STUDY OF NONLINEAR DYNAMICS: BEHAVIOUR OF DIFFERENT NON LINEAR SYSTEMS

Project report required

In the completion of the degree of

Masters of Science, in Physics

SUBMITTED BY:

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UNDER THE SUPERVISION OF

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CERTIFICATE

This is to certify that**, Miss Olivia Dey** (Roll number: 413PH2083), a final year student of M.sc (2 years), batch 2013-2015, of this institute, has successfully completed the project titled as "**STUDY OF NONLINEAR DYNAMICS: BEHAVIOUR OF DIFFERENT NON LINEAR SYSTEMS"**, under my supervision. She was successfully able to establish the results which absolutely went with the theoretical notion. The programs written and graphs generated are totally authenticated.

Dr. BIPLAB GANGULI

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ABSTRACT

Study of non linear dynamics has developed a lot in the mid 20th century and since then many scientists have contributed in this particular branch of science. In this project, the basic idea behind this branch has been studied. Lorenz attractor graph which gives us the idea of a chaotic system has been studied and obtained. Next, the phase space graphs of different non linear systems like the double pendulum and the coupled oscillator were virtually obtained. There are different phases of these systems and so are there different methods to find the state of chaos. In this report, one of those methods has been mentioned, which is the Poincare section graph. The graphs obtained help us to determine the exact state of the dynamical system. In addition, the change in the system with varying parameters has also been studied. If the initial parameters are varied, the whole system might go topsy-turvy for a particular value. This change of system with parameter gives us a particular graph known as the bifurcation diagram, from which it has been known exactly for which value of the parameter, the system changes. So overall in this dissertation, both qualitative and quantitative study of non linear systems and their corresponding dynamical behaviors has been given.

INTRODUCTION

The dynamics study started in the mid 1600s when Newton started is works on differential equations discovered his laws of motion and united them to elucidate Kepler's laws of planetary motion. Dynamics is the learning of change and the dynamical system is just a method for saying how a system of variables reacts and changes with time. A dynamical system is a notion of mathematics where a fixed rule shows how a point in a geometrical shape depends on time. Examples include mathematical methods that describe fluctuation of a clock pendulum, running of water in a pipe, LCR circuit equation etc.

At any given time a dynamical system has a state specified by a set of real numbers (a vector) that can be denoted by a point in an appropriate state space. Minute changes in the situation of the structure produce small changes in the statistics. The progression rule of the dynamical format is a fixed rule that describes what outlook states follow from the present state. The rule is deterministic; in other language, for a known time gap only one future condition follows from the existing state. Before the opening of computers, finding an orbit was necessary for complicated mathematical techniques and could be competently completed only for a small group of dynamical systems. Numerical techniques implemented on electronic computing equipment have cut down the job of knowing the orbits of a dynamical state. The concept of softness changes with applications and the kind of manifold.

HISTORY OF DYNAMICS

The subject began when Newton invented differential equations. Succeeding generations of scientists tried to expand Newton's analytical methods to the three body problem but inquisitively this problem turned out to be much harder to solve. After years of hard work, it was finally realized that three body problems were basically unfeasible to solve, in the sense of obtaining clear formulas for the motions of the three bodies. At this stage, situation seemed impossible.

The breakthrough came with the work of Poincare in the late 1800s. He discovered a new point of view that emphasized qualitative, rather than quantitative questions. That approach was inserted into the modern subject of dynamics, with applications getting far beyond celestial mechanics. Poincare was also the first person to sight the likelihood of chaos in which a deterministic system exhibits aperiodic actions that depends delicately on the early conditions, and hence predicting long term forecast not possible.

But chaos stayed in the backdrop for the first half of this century; in its place dynamics was largely worried with nonlinear systems and their applications in science. Non linear oscillators played a fundamental role in the progress of such technologies as radio, phase locked loops and radars. Theoretically, non linear oscillators stirred the creation of new mathematical methods.

The high speed computer invented then allowed one to test with equations in a way that was unworkable before and hence to develop some instinct about non linear systems. Those experiments guided to Lorenz's discovery of chaotic motion in 1963 on a strange attractor. Lorenz's labor had slight blow until the 1970s, the bang years for chaos. New theory for the onset of turmoil in fluids based on conceptual considerations about strange attractors was planned. May established examples of chaos in iterated mappings arising in population biology, and wrote a powerful article that hassled out the significance of studying straightforward non linear systems, to balance the often deceptive linear perception fostered by conventional education. Next, physicist Feigenbaum exposed that there are positive universal laws leading the change from usual to chaotic actions. His work recognized a connection between chaos and phase transitions, and enticed a cohort of scientists to the study of dynamics.

Though chaos stole the attention, two other main developments arose in dynamics in the 1970s. Mandelbrot codified the popularized fractals, shaped splendid computer graphics of them, and showed how they could be a functional use in a range of subjects. And in the rising are of mathematical biology, Winfree applied the geometric tactics of dynamics to biological oscillations, especially heart beats. By the 1980s many scientists were operational on dynamics, with offerings too many to catalog.

NONLINEARITY

Though Newton productively solved the 2 body problem, other scientists established that it was very difficult to solve a three body problem, while obtaining open formulas for the motions of the three bodies. In the late 1800s Poincare introduced a fresh point of view that emphasized qualitative, rather than quantitative queries. For example, he asked, "Is the solar system stable forever, or will some planets fly off to infinity?" Poincare created an influential geometric approach to examine such questions. He was also the first person to sight the likelihood of chaos, in which a deterministic scheme exhibits aperiodic performance that depends sensitively on the preliminary conditions and hence depicting long term prediction impracticable.

In this real world almost all and every model is 'nonlinear' in nature. Dynamics was mainly concerned with nonlinear pendulums and their applications. Lorenz was studying a basic model of convection rolls in the atmosphere to increase inside into the infamous randomness of the weather. Lorenz found that the solutions to his equations never established down to equilibrium or a periodic state. Instead they sustained to oscillate in an irregular, aperiodic style. Also if he started his simulations from two a little different initial conditions, the ensuing behavior soon become completely different. The insinuation was that the system was intrinsically unpredictable. Not only this weather problem, there are many other problems and models in nature which have this kind of chaotic and random result. All these arise due to the nonlinearity of the state.

LORENZ ATTRACTOR:

Lorenz attractor was first considered by Ed N. Lorenz, a meteorologist, in 1963. It was derived from a simple model of convection in the earth's atmosphere. The system is most usually explained as 3 coupled non linear differential equations.

