PROJECT ON

EXISTENCE OF WEAK SOLUTIONS OF p-LAPLACIAN PROBLEM

DEPARTMENT OF MATHEMATICS

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CERTIFICATE

This is to certify that the dissertation entitled " Existence of weak solutions of the p-Laplacian problem" is a bonafide record of independent research work done by Asim Patra(Roll no-413ma2066) under the guidance and supervision of Dr. Debajyoti Choudhuri and submitted to National Institute of Technology, Rourkela in partial fulfilment of the award of the degree of Master Of Science in Mathematics.

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INTRODUCTION:

In the given project titled "Existence of weak solutions of the p-Laplacian problem" , unique methodology is used to find the existence of weak solution to the given p-Laplacian problem. There are several ways of solving the problem out of which the Nehari method of finding solutions is used in this project , specifically which uses the concept of the mountain pass theorem. In other words, the Nehari hypothesis comes into existence ,or we can say, can be derived from the famous mountain pass theorem which is very helpful in case of finding solutions to the linear or non-linear partial differential equations. The study of differential equations and variational problems with variable exponent has been a new and interesting topic. Its interest is widely justified with many physical examples, such as nonlinear elasticity theory, electrorheological fluids, etc. The study on variable exponent problems is attracting more and more interest in recent years, for example, there have been many contributions to nonlinear elliptic problems associated with the p(x)-Laplacian for a thorough overview of the recent advantages from various view points.

Literature Survey:

In all the papers, information about the p-Laplacian as well as different techniques of solving it are provided which is actually, in mathematics, a quasilinear elliptic partial differential operator of second order . In the paper by Bin Ge(2013) , a method of formulating the sign changing solutions was obtained which also included the techniques of finding the weak solutions of a given p-Laplacian problem. Lorenzo,Enea P,Marco(2015) had introduced methodologies for stability of the variational eigenvalues for the fravtional p-Laplacian. In the paper by Dragomir(2005) , the first eigenvalue came into existence for the p-Laplacian operator.In this paper, he specifically mentioned the variational techniques of finding the eigenvalues.Kovacik and Rakosuik(1991) gave the different properties of the spaces including the Sobolev spaces and also the L^p spaces.This paper thoroughly gives a clear understanding of both the spaces with some new forms.While Fan X L , Zhang Q H(2003) discussed about the existence of the p(x)-Laplacian dirichlet problems. Also the embedding theorems i.e. the Sobolev embedding theorems for the spaces $W^{1,p}$ are tackled in a very unique manner by Fan X L,Shen J,Zhao D(2001)

ABSTRACT

This project deals with the variational and the Nehari manifold method or by the Nehari hypothesis for the p-Laplacian equations in a bounded domain or in the whole space.Then a proof of the existence of the weak solutions of the given p-Laplacian problem is given under certain specific conditions. In this project ,a different approach is used to tackle the proposed p-Laplacian problem which is by the variational method by using the mountain pass theorem which is used to guarantee the existence of solutions of the non-linear partial differential equation. Few information about various topics which is required to solve the proposed problem like the Sobolev spaces,Sobolev embeddings,distribution theory,Sobolev inequalities etc, are provided to have better understanding of the given p-Laplacian problem.The hypothesis by Nehari along with the techniques to solve non-linear partial differential equations is given along with some of the theorem like the Lax-Milgram theoerem, mountain pass theorem and the Banach fixed point theorem.

CONTENTS

1 DISTRIBUTIONAL THEORY

1.1 Distributions

Distributions or the generalized functions are objects that generalize the classical notion of functions in mathematical analysis. They make it possible to differentiate functions for which derivatives doesnot exist. In particular the locally integrable functions have distributional derivative. They are generally encountered in case of the partial differential equations(PDE) . The basic idea in distribution theory is to reinterpret functions as linear functionals acting on a space of test functions. Standard functions act by integration against a test function, but many other linear functionals do not arise in this way, and these are the "generalized functions". There are different possible choices for the space of test functions, leading to different spaces of distributions. The basic space of test function consists of smooth functions with compact support, leading to standard distributions. They are in mathematical sense a class of linear functionals which are a mapping from the set of test functions to set of real numbers.

$$T : D(R) \rightarrow R$$

1.2 Test functions

In the simplest case, the set of test functions considered is D(R), which is the set of functions $\varphi : R \rightarrow R$ having two properties:

- φ is smooth(infinitely differentiable);
- φ has a compact support(is identically zero outside some bounded interval).

