right)^{a-1}}{\Gamma(a)}. \end{aligned} \tag{2.4}$$

Survival Function :

$$\begin{aligned} S_{GGu}(x) &= 1-F_{GGu}(x) \\ &= \frac{\Gamma\left(a, exp\left(\frac{x-\mu}{\sigma}\right)\right)^{a-1}}{\Gamma(a)}. \end{aligned} \tag{2.5}$$

$$\begin{aligned} s_{GGu}(x) &= -f_{GGu}(x) \\ &= -\frac{1}{\sigma\Gamma(a)}\left\{exp\left\{a\left(\frac{x-\mu}{\sigma}\right)-exp\left(\frac{x-\mu}{\sigma}\right)\right\}\right\}. \end{aligned} \tag{2.6}$$

Hazard Function :

$$h_{GGu}(x) = \frac{f_{GGu}(x)}{S_{GGu}(x)} = \frac{\frac{1}{\sigma}\left\{exp\left\{a\left(\frac{x-\mu}{\sigma}\right)-exp\left(\frac{x-\mu}{\sigma}\right)\right\}\right\}}{\Gamma\left(a, exp\left(\frac{x-\mu}{\sigma}\right)\right)^{a-1}}. \tag{2.7}$$

Asymtote :

For f_{GGu} as $x \longrightarrow \pm\infty$

$$\begin{aligned} f_{GGu}(x) &\sim \frac{G^{a-1}(x)g(x)}{\Gamma(a)} \\ &\sim \frac{1}{\sigma\Gamma(a)}\left\{\left\{1-exp\left\{-exp\left(\frac{x-\mu}{\sigma}\right)\right\}\right\}^{a-1} exp\left\{\left(\frac{x-\mu}{\sigma}\right)-exp\left(\frac{x-\mu}{\sigma}\right)\right\}\right\} \end{aligned} \tag{2.8}$$

For F_{GGu}

As $x \longrightarrow -\infty$

$$F_{GGu}(x) \sim \frac{1}{\Gamma(a+1)}\left\{-log\left[1-G(x)\right]^{a}\right\} \sim \frac{1}{\Gamma(a+1)}\left\{exp\left(\frac{x-\mu}{\sigma}\right)\right\}^{a} \tag{2.9}$$

As $x \longrightarrow +\infty$

$$1-F_{GGu}(x) \sim \frac{1}{\Gamma(a)}\left\{\left(exp\left(\frac{x-\mu}{\sigma}\right)\right)^{a-1} exp\left\{-exp\left(\frac{x-\mu}{\sigma}\right)\right\}\right\} \tag{2.10}$$

For h_{GGu}

As $x \longrightarrow -\infty$

$$\begin{aligned} h_{GGu}(x) &\sim \frac{G^{a-1}(x)g(x)}{\Gamma(a)} \\ &\sim \frac{1}{\sigma\Gamma(a)}\left\{\left\{1-exp\left\{-exp\left(\frac{x-\mu}{\sigma}\right)\right\}\right\}^{a-1} exp\left\{\left(\frac{x-\mu}{\sigma}\right)-exp\left(\frac{x-\mu}{\sigma}\right)\right\}\right\} \end{aligned} \tag{2.11}$$

As $x \longrightarrow -\infty$

$$h_{GGu}(x) \sim \frac{g(x)}{1-G(x)} \sim \frac{1}{\sigma}exp\left(\frac{x-\mu}{\sigma}\right) \tag{2.12}$$

Remark 2.1 *The asymptote for Hazard rate when $x \longrightarrow -\infty$ is similar to the exponential distribution.*

2.2 Moment of Gamma-Gumbel

The expectation of Gamma family is given as :

$$E(X^n) = \sum_{k=0}^{\infty} (a+k)\Gamma(na+k). \tag{2.13}$$

where,

$$\begin{aligned} \Gamma(n,a) &= \int_{-\infty}^{+\infty} x^n G(x)^n g(x) dx \\ \Gamma(n,a) &= \int_0^1 Q_G(u)^n u^a du. \end{aligned} \tag{2.14}$$

Now, inverse of Gamma-Gumbel can be determine as :

$$\begin{aligned} F_{GGu} &= 1 - exp\left\{-exp\left(\frac{\mu - y}{\sigma}\right)\right\} \\ y &= 1 - exp\left\{-exp\left(\frac{x - y}{\sigma}\right)\right\} \\ (1-y) &= exp\left\{-exp\left(\frac{x - y}{\sigma}\right)\right\} \\ log(1-y) &= -exp\left(\frac{x - y}{\sigma}\right). \end{aligned} \tag{2.15}$$

Thus we get the inverse as:

$$x = \left\{\frac{\mu}{\sigma} + loglog\left(\frac{1}{1-y}\right)\right\}. \tag{2.16}$$

Therefore we get the Quantile as :

$$Q_G(U) = \left\{\frac{\mu}{\sigma} + loglog\left(\frac{1}{1-y}\right)\right\}. \tag{2.17}$$