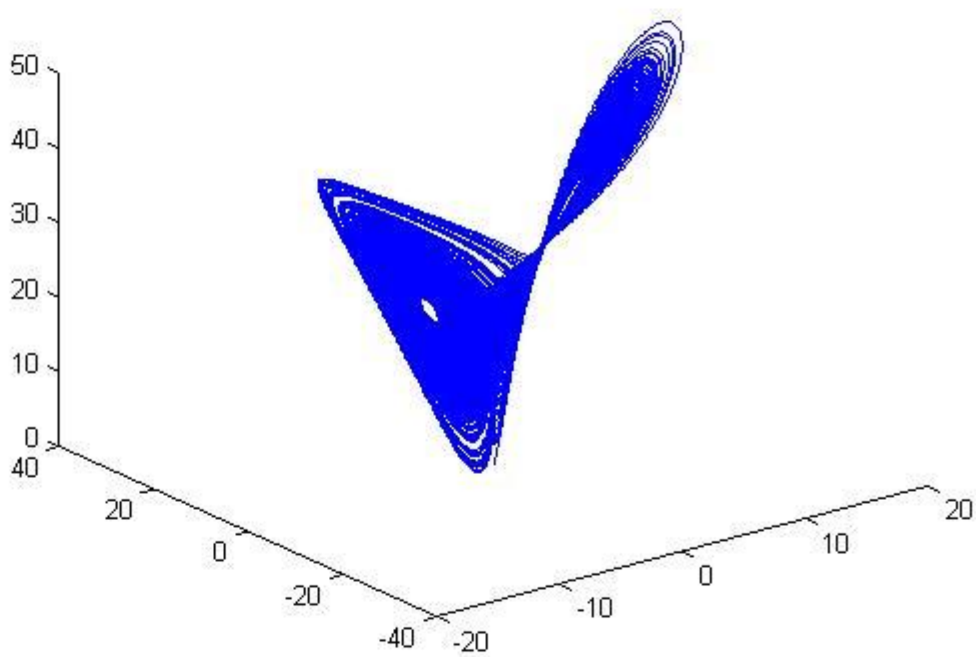
$$dx/dt=a(y-x)$$

$$dy/dt=x(b-z)-y$$

$$dz/dt=xy-cz$$

Our normally used set of constants is: a=10, b=28, c=8/3. This sequence does not figure limit cycles nor does it ever attain a fixed state. Instead it is an instance of deterministic chaos. This system is chaotic to the early conditions, two initial states no matter how close, will diverge, generally earlier rather than afterward.

Though the equations seem simple enough, they direct to amazing trajectories. The program was written in MATLAB with proper initial conditions and the graphical illustration is as shown below:



The two wing like structures are called the ‘attractors’, or rather ‘strange’ attractors, because no one knows when the solution is going to skip from one attractor to the other during moving through the trajectory.

DOUBLE PENDULUM

Now let us move on from a parametric or damped pendulum to another system which is the double pendulum. The double pendulum is basically a system where one pendulum attaches from the base of another. The figure is shown below.

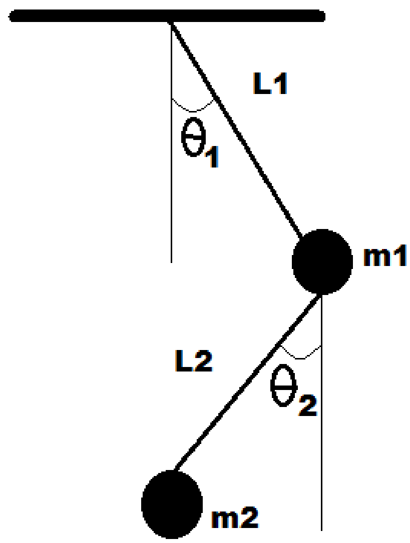


Fig 1: Double pendulum

I would not be deriving the whole equation of motion since this is not the major part of my project. Instead I would directly write the formulas which have been used to write program in FORTRAN 90 to obtain the phase space graph followed by the Poincare section and the Bifurcation diagram. Before going into detailed analysis of the phase space, Poincare section and the bifurcation diagram, I would like to go into a detailed theoretical study of what exactly Poincare section and Bifurcation means in Non linear dynamics.

POINCARE MAP

The Poincare map is the cutting of an orbit which is periodic in the phase space of a continual dynamical system by a certain lower dimensional subspace. This section is called the Poincare section. This section is perpendicular the flow of the system. In this case we take into account those points in the section which are obtained by multiple intersections of the flow. A map is then created to send the first leaving point to the next and hence it is also known as the first recurrence map. In case of a three dimensional system, the Poincare section is a two dimensional one, which is also perpendicular to it.

So basically in Poincare map, we take a section through the system or the attractor and analyze the intersections of the lines with this section. These Poincare maps are used to determine the stability of a system. It can be easily determined whether a system is periodic, stable or chaotic just by the detailed analysis of these maps.

Mathematically we take an n-dimensional system and we take 'S' to be an n-1 dimensional surface of section. S is perpendicular to the flow of the trajectories. Then a Poincare map, 'P' is a mapping which is drawn considering trajectories from S to itself. This is gotten by taking trajectories from one intersection to the other. By keeping one phase element constant and plotting other elements, each time a surface intersection is obtained. If we take an attractor, the Poincare method consists of sectioning the attractors at regular intervals. Then the phase plane is looked upon. If the sectioning is done at intervals corresponding to the driving force parameter, we will get one point. If the phase is increased by 2π, the motion will come back to the same coordinates. In case of a limit cycle or a periodic bounded motion, only one point in the Poincare map will be obtained. But if the section is chaotic, then we will get many numbers of points in the Poincare map. The Poincare map thus help us to study a dynamical system and also tells us about the state of the system whether periodic or chaotic. These kinds of mappings are used to study many dynamical systems including swirling flows and whirlpools.

BIFURCATION

Other than Poincare maps, I have also studied about bifurcations and bifurcation diagrams. Bifurcations are scientifically significant. Now when we study, non linear dynamics, the first question arises is that what is the essential difference between linear and non linear systems. The answer is 'parameters' and dependence of the entire dynamics of the structure on the parameters. The qualitative structure of the flow can vary as parameters are varied. In particular, fixed points can be distorted, created or shattered. These qualitative changes in the dynamics are called bifurcations. The parameter values at which they happen are known as bifurcation values.

Bifurcations offer models of transitions and instabilities as some restricted parameter is changed. There are distinct kinds of bifurcations in nature like Saddle-Node bifurcation, Transcritical bifurcations, Pitchfork bifurcations etc.

Saddle-node bifurcation is the fundamental mechanism by which permanent points are created or destructed. As a parameter is changed, two fixed points shift towards each other, have a collision and jointly destroy.

Transcritical bifurcation is that case where a fixed point should subsist for all values of a parameter and can never be destructed. The important disparity between saddle-node and Transcritical bifurcation is that in Transcritical, the two fixed points don't vanish after the bifurcation; in its place they just switch their stability.

Pitchfork bifurcation is frequent in physical problems that have symmetry. Most problems have spatial symmetry between left and right. In those cases fixed points tend to appear and vanish in symmetrical pairs.

Equation of motion for double pendulum:

There are basically 2 major equations of motion for a double pendulum one for each. To write the program in FORTRAN 90, I have used the Euler method of differential equations to solve the equation of motion for the double pendulum. Since it is a second order differential equation, I have broken up into four first order differential equations and then have solved them using Euler method. The four differential equations are as follows:

$$\frac{dx1}{dt} = v1 \qquad (1)$$

$$\frac{dx2}{dt} = v2 \qquad (2)$$

$$\frac{dv1}{dt} = -av1 - \sin(x1) + bcos(wt) \quad (3)$$

$$\frac{dv2}{dt} = -av2 - \sin(x2) + bcos(wt) \quad (4)$$

Where x1, x2 are positions of the two pendulums respectively and v1, v2 are the velocities of the two pendulums respectively. The constants a and b contain all the combined values of mass of the pendulums, acceleration due to gravity, lengths of the pendulums etc. w is the frequency of oscillation and t denotes time. Now these four equations have been solved using Euler method. Again I won't be going into descriptive part of the Euler equation since this is out of the scope of my project.

PHASE SPACE

The phase space of a dynamical system is a mathematical space with orthogonal coordinate direction representing each of the variables needed to specify the instantaneous state of the system. For example, the state of a particle moving in one dimension is specified by its position (x) and velocity (v); hence its phase space is a plane. On the other hand a particle moving in three dimensions would have six dimensional phase space with three position and three velocity directions. A phase space may be constructed in several different ways. For example, momenta can be used instead of velocity.