A distribution T is a linear mapping $T : D(R) \rightarrow R$. Instead of writing $T(\varphi)$, it is conventional to write $\langle T, \varphi\rangle$ for the value of T acting on a test function φ. A simple example of a distribution is the δ called as the Dirac delta , defined by

$$\langle \delta , \varphi\rangle = \varphi(0)$$

meaning that δ evaluates a test function at 0. Its physical interpretation is as the density of a point source.In case of test functions, there is a space called the test function space. The space D(U) of test functions on U is defined as follows. A function : $U \rightarrow R$ is said to have compact support if there exists a compact subset K of U such that $\varphi(x) = 0$ for all x in $U \setminus K$. The elements of D(U) are the infinitely differentiable functions $\varphi : U \rightarrow R$ with compact support – also known as bump functions. This is a real vector space. It can be given a topology by defining the limit of a sequence of elements of D(U). A sequence (φ_k) in D(U) is said to converge to $\varphi \in D(U)$ if the following two conditions hold :

- There is a compact set $K \subset U$ containing the supports of all φ_k:

$$\cup_k supp(\varphi_k) \subset K$$

- For each multi-index α, the sequence of partial derivatives $\partial^\alpha \varphi_k$ tends uniformly to $\partial^\alpha \varphi$

1.3 Functions as distributions

The function $f : U \to R$ is called locally integrable if it is Lebesgue integrable over every compact subset K of U. This is a large class of functions which includes all continuous functions and all L^p functions. The topology on D(U) is defined in such a fashion that any locally integrable function f yields a continuous linear functional on D(U) – that is, an element of D′(U) – denoted here by T_f, whose value on the test function φ is given by the Lebesgue integral :

$$\langle T_f, \varphi \rangle = \int_U f \, \varphi \, dx$$

Conventionally, one abuses notation by identifying T_f with f, provided no confusion can arise, and thus the pairing between T_f and φ is often written:

$$\langle f, \varphi \rangle = \langle T_f, \varphi \rangle$$

If f and g are two locally integrable functions, then the associated distributions T_f and T_g are equal to the same element of D′(U) if and only if f and g are equal almost everywhere . In a similar manner, every Radon measure μ on U defines an element of D′(U) whose value on the test function φ is $\int \varphi d\mu$. As above, it is conventional to abuse notation and write the pairing between a Radon measure μ and a test function φ as $\langle \mu, \varphi \rangle$. Conversely, as shown in a theorem by Schwartz (similar to the Riesz representation theorem), every distribution which is non-negative on non-negative functions is of this form for some (positive) Radon measure.The test functions are themselves locally integrable, and so define distributions. As such they are dense in D′(U) with respect to the topology on D′(U) in the sense that for any distribution $T \in$ D′(U), there is a sequence $\varphi_n \in$ D(U) such that

$$\langle \varphi_n, \psi \rangle \to \langle T, \psi \rangle \ , \quad \forall \ \psi \in D(U)$$

1.4 Derivatives of distributions

It is desirable to choose a definition for the derivative of a distribution which, at least for distributions derived from smooth functions. Then the disributional derivative of a locally integrable function say f is given by

$$< f' , \emptyset > = \int_R f'(x) \, \emptyset(x) dx$$

$$= - < f , \emptyset' >$$

Let us take the example of the Heaviside function denoted by H(x) and which is defined by

$$H(x) = \begin{cases} 1 & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

Then the distributional derivative of a Heaviside function is found as:

$$< H' , \emptyset > = - < H , \emptyset' >$$

$$= \int_R H(x)\emptyset'(x)dx$$

$$= \int_{-\infty}^{\infty} H(x)\emptyset'(x)dx$$

$$= - \int_0^{\infty} \emptyset'(x)dx$$

$$= -[\emptyset(\infty) - \emptyset(0)] = \emptyset(0) = <\delta , \emptyset >$$

So we will get that the dirac delta function is the distributional derivative of the Heaviside function.

1.5 Convolutions

In mathematics and, in particular, functional analysis, convolution is a mathematical operation on two functions *f* and *g*, producing a third function that is typically viewed as a modified version of one of the original functions, giving the area overlap between the two functions as a function of the amount that one of the original functions is translated. Convolution is similar to cross-correlation. It has applications that include probability, statistics, computer vision, image and signal processing, electrical engineering, and differential equations.