Substituting it in 2.14 we get

$$\Gamma(n,a) = \int_0^1 \left\{\frac{\mu}{\sigma} + loglog\left(\frac{1}{1-y}\right)\right\}^n u^a du. \tag{2.18}$$

by use of equation 2.18 can be reduced to

$$\Gamma(n,a) = \left(\frac{\mu}{\sigma}\right)^n \int_0^1 \left\{1 + loglog\left(\frac{1}{1-y}\right)^{\frac{\sigma}{\mu}}\right\}^n u^a du \tag{2.19}$$

$$\Gamma(n,a) = \left(\frac{\mu}{\sigma}\right)^n \int_0^1 \left\{1 + loglog\left(\frac{1}{1-y}\right)^{\frac{\sigma}{\mu}}\right\}^n u^a du \tag{2.20}$$

Now use series expansion of

$$\left\{1 + loglog\left(\frac{1}{1-y}\right)^{\frac{\sigma}{\mu}}\right\}^n$$

. Then we know,

$$\left\{-log[1-G(x)]^{a-1}\right\} = (a-1)\sum_{k=0}^{\infty}\binom{k+1-a}{k}\sum_{j=0}^{k}\frac{(-1)^{j+k}\binom{k}{j}p_{j,k}}{a-1-j}G(x)^{a+k-1}$$

Then after using expansion of log x, the expression for moment reduces to :

$$\Gamma(n,a) = \left(\frac{\mu}{\sigma}\right)^n \int_0^1 \left\{(a-1)\sum_{k=0}^{\infty}\binom{k+1-a}{k}A_k u^a\right\} du \tag{2.21}$$

where A_k is given by

$$\sum_{j=0}^{k}\frac{(-1)^{j+k}\binom{k}{j}p_{j,k}}{a-1-j}\left(\log\log\left(\frac{1}{1-y}\right)^{\frac{\sigma}{\mu}}\right)^{a+k-1}$$

and the constants $p_{j,k}$ can be calculated recursively by

$$p_{j,k} = k^{-1}\sum_{m=1}^{k}\left[k - m(j+1)\right] c_m p_{j,k-m} \tag{2.22}$$

for k = 1, 2, ... with $p_{j,0} = 1$ and $c_k = (-1)^{k+1}(k+1)^{-1}$.

We can also find the expectation by use of exponentiated-Gumbel distribution. Then the central moment of the distribution is given by the formula

$$E(X^n) = \sum_{i=1}^{n} b_k Y_k^n$$

. Where Y_k^n are the Exponential Gumbel Distribution with parameter a + k, now we find the expectation of $E(Y^n)$ where Y is Exponentiated Gumbel. Then by use of central moments described by Nadarajah (2006) expectation of exponentiated-Gumbel is given as

$$E(Y^n) = \alpha\mu^n \sum_{k=0}^{n}\sum_{l=0}^{n} \frac{(-1)^{k+l}\Gamma(\alpha)\Gamma(n)}{k!l!\Gamma(\alpha-1)\Gamma(n-k+1)} \left(\frac{\sigma}{\mu}\right)^k \left(\frac{\partial}{\partial a}\right)^k (l+1)^{-a}\Gamma(a)\,|_{a=1} \quad (2.23)$$

Now we substitute this value in

$$E(X^n) = \sum_{i=1}^{n} b_k Y_k^n$$

Then we get, Expectation as:

$$E(X^n) = \sum_{i=1}^{n} b_k \alpha\mu^n \sum_{k=0}^{n}\sum_{l=0}^{n} \frac{(-1)^{k+l}\Gamma(\alpha)\Gamma(n)}{k!l!\Gamma(\alpha-1)\Gamma(n-k+1)} \left(\frac{\sigma}{\mu}\right)^k \left(\frac{\partial}{\partial a}\right)^k (l+1)^{-a}\Gamma(a)\,|_{a=1} \quad (2.24)$$

Theorem 2.1 *The mode of Gamma Gumbel distribution is given by*

$$x = \mu + \sigma loga$$

Proof 2.1

$$
\begin{aligned}
f_{GGu}(x) &= \frac{1}{\Gamma(a)}\left\{-log\left[1-G(x)\right]^{a-1}\right\} g(x)\\
g(x) &= \frac{1}{\sigma}exp\left\{\left(\frac{x-\mu}{\sigma}\right)-exp\left(\frac{x-\mu}{\sigma}\right)\right\}\\
f_{GGu}(x) &= \frac{1}{\Gamma(a)}\left\{-log\left(exp\left\{-exp\left(\frac{x-\mu}{\sigma}\right)\right\}\right)\right\}^{a-1} g(x)\\
&= \frac{1}{\Gamma(a)}\left(exp\left(\frac{x-\mu}{\sigma}\right)\right)^{a-1}\frac{1}{\sigma}exp\left\{\left(\frac{x-\mu}{\sigma}\right)-exp\left(\frac{x-\mu}{\sigma}\right)\right\}\\
&= \frac{1}{\sigma\Gamma(a)}\left(exp\left(\frac{x-\mu}{\sigma}\right)\right)^{a-1} exp\left\{\left(\frac{x-\mu}{\sigma}\right)-exp\left(\frac{x-\mu}{\sigma}\right)\right\}\\
&= \frac{1}{\sigma\Gamma(a)}exp\left\{a\left(\frac{x-\mu}{\sigma}\right)--exp\left(\frac{x-\mu}{\sigma}\right)\right\}
\end{aligned}
\tag{2.25}
$$