Below is shown the "phase space" graph of the double pendulum for various different cases. The double pendulum, shows a complete chaotic behavior for a=0.5, b=1.5 and w=0.66. These values are idealistic values to get a total state of chaos in case of the double pendulum.

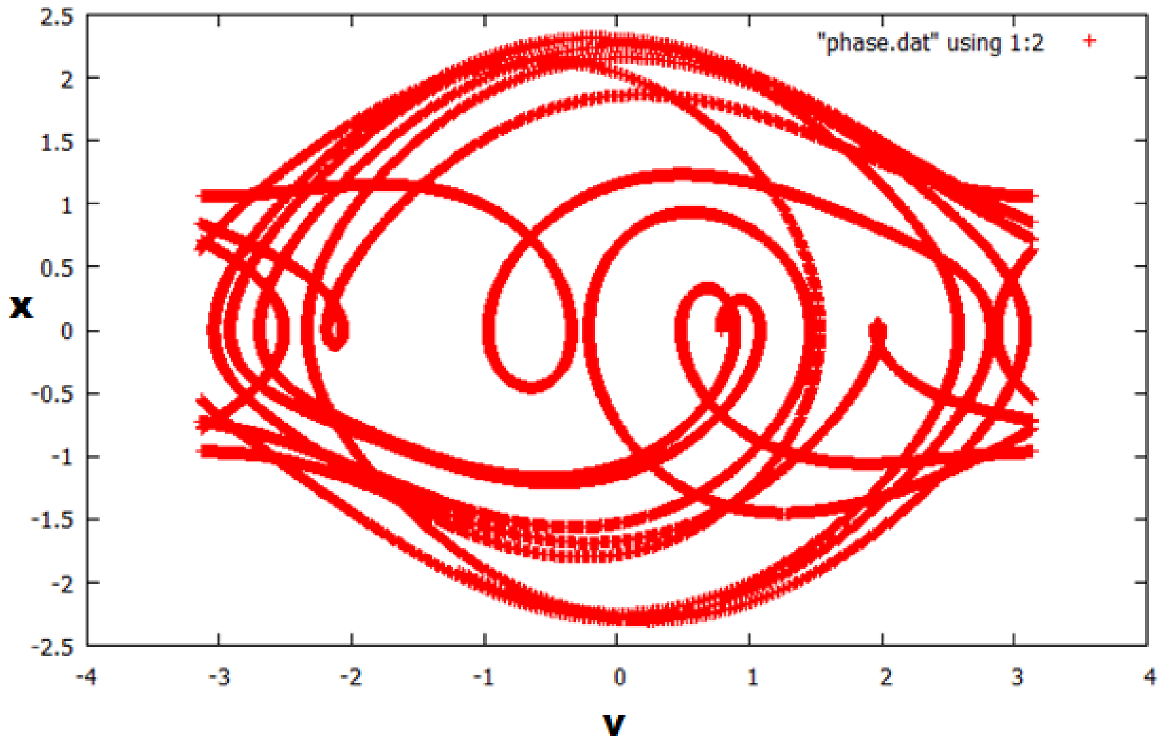


Fig 2: Phase space graph over a time span of 10000.

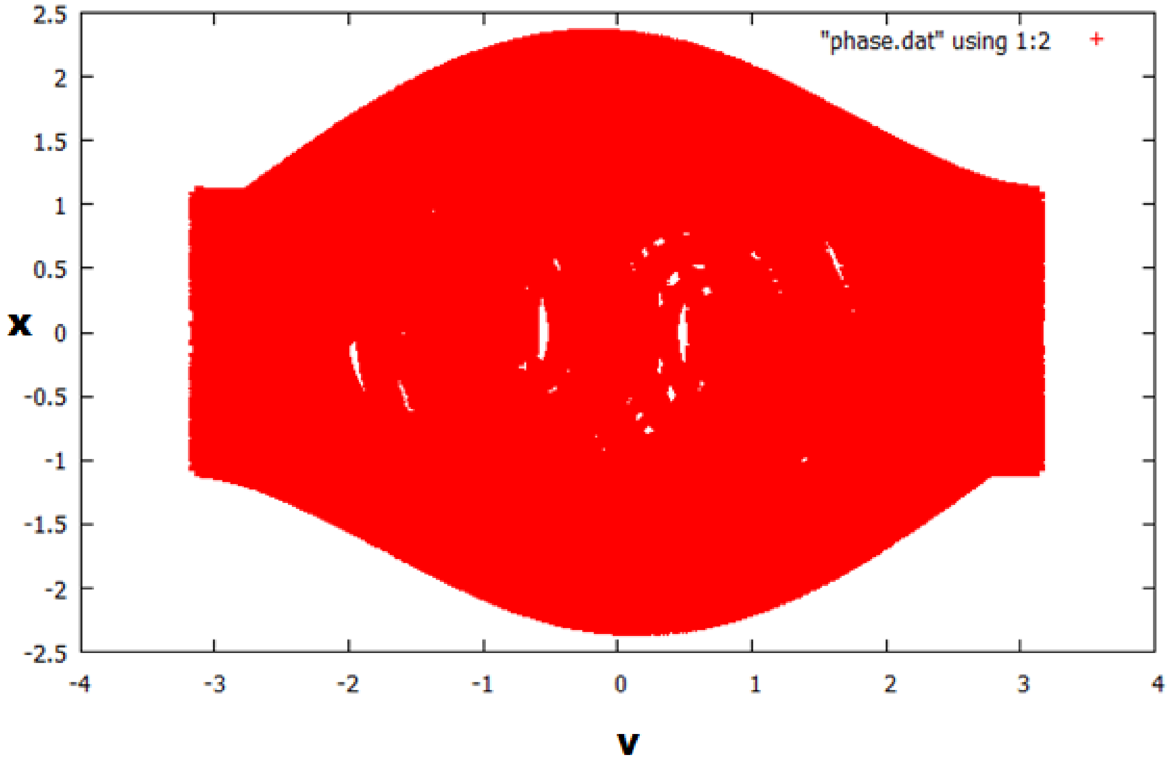


Fig 3: Phase space diagram over a time span of 1000000

CHAOS:

A dynamical system is thought to be chaotic, if it has the subsequent properties:

(a) It must be responsive to initial conditions.

(b) It must be topologically mixing.

(c) Its periodic orbits should be intense.

The essential requirements for a deterministic incessant dynamical system to be chaotic are that the system have to be non linear and be at least three dimensional. Sensitivity to early condition implies that each point in such a structure is randomly approximated by other points with considerably dissimilar future trajectories. Thus an arbitrarily small disturbance of the present trajectory may lead to significantly different future performance.

Sensitivity to primary conditions is commonly known as the butterfly effect, so called because of the title of a paper given by Edward Lorenz in 1979. Does the flap of a butterfly wing in Brazil set off a tornado in Texas? The flapping wing represents a minute change in the initial condition of the system, which causes a series of events leading to great scale phenomena. Had the butterfly not flapped its wings, the trajectory of the structure would have been enormously different.

PHASE SPACE GRAPH ANALYSIS

From the phase space graphs shown, we can see the chaotic nature of the double pendulum. Based on the initial conditions we finally got the desired result. Initially in Fig 1, we can see that the phase space is slowly getting filled up. The graph is plotted for a time span of 10000. Next I plotted the graph for a time span of 1000000. In Fig 2, thus we can see the dense filling of the phase space. The phase space is almost filled up and according to the definition of chaos; we can say that the motion of the double pendulum is unpredictable and totally chaotic.