The convolution of *f* and *g* is written *f***g*, using an asterisk or star. It is defined as the integral of the product of the two functions after one is reversed and shifted. As such, it is a particular kind of integral transform:

$$f * g\,(t) = \int_{-\infty}^{\infty} f(x)g(t-x)dx$$

$$= \int_{-\infty}^{\infty} f(t-x)g(x)dx \qquad \text{(commutativity)}$$

While the symbol *t* is used above, it need not represent the time domain. But in that context, the convolution formula can be described as a weighted average of the function *f*(*x*) at the moment *t* where the weighting is given by *g*(−*x*) simply shifted by amount *t*. As *t* changes, the weighting function emphasizes different parts of the input function.

2 WEAK DERIVATIVES

In mathematics, a weak derivative is a generalization of the concept of the derivative of a function(strong derivative) for functions not assumed differentiable, but only integrable, i.e. to lie in the L^p space . Let v be a function in the Lebesgue space $L^1(a,b)$. We say that a function f in space $L^1(a,b)$ is a weak derivative of v if,

$$\int_a^b v(x)\emptyset'(x)dx \quad = \quad -\int_a^b f(x)\emptyset(x)dx \qquad , \quad \forall \quad \emptyset \in C_\infty^0(\Omega)$$

Generalizing to n dimensions, if u and v are in the space $L_{loc}^1(U)$ of locally integrable functions for some open set $U \subset R^n$ and if α is the multi-index then we say, v is the α^{th} weak derivative of u if,

$$\int_U u\, D^\alpha \varphi \quad = \quad -1^{|\alpha|} \int_U v\varphi \quad , \qquad \forall \quad \emptyset \in C_\infty^0(\Omega)$$

Let us see one example , for the absolute value function u:[-1,1]→[0,1] defined by u(t) = $|t|$ which is not differentiable at t=0 has a weak derivative v known as the sign function given by

$$v\text{:[-1,1]} \rightarrow [-1,1]: t \mapsto v(t) = \begin{cases} -1\,, & t < 0 \\ 0\,, & t = 0 \\ 1\,, & t > 0 \end{cases}$$

This is not the only weak derivative for *u*: any *w* that is equal to *v* almost everywhere is also a weak derivative for *u*. Usually, this is not a problem, since in the theory of L^p spaces and Sobolev space, functions that are equal almost everywhere are identified.

2.1 Weak formulation

Δu = f in weak formulation can be written as

$$\int_\Omega \Delta u\, v(x)dx \quad = \quad \int_\Omega f(x,u)v(x)dx \qquad , \quad \forall \text{ v} \in C_\infty^0(\Omega)$$

$$\Rightarrow \quad -\int_\Omega \nabla u\, \nabla v\, dx \quad = \quad \int_\Omega f(x,u)v(x)dx$$

If u satisfies the second equation ,then it is said to be the weak formulation of first equation.

2.2 Weak solutions

In mathematics, a weak solution (also called a generalized solution) to an ordinary or partial differential equation is a function for which the derivatives may not all exist but which is nonetheless deemed to satisfy the equation in some precisely defined sense. There are many different definitions of weak solution, appropriate for different classes of equations. One of the most important is based on the notion of distributions.Avoiding the language of distributions, one starts with a differential equation and rewrites it in such a way that no derivatives of the solution of the equation show up (the new form is called the weak formulation, and the solutions to it are called weak solutions). Somewhat surprisingly, a differential equation may have solutions which are not differentiable; and the weak formulation allows one to find such solutions.Weak solutions are important because a great many differential equations encountered in modelling real world phenomena do not admit sufficiently smooth solutions and then the only way of solving such equations is using the weak formulation. Even in situations where an equation does have differentiable solutions, it is often convenient to first prove the existence of weak solutions and only later show that those solutions are in fact smooth enough.

As an illustration of the concept, consider the first-order wave equation,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

where $u = u(t, x)$ is a function of two real variables. Assume that u is continuously differentiable on the Euclidean space R^2, multiply this equation (1) by a smooth function φ of compact support, and integrate. One obtains ,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial u(t,x)}{\partial t}\varphi(t,x)\,dt\,dx + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial u(t,x)}{\partial x}\varphi(t,x)\,dt\,dx = 0$$

Using Fubini's theorem which allows one to interchange the order of integration, as well as integration by parts (in t for the first term and in x for the second term) this equation becomes

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(t,x)\frac{\partial \varphi(t,x)}{\partial t}\,dt\,dx \;-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(t,x)\frac{\partial \varphi(t,x)}{\partial x}\,dt\,dx = 0$$

(Notice that while the integrals go from $-\infty$ to ∞, the integrals are essentially over a finite box because φ has compact support, and it is this observation which also allows for integration by parts without the introduction of boundary terms.)