$$
\begin{aligned}
\frac{a}{\sigma}-e^{\frac{x-\mu}{\sigma}}(\frac{1}{\sigma}) &= 0\\
\frac{a-e^{\frac{x-\mu}{\sigma}}}{\sigma} &= 0
\end{aligned}
\tag{2.26}
$$

$$x=\mu+\sigma\log a \tag{2.27}$$

2.3 Expansion of CDF and PDF

Some useful expansions for p.d.f and c.d.f of Gamma-gumbel can be derived by use of exponentiated distributions. If we have an arbitrary baseline c.d.f G(x), a RV is said to have the exponentiated-G distribution with parameter a > 0, say X $\sim$ exp-G(a), if its pdf and cdf are given by the equations:

$$f_a^*(x) = aG^{a-1}(x)g(x) \tag{2.28}$$

and

$$F_a^*(x) = G^a(x), \tag{2.29}$$

respectively.

We note that for a > 1 and a < 1 and for larger values of x, the multiplicative factor $aG(x)^{a-1}$ is greater and smaller than one, respectively. The reverse assertion is also true for smaller values of x. The latter immediately implies that the ordinary moments associated with the pdf $f_a^*(x)$ are strictly larger (smaller) than those associated with the pdf $g(x)$ when a > 1 (a < 1).

The binomial coefficient generalized to real arguments is given by $\binom{x}{y} = \Gamma(x+1)/\{\Gamma(y+1)\Gamma(x-y+1)\}$

For any real parameter a > 0, the following formula holds (http:// functions. wolfram.com/ ElementaryFunctions/ Log/ 06/ 01/ 04/ 03/)

$$\left\{-log\,[1-G(x)]^{a-1}\right\} = (a-1)\sum_{k=0}^{\infty}\binom{k+1-a}{k}\sum_{j=0}^{k}\frac{(-1)^{j+k}\binom{k}{j}p_{j,k}}{a-1-j}G(x)^{a+k-1} \qquad (2.30)$$

where the constants $p_{j,k}$ can be calculated recursively by

$$p_{j,k} = k^{-1}\sum_{m=1}^{k}\left[k-m(j+1)\right]c_m p_{j,k-m} \qquad (2.31)$$

for k = 1, 2, ... with $p_{j,0} = 1$ and $c_k = (-1)^{k+1}(k+1)^{-1}$.
For a real parameter a > 0, we define

$$b_k = \frac{\binom{k+1-a}{k}}{(a+k)\Gamma(a-1)}\sum_{k=0}^{\infty}\binom{k+1-a}{k}\sum_{j=0}^{k}\frac{(-1)^{j+k}\binom{k}{j}p_{j,k}}{a-1-j} \qquad (2.32)$$

and then (1) can be expressed as

$$f(x) = \sum_{k=0}^{\infty} b_k f_{a+k}^*(x) \qquad (2.33)$$

where $f_{a+k}^*(x)$ denotes the pdf of the exp-G(a + k) distribution. The corresponding (2) can be expressed as

$$F(x) = \sum_{k=0}^{\infty} b_k F_{a+k}^*(x) \qquad (2.34)$$

where $F^*_{a+k}(x)$ denotes the cdf of the exp-G(a + k) distribution. So, several properties of the Zografos-Balakrishnan-G distribution can be obtained by knowing those of the exp-G distribution, see, for example, Mudholkar et al. (1995), Gupta and Kundu (1999), Nadarajah and Kotz (2006), among others.

2.4 Reliability function

In this section, we derive the reliability for the gamma-gumbel distribution by making use of reliability of exponentiated-gumbel distribution. Now, reliability is defined as: R = $\Pr(X_2 < X_1)$, when $X_1 \sim$ Zografos-Balakrishnan-G(a_1) and $X_2 \sim$ Zografos-Balakrishnan-G(a_2) are independent random variables. Such Probabilities have many applications especially in engineering.

Theorem 2.2 (Reliability function) *: The reliability R of Z-B Gumbel distribution is then given by:*

$$R = \sum_{j,k}^{\infty} c_{jk} R_{jk} \tag{2.35}$$

where,

$$c_{jk} = \frac{\binom{k+1-a_1}{k}}{(a_1+k)\Gamma(a_1-1)} \frac{\binom{j+1-a_2}{k}}{(a_2+k)\Gamma(a_2-1)} \left[\sum_{i=0}^{k} \frac{(-1)^{i+k}\binom{k}{i} p_{i,k}}{a_1-1-i} \right] \left[\sum_{i=0}^{j} \frac{(-1)^{i+j}\binom{j}{i} p_{i,j}}{a_2-1-i} \right]$$

and

$$R_{jk} = Pr(Y_j < Y_k)$$

is the reliability between exponentiated-G distributions with parameters a+k and a+j and when $a_1 = a_2$ then R = 0.5.