POINCARE SECTION

Below is plotted the Poincare section for the double pendulum. Poincare section here is a surface where only those points are plotted for which time is equal to ($2\pi/w$) where w is the frequency. Theoretically it means that we are plotting only those points where the trajectory is crossing those points for which the time period is ($2\pi/w$). Here the motion repeats itself exactly every period of the external drive.

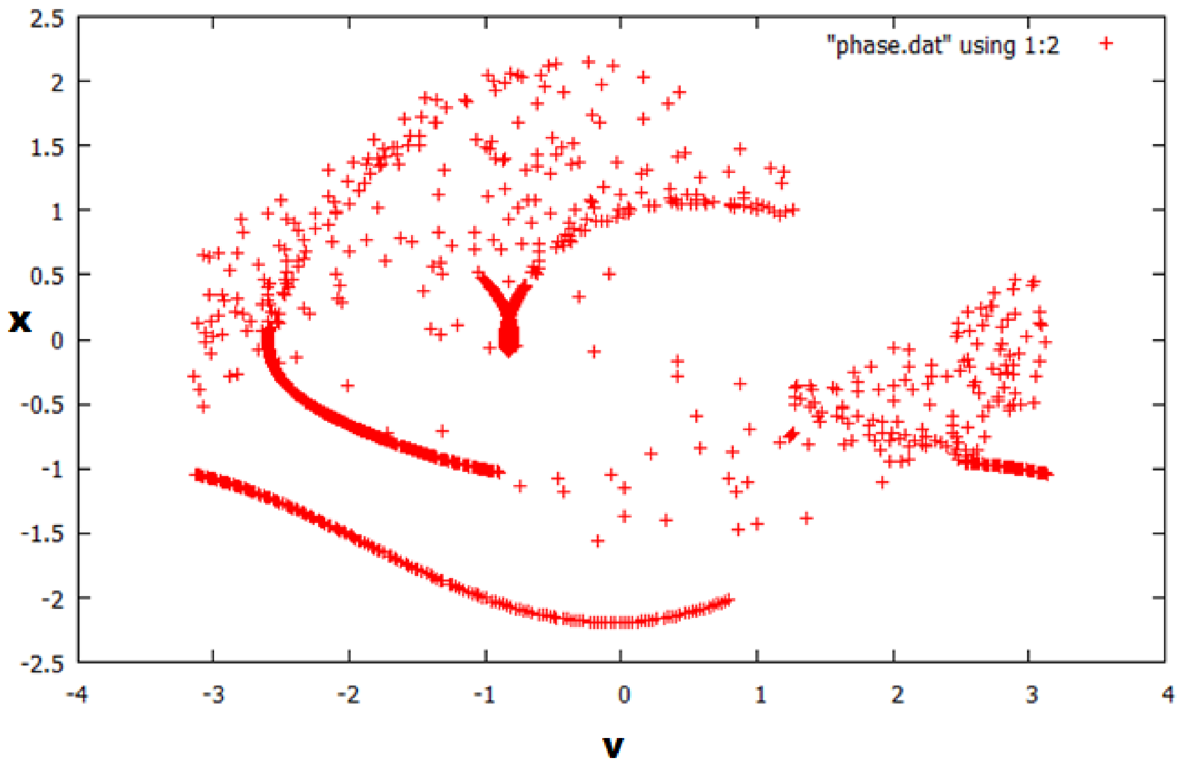


Fig: 4, Poincare section for a double pendulum for time span of 1000000

The Poincare section is a measure or indicator of how chaotic a system is. The dense filling of the Poincare map is also an indication of chaos. If the Poincare map had only few (2-4) points, then it would have indicated a more stable and bounded motion as will be seen later while studying the coupled oscillator.

PARAMETRIC COUPLED OSCILLATOR

Now we would move on to the next non linear system which is the parametric coupled oscillator, whose various phases and behaviors would be studied. Again the derivation of the equation is beyond the scope of this project and hence I would directly write the four equations which have been used in Euler method to write the code in FORTRAN and solve it. Generally there are two second order differential equations; each for the two pendulums. The two equations are broken up into four first order differential equations and fed into the Euler equation. The four differential equations are:

$$\frac{dx1}{dt} = v1 \tag{5}$$

$$\frac{dx2}{dt} = v2 \tag{6}$$

$$\frac{dv1}{dt} = -2\lambda x1 - w0^2(1 + \text{acos}(wt))\sin(x1t)) - k(x1 - x2) \tag{7}$$

$$\frac{dv2}{dt} = -2\lambda x2 - w0^2(1 + \text{acos}(wt))\sin(x2t)) - k(x2 - x1) \tag{8}$$

The diagram for the coupled oscillator is shown below, the coupling constant being ‘k’.

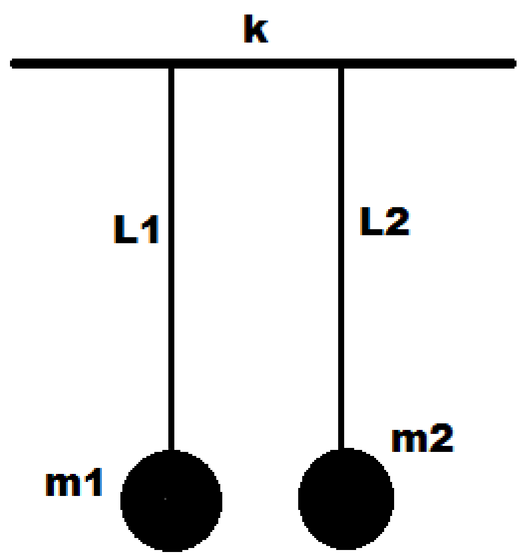


Fig 5: The coupled oscillator

Different phases for this system were plotted. In this case the parameters are being a, λ, and k. Now I have kept the value of λ to be constant at 0.5. This is because the term λ consists of the combined factors of mass, length and acceleration due to gravity. 'w0' here is another constant comprising of acceleration due to gravity and lengths of the pendulums. I have kept the value of w0 constant at 1 for the sake of smooth running of the code and better graphs. The value of 'w' is taken to be equal to 1, where 'w' is the damping frequency given to the pendulums. 'k' is the coupling constant and 'a' is a parameter whose value I have constantly changed in order to get different phase space graphs which are shown below.