We have shown that first equation implies second equation as long as u is continuously differentiable. The key to the concept of weak solution is that there exist functions u which satisfy second equation for any φ, and such u may not be differentiable and thus, they do not satisfy first equation . A simple example of such function is $u(t, x) = |t - x|$ for all t and x. (That u defined in this way satisfies second equation is easy enough to check, one needs to integrate separately on the regions above and below the line $x = t$ and use integration by parts.) A solution u of second equation is called a weak solution of the first equation .

3 SOBOLEV SPACES

Sobolev spaces are the space of functions whose distributional derivatives exists in the L^p spaces. In other words, in mathematics, a Sobolev space is a vector space of functions equipped with a norm that is a combination of L^p-norms of the function itself as well as its derivatives up to a given order. The derivatives are understood in a suitable weak sense to make the space complete, thus a Banach space. Intuitively, a Sobolev space is a space of functions with sufficiently many derivatives for some application domain, such as partial differential equations, and equipped with a norm that measures both the size and regularity of a function.

Sobolev spaces are named after the Russian mathematician Sergei Sobolev. Their importance comes from the fact that solutions of partial differential equations are naturally found in Sobolev spaces, rather than in spaces of continuous functions and with the derivatives understood in the classical sense.

They are generally denoted by $W^{1,p}(\Omega)$ where,

$$W^{1,p}(\Omega) = \{ u \in L^p(\Omega) : |\nabla u| \in L^p(\Omega)\}$$

Here 1 represents the appearance of the first order derivative and p refers to the existence of the p norms.

In the one-dimensional case (functions on R) the Sobolev space $W^{k,p}$ is defined to be the subset of functions f in $L^p(R)$ such that the function f and its weak derivatives up to some order k have a finite p norm, for given p ($1 \leq p \leq +\infty$). As mentioned above, some care must be taken to define derivatives in the proper sense .

With the above definition Sobolev spaces admits a natural norm

$$\|f\|_{k,p} = \sqrt[p]{\sum_{i=0}^{k}\|f^i\|_p^p} = \left(\sum_{i=0}^{k}\left|f^i(t)\right|^p dt\right)^{1/p}$$

One of the type of Sobolev spaces where p=2 is the Hilbert space .

3.1 Sobolev Embeddings

For any two spaces X and Y such that $X \subset Y$ and

$$\|u\|_Y \leq C\,\|u\|_X \quad , \quad \forall\ u \in X$$

then we say X is embedded in Y .

3.2 Sobolev Inequalities

3.2.1 General Sobolev inequalities

Let U be a bounded open subset of R^n , with a C^1 boundary (U may also be unbounded, but in this case its boundary, if it exists, must be sufficiently well-behaved.) .Assume u ∈ $W^{k,p}(U)$ then we consider the two cases,

- For k < $\frac{n}{p}$, In this case u ∈ $L^q(U)$ where , $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$

  We have in addition the estimate,

$$\|u\|_{L^q(U)} \leq C \|u\|_{W^{k,p}(U)}$$

- For k > $\frac{n}{p}$, In this case u belongs to holders space,

  We have in addition the estimate ,

$$\|u\|_{C^{k-[\frac{n}{p}]-1,\gamma}(U)} \leq C \|u\|_{W^{k,p}(U)}$$

3.2.2 Gagliardo Nirenberg Sobolev inequality

Let us 1≤p<n , first to see we can establish an inequality of the form

$$\|u\|_{L^q(R^n)} \leq C \|Du\|_{L^p(R^n)} \qquad \forall \quad u \in C^0_\infty(R^n)$$

for certain constant C >0 , 1≤q<∞

Motivation:

Let us first demonstrate if any inequality holds then q cannot be arbitrary but have a very specific form.

