ANALYTIC FUNCTIONS WITH RING THEORY

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DECLARATION

I hereby certify that the revised work is being presented in this thesis entitled "ANALYTIC FUNCTIONS WITH RING THEORY"in partial fulfillment of the requirement for the degree of Master of Science. This revise work carried out by me and the thesis has not formed the basis for the award of any other degree.

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CERTIFICATE

This is to certify that the thesis entitled "ANALYTIC FUNCTIONS WITH RING THEORY" which is being submitted by Mr. Balaram Sahu having Roll No.413MA2070, for the award of the degree of Master of Science from National Institute of Technology, Rourkela is a record of bonafide research work, carried out by him under my supervision. The results embodied in this thesis are modified and have not been submitted to any other university or institution for the award of any degree or diploma.

To the best of my knowledge, Mr. Balaram Sahu bears a good moral character and is mentally and physically fit to get the degree.

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ABSTRACT

Complex Analysis is a subject which has something important and interesting for all mathematicians. In addition to having applications to other parts of analysis, it can rightly claim to be an ancestor of many areas of mathematics like pure mathematics and applied mathematics. We know that the Ring theory have nice properties. In this thesis we will discuss some definitions and notations of some elementary terms like integral domain, ideals, and Maximal ideals. This view of complex analysis and Ring theory the thesis submitted by me has named as Entire functions with Ring theory has influenced me to writing and selection of subjects matter. In each chapter all concepts and definitions have been discussed in detail and in lucid manner i.e clear and easy to understand, so that some one should fell no difficulties. Chapter 1 includes only the introduction of this thesis paper that whatever work done by me and also what is the consequence that how the results are modified simply (no new results) Chapter 2 is the basic of Holomorphic function and Meromorphic functions and Entire function and some definitions are given. By using the definitions of these we will also discussed about Univalent functions and the area theorem and its consequence. Chapter 3 contains the basic idea of preparation for the ideal theory in the rings of Analytic functions and also ideal structure of the rings of Analytic functions. In this chapter we use the concept of divisibility of an integral domain and the theorem based on homomorphism.

Contents

CHAPTER 1

1 INTRODUCTION

In the field of complex analysis there is a important and useful concept which is the heart of complex analysis is called as Analytic functions. We have a idea about the limit, continuity and differentiability of functions of complex variables. After then by using all these a basic and heart of the complex analysis arises in our mind that is Analytic functions.The analytic functions have nice properties which are given details in chapter 1. We know that a group is abasic structure with only one bianary operation. We shall study another basic structure 'Ring' with two binary operations. Such basic structures proved the fundamental theorem of algebra or the solvability of the problem of trisection of an angle. This makes the study of rings, fields and integral domains more significant and interesting.In this thesis it is shown that a ring homomorphism on the set of all analytic functions.The algebra of analytic functions on a regular region G in the complex plane is either linear or conjugate linear provided that the ring homomorphism takes the identity function into a nonconstant function. This thesis is the combination of Complex analysis and Abstract algebra. There is a good relation and some important results between the Analytic functions and the Ring theory. The divisibility property plays an important role to define the relation. The role of Ideal of a ring helps to the preparation of ideals in $A = A(G)$ is the set of all analytic functions. The set $A = A(G)$ form an integral domain, for which we have some results in this thesis.To study the analyticity and uses of ring theory we have the following definitions and notations. The following definitions are (given in [1],[2]).

1.1 Functions of a complex variable

If certain rules are given by means of which it is possible to find one or more complex numbers w for every value of z in a certain domain G , w is said to be a function of z defined on the domain G and we write $w = f(z)$ Since $z = x + iy$, $f(z)$ will be of the form $u + iv$, where $u = u(x, y)$ and $v = v(x, y)$.

1.2 Neighborhood of a point

A neighborhood of a point z_0 in the argand plane is the set of all points z such that $|z - z_0| < \delta$, where δ is an arbitrary small positive number.

