# Flexural Analysis of FRP Strengthened RCC Beams Using Meshless Local Petrov Galerkin Method (MLPG)

**A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF DEGREE**

**OF**

**MASTER OF TECHNOLOGY**

**IN**

**STRUCTURAL ENGINEERING**

**BY**

**VIJAY KUMAR POLIMERU**

**(ROLL NO. 213CE2069)**

**DEPARTMENT OF CIVIL ENGINEERING**

**NATIONAL INSTITUTE OF TECHNOLOGY, ROURKELA**

**MAY 2015**

# Flexural Analysis of FRP Strengthened RCC Beams Using Meshless Local Petrov Galerkin Method (MLPG)

**A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF DEGREE**

**OF**

**MASTER OF TECHNOLOGY**

**IN**

**STRUCTURAL ENGINEERING**

**BY**

**VIJAY KUMAR POLIMERU**

**(ROLL NO. 213CE2069)**

**Under the guidance of**

**Prof. K. C. BISWAL**

**DEPARTMENT OF CIVIL ENGINEERING**

**NATIONAL INSTITUTE OF TECHNOLOGY, ROURKELA**

**MAY 2015**

**NATIONAL INSTITUTE OF TECHNOLOGY**
**Rourkela-769008, Odisha, India**

# CERTIFICATE

This is to certify that the thesis entitled "**Flexural Analysis of FRP Strengthened RCC Beams Using Meshless Local Petrov Galerkin Method (MLPG)**" submitted by **Vijay Kumar Polimeru** bearing **Roll No. 213CE2069** in partial fulfillment of the requirements for the award of **Master of Technology** Degree in **Civil Engineering** with specialization in **Structural Engineering** during **2013-2015** session to the **National Institute of Technology Rourkela** is an authentic work carried out by him under my supervision and guidance. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any Degree or Diploma.

**Date:**

**Prof. K. C Biswal**
Department of Civil Engineering
National Institute of Technology
Rourkela - 769008

# ACKNOWLEDGEMENT

To remember and respect the contributions-small and large- of every individual in my stay here is in a way more difficult than remembering by rote the countless formulae and expressions I have come across in my two year sojourn in our Institute. I list the ones that I must not miss at any rate, and apologize beforehand for omissions on my part.

I must acknowledge the **MHRD scholarship** which encouraged and enabled me to pursue my post graduate studies without burning my pocket much.

I thank and admire my supervisor, **Prof. K.C. Biswal** for the infinite patience and cheerfulness he possesses- which had a calming effect on me whenever things went the other way. I owe him a lot for his relentless encouragement, faith he reposed in me and the mild and warm rebukes that put me on the right track many a times.

I express my sincere regards to Prof. **S.K. Sarangi**, Director of NIT, Rourkela & our head of the department, **Prof. Shishir Kumar Sahu**, Civil Engineering NIT, Rourkela for giving me the needful resources in the department. Special mention must be made of **Prof. Pradip Sarkar** and **Prof. Robin Davis** for keeping their doors open to us for any help.

I must thank all the Professors of the Department of Civil engineering for being so courteous and supportive all through these days. Their infectious enthusiasm and friendly demeanour have left a lasting impression on me. Especially, I thank **Prof. M.R. Barik**, who not only taught us MATLAB passionately, but also shared loads of wisdom in a humorous and charismatic way. One such piece of wisdom is apt here- that the worth of a thesis may be gauged from the care given to the acknowledgements.

I must mention how all the staff and others in our Institute- from the industrious Institute management to the cycle repairers- have influence me and my work in their own ways.

I thank **Sachin. D. Dixini**, Barbia institute of technology and **Jeetender Singh Kushawaha**, U.P Technical University, Lucknow for granting me timely access to their papers, which helped me a lot in my work.

I owe my little successes and moments of happiness to my family and friends, whose love and support made my life here easy, literally.