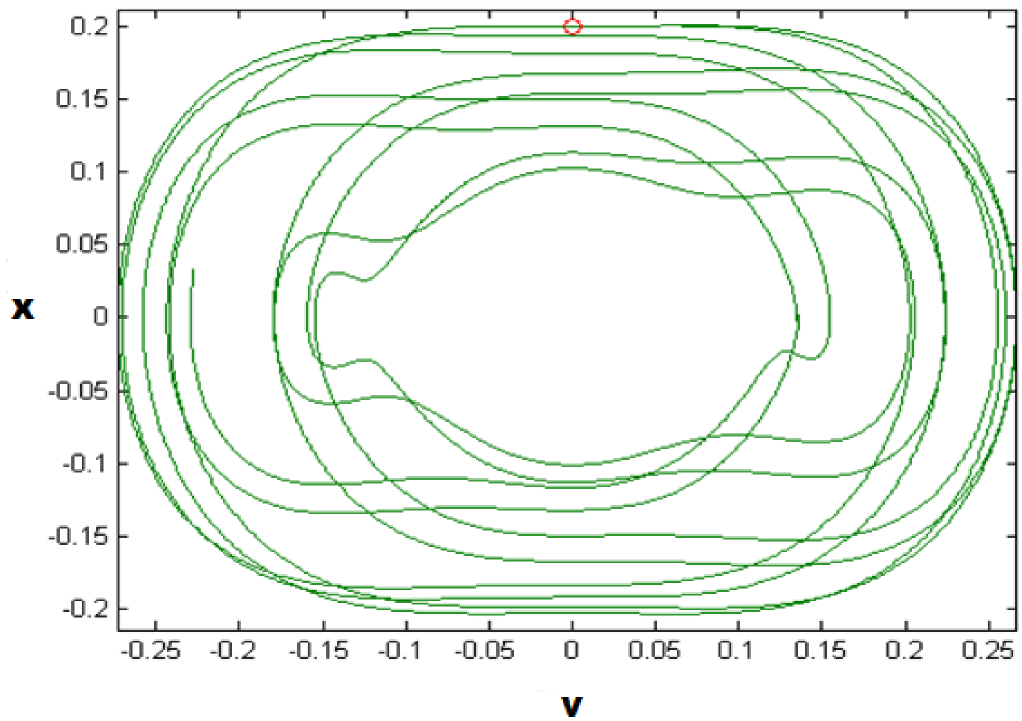


Fig 6: Phase space graph for a=8.5, k=0.1, taken over time span of 10000

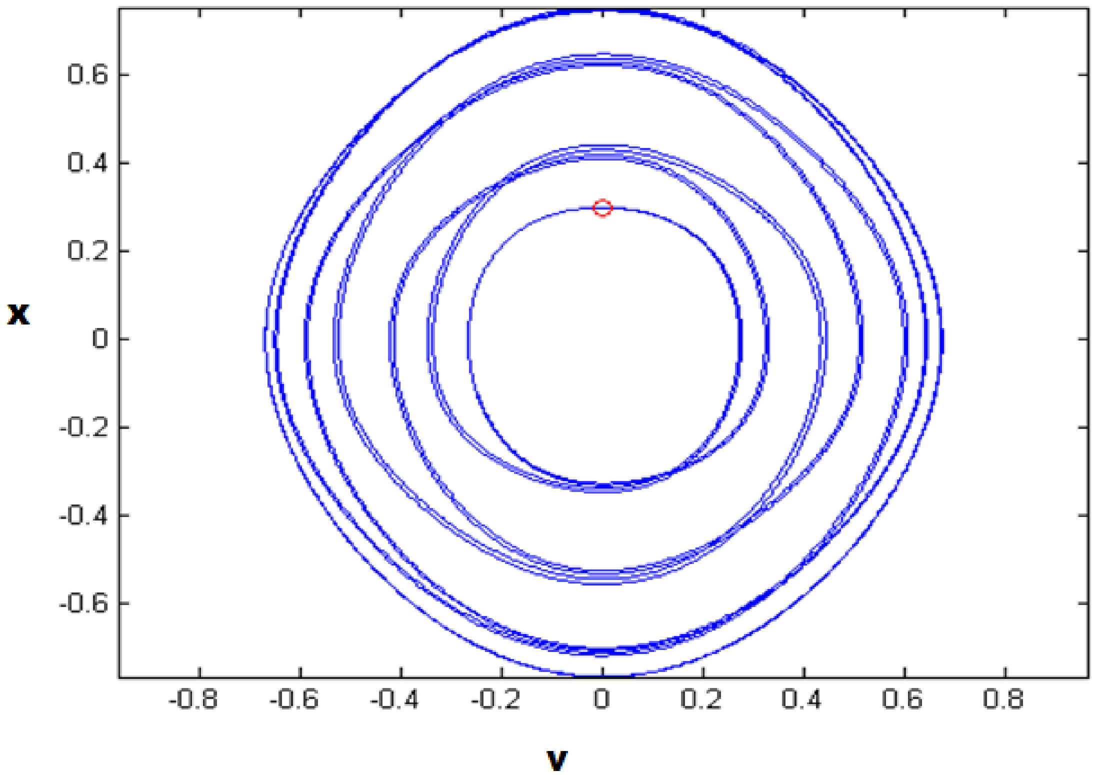


Fig 7: Phase space diagram for a= 4, k=0.1 and for a time span of 10000

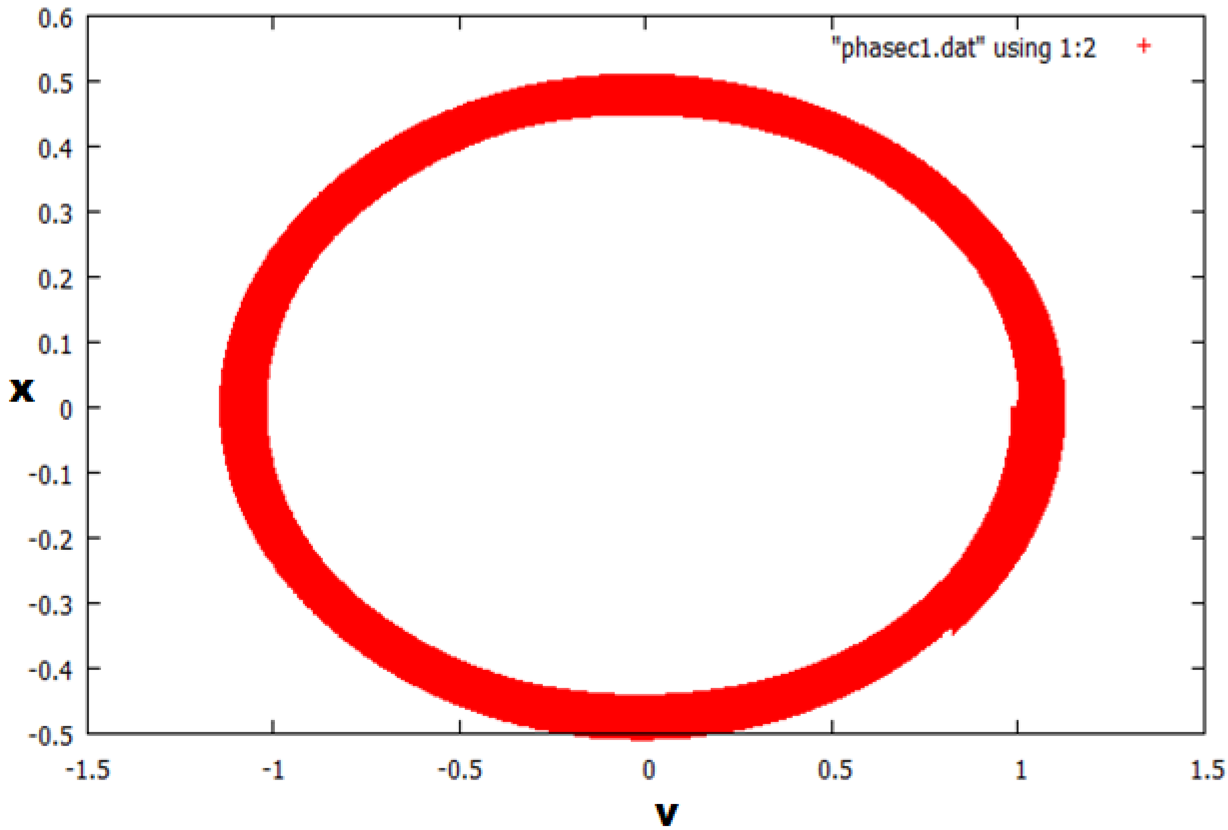


Fig 8: Phase space graph for a=0.1, k=0.1 and for a time span of 10000.