1.3 Limit of a function

Let $f(z)$ be any function of the complex variable z defined in a open connected set G. Then $'l'$ is said to be the limit point of $f(z)$ as z approaches along any path in G if for any arbitrary chosen positive number ϵ(small but not zero), then there exists a corresponding number $\delta > 0$ such that

$$|f(z) - l| < \epsilon$$

$\forall$ values of z for which $0 < |z - a| < \delta$. Symbolically,

$$\lim_{z \to a} f(z) = l$$

1.4 Continuity

Let $f(z)$ be any function of the complex variable z defined in a open connected set G is said to be continuous at $a \in G$ if and only if for ϵ then there exists a corresponding number $\delta > 0$ such that $|f(z) - f(a)| < \epsilon$, whenever $|z - a| < \delta$. It follows that from the definition of limit and continuity that $f(z)$ is continuous at $z = a$ iff

$$\lim_{z \to a} f(z) = f(a)$$

CHAPTER 2

2 ANALYTIC FUNCTIONS AND UNIVALENT FUNCTIONS

2.1 Introduction

In this chapter we recall some definitions and known results on Analytic functions in a complex plane, zeros of analytic functions, entire functions etc. This chapter serves as base and background for the study of upcoming chapters chapters. We have to keep on referring back to it as and when required.

2.2 Analytic functions

We know about functions of complex variables and also know about limit, continuity and differentiability of complex functions. Now we will have to introduce the concept of an entire function. A function of the complex variable z is analytic at a point if it has the derivative at each point in some neighborhood of. It also follows that if is analytic at a point; it must be analytic at each point in some neighborhood of. A function is analytic in an open set S if it has a derivative everywhere in that set S. If the set S is closed, it is to be remember that is analytic in an open set containing S.

Is it true that the analyticity at each points which are greater than zero in the finite plane?. If a function is differentiable only at a point but not throughout of some neighborhood, then can not say that the function is analytic. The necessary condition for a function should be analytic in a domain G if it is continuous of throughout G, but the sufficient condition is not true. Satisfactions of Cauchy Riemann equation is a necessary condition but not sufficient condition. Let two functions are analytic in the domain G, then their sum and their product are both analytic in G according to the properties of differential coefficient. We can say a function $f \in G$ which is a combination of two functions in the form quotient is analytic if the denominator does not vanishes at any point. We will see that the composition of two analytic functions is analytic if the two functions are analytic according to the chain rule of derivative. Note that if f and g are conjugate each other and both are analytic in the same domain, and then the function is constant.

2.3 Entire functions

A function $f$is called entire function if it is analytic in every finite region G of the complex plane. It is also called as integral function. We can expand such functions in power series about any point in the complex plane. The simplest form of an entire function is where the radius of convergence can be taken as large as we wish. The power series is convergent in the whole plane. The only singularity of an entire function may be at infinity. An entire function is divided by into three categories:

(i) An entire function is a constant if it has no singularity at infinity.

(ii) An entire function is said to be an entire transcendental function if it has an essential singularity at infinity.

(iii) The entire function f is said to be a polynomial of degree n if a f has a pole of order n at infinity.

2.4 Univalent functions

Definition 2.3.1 Let $w = f(z)$. If w takes only one value for each value of z in the region G, then w is said to be a uniform or single valued function.

Definition 2.3.2 A single valued function f is said to be $univalent function$ in a domain $G \in \mathbb{C}$ if it does not take the same value twice. That means if $f(z_1) \neq f(z_2)$ for all points z_1 and z_2 in G with $z_1 \neq z_2$. It is also called as Schlicht function.

The function f is said to be locally equivalent at a point $z_0 \in G$ if it is univalent in some neighborhood of z_0.

Difinition 2.3.3

A transformation $w = f(z)$ defined on a domain G is said to be conformal mapping or transformation , when it is conformal at each point in G. It means the mapping is conformal in G if f is analytic in G and its derivative f' has no zeros there.

Note:

(i) For analytic function f, the condition is equvalent to local univalent at z_0.

(ii) Since an analytic univalent function satisfies the angle preserving property thats why it is called conformal mapping.