Proof 2.2 *Let f_i denote the pdf of X_i and F_i denote the cdf of X_i. By use of equations, (3.27) and (3.28), we can then write*

$$R = \sum_{j,k=0}^{\infty} c_{jk} \int_{-\infty}^{\infty} F^{*}_{a_2+j}(x) f^{*}_{a_1+k}(x) dx = \sum_{j,k=0}^{\infty} c_{jk} R_{jk} \tag{2.36}$$

where

$$c_{jk} = \frac{\binom{k+1-a_1}{k}}{(a_1+k)\Gamma(a_1-1)} \frac{\binom{j+1-a_2}{k}}{(a_2+k)\Gamma(a_2-1)} \left[\sum_{i=0}^{k} \frac{(-1)^{i+k}\binom{k}{i} p_{i,k}}{a_1-1-i}\right] \left[\sum_{i=0}^{j} \frac{(-1)^{i+j}\binom{j}{i} p_{i,j}}{a_2-1-i}\right]$$

and $R_{jk} = Pr(Y_j < Y_k)$ *is the reliability between the independent random variables* $Y_j \sim exp{-}G(a_2+j)$ *and* $Y_k \sim exp{-}G(a_1+k)$. *Hence, the reliability for Zografos-Balakrishnan-G random variables is a linear combination of those for exp-G random variables. In the particular case when* $a_1 = a_2$, *then equation (3.30) reduces to* $R = 0.5$.

Chapter 3

Zogarafos-Balakrishnan Normal Distribution

3.1 Introduction

Normal distribution is the most commonly used probability distribution. It was first used by deMoivre (1733) as an approximation of the binomial distribution. The main credit for its development goes to Gauss (1809, 1816). Properties of the normal distribution have been well developed refer Johnson et al. (1994); Patel and Read (1996). Azzalini (1985) developed generalized normal distributions. After that a lot of distributions have been discovered such as skew normal family, half normal distributions, beta normal distribution, gamma- normal distribution. We here propose Zografos-Balakrishnan(ZB)-normal distribution and discuss it's properties.

ZB-G family has pdf and cdf given by

$$\begin{aligned} f(x) &= \frac{1}{\Gamma(a)}\{-\log[1-G(x)]\}^{a-1}g(x) \\ F(x) &= \frac{\gamma(a,-\log[1-G(x)])}{\Gamma(a)} = \int_{0}^{-\log[1-G(x)]} t^{a-1}\exp(-t)dt \end{aligned} \tag{3.1}$$

Now for normal distribution,

$$\begin{aligned} g(x) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} = \phi(x) \\ G(x) &= \Phi(x) \end{aligned} \tag{3.2}$$

Then we get pdf and cdf of Zografos-Balakrishnan normal Distribution as

$$\begin{aligned} f(x) &= \frac{1}{\Gamma(a)}\{-\log[1-\Phi(x)]\}^{a-1}\phi(x) \\ F(x) &= \frac{\gamma(a, -\log[1-\Phi(x)])}{\Gamma(a)} = \int\limits_{0}^{-\log[1-\Phi(x)]} t^{a-1}\exp(-t)dt \qquad \alpha > 0, \sigma > 0, -\infty < \mu < \infty \end{aligned} \tag{3.3}$$

The hazard function is given by :

$$h(x) = \frac{f_{ggu}(x)}{1 - F_{ggu}(x)} = \frac{f_{ggu}(x)}{S_{ggu}(x)} \tag{3.4}$$

Then we get hazard function as:

$$h(x) = \frac{\{-log[1-\Phi(x)]\}^{a-1}\phi(x)}{\Gamma(a, -log[1-\Phi(x)])} \tag{3.5}$$

Quantile function is then given as:

$$F^{-1}(u) = G^{-1}\left\{1 - \exp\left(-Q^{-1}(a, 1-u)\right)\right\} \qquad for, 0 < u < 1 \tag{3.6}$$

where,

$$Q^{-1}(u) = 1 - \frac{\gamma(a, x)}{\Gamma(a)}$$

Now, we find the inverse of normal distribution as:

$$x = \mu + \sigma\sqrt{2}erf^{-1}(2y - 1) \tag{3.7}$$

3.2 Properties of ZB-Normal

Lemma 3.1 *The quantile function for ZB-Normal distribution is given as:*

$$F^{-1}(u) = \mu + \sigma\sqrt{2}erf^{-1}\left(2\left(1 - \exp\left(-Q^{-1}\left(a, 1-u\right)\right)\right) - 1\right) \tag{3.8}$$

Proof : Proof is obvious by use of inverse of normal distribution and quantile function.