For this let us define for λ>0, the rescaled function

$$u_\lambda(x) = u(\lambda x)$$

So we get :

$$\|u(\lambda x)\|_{L^q(R^n)} = \frac{1}{\lambda^{\frac{n}{q}}}\left(\int_{R^n} |u(y)|^q\, dy\right)^{1/q} \quad \text{, here} \quad y = \lambda\, x$$

$$\|Du(\lambda x)\|_{L^p(R^n)} = \frac{\lambda}{\lambda^{\frac{n}{p}}}\left(\int_{R^n} |Du(y)|^p\, dy\right)^{1/p}$$

Putting in the equation ,

$$\frac{1}{\lambda^{\frac{n}{q}}}\|u\|_{L^q(R^n)} \leq C\frac{\lambda}{\lambda^{\frac{n}{p}}}\|Du\|_{L^p(R^n)}$$

$$\Rightarrow \quad \|u\|_{L^q(R^n)} \leq C\lambda^{1-\frac{n}{p}+\frac{n}{q}}\|Du\|_{L^p(R^n)}$$

So in order that the estimate should hold,

$1-\frac{n}{p}+\frac{n}{q}$ must be 0

$$\Rightarrow \quad q = \frac{np}{n-p}$$

This observation motivates the following.

3.3 Sobolev embedding theorem

If is Ω a domain in R^n , $H_0^{1,p}(\Omega) \subset L^q(\Omega)$ is a continuous embedding provided p< n

and $p \leq q \leq \frac{np}{n-p}$, $\|u\|_q \leq C\,\|u\|_{1,p}$ holds

$\forall$ $u \in C_0^1(\Omega)$ and by completion $\forall$ $u \in H_0^{1,p}(\Omega)$

3.4 Sobolev conjugate

By definition if $1 \leq p < n$, then the sobolev conjugate of p ,i.e. p* is defined as $p^* = \frac{np}{n-p}$

So the foregoing analysis shows that the estimate or the inequality can only be true if q=p*.This special case of the Sobolev embedding is a direct consequence of the Gagliardo–Nirenberg–Sobolev inequality.

4 THE p-LAPLACIAN PROBLEM

Here we are concerned with the elliptic problem or specifically the p-laplacian problem

$$\Delta_p u = f(x,u)$$

$$u|_{\partial\Omega} = 0$$

where $\Delta_p u$ = -div($|\nabla u|^{p-2}\nabla u$)

First of all by the definition of the weak solution , u ∈ $W_0^{1,p}(\Omega)$ is a weak solution of P if

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx = \int_\Omega f(x,u)v(x)dx$$

Now we define **,**

$$\emptyset(u) = \int_\Omega \frac{1}{p}|\nabla u|^p \, dx$$

and $$\psi(u) = \int_\Omega F(x,u)dx$$

where $F(x\,,u) = \int_0^u f(x,t)dt$

Now the energy function $\varphi = \emptyset - \psi$: $W_0^{1,p}(\Omega)$ → R associated with the problem is well defined. Then it is easy to see that $C^1(W_0^{1,p}(\Omega))$ is weakly lower semi-continuous and u∈ $W_0^{1,p}(\Omega)$ isa weak solution of the proposed proble if and only if u is a critical point of the energy function.

Indeed we have,

$$\varphi'(u)\, v = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx - \int_\Omega f(x,u)v dx$$

$$= \emptyset'(u)\, v - \psi'(u)\, v$$

Now we have a method to study the existence of solutions of the given problem whose hypothesis on non- smooth potential f(x,t) are as follows:

H(f): f:Ωx R→ R is a continuous function satisfying ,

i) f is C^1 in t.

ii) f(x,t) = o($|t|^{p^+-1}$) as $|t|$ →0 uniformly in x.

iii) $\exists\ \mu > p^+$ and $p^+ < q < p^*$ such that

$$\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^{\mu}} = +\infty$$

$$\lim_{|t|\to\infty} \frac{|f(x,t)|}{|t|^{q-1}} = 0 \text{ uniformly in } x \in \Omega \text{ where}$$

$$p^*(x) = \frac{Np(x)}{N-p(x)}$$

and $F(x,t) = \int_0^t f(x,s)ds$

iv) For each $x \in \Omega$,

$$\frac{\partial}{\partial t}\left(\frac{f(x,t)}{|t|^{p^+-1}}\right) > 0 \quad \text{for} \quad |t| > 0$$