Notation:

Let S denote the set of analytic, univalent functions on the disk G normalized by the condition that $f(0) = 0$ and $f'(0) = 1$. That is

$$S = \{f : G \to \mathbb{C} : f \text{ is analytic and univalent on } G, f(0) = 0, f'(0) = 0\}$$

It follows that every $f \in S$ has a Taylor expansion of the form $f(z) = z + a_2 z^2 + a_3 z^3 + ...\ ;\ |z| < 1$. Where $a_n \in \mathbb{C}$, $n = 2, 3, ...$

In order to simplify certain formulae, we will sometimes follow the convention of setting $a_1 = 1$ for $f \in S$.

Example 2.3.1 The best example of a function of class S is the Koebe function given as follows:

$$k(z) = \frac{z}{(1-z)^2}$$

Applying the Binomial expansion theorem, we will get

$$k(z) = z[1 + 2z + 3z^2 + ...]$$

$$\Rightarrow k(z) = z + 2z^2 + 3z^3 + ...$$

Example 2.3.2 The example shows that $f \in S$ and $g \in S$ need not imply that $f + g \in S$. So that is not closed under addition.

Solution : Let $f(z) = \frac{z}{1-z}$ and $g(z) = \frac{z}{1+iz}$ so that $f, g \in S$

Again,

$$f(z) = \frac{z}{1-z} = f \quad (say)$$

First take logarthim in both side and then differentiate both side we will get,

$$f'(z) = \frac{1}{(1-z)^2}$$

Similarly let

$$g(z) = \frac{z}{(1+iz)^2} = g \quad (say)$$

First take logarthim in both side and then differentiate both side we will get,

$$g'(z) = \frac{1}{(1+iz)^2}$$

Hence

$$f'(z) + g'(z) = \frac{1}{(1-z)^2} + \frac{1}{(1+iz)^2}$$

$$\Rightarrow f'(z) + g'(z) = \frac{1 - z^2 + 2iz + 1 - 2z + z^2}{(1-z)^2(1+iz)^2}$$

$$\Rightarrow f'(z) + g'(z) = \frac{2 - 2z(1-i)}{(1-z)^2(1+iz)^2}$$

From which we conclude that if $f'(z)+g'(z) = 0$ if $z = \frac{1}{1-i} = \frac{1+i}{2}$. This shows that $f+g \notin S$.That means S is not closed under addition.

This example is a counter example from which we conclude that S can not form a group, hence not a ring and hence not an Integral domain.

Theorem 2.3.1 The class S is preserved under the following transformations:

(i) Ratation: If $f \in S$ and $g(z) = e^{-i\theta} f(e^{i\theta} z)$, then $g \in S$, then $g \in S$.

(ii) Dilation: If $f \in S$ and $g(z) = r^{-1} f(rz)$, where $0 < r < 1$, the $g \in S$. (iii) Conjugation: If $f \in S$ and $g(z) = \overline{f(\overline{z})} = z + \overline{a_2} z^2 + ...$, then $g \in S$.

Proof: To prove this theorem that S is preserved under rotation we begin by noting that the composition of one-to-one mapping.

(i)Let $f \in S$

Let $R(z) = e^{i\theta}$ and $T(z) = e^{-i\theta}$.

Clearly the maps

$R : \mathbb{C} \to \mathbb{C}$ and $T : \mathbb{C} \to \mathbb{C}$ are one-to-one. Since $g(z) = e^{i\theta} f(e^{i\theta})$

$$\Rightarrow g(z) = (T \circ f \circ R)(z)$$

is a composition of one-to-one mapping, we conclude that $'g'$ is univalent on G.

Here

$$g(z) = e^{i\theta} f(e^{i\theta})$$

$$\Rightarrow g'(z) = e^{i\theta} f'(e^{i\theta} z) e^{i\theta}$$

$$\Rightarrow g'(z) = f'(e^{i\theta}z)$$

We will see that g is analytic on G. Furthermore, $g(0) = 0$ and $g'(0) = f'(0) = 1$

So that $g \in S$. We also note that the Taylor expanssion of g is given by

$g(z) = e^{-i\theta}(e^{i\theta}z + a_2e^{2i\theta}z^2 + ...) \Rightarrow g(z) = z + a_2e^{i\theta}z^2 + ...$

Hence the result proved.

(ii) Suppose that $f \in S$ and $0 < r < 1$.