GRAPH ANALYSIS:

The three graphs shown above, show three different cases with different values of ‘a’. Fig 6 shows a trajectory which tells us that the motion is chaotic. It means when there are two pendulums, coupled together, initially with high damping constant; the motion gives us a state of chaos.

Fig 7 tells us about an interesting case of period doubling. The graph obtained is thus shown.

Fig 8 gives us a state which is completely periodic. This means that both the pendulums will oscillate with the same phase and hence a smooth ellipse is formed as the graph. The trajectory is bounded.

The whole scenario can also be explained by taking the **Poincare maps.**

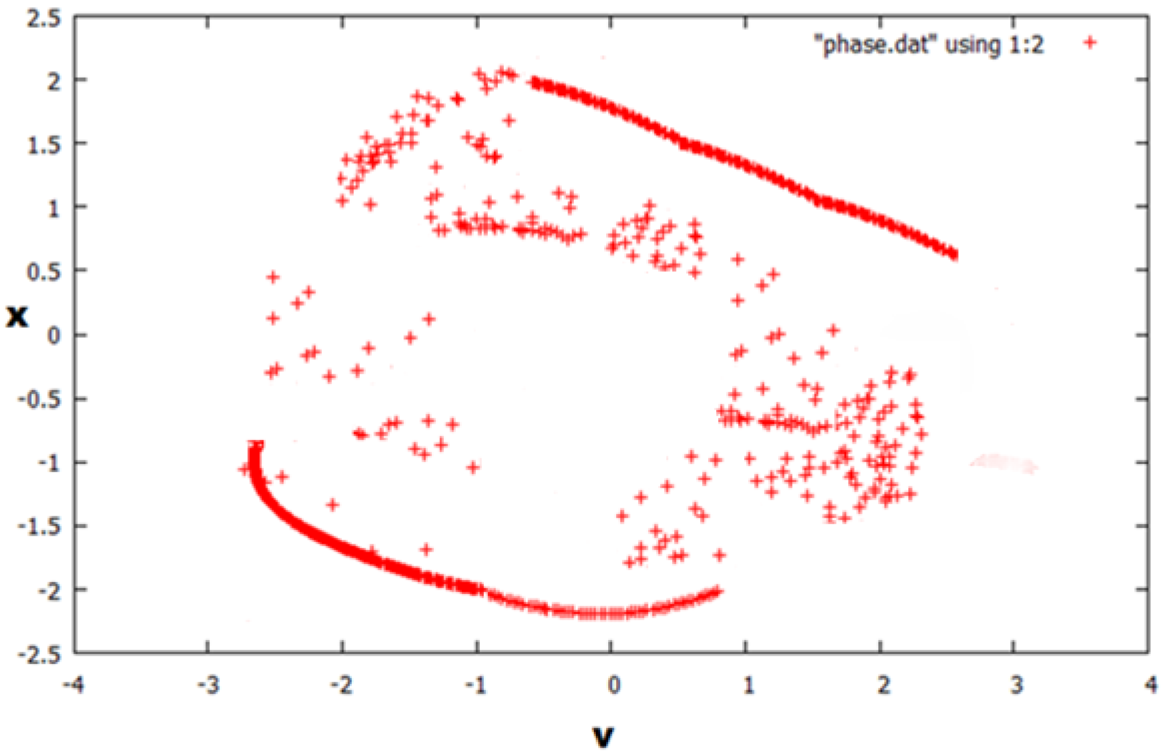


Fig 9: Poincare map for chaotic motion

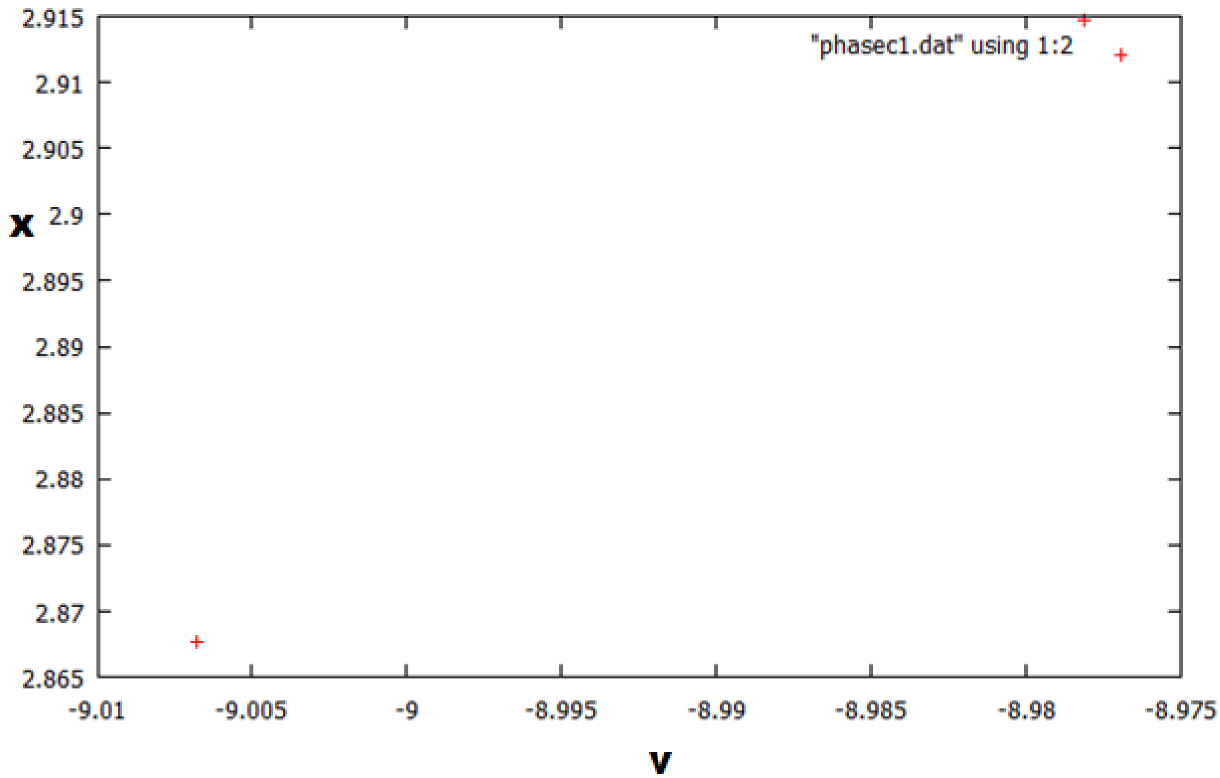


Fig 9: Poincare map for periodic motion

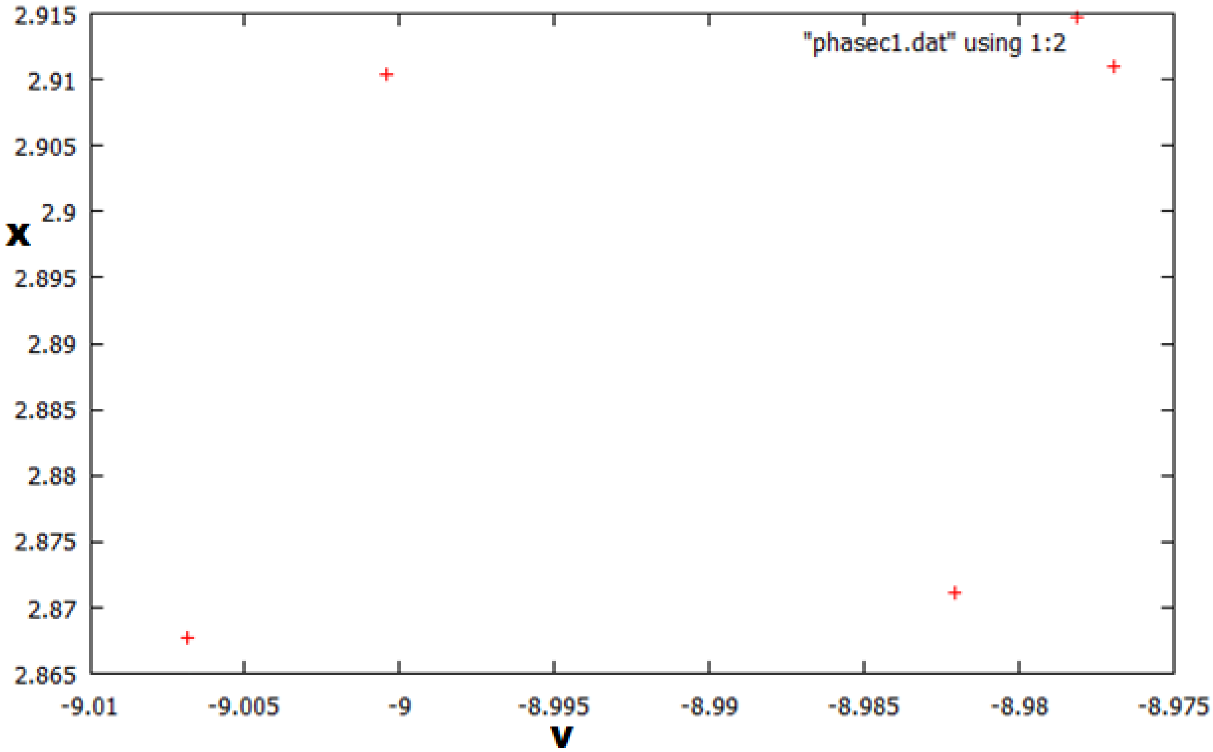


Fig 10: Poincare map for period doubling.

In the two Poincare maps shown above, we can see two distinct cases of periodic trajectory and period doubling. If we think theoretically, then the phase space diagram for a periodic orbit will indeed show only two points. This is because in that case, the trajectory will intersect the points with time period ($2\pi/w$) twice and only twice.

In the second case, we see the scenario of period doubling. Hence as expected, we got four points in the Poincare section which absolutely corresponds to its respective phase space graph.

Thus we can say that Poincare maps are indeed the indicators of study of non linear systems. Looking at the Poincare section, we can easily conclude whether the dynamical state of a system is chaotic, periodic etc.

BIFURCATION DIAGRAM

Now I would like to discuss this important factor in the study of non-linear dynamics which is called bifurcation. As stated earlier, we theoretically know the definition of bifurcation and the way the graph should look like. Here I would be showing the actual obtained graph for the case of coupled oscillator. The figure below shows the same.

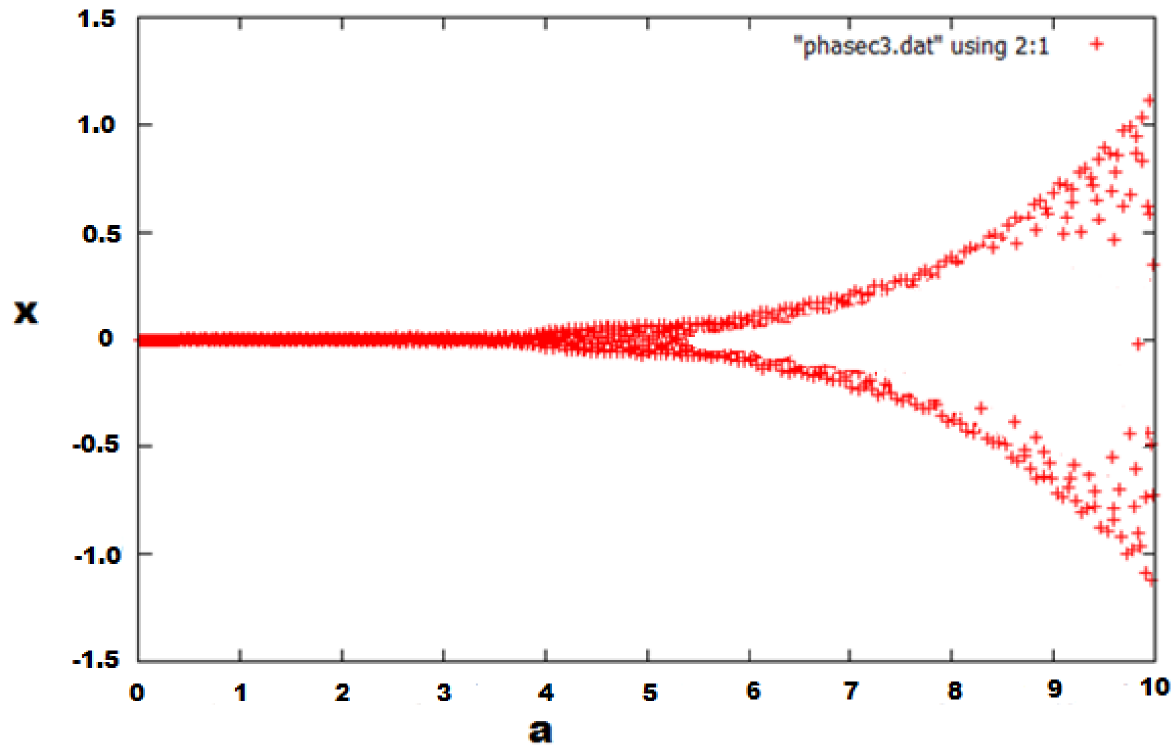


Fig 11: Bifurcation diagram for coupled oscillator, for a time span of 100000