So if the hypothesis H(f) holds , then the problem has a weak solution $u \in W_0^{1,p}(\Omega)$ such that

$$\varphi(u) = \max_{t>0} \varphi(tu) > 0$$

Now the proof of the existence of solution to the given problem is considered in different steps:

STEP 1: In the first step we will show that 0 is a strict local minimum of φ

Now by the conditions h(f) i to iii for any $\epsilon > 0$, $\exists\ C_\epsilon > 0$ such that

$$|F(x,t)| \leq \epsilon |t|^{p^+} + C_\epsilon |t|^q$$

So now ,

$$\varphi(u) = \int_\Omega \frac{1}{p}|\nabla u|^p\,dx - \int_\Omega F(x,u)dx$$

$$\geq \frac{1}{p^+}\int_\Omega |\nabla u|^p\,dx - \epsilon\int_\Omega |u|^{p^+}dx - C_\epsilon\int_\Omega |u|^q dx$$

Note that $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, so $\exists$ a $c_0 > 0$ such that

$$|u|_q \leq c_0\|u\|$$

Hence for $\|u\| = p(\leq \frac{1}{c_0})$ we have

$$|u|_q \leq 1,$$

$$|u|_q^{q^+} \leq \int_\Omega |u|^q dx \leq |u|_q^{q^-}$$

Thus $\varphi(u) \geq \frac{1}{p^+}\int_\Omega |\nabla u|^p\, dx - \epsilon\, |u|_{p^+}^{p^+} - C_\epsilon \|u\|^{q^-}$

$$\geq \frac{1}{p^+}\|u\|^{p^+} - {c_0}^{-p^+}\epsilon\, \|u\|^{p^+} - C_\epsilon\|u\|^{q^-}$$

Here we use the Sobolev embedding with constant c_0 and choosing

$${c_0}^{-p^+}\epsilon = \frac{1}{2p^+} \quad \text{then ,}$$

$$\varphi(u) \geq \|u\|^{p^+}\left(\frac{1}{2p^+} - C_\epsilon\|u\|^{q^- - p^+}\right)$$

which shows that $\varphi(u) > 0$ if

$$0 \leq \|u\| < \min\{\frac{1}{2p + C_\epsilon}^{\frac{1}{q^- - p^+}}, \frac{1}{c_0}, 1\}$$

STEP 2: Here we will show that for any u≠ 0 , $\varphi(tu) \to -\infty$ as $t \to +\infty$

By H(f) ii and iii , ∃ l >0 , such that

F(x,t) ≥ $l|t|^\mu$ – C for any x ∈ R^N and t ∈ R

Hence for u ≠ 0 ,

$$\varphi(u) \leq \frac{1}{p^-}\int_\Omega |t|^p\, |\nabla u|^p dx - l\int_\Omega t^\mu |u|^\mu dx - C_{meas}(\Omega)$$

$$\leq \frac{t^{p^+}}{p^-}\int_\Omega |\nabla u|^p dx - l\int_\Omega t^\mu |u|^{p^+} dx - C_{meas}(\Omega)$$

$\to -\infty$ as $t \to +\infty$

Thus by step 1 and step 2 , we get

c > 0 is well defined.

Now let $\{u_n\}$ be a minimizing sequence of c such that,

$$\varphi(u_n) = \max_{t>0}\varphi(tu_n) \to c \quad \text{as} \quad n \to \infty$$

We first prove that $v_n \to v$ in $W_0^{1,p}(\Omega)$, $v_n \to$ v in $L^p(\Omega)$ and $v_n(x) \to v(x)$ almost everywhere .

- If $v(x) \neq 0$,

we have $|u_n(x)| \to +\infty$ almost everywhere , $x \in \Omega$,

then using H(f) iii we obtain,

$\frac{F(xu_n(x))}{|u_n(x)|^{\mu}} |v_n(x)|^{\mu} \to +\infty$ almost everywhere , $x \in \Omega$,

Since , $\|u_n\| > 1$ for large value of n , then by H(f) iii and also by Fatou's Lemma

we have,

$$\frac{1}{p^-} \geq \lim_{n \to \infty} \frac{1}{p^- \|u\|^{p^+}} (\int_\Omega |\nabla u_n|^p dx + \int_\Omega |u_n|^p dx)$$

$$\geq \lim_{n \to \infty} \frac{1}{\|u\|^{p^+}} (\int_\Omega \frac{1}{p} |\nabla u_n|^p dx)$$

$$\geq \lim_{n \to \infty} \frac{1}{\|u\|^{\mu}} (\int_\Omega \frac{1}{p} |\nabla u_n|^p dx)$$

$$= \lim_{n \to \infty} \frac{1}{\|u\|^{\mu}} (\varphi\left(u_{n)}\right) + \int_\Omega F(x, u_n) dx\right)$$

$$\geq \int_\Omega \lim_{n \to \infty} \frac{F(xu_n)}{|u_n|^{\mu}} |v_n|^{\mu} \, dx - 1$$

$$\to +\infty \text{ as } n \to +\infty$$

which is impossible.