Let $R(z) = rz$ and $T(z) = \frac{z}{r}$, so that the maps $R : \mathbb{C} \to \mathbb{C}$ and $T : \mathbb{C} \to \mathbb{C}$ are one-to-one. Since $g(z) = \frac{1}{r}f(rz) = (T \circ f \circ R)(z)$ is a composition of one-to-one mapping. From which we conclude that g is univalent on G.

Since $g'(z) = \frac{1}{r}.rf'(rz) = f'(rz)$

We will see that g is analytic on G. Furthermore, $g(0) = f(0)$ and $g'(0) = f'(0) = 1$, so that $g \in S$ as required.

We also note that the Taylor expansion of g is given by

$$g(z) = \frac{1}{r}(rz + a_2r^2z^2 + ...)$$

$$\Rightarrow g(z) = z + a_2rz^2 + a_3r^2z^3 + ...$$

(iii) Suppose that $f \in S$.

Let $w(z) = \overline{z}$ so that $w : \mathbb{C} \to \mathbb{C}$ is clearly one-to-one. Since $g(z) = \overline{f(z)} = (w \circ f \circ w)(z)$ is a composition of one-to-one mappings. Then we conclude that g is univalent on G. Note that $w(z)$ is not analytic on G, and so we can not simply use the fact that a composition of analytic functions is analytic as was done in (i) and (ii). Again, we note that the Taylor series for f, given by

$$z + \sum_{n=2}^{\infty} a_n z^n$$

has radius of convergence 1. That is, the above Taylor series converges to $f(z)$ for all $|z| < 1$ with the convergence uniform on every closed disk $|z| \leq r < 1$. It is then follows that the Taylor series,

$$z + \sum_{n=2}^{\infty} \overline{a_n} z^n$$

has radius of convergence 1, and so it is defined an analytic function on G.

Hence we coclude that

$$g(z) = \overline{f(z)} = z + \overline{a_2}z^2 + \overline{a_3}z^3 + ...$$

is analytic on G with $g(0) = 0$ and $g^{(\prime)}(0) = 1$. Thus $g \in S$ as required.

Taking (i) , (ii) and (iii) together computes the proof of theorem (2.3.1).

Hence the theorem proved. □

2.5 The Area theorem and its consequence

The area theorem helps us to calculate the area of a region easily. As to find the area of a region we were using the Green's theorem, but instead it we can also get the same area of the rigion by using the Area theorem.

Theorem: If $f : G \to f(G)$ is a conformal mapping of G with $f(0) = 0$ and $|f'(0)| > 0$ so that f has a Taylor expansion

$$f(z) = a_1 z + a^2 z^2 + ... \tag{1}$$

where $|z| < 1$ with $a_1 \in R$, $a_1 > 0$ then

$$Area((f(G)) = \pi \sum_{n=1}^{\infty} n(|z|)^2$$

. Where G is in Z plane and $f(G)$ is in G plane.

Proof Suppose let $G = f(G)$. We know that from Green's theorem

$$Area(G) = \frac{1}{2i} \int_{\delta G} \overline{w} dw$$

Let $w = f(z)$. So by changing variable gives

$$Area(G) = \frac{1}{2i} \int_{\delta G} \overline{f(z)} f'(z) dz \tag{2}$$

From $eq.(1)$ and $eq.(2)$ we will get,

$$Area(G) = \frac{1}{2i} \int_{\delta G} (\sum_{n=1}^{\infty} \overline{a_n} z^n)(\sum_{m=1}^{\infty} m a_m z^{m-1}) dz$$

Put

$$z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta$$

$$\Rightarrow Area(G) = \frac{1}{2i}\int_0^{2\pi}(\sum_{n=1}^{\infty}\overline{a_n}e^{-in\theta})(\sum_{m=1}^{\infty}ma_m z^{i(m-1)\theta})d\theta$$