The bifurcation graph is plotted between the position (x) and the damping parameter (a). The splitting is visible in the graph for around ‘a’ equal to 4. The scale is shown on the two axes. The splitting of the line indicates period doubling. This bifurcation is not a perfect one due to the presence of erroneous terms in the code, but it perfectly shows the period doubling along with the change in damping parameter. This is a type of pitchfork bifurcation with a bit different curvature as expected. The total system is symmetric and hence pitchfork bifurcation is obtained as expected.

CONCLUSIONS:

In this project, the theoretical part of this branch of physics known as non linear dynamics was studied. Periodic and chaotic behavior of different non linear systems was studied. The double pendulum totally showed a chaotic behavior. Its behavior was confirmed by the Poincare map which was obtained. The filling up of the Poincare section proved the chaotic behavior. Next, another non linear system which is the coupled oscillator was considered. In this case with different values of the parameter, different results were obtained. I obtained a periodic case, a period doubling case and a chaotic case. The corresponding Poincare maps were also obtained. The Poincare maps absolutely went with the theoretical results. Next I generated the bifurcation graphs. Bifurcation is a plot between the position and the parameter whose value we are changing. Bifurcation in the case of coupled oscillator confirmed about the period doubling and from the graph we came to know about the value of the parameter at which the period doubled. In future I have intentions to work in synchronization of coupled oscillator, but for now it's beyond the scope of my project.

It was started from scratch and finally I was able to do a systematic study of non linear dynamics where I learnt about the basics of this dynamics, attractors and their behaviors and then I virtually designed codes to get the trajectory of two different non linear systems, study their behavior and match the results with expected theoretical notion. All the programs were written in FORTRAN 90 and the graphs were plotted in GNUPLOT.

FORTRAN CODES:

Following are the important codes used in my project.