- If $v(x) = 0$, then fixing an $R > \max(1, p^+ c^{\frac{1}{p^-}})$, we have

$$c \leftarrow \varphi(u_{n)} \geq \varphi(\frac{Ru_n}{\|u_n\|}) = \varphi(R\, v_n)$$

$$= (\int_\Omega \frac{1}{p} |R\nabla v_n|^p dx) - \int_\Omega F(x, Rv_n) dx$$

$$\geq \frac{1}{p^+} R^{p^-} \int_\Omega |\nabla v_n|^p dx) - \int_\Omega F(x, Rv_n) dx$$

$$= \frac{1}{p^+} R^{p^-} - \int_\Omega F(x, Rv_n) dx$$

$$\to \frac{1}{p^+} R^{p^-} \quad \text{as} \quad n \to +\infty$$

So, $c \geq \frac{1}{p^+} R^{p^-}$

$$\Rightarrow R^{p^-} \leq p^+ c$$

$$\Rightarrow R^{p^-/p^-} \leq (p^+ c)^{1/p^-}$$

$\Rightarrow R \leq (p^+ c)^{1/p^-}$ which is impossible . So $\{u_n\}$ is bounded .

STEP 3: Now we will show that su is a critical point of φ .

Since $\max_{t>0} \varphi(tu)$ is achieved at only one point t=s , it is only the unique point at which $<\varphi'(su), u> = 0$. Next we claim that su is the critical point of φ.

Without any loss of generality , we can assume that s=1 . If u is not a critical point , then there is v $\in C_\infty^0(\Omega)$, such that $<\varphi'(u), v> = -2$. There is some $\epsilon>0$ such that

$$<\varphi'(tu+\epsilon v), v> \leq -1$$

for $|t-1| + |\epsilon| \leq \epsilon_0$.

Now consider the two dimensional plane spanned by u and v. For small $\epsilon > 0$,

let $t_\epsilon > 0$ be the unique number such that

$$\max_{t>0} \varphi(t(u+\epsilon v)$$

$$= \varphi(t_\epsilon(u+\epsilon v)$$

Then $t_\epsilon \to 1$ as $\epsilon \to 0$.

For ϵ small such that $|t_\epsilon - 1| + t_\epsilon \epsilon \leq \epsilon_0$,we have contradiction as follows,

On one hand, $\varphi(t_\epsilon(u+\epsilon v) \geq c$

On the other hand,

$$\varphi(t_\epsilon(u+\epsilon v)$$

$$= \varphi(t_\epsilon u) + \int_0^1 \langle \varphi'\big(t_\epsilon(u + s t_\epsilon \epsilon v)\big), t_\epsilon \epsilon v \rangle ds$$

$$\leq c - t_\epsilon \epsilon$$

$$< c$$

So s is a critical point of φ .

APPENDIX

Nehari techniques to solve non-linear PDEs

H(f): f:Ωx R→ R is a continuous function satisfying ,

i) f is C^1 in t.

ii) f(x,t) = o($|t|^{p^+-1}$) as $|t|$ →0 uniformly in x.

iii) ∃ μ>p^+ and p^+< q <p* such that

$$\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^{\mu}} = +\infty$$

$$\lim_{|t|\to\infty} \frac{|f(x,t)|}{|t|^{q-1}} = 0 \text{ uniformly in } x\in \Omega \text{ where}$$

$$p^*(x) = \frac{Np(x)}{N-p(x)}$$

and $$F(x,t) = \int_0^t f(x,s)ds$$

iv) For each x ∈ Ω ,

$$\frac{\partial}{\partial t}\left(\frac{f(x,t)}{|t|^{p^+-1}}\right) > 0 \qquad \text{for} \quad |t| > 0$$