Where $0 \leq \theta \leq 2\pi$

$$\begin{aligned}
\Rightarrow Area(G) &= \frac{1}{2}\int_0^{2\pi}(\sum_{k=1}^{infty}k\overline{a_k}a_k)d\theta \\
\Rightarrow Area(G) &= (\sum_{k=1}^{\infty}k|a_k|^2)\frac{1}{2}\int_0^{2\pi}d\theta \\
\Rightarrow Area(G) &= \frac{1}{2}\sum_{k=1}^{\infty}k|a_k|^2[\theta]_0^{2\pi} \\
\Rightarrow Area(G) &= \frac{1}{2}\sum_{k=1}^{\infty}k|a_k|^2 2\pi \\
\Rightarrow Area(G) &= \sum_{k=1}^{\infty}k|a_k|^2\pi \\
\Rightarrow Area(G) &= \pi\sum_{k=1}^{\infty}k|a_k|^2
\end{aligned}$$

Taking $k = n$, so

$$Areaf((G)) = \pi\sum_{n=1}^{\infty}n|a_n|^2$$

Example: Consider the function $f(z) = 2z - \frac{2}{3}z^2$ so that $f \in S$, f maps the unit disk onto the interior of a Cardiod. Determine the area of the Cardiod and verify that f is bounded. That is $a_1 = 2$ and $a_2 = \frac{-2}{3}$. □

Solution: Here given that $f(z) = 2z - \frac{2}{3}z^2$. So

$$Areaf((G)) = \pi\sum_{n=1}^{\infty}n|a_n|^2$$

$$\Rightarrow Area(G) = \pi[1.|2|^2 + 2.|\frac{-2}{3}|^2] = \pi(4 + \frac{8}{9}) = \frac{44\pi}{9}$$

Which is the required area of the cardiode. ($Ans.$)

CHAPTER 3

3 THE RING THEORY IN ANALYTIC FUNCTIONS

3.1 Introduction

In this chapter we will discuss about the preparation of ideals for $A = A(G)$, the set of all analytic functions and also A forms an Integral domain. By using the divisibility properties and ideal concepts we shall discuss some important results and its consequences. In this section (f) denotes the generator, i.e. it generates $A = A(G)$ and denoted by (f).

3.2 Greatest common divisor

A function $f \in A(G)$ is called a divisor of $g \in A(G)$ if $f = g.h$, where $h \in A(G)$.

There is a relation between divisibility for elements $f, g \neq 0$ and their principal divisors (f), (g).

Theorem 3.1.1 Suppose $f, g \in A(G)\backslash\{0\}$. Then f divides g if and only if $(f) \subseteq (g)$.

Proof: $(\Rightarrow:)$ Let the function f divides g. So there exists $h \in A(G)$ such that

$$f = g.h \in A(G)$$

$$\Rightarrow h = g\backslash f$$

$$\Rightarrow O_z(h) = O_z(g) - O_z(f)$$

$$\Rightarrow O_z(g) \geq O_z(f)$$

$$\Rightarrow O_z(f) \leq O_z(g) \Rightarrow (f) \subseteq (g)$$

. $(:\ \Leftarrow)$ Let $O_z(f) \subseteq O_z(g) \Rightarrow O_z(g) - O_z(f) \geq 0 \Rightarrow O_z(h) = O_z(g) - O_z(f)$

So $h = g\backslash f \Rightarrow f = g.h \in A(G)$

That is f is a divisor of g. $(Proved)$

Definition: Let $X \subset A(G)$ and $X \neq \phi$, then $f \in A(G)$ is called a common divisor of X if f divides every element g of X. Let f be a common divisor of X. If every common divisor of X is a divisor of f, then f is called a greatest common divisor of X . It is denoted by $f = gcd(X)$.

Note: The Greatest common divisors are unique.

Definition: Let $X \subset A(G)$ and $A \neq \phi$. Then A is called relatively prime if $gcd(X) = 1$.

Theorem

Let $f \in A(G)$ and a set $Y \subset A(G)$. Suppose $(f) = \{(g) : g \in Y; g \neq 0\}$. Then f is greatest common divisor of $Y \neq \phi$.

Proof: Suppose $f \in A(G)$.

Given that

$$(f) = \{(g) : g \in Y; g \neq 0\}$$

$$\Rightarrow (f) \leq (g)$$

i.e. f divides g for all $g \in Y$

i.e. f is a common divisor to Y.

Consider another divisor h of Y, so $(h) \leq (g) \Rightarrow (h) \leq (f)$.i.e. h divides f. Hence f is greatest common divisor of (Y). i.e. $f = gcd(Y)$.