Lastly, I must thank Google Inc., and Microsoft Inc., (among others) for making my work unimaginably easy and made it possible for me to do it from the comfort of my armchair. This thesis, if it is to be dedicated to anyone, must be dedicated to them!

*Vijay Kumar Polimeru*

# Abstract

In this project Meshless local Petrov Galerkin (MLPG) method is utilized for the flexural analysis of simply supported RCC beams strengthened with FRP laminates. This method uses the moving least-squares (MLS) approximation with different weight functions to interpolate the field variables and uses a local symmetric weak form (LSWF). The beams under consideration are rectangular and T-beams reinforced either on tension face or on both faces as per IS 456:2000. The proposed method is first applied to unstrengthened beam to check its applicability. The computed displacements are in good accord with the displacements attained using code formula. Then, it is extended to beams strengthened with FRP laminate. A parametric study is carried out to study the effect of disparity of field nodes in the global domain, integration cells in the sub domain and young`s modulus on the displacement. The efficiency of the algorithm developed is verified.

***Keywords:*** RCC beam; FRP strengthening; ODE; meshfree; Meshless Local Petrov Galerkin; moving least squares approximation.

# TABLE OF CONTENTS

# LIST OF FIGURES

# LIST OF TABLES

# CHAPTER 1

# INTRODUCTION

## 1. 1 Introduction

Numerical simulation has been proved to be a good alternative scientific investigation tool for expensive, time consuming, and sometimes risky experiments in intricate engineering problems. Meshfree methods are among the breed of numerical simulation techniques that are being vigorously developed to avoid the drawbacks that traditional methods like Finite Element method possess. FEM, with half a century of passionate research behind it is versatile, time tested and trustworthy. Yet when it comes to specific areas like fracture mechanics and crack propagation, FEM has disadvantages which necessitates the need to have specialist methods dealing with such problems.

The structural aspect of ‘element’ in FEM was found to be restrictive in nature when it comes to the implementation in such problems. Meshfree methods permit an alternative implementation based totally on nodes and devoid of the restriction of the element. The field of meshfree analysis is in budding stage and there have not been any single method that could be versatile enough to compete with FEM. Hence it becomes important for developing methods or applying existing methods for each kind of problem. Until a general meshfree framework is formulated, more and more specialized methods would be conceived and applied to niche problems. The main purpose of the present disclosure is to study the flexural characteristics of simply supported FRP strengthened RCC beam using Meshless local petrov galerkin method using Local symmetric weak form (LSWF) and moving least squares approximation. The beams under consideration are rectangular and T-beams reinforced either on tension face or on both faces as per IS 456:2000. The proposed method is first applied to unstrengthened beam to check its applicability. The computed displacements are in good accord with the displacements attained using code formula. Then, it is extended to beams strengthened with FRP laminate. In IS 456:2000 the deflection formula is confined only to RCC. There are no specifications mentioned for FRP strengthened RCC. Hence we made an attempt to find the displacements in FRP strengthened RCC beams using MLPG. A parametric study is carried out to study the effect of disparity in field nodes in the global domain, integration cells in the sub domain and young`s modulus on the displacement. Computer programs are developed using MATLAB.

## 1.2 Objective

Objectives of the study may be summarized as follows:

1. To study the flexural behavior of FRP strengthened RCC beams.
2. To study the effect of different FRP composites and different cross sections of beam on displacement.

## 1.3 Scope of the Present Study

Scope of present study is summarized as follows:

1. Unstrengthened RCC beams are designed as per IS 456-2000
2. Analysis is limited to linear static
3. Simple moving least squares and Local symmetric weak form is utilized for the analysis

4. Beams are strengthened with Carbon/Epoxy, Boron/Epoxy, E-Glass/Epoxy laminates

### 1.4 Thesis Layout

The organization of the thesis is as follows. Chapter 2 gives a detailed review of literature in the field of meshfree methods. Emphasis is on the MLPG and its application to strengthened and unstrengthened beams. The papers followed to arrive at the formulation utilized in the current work are also reviewed.