Theorem 3.1 *The mode of the ZB-normal distribution is given as:*

$$x = \mu + \sigma^2 h_N(x)\left\{\frac{a-1}{H_N(x)}\right\} \tag{3.9}$$

where,

$$h_N(x) = \frac{\phi(x)}{[1-\Phi(x)]}$$
$$H_N(x) = -\log\left[1-\Phi(x)\right]$$

Proof 3.1 *We have density function of ZB-Normal as:*

$$f(x) = \frac{1}{\Gamma(a)}\{-\log\left[1-\Phi(x)\right]\}^{a-1}\phi(x) \tag{3.10}$$

then,

$$\begin{aligned} f^{'}(x) &= \frac{a-1}{\Gamma(a)}\frac{\left(-log\left[1-\Phi(x)\right]\right)^{a-2}}{\left[1-\Phi(x)\right]}\left(-\phi(x)\right) + \frac{\phi^{'}(x)}{\Gamma(a)}\left(-log\left[1-\Phi(x)\right]\right)^{a-1} \\ &= \frac{\left(-log\left[1-\Phi(x)\right]\right)^{a-2}}{\Gamma(a)}\left\{\left(-\phi(x)\right) + \phi^{'}(x)\left(-log\left[1-\Phi(x)\right]\right)\right\} \end{aligned} \tag{3.11}$$

Now,

$$\frac{1}{\sigma\sqrt{2\pi}}e^{-\left(\frac{x-\mu}{\sigma}\right)^2} = \phi(x) \tag{3.12}$$

then,

$$\phi^{'}(x) = \frac{2}{\sigma\sqrt{2\pi}}e^{-\left(\frac{x-\mu}{\sigma}\right)^2\frac{x-\mu}{\sigma} = 2\phi(x)\left(\frac{x-\mu}{\sigma}\right)}(3.13)$$

Substituting the above values in equation [4.11]. We get:

$$f^{'}(x) = \frac{1}{\Gamma(a)} \frac{(-\log[1-\Phi(x)])^{a-2}}{[1-\Phi(x)]} \phi(x) t(x) \tag{3.14}$$

where,

$$t(x) = \frac{1}{\sigma^2}(x-\mu)[1-\Phi(x)]\log[1-\Phi(x)] + (a-1)\phi(x)$$

Now for mode,

$$f^{'}(x) = 0 \tag{3.15}$$

Then, this implies

$$t(x) = \frac{1}{\sigma^2}(x-\mu)[1-\Phi(x)]\log[1-\Phi(x)] + (a-1)\phi(x) = 0 \tag{3.16}$$

$$\begin{aligned} x &= \mu - \sigma^2 \frac{\phi(x)}{[1-\Phi(x)]\log[1-\Phi(x)]} \\ x &= \mu + \sigma^2 h_N(x) \left\{ \frac{a-1}{H_N(x)} \right\} \end{aligned} \tag{3.17}$$

Where,

$$\begin{aligned} h_N(x) &= \frac{\phi(x)}{[1-\Phi(x)]} \\ H_N(x) &= -\log[1-\Phi(x)] \end{aligned}$$

Now for a=1,the result reduces to Normal distribution, as a special case. For a > 0, mode is non-negative. While for a < 1 mode will be negative. Now if we consider various values of the parameter we can see the distribution to be unimodal. Also we know that beta-normal is bimodal, but ZB-Normal is not so.

Theorem 3.2 *The moment of ZB-Normal distribution is given by:*

$$E(X^r) = \sum_{j=0}^{r} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} ... \sum_{k_3=0}^{\infty} \sum_{i=0}^{2s_j+j} 2^{\frac{j}{2}} \sigma^j \mu^{r-j} A(k_1, k_2, ..., k_n) \binom{r}{j} \binom{2s_j+j}{i} (-2)^i E\left(e^{-iT}\right) \tag{3.18}$$

where

$$A\left(k_1, k_2, ..., k_n\right) = \left(\frac{\sqrt{\pi}}{2}\right)^{2s_j+j} a_{k_1} a_{k_2} a_{k_3} ... a_{k_n}$$
$$s_j = k_1 + k_2 + ... + k_n$$
$$a_k = \frac{c_k}{2k+1}$$
$$c_k = \sum_{j=0}^{k-1} \frac{c_j c_{k-1-j}}{(j+1)(2j+1)}$$
$$c_0 = 1$$

Proof 3.2 *If a variable follows the gamma distribution with parameter (say a) and F be its CDF, then random variable X follows gamma-X distribution with the parameter a of Gamma. Where,*

$$X = F^{-1}\left(1 - e^{Y}\right)$$

Then Non Central Moments can be found as Expectation of this X.

$$E\left(X^r\right) = E\Big(\Phi^{-1}\left(1 - e^{T}\right)\Big)^r \tag{3.19}$$

Where T follows gamma with one parameter a. Now inverse of Normal is given as:

$$x = \mu + \sigma\sqrt{2} erf^{-1}\left(2y - 1\right) \tag{3.20}$$

Then, substituting the values, we get

$$\Phi^{-1}\left(1 - e^{T}\right) = \mu + \sigma\sqrt{2} erf^{-1}\left(1 - 2e^{-T}\right) \tag{3.21}$$