Theorem: Suppose $Y \subset A(G)$ and $Y \neq \phi$. Then Y is a relative prime if and only if the functions in Y have no common zeros in G.

Proof: Let $Y \subset A(G)$ and $Y \neq \phi$.

($\Rightarrow$:) Given that Y is relatively prime

To show : The functions of Y have no common zeros in G.

Let g_1,g_2,...,g_n are in Y. Consider $\bigcap_{g \in Y} Z(g) = Z$.

To Show: $Z = \phi$. We will prove this by method of contradiction.

Let $Z \neq \phi$. Then a function exists on a point on $Z \neq \phi$ with order 1 and also have order less or equal to every $g \in Y$. Hence it has be a common divisor of Y and thus divides one. Which is a contradiction.

(: $\Leftarrow$) Let $Z = \phi = Z(1)$.Then 1 has order less than or equal to every $g \in Y$ and hence $1 = gcd(Y)$. that is Y is relatively prime.

Hence the result. $\square$

3.3 Preparation for the ideal theory in $A = A(G)$

To prove some important result about the ideal structure in the integral domain $A(G)$ we make use of some important definitions and notations. We will start with ideals.The following definitions are given in ([3],[4],[5],[6])

3.3.1 Ideals

Since we are going to consider the ideal structure in the ring of functions holomorphic in open connected subsets of $\mathbb{C}$ we will give some basic definitions.

In this subsection R is a commutative ring with unity.

Definitions:

Let R be the ring. A subset $U \subset R$ and $U \neq \phi$ is called an ideal in R if U is an additive subgroup and $ur, ru \in U$ for all $u \in U$ and $r \in R$.

Now, let $N \neq \phi$ be any subset of R and L be the set of all finite linear combinations i.e.

$$L = \{u = \sum_{i=1}^{n} r_i f_i, r_i \in R, f_i \in N\}$$

. Then L is an ideal in R and N is said to generate L.

Definition An ideal U in a ring R is called finitely generated if there is a finite set that generates U.

If we can choose N with only one element, we will a special case.

Definition: An ideal U in a ring R is called principal if it is generated by an element f, i.e. if $U = \{rf : r \in R\}$ for some $f \in R$. A integral domain R is called a principal ideal domain if every ideal is a principal ideal.

Remark: A finitely generated ideal generated by $N = \{f_1, f_2 ..., f_n\}$ is usually denoted $U = \{r_1 f_1 + ... + r_n f_n\}$. Where $r_i \in R$ for $i = 1, 2, ..., n$. The principal ideal generated by f is usually denoted (f) or Rf.

Definition: An ideal M in a ring R is said tobe maximal ideal if whenever U is an ideal of R such that $U \subset M \subset R$, the either $R = U$ or $M = U$.

Definition: An ideal U is called a prime ideal if $u_1 u_2 \in U$, where $u_1 \in U$ implies that either $u_1 \in U$ or $u_2 \in U$ for $u_1, u_2 \in R$.

Definition: An Ideal $U \in R$ of $\bigcap_{f \in U} A(f)$ is called fixed if it is non empty. Otherwise it is called a free Ideal.

3.4 The Ring of analytic functions

We consider on open Riemann surface G and the ring $A = A(G)$ of all analytic functions on G. In this section we will discuss some of the divisibility properties and ideal theory of this ring. We first note that an element f of A is a unit if and only if it has no zeros. This suggests the introduction of the zero set $N(f) = \{w \in G : f(w) = 0\}$. Any finite number of functions in have a greatest common divisor,that is, a function which divides each and which is divisible by other functions which does so.An important property of greatest common divisors of the.

Proposition 1 Let f_1,f_2,...,f_n be functions in the ring A .Then they have a greatest common divisor q, and if q is any greatest common divisor, then there exist functions e_1,...,e_n in A such that $q = e_1 f_1 + ... + e_n f_n$.

Proposition 2 Let G be an open Riemann Surface and $q \in G$.Consider an ideal U consisting of all analytic functions of W. Then U vanishes at q if ana only if U is the Kernel of ψ which is homomorphism of $A = A(G)$ into the complex numbers such that for all complex constants c $\psi(c) = c$.