Chapter 3 deals with the Meshless Local Petrov Galerkin formulation behind the analyses. Moving least squares and Local symmetric weak formulation are discussed in detail. The requirements of numerical implementation of these theories and difficulties involved are also discussed.

Chapter 4 presents a thorough discussion on theoretical formulation for determining beam deflection as IS: 456-2000 codal provisions into consideration. The formulations generally utilized and the formulation is utilized in this work have been discussed elaborately and the issues involved in their implementation are briefly touched upon.

Chapter 5 presents the results of the work and also the inferences that are discerned from these results. The performance of MLPG is analyzed and the parametric behavior of the problem is investigated. The results of these meshfree methods are compared with the analytical solution of unstrengthened RCC beams and then it is extended to strengthened RCC beams.

Chapter 6 concludes the discussion and presents a holistic view on the results. The broad lines of understanding arrived due to the work are elaborated upon. The future possibilities in the work are also dealt with.

# CHAPTER 2

# REVIEW OF LITERATURE

## 2.1. Introduction

An extensive survey of literature relevant to the objectives outlined in the previous chapter is presented. A historical overview of the significant research works dealing with different formulations and approximation procedures leading to the chapters of this thesis are discussed. The meshfree methods had their initial stages in the early 1970s when Finite Element Method was in the peak of its popularity. With the advent of higher computational power, FEM became ubiquitous. However as FEM was being applied to great variety of fields, its limitations and inhibitive features were also understood. The meshfree methods began as one among the many lines of thought to resolve this issue and to replace or complement FEM in such problems. The first, method to be developed was the Smooth particle hydrodynamics from the works of Lucy (1977), Gingold and Monaghan (1977) the method utilized the global strong form. The trial function is assumed as an integral representation. The ideas like support domain etc. were first introduced in this work. Liszka and Orkisz (1980) proposed the Finite point method which was also based on the strong form. The finite differential representation involves Taylor series. Also they utilized Moving least squares method for approximation. Nayroles et al (1992) developed the Diffuse Element Method. The method was based on weak form and utilized Moving Least squares method for approximation of the field variable. They called it by the name Diffuse approximation and it was expected tom be complementary to the FEM. It was expected that the approximation would find use as the smoothing function and also for approximating functions. Belytschko et al (1994) refined the ideas of Nayroles and developed the Element Free Galerkin Method, the most popular meshfree method till date. They employed the weak form of the governing equation and utilized moving least squares for approximating the shape function. They considered certain derivatives that Nayroles had discarded in the interpolation. Also they applied Lagrange Multipliers for the imposition of boundary conditions. However the EFGM was not totally meshfree as simple shaped cells were utilized for integration. Slowly the trend veered towards using local weak form for arriving at the system algebraic equations. Mukherjee and Mukherjee (1997) introduced boundary node methods, followed by point interpolation method of Liu et al (1999). Atluri et al (1999) formulated a meshfree method based on local weak form using Petrov Galerkin approach. The approach attempted to eliminate the need of a mesh for integration cells on a global scale. They called this method as Meshless Local Petrov Galerkin method. Other methods followed, like the kinds of XFEM (Belytschko, 1999) and Natural Element Method (Sukumar, 1998). There is a push towards computationally efficient, reasonably accurate meshfree methods. However the most popular ones are still the methods of EFGM and MLPG.

G. R. Liu (2010) did an extensive work in abridging all meshfree methods developed so far and their major features are listed in the table: 01 and the procedures in FEM and the other Meshfree methods which use weak formulation are outlined in Fig. 01.