Then,

$$E\left(X^r\right) = E\Big(\mu + \sigma\sqrt{2} erf^{-1}\left(1 - 2e^{-T}\right)\Big)^r \tag{3.22}$$

By use of binomial series expansion we get,

$$E\left(X^r\right) = \sum_{j=0}^{r} E\left(erf^{-1}\left(1 - 2e^{-T}\right)\right) \tag{3.23}$$

Now we use series expansion of $erf^{-1}\left(1 - 2e^{-T}\right)$ *as given in Wolfram.com, 2012*

$$erf^{-1}\left(1 - 2e^{-T}\right) = \sum_{i=1}^{n} \left(\frac{\sqrt{\pi}}{2}\right)^{2k+1} a_k \left(1 - 2e^{-T}\right)^{2k+1} \tag{3.24}$$

Where,

$$\begin{aligned} a_k &= \frac{c_k}{2k+1} \\ c_k &= \sum_{j=0}^{k-1} \frac{c_j c_{k-1-j}}{(j+1)(2j+1)} \\ c_0 &= 1 \end{aligned} \tag{3.25}$$

Then, we get

$$erf^{-1}\left(1-2e^{-T}\right)^j = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}...\sum_{k_j=0}^{\infty} A\left(k_1,k_2,...,k_j\right)\left(1-2e^{-T}\right)^{2s_j+j} \tag{3.26}$$

where,

$$\begin{gathered} A\left(k_1,k_2,...,k_n\right) = \left(\frac{\sqrt{\pi}}{2}\right)^{2s_j+j} a_{k_1}a_{k_2}a_{k_3}...a_{k_n} \\ s_j = k_1+k_2+...+k_n \end{gathered}$$

Now use binomial expansion of the expression $\left(1-2e^{-T}\right)^{2s_j+j}$, *the equation reduces to:*

$$erf^{-1}\left(1-2e^{-T}\right)^j = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}...\sum_{k_3=0}^{\infty}\sum_{i=0}^{2s_j+j} 2^{\frac{j}{2}}\sigma^j\mu^{r-j}A\left(k_1,k_2,...,k_n\right)\binom{r}{j}\binom{2s_j+j}{i}(-2)^i\left(e^{-iT}\right) \tag{3.27}$$

Now we want expectation of this, substituting above we get the equation as:

$$E\left(X^r\right) = \sum_{j=0}^{r}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}...\sum_{k_3=0}^{\infty}\sum_{i=0}^{2s_j+j} 2^{\frac{j}{2}}\sigma^j\mu^{r-j}A\left(k_1,k_2,...,k_n\right)\binom{r}{j}\binom{2s_j+j}{i}(-2)^i E\left(e^{-iT}\right) \tag{3.28}$$

Now, we need to find

$$E\left(e^{-iT}\right)$$

where T *is gamma with single parameter* a*:- Then, we find it to be*

$$E\left(e^{-iT}\right) = (1+i)^{-a} \tag{3.29}$$

So, the final expression for non-central moment is then,

$$E\left(X^r\right) = \sum_{j=0}^{r}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}...\sum_{k_3=0}^{\infty}\sum_{i=0}^{2s_j+j} 2^{\frac{j}{2}}\sigma^j\mu^{r-j}A\left(k_1,k_2,...,k_n\right)\binom{r}{j}\binom{2s_j+j}{i}(-2)^i E\left(1+i\right)^{-a} \tag{3.30}$$

Theorem 3.3 *Let mean and median of the distribution be and M respectively, then Let mean deviation from mean and median be D (μ) and D (M) then,*

$$D(\mu) = 2\mu F(\mu) - 2I_\mu \tag{3.31}$$

$$D(M) = \mu - 2I_M \tag{3.32}$$

where,

$$F(\mu) = \frac{\gamma(a, -\log[1 - G(\mu)])}{\Gamma(a)}$$

and

$$I_c = \frac{1}{\Gamma(a)} \left(\int_0^{-\log[1-F(c)]} \left(G^{-1}\left(1 - \mathrm{e}^{-u}\right)\right) u^{a-1} e^{-u} du \right) \tag{3.33}$$

Proof 3.3 *The deviation from mean and median are given as*

$$\begin{aligned} D(\mu) &= \int_{-\infty}^{\mu} (\mu - x) f(x)dx + \int_{\mu}^{\infty} (x - \mu) f(x)dx \\ &= 2\int_{-\infty}^{\mu} (\mu - x) f(x)dx \\ &= 2\mu F(x) - 2\int_{-\infty}^{\mu} x f(x)dx \end{aligned} \tag{3.34}$$

$$\begin{aligned} D(M) &= \int_{-\infty}^{M} (M - x) f(x)dx + \int_{M}^{\infty} (x - M) f(x)dx \\ &= 2\int_{-\infty}^{M} (M - x) f(x)dx + E(X) - M = \mu - 2\int_{-\infty}^{M} x f(x)dx \end{aligned} \tag{3.35}$$