Solutions corresponds to the critical pont of the C^1 functional ,

$$\varphi(u) = \int_{\Omega} \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \int_{\Omega} F(x,u)dx \quad , \quad u \in W_0^{1,p}(\Omega)$$

Now for a function u(x) we use,

$$u_+(x) = \max\{ u(x) , 0\}$$

$$u_-(x) = \min\{ u(x) , 0\}$$

So if the hypothesis H(f) holds , then the problem has a weak solution u ∈ $W_0^{1,p}$(Ω) such that

$$\varphi(u) = \max_{t>0} \varphi(tu)$$

$$> 0$$

Lax- Milgram theorem

In mathematics, the Babuška–Lax–Milgram theorem is a generalization of the famous Lax–Milgram theorem, which gives conditions under which a bilinear form can be "inverted" to show the existence and uniqueness of a weak solution to a given boundary value problem. The result is named after the mathematicians Ivo Babuška, Peter Lax and Arthur Milgram.

Statement of the theorem

In 1971, Babuška provided the following generalization of Lax and Milgram's earlier result, which begins by dispensing with the requirement that U and V be the same space. Let U and V be two real Hilbert spaces and let $B : U \times V \to \mathbf{R}$ be a continuous bilinear functional. Suppose also that B is weakly coercive: for some constant $c > 0$ and all $u \in U$

$$\sup|B(u,v)| \geq c\|u\|$$

and for all $0 \neq v \in V$, $\sup|B(u,v)| > 0$

Then, for all $f \in V^*$, there exists a unique solution $u = u_f \in U$ to the weak problem ,

$$B(u_f, v) = \langle f, v\rangle \quad , \ \forall \ v \in V$$

Moreover, the solution depends continuously on the given datum:

$$\|u_f\| \leq \frac{1}{c}\|f\|$$

Banach fixed point theorem

In mathematics, the Banach fixed-point theorem (also known as the contraction mapping theorem or contraction mapping principle) is an important tool in the theory of metric spaces; it guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method to find those fixed points. The theorem is named after Stefan Banach (1892–1945), and was first stated by him in 1922.

Statement:

Let (X,d) is a metric space. Then a map $T : X \to X$ is called a contraction mapping on X if there exists $q \in [0, 1)$ such that $d(T(x),T(y)) \leq q d(x,y)$, $\forall$ $x,y \in X$

Let (X, d) be a non-empty complete metric space with a contraction mapping $T : X \to X$. Then T admits a unique fixed-point x^* in X (i.e. $T(x^*) = x^*$). Furthermore, x^* can be found as follows: start with an arbitrary element x_0 in X and define a sequence $\{x_n\}$ by $x_n = T(x_{n-1})$, then $x_n \to x^*$.

The following inequalities are equivalent and describe the speed of convergence:

$$d(x^*, x_n) \le \frac{q^n}{1-q} d(x_1, x_0)$$

$$d(x^*, x_{n+1}) \le \frac{q}{1-q} d(x_{n+1}, x_n)$$

$$d(x^*, x_{n+1}) \le q\, d(x^*, x_n)$$

Any such value of q is called a Lipschitz constant for *T*, and the smallest one is sometimes called "the best Lipschitz constant" of *T*.

Mountain pass theorem

The mountain pass theorem is an existence theorem from the calculus of variations. Given certain conditions on a function, the theorem demonstrates the existence of a saddle point. The theorem is unusual in that there are many other theorems regarding the existence of extrema, but few regarding saddle points.

Theorem statement:

The assumptions of the theorem are:

i) *I* is a functional from a Hilbert space *H* to the reals .
ii) $I \in C^1(H, R)$ and L' is Lipschitz continuous on bounded subsets of *H*,
iii) *I* satisfies the Palais-Smale compactness condition,
iv) $I[0]=0$
v) there exists positive constants *r* and *a* such that $I[u] \ge a$ if $\|u\|$ = r and
vi) There exists $v \in$ H with $\|v\| >$ r

CONCLUSION

A partial differential equation involving the p-Laplacian operator which is of the following type

$$\Delta_p u = f(x,u) \; for \; x \in \Omega,$$

$$u|_{\partial\Omega} = 0,$$

has been studied using the Nehari hypothesis. To understand the problem, a thorough knowledge on the Sobolev spaces was gained which will be useful in extending this work in the future. As a future plan, we propose to obtain an existence result for the above problem using the classical fixed point theorems.

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