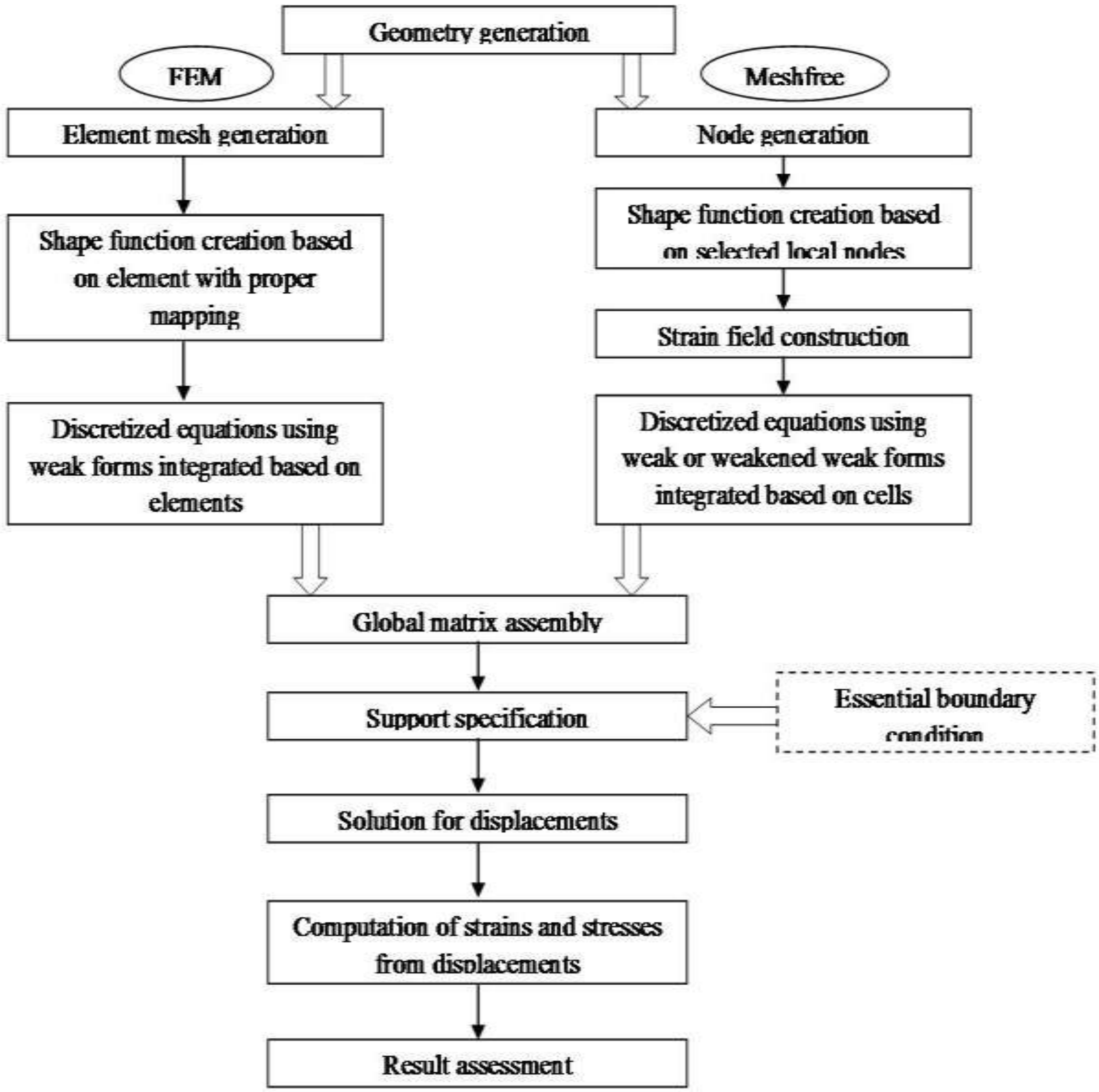



Fig. 01. Flowchart for FEM and meshfree method procedures

Table: 01. Some Meshfree Methods Using Local Irregular Nodes For Approximation

| Method | References | Formulation procedure | Local function approximation |
|---|---|---|---|
| Diffuse element method | Nayroles et. al (1992) | Galerkin weak form | MLS Approximation |
| EFG Method | Belytschko et. al (1994b) | Galerkin weak form | MLS Approximation |
| MLPG Method | Atluri and Zhu (1998) | Local Petrov Galerkin | MLS Approximation |
| PIMs | Liu et. al (1999,2000b, 2001 a,b,c,d) | Weakened weak forms | Point interpolation using polynomial and radial basis functions (RBFs) |
| SPH | Lucy (1997), Gingold and Monaghan (1997) | Weak form Like | Integral representation, particle approximation |
| GSM | Liu et. al (2008) | Weak form Like | Point interpolation and gradient smoothing |
| Finite point method | Onate et. al. (1996) | Strong form | MLS Approximation |
| FDM using irregular grids | Liszka and Orkisz (1980), Jensen (1980), Liu et. al (2006) | Strong form | Differential representation |
| Hp- clouds | Oden and Abani (1994), Armando and Oden (1995) | Strong or weak form | Partition of unity |
| Partition of unity FEM | Babuska and Melenk (1995) | Weak form | Partition of unity |
| Boundary node methods (BNM) | Mukherjee and Mukherjee (1997a, b) | Boundary integral formulation | MLS Approximation |
| BPIM | Liu and Gu (2000d, 2001 a, e) | Boundary integral equation | Point interpolation RBF |

## 2.2. Defining Meshfree Methods:

In conventional FEM, FDM, and FVM problems global domain is discretized into elements. A element can be defined as interstices between the strings of web that is formed by connecting the nodes in a particular manner. It must be predefined in order to provide certain relationship between nodes; finally these elements became the building blocks of the conventional numerical methods.

Here a Meshfree method tries to create a system of algebraic equations for the entire global domain without the use of predefined mesh, or uses elements that can be generated very easily in much more flexible manner.