Now consider the integral,

$$I_c = \int_{-\infty}^{c} x f(x)dx = \int_{-\infty}^{\mu} x \frac{1}{\Gamma(a)} \{-\log[1 - G(x)]\}^{a-1} g(x)dx \tag{3.36}$$

Take u as

$$u = -\log\left[1 - G(x)\right]$$

Then we get,

$$I_c = \frac{1}{\Gamma(a)}\left(\int\limits_0^{-\log[1-F(c)]} \left(G^{-1}\left(1 - \mathrm{e}^{-u}\right)\right) u^{a-1} e^{-u} du\right) \tag{3.37}$$

Now

$$F(x) = \frac{\gamma(a, -\log\left[1 - G(x)\right])}{\Gamma(a)}$$

Then by use of above, we get the deviation from mean and median as:

$$\begin{aligned} D\left(\mu\right) &= 2\mu F\left(\mu\right) - 2I_\mu \\ D\left(M\right) &= \mu - 2I_M \end{aligned} \tag{3.38}$$

In Particular for Normal-Distribution, We know inverse is given as:

$$x = \mu + \sigma\sqrt{2} erf^{-1}\left(2y - 1\right)$$

Then,

$$G^{-1}\left(1 - \mathrm{e}^{-u}\right) = \mu + \sigma\sqrt{2} erf^{-1}\left(1 - 2e^{-u}\right) \tag{3.39}$$

Substituting it in I_c we then get

$$I_c = \frac{\sigma\sqrt{2}}{\Gamma(a)}\left(\int\limits_0^{-\log[1-F(c)]} \left(erf^{-1}\left(1 - 2e^{-u}\right)\right) u^{a-1} e^{-u} du\right) + \frac{\mu}{\Gamma(a)}\gamma(a, -\log\left[1 - F(c)\right]) \tag{3.40}$$

Using the series expansion of error function and Binomial expansion we get,

$$erf^{-1}\left(1 - 2e^{-u}\right) = \sum_{k=0}^{\infty}\sum_{i=0}^{2k+1} a_k \left(\frac{\sqrt{\pi}}{2}\right)^{2k+1} \frac{(2k+1)!}{i!\,(2k+1-i)!}(-2)^i\left(e^{-iu}\right) \tag{3.41}$$

Substituting that in I_c we then get after some simple calculation and use of Incomplete gamma function, We get the I_c as:

$$I_c = \left\{\frac{\sigma\sqrt{2}}{\Gamma(a)}\sum_{i=0}^{2k+1} a_k \left(\frac{\sqrt{\pi}}{2}\right)^{2k+1}\left(\frac{(2k+1)!}{i!\,(2k+1-i)!}(-2)^i \frac{(-1)^a}{(1+i)^a}\gamma(a, (1+i)\log\left[1 - F(c)\right])\right)\right\}$$

$$+\frac{\mu}{\Gamma(a)}\gamma\left(a, -\log\left[1-F(c)\right]\right) \tag{3.42}$$

Theorem 3.4 (Boundedness of expectation) *For a>1/2 we have*

$$E(X) \leqslant \sigma\sqrt{\frac{\Gamma(2a-1)}{\Gamma^2(a)}} + \mu \tag{3.43}$$

Proof 3.4 *The given inequality is sharp because for a=1, we have then*

$$E(X) \leqslant \mu$$

, Then it reduces to:

$$E(X) = \mu$$

Now consider,

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\mu} x\frac{1}{\Gamma(a)}\{-\log\left[1-\Phi(x)\right]\}^{a-1}\phi(x)dx \tag{3.44}$$

Let

$$u = \Phi(x)$$

Then we get,

$$E(X) = \frac{1}{\Gamma(a)}\int_{0}^{1} x(u)\{-\log\left[1-u\right]\}^{a-1}du \tag{3.45}$$

Now we know that if

$$x(u) = \Phi^{-1}(u)$$

Then,

$$\begin{aligned} \int_0^1 x(u) &= \mu \\ \int_0^1 x^2(u) &= \mu^2+\sigma^2 \end{aligned} \tag{3.46}$$

As Gumbel (1954) used method of variations then we have,

$$E(X) = \frac{1}{\Gamma(a)} \int_0^1 \left(x(u)\{-\log[1-u]\}^{a-1} - \lambda_1 x^2(u) - \lambda_2 x(u) \right) du \qquad Where, 0 \leqslant u \leqslant 1 \tag{3.47}$$

Now, stationary solution of

$$\int_0^1 x(u) = \mu$$
$$\int_0^1 x^2(u) = \mu^2 + \sigma^2 \tag{3.48}$$

Satisfies the equation,

$$E(X) = \frac{1}{\Gamma(a)} \int_0^1 \frac{\partial}{\partial x} \left(x(u)\{-\log[1-u]\}^{a-1} - \lambda_1 x^2(u) - \lambda_2 x(u) \right) du = 0 \tag{3.49}$$