1. **To obtain the phase space graph of the double pendulum.**

```
program double
implicit none
real :: x0,v0,t0,a,b,w0,dt
real, dimension(10000) :: v1,x1,t,v2,x2
integer :: i,k
open(10,file="phase.dat")
dt=0.01
x0=0.7854
v0=0.0
t0=0
x1(1)=x0
v1(1)=v0
x2(1)=x0+.001
v2(1)=v0
t(1)=t0
do i=2,10000
x1(i)=x1(i-1)+dt*v1(i-1)
if(x1(i)>3.14) x1(i)=x1(i)-6.28
if(x1(i)<-3.14) x1(i)=x1(i)+6.28
v1(i)=v1(i-1)+dt*(-0.5*v1(i-1)-sin(x1(i-1))+1.15*cos(2*t(i-
1)/3))
x2(i)=x2(i-1)+dt*v2(i-1)
if(x2(i)>3.14) x2(i)=x2(i)-6.28
if(x2(i)<-3.14) x2(i)=x2(i)+6.28
```

```
v2(i)=v2(i-1)+dt*(-0.5*v2(i-1)-sin(x2(i-1))+1.15*cos(2*t(i-
1)/3))
t(i)=t(i-1)+dt
write(10,*) x1(i-1),v1(i-1),x2(i-1),v2(i-1)
enddo
end program
```

2. To find the Poincare section of the double pendulum.

```
program double
implicit none
real :: x0,v0,t0,a,b,w0,dt
real, dimension(1000000) :: v1,x1,t,v2,x2
integer :: i,k=1
open(10,file="phase.dat")
dt=0.01
x0=0.7854
v0=0.0
t0=0
x1(1)=x0
v1(1)=v0
x2(1)=x0+.001
v2(1)=v0
t(1)=t0
do i=2,1000000
x1(i)=x1(i-1)+dt*v1(i-1)
if(x1(i)>3.14) x1(i)=x1(i)-6.28
if(x1(i)<-3.14) x1(i)=x1(i)+6.28
```

```
v1(i)=v1(i-1)+dt*(-0.5*v1(i-1)-sin(x1(i-1))+1.15*cos(2*t(i-
1)/3))
x2(i)=x2(i-1)+dt*v2(i-1)
if(x2(i)>3.14) x2(i)=x2(i)-6.28
if(x2(i)<-3.14) x2(i)=x2(i)+6.28
v2(i)=v2(i-1)+dt*(-0.5*v2(i-1)-sin(x2(i-1))+1.15*cos(2*t(i-
1)/3))
if(abs((3*3.14*k)-t(i-1))<=dt)then
  write(10,*)x1(i-1),v1(i-1)
  k=k+1
  endif
 t(i)=t(i-1)+dt
enddo
end program
```

3. To find the phase space graph for the parametric coupled oscillator.

```
program pendulum
implicit none
real :: x0,v0,t0,a,b,w0,dt,lam,w,t1
real, dimension(10000) :: v1,x1,t,v2,x2
integer :: i,k=1
open(15,file="phasec1.dat")
open(16,file="phasec2.dat")
print*, "enter the value of lambda"
read*,lam
print*, "enter the value of ohmega zero"
```

```
read*,w0
print*, "enter the value of ohmega"
read*,w
print*, "enter the value of a"
read*,a
print*, "enter delta t"
read*,dt
print*, "enter value of x0"
read*,x0
v0=0.0
t0=0
x1(1)=x0
v1(1)=v0
x2(1)=x0+.001
v2(1)=v0
t(1)=t0
do i=2,10000
x1(i)=x1(i-1)+dt*v1(i-1)
v1(i)=v1(i-1)+dt*((-2*lam*x1(i-1))-((w0**2)*(1+(a*cos(w*t(i-
1)))*sin(x1(i-1)*t(i-1))))-(k*(x1(i-1)-x2(i-1))))
x2(i)=x2(i-1)+dt*v2(i-1)
v2(i)=v2(i-1)+dt*((-2*lam*x2(i-1))-((w0**2)*(1+(a*cos(w*t(i-
1)))*sin(x2(i-1)*t(i-1))))-(k*(x2(i-1)-x1(i-1))))
  write(15,*)x1(i-1),v1(i-1)
  write(16,*)x2(i-1),v2(i-1)
t(i)=t(i-1)+dt
enddo
end program
```

4. **To find the Poincare section of the parametric coupled oscillator.**

```
program pendulum
implicit none
real :: x0,v0,t0,a,b,w0,dt=0.01,lam,w,t1
real, dimension(100000) :: v1,x1,t,v2,x2
integer :: i,k
open(10,file="phasec1.dat")
open(11,file="phasec2.dat")
print*, "enter the value of lambda"
read*,lam
print*, "enter the value of ohmega zero"
read*,w0
print*, "enter the value of ohmega"
read*,w
print*, "enter the value of a"
read*,a
print*, "enter the value of k"
read*,k
print*, "enter value of x0"
read*,x0
v0=0.0
t0=0
x1(1)=x0
v1(1)=v0
x2(1)=x0+.001
v2(1)=v0
t(1)=t0
do i=2,100000
```

```
x1(i)=x1(i-1)+dt*v1(i-1)
v1(i)=v1(i-1)+dt*((-2*lam*x1(i-1))-((w0**2)*(1+(a*cos(w*t(i-
1)))*sin(x1(i-1)*t(i-1)))))-(k*(x1(i-1)-x2(i-1))))
x2(i)=x2(i-1)+dt*v2(i-1)
v2(i)=v2(i-1)+dt*((-2*lam*x2(i-1))-((w0**2)*(1+(a*cos(w*t(i-
1)))*sin(x2(i-1)*t(i-1)))))-(k*(x2(i-1)-x1(i-1))))
if(abs((2*3.14/w)-t(i-1))<=dt)then
  write(10,*)x1(i-1),v1(i-1)
  write(11,*)x2(i-1),v2(i-1)
  k=k+1
endif
t(i)=t(i-1)+dt
enddo
end program
```

5. **To find the bifurcation graph of the parametric coupled oscillator.**

```
program pendulum
implicit none
real :: x0,v0,t0,b,w0=1,dt=0.01,lam,w,t1,k
real, dimension(500000) :: v1,x1,t,v2,x2
integer :: i,a=1
open(15,file="phasec11.dat")
open(16,file="phasec2.dat")
open(17,file="phasec3.dat")
print*, "enter the value of lambda"
read*,lam
```

```
print*, "enter the value of ohmega"
read*,w
print*, "enter the value of k"
read*,k
print*, "enter the value of x0"
read*,x0
v0=0.0
t0=0
x1(1)=x0
v1(1)=v0
x2(1)=x0+.001
v2(1)=v0
t(1)=t0
do i=2,10000
x1(i)=x1(i-1)+dt*v1(i-1)
v1(i)=v1(i-1)+dt*((-2*lam*x1(i-1))-((w0**2)*(1+(a*cos(w*t(i-
1)))*sin(x1(i-1)*t(i-1))))-(k*(x1(i-1)-x2(i-1))))
x2(i)=x2(i-1)+dt*v2(i-1)
v2(i)=v2(i-1)+dt*((-2*lam*x2(i-1))-((w0**2)*(1+(a*cos(w*t(i-
1)))*sin(x2(i-1)*t(i-1))))-(k*(x2(i-1)-x1(i-1))))
  write(15,*)x1(i-1),v1(i-1)
  write(16,*)x2(i-1),v2(i-1)
if(MOD(a,50)==0)then
write(17,*)x1(i-1),a
endif
  a=a+1
t(i)=t(i-1)+dt
enddo
end program
```

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