Theorem: Let G_1 and G_2 be two open Riemann surfaces. Suppose $A(G_1)$ and $A(G_2)$ be the rings of all analytic functions of G_1 and G_2 respectively. Define a map

$$\phi : A(G_2) \longrightarrow A(G_1)$$

and ϕ is a homomorphism by the rule $\phi(c) = c$ for all complex constants c. Then there exists a unique analytic map

$$\psi : G_1 \longrightarrow G_2$$

by the rule

$$\phi(f) = f \circ \psi$$

. In particular if $A(G_1) \simeq A(G_2)$, then G_1 and G_2 are conformally equivalent.

Proof: Let $q \in G_1$. The ideal containing all functions of $A(G)$ which is vanishes at q. Let this ideal is U_q. By proposition 2 we will see that U_q is the Kernel of homomorphism ψ with $\psi(c) = c.$, where $c \in C$, c is the constant and C is the complex field. So clearly U_q is Kernel.

Let $f \in G_2$ and suppose the value of $\phi(f)$ at q is c. Then,

$$\phi f - c \in U_q$$

$$\Rightarrow (f - c) \in U_{\psi(q)}$$

Hence the value of f at $\psi(q)$ is also c and we will see that

$$\phi(f) = f \circ \psi$$

To Show: ψ is continuous.

Clearly G_1 and G_2 are satisfied the second axiom of countability.

To Show: $q_n \longrightarrow q$ on $G_1 \Rightarrow \psi(q_n) \longrightarrow \psi(q)$ on G_2.

We will prove this by method of contradiction.

Let $f \in A(G_2)$ with $f = 0$ at $\psi(q)$ and $f \neq 0$ at any point z. Then;

$\phi(f(q_n)) \longrightarrow \phi(f(q)) = 0$ while $\phi(f(q_n)) \longrightarrow \phi(f(z)) \neq 0$. Which is a contradiction as $\phi f = f \circ \psi$.

Thus ψ must be continuous.Next

To Show: ψ is analytic.

Let $x \in G_1$ be point. Suppose $f \in A(G_2)$ be the function with a simple zero at $\psi(x)$. Consider $h = \phi f$. Take a neighborhood P of $\psi(x)$ in which the f is univalent. Then there is a neighborhood Q of x such that $Q \subset \psi^{-1}(P)$ and $h(Q) \subset f(P)$. Then in Q we have the representation $f^{-1} \circ h$ for the mapping ψ. Hence ψ is analytic. Next,

To Show: ψ is unique.

Let ψ_1 and ψ_2 be two maps. So

$$\phi f = f \circ \psi_1 = f \circ \psi_2$$

Let q be a point where $\psi_1(q) \neq \psi_2(q)$. Then we have a function $f \in A(G_2)$ with different values at q. Which is a contradiction. So ψ is unique.

Suppose ϕ is isomorphism with onto, then it has the inverse ϕ^{-1}. Let ψ_1 and ψ_2 be the analytic transformations associated with them. Then $\psi_1 \circ \psi_2$ and $\psi_2 \circ \psi_1$ are analytic transformations of G_1 and G_2 respectively. By uniqueness we will see that $\psi_1 \circ \psi_2$ and $\psi_2 \circ \psi_1$ are identity maps on

G_1 and G_2. Hence ψ is one-to-one correspondence between G_1 and G_2. This completes the proof of the theorem. $\square$

References

[1] J.W. Brown and R.V. Churchill, Complex Variables and Applications, McGraw Hill Education,8e, 2014.

[2] J. B. Conway, Functions of One Complex Variable, 2nd edition, 1978.

[3] J. B. Fraleigh, A first course in abstract algebra, Addison-Wesley Publishing Company, 1999.

[4] O. Helmer, Divisibility properties of integral functions, Duke Math. J. 6(1940), 345-356.

[5] M. Henrikssen, On the ideal structure of the ring of entire functions, Pac. Journ. Math. 2(1952), 179-184.

[6] M. Henrikssen, On the prime ideals of the ring of entire functions, Pac. Journ. Math. 3(1953), 711-720.

[7] M.S.Robertson, On the theory of Univalent functions, Ann of Math. 37 (1936), 376-408.