Then we get,

$$\frac{1}{\Gamma(a)}\{-\log[1-u]\}^{a-1} - 2\lambda_1 x(u) - \lambda_2 = 0 \tag{3.50}$$

This implies,

$$x(u) = \frac{\frac{1}{\Gamma(a)}\{-\log[1-u]\}^{a-1} - \lambda_2}{2\lambda_1} \tag{3.51}$$

Upon integrating above equations:

$$\int_0^1 \frac{1}{\Gamma(a)}\{-\log[1-u]\}^{a-1} - 2\lambda_1 \int_0^1 x(u) - \int_0^1 \lambda_2 = 0 \tag{3.52}$$

Now, use the substitution

$$v = -\log(1-u)$$

then, We get,

$$\lambda_2 = 1 - 2\mu\lambda_1$$

And by use of above equation and earlier, we get:

$$\lambda_1 = \pm\frac{1}{2\sigma}\sqrt{\frac{\Gamma(2a-1)}{\Gamma^2(a)} - 1} \tag{3.53}$$

Where a>1/2 Substituting the above values in x(u) we get,

$$x(u) = \mu \pm \sigma \sqrt{\frac{\Gamma^2(a)}{\Gamma(2a-1) - \Gamma^2(a)}} \left(\frac{1}{\Gamma(a)} \{-\log[1-u]\}^{a-1} - 1 \right) \tag{3.54}$$

Using it in expectation expression, Further note that minimum occurs at - λ_1 *and maximum occurs at +* λ_2 *Then we get,*

$$E(X) \leqslant \sigma \sqrt{\frac{\Gamma^2(a)}{\Gamma(2a-1) - \Gamma^2(a)}} \left\{ \int_0^1 \frac{1}{\Gamma^2(a)} \{-\log[1-u]\}^{2a-2} du \right\} \tag{3.55}$$

Let,

$$v = -\log(1-u)$$

, then we get

$$E(X) \leqslant \sigma \sqrt{\frac{\Gamma(2a-1)}{\Gamma^2(a)}} + \mu \qquad wherea > 1/2 \tag{3.56}$$

Theorem 3.5 *Shannon entropy for Z-B Normal distribution is given by :*

$$\eta_X = \log\left(\sigma\sqrt{2\pi}\right) + 0.5(v+m)^2 + \log(\Gamma(a)) + (1-a)\,\psi(a) \tag{3.57}$$

Where m, v are mean and variance of the ZB-Normal with parameter a, 0, 1.

Proof 3.5 *For Shannon entropy we need to find,*

$$-E\left(\log\left(f\left(F^{-1}\left(1-e^{-T}\right)\right)\right)\right) \tag{3.58}$$

Now,

$$f(x) = \phi(x)$$
$$F(x) = \Phi(x)$$

Then,

$$\log(\Phi(x)) = -\log\left(\sigma\sqrt{2\pi}\right) - \left(\frac{x-\mu}{\sigma}\right)^2 \tag{3.59}$$

, where T follows the Gamma Distribution(with one parameter a) Then, $\Phi^{-1}\left(1-\mathrm{e}^{-T}\right)$ *follows ZB-Normal with parameter a,* μ, σ *Also this implies then,*

$$\frac{\Phi^{-1}\left(1-\mathrm{e}^{-T}\right)-\mu}{\sigma}$$

follows ZB-Normal with parameter a, 0, 1
Then,

$$E\left(\log\left(f\left(F^{-1}\left(1-e^{-T}\right)\right)\right)\right)=\log\left(\sigma\sqrt{2\pi}\right)+0.5(v+m)^2 \tag{3.60}$$

Now substituting this in formula for shannon entropy we get the desired result i.e.

$$\eta_X=\log\left(\sigma\sqrt{2\pi}\right)+0.5(v+m)^2+\log\left(\Gamma(a)\right)+(1-a)\,\psi\left(a\right) \tag{3.61}$$

where $\psi(a)$ *is digamma-function.*

Conclusions and Scope of Future Works

We have derived some basic properties of ZB-Gumbel and ZB-normal distributions such as pdf, cdf, hazard and survival function, central moments, moment-generating function, mode, Renyi and Shannon entropy etc. Extreme values, reliability function, order statistics and bivariate generalizations have been done along with various applications. A few theorems have been proposed regarding boundedness of moment and deviation from mean and median. These distributions have vivid applications because they always tend to give better fit than the parent distribution. Applications are possible in stress of carbon fibers and environmental pollution for gamma-normal and gamma-Gumbel respectively. It can be seen that if we use residuals of the data from certain experiment, and if data is skewed we do not get better fit. This problem can be eliminated by use of ZB-normal distribution and ZB-Gumbel distribution.

A possible future work in which direction can be estimation of parameters of ZB-Gumbel and ZB-normal distribution. The MLE can be obtained by solving system of equations numerically. Also some Bayesian estimation can be done by using some approximation techniques. Further interval estimation can also be done for this parameters. An attempt can be made to standardize the gamma-normal and use it in methods like central limit theorem and Analysis of Variance(ANOVA).

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