In we look these meshfree methods in distinction to FEM the term Element Free is ideal and finite difference method using non uniform or random grids is preferred in contrast to Finite difference method. Number of meshless methods are often termed as truly meshfree methods, hence these don`t even bother about nodal connectivity. Thus the perfect requirement for a meshless method is “No mesh is essential at all right through the process of solving the governing partial differential equations subjected to all varieties of boundary conditions”.

## 2.3. Need for Meshless Methods:

FEM is very versatile tool and methodically developed for static and dynamic, linear and non linear stress analysis of solids, structures, fluid flows, and interaction problems. Almost all engineering problems can be solved using FEM packages. But there are certain limitations to FEM.

1. An eminent mesh creation for the entire global domain of the problem in preprocessing work involved in any FEM packages. An analyst has to spend more time in generating quality that covers the entire problem domain and boundaries. Hence the processing power of computer also increases drastically.
2. When handling problems that involve large deformation, moving crack growth in a complex random path the accuracy can be lost.
3. There are several stress and strain smoothing post processing techniques and re meshing techniques are developed for FEM in order to show continuous disparity. Generally meshfree methods do not require these post processing techniques.
4. The simulation of FEM is built on continuum mechanics hence it is very difficult to simulate the breakage of material since the elements formed once cannot be broken.

**2.4. Meshless Local Petrov Galerkin Method:**

A background mesh is rewired in Element Free Galerkin method for integration in calculating the global system equations. Galerkin weak form is the main reason behind this requirement. Meshfree methods that function on strong form such as Finite Point Method, non uniform Finite Difference method, Local point collocation methods etc do not require back ground integration meshes.

The Meshless Local Petrov Galerkin (MLPG) method was first introduced by Atluri and Zhu (1998) utilized the Local Petrov Galerkin form. It has been tuned up and extended for several years. The MLPG method is one of the several meshfree schemes. The major benefit of this method compared with other meshfree methods is that no background mesh is utilized to assess various integrals appear in the local weak formulation of a problem. Therefore, this method is “truly meshfree” method in terms of both interpolation of field variables and integration of energy function and is quite successful in solving boundary value problems. Atluri et. al. (1999) formulated a methodology using generalized moving least squares to analyse bending behavior of Euler Bernoulli beam. After that Prof. S. N. Atluri, the author of Meshless Local Petrov Galerkin method proposed several formulations along with other pioneering researchers in the field of computations mechanics to variety of problems in the year 2000 an application to solve the problems in elasto statics is proposed, in the year 2001 proposed an MLPG formulation for solving incompressible Navier Stokes equations. Y. T. Gu et. al (2001a,b) proposed an MLPG formulation for free and forced vibration analysis for solids and thin plates. Shuyao Long et. al (2002) proposed MLPG formulation for bending analysis of thin plates. A nonlinear formulation of the Meshless Local Petrov-Galerkin (MLPG) finite-volume mixed method is generated for the large deformation analysis of static and dynamic problems by Z. D Han et. al. (2005).

However, in all the above studies the numerical analysis of reinforced cement concrete beams (RCC) strengthened with FRP laminate using MLPG is not published till date. Hence, the current study mainly focus on developing a MLPG formulation with local symmetric weak form (LSWF) and MLS approximation for flexural analysis of RCC beams strengthened with FRP laminates. The beams under consideration are rectangular and T-beams reinforced either on tension face or on both faces as per IS 456:2000. The proposed method is first applied to unstrengthened beam to check its applicability. The computed displacements are in good accord with the displacements attained using code formula. Then, it is extended to beams strengthened with FRP laminate. In IS 456:2000 the deflection formula is confined only to RCC. There are no specifications mentioned for FRP strengthened RCC. Hence we made an attempt to find the displacements in FRP strengthened RCC beams using MLPG. A parametric study is carried out to study the effect of disparity in field nodes in the global domain, integration cells in the sub domain and young`s modulus on the displacement. Computer programs are developed using MATLAB.

## 2.5. Basic Techniques of Meshfree Methods:

We now brief the general procedure and basic steps involved in meshfree methods

### Step 1: Domain Representation

The geometry of the global domain is first modeled and is represented using sets of nodes scattered in the problem domain and its boundary. Boundary conditions and loading conditions are then specified. The density of the nodes depends on the accuracy requirement of the analysis and the resources available. The nodal distribution may be regular or non uniform. These nodes are often called field nodes.

### Step 2: Displacement Interpolation

The field variable $u$ at any point $X = (x, y, z)$ within the problem domain is interpolated using the displacements at its nodes within the support domain of the point at $x$, i.e.

$$u^h(x) = \sum_{i=1}^{S_n} \phi_i(x) u_i = \Phi(x) d_s$$

Where,

$S_n$ = Set of nodes in the small local domain

$\phi_i(x)$ = Shape function the $i$ th node

$d_s$ = Nodal field variables in the support domain

**Step 3: Formation of System Equations**

The discrete equations of a Meshfree method can be formulated using strong or weak form system equation and shape functions. These equations are often written in nodal matrix form and are assembled into the global system matrices for the entire problem domain. Generally for static analysis we will have set of algebraic equations whereas for dynamic analysis we will have eigenvalue equations.

**Step 4: Solving the Global Meshfree Equations**

We attain solutions for different types of problems after solving the set of global meshfree equations. For static problems, the displacements at all nodes in the entire global domain are attained first. The stress and strain at a specific point can then be retrieved. A standard linear algebraic equation solver such as gauss elimination method, LU decomposition methods etc are utilized for this process. For dynamic and bucking problems the eigenvalues and corresponding eigenvectors are attained using the standard eigenvalue solvers. Commonly utilized methods are Jacobi`s method, Given`s method and Householder`s method etc.

**2.5. 1. Determination of the Support Domain Dimension:**

The accuracy of the interpolation depends on the nodes in the support domain of point. A suitable support domain should therefore be chosen to ensure proper area of coverage for interpolation. Therefore the support domain dimension $d_s$ for a point $x$ is determined by

$$d_s = \alpha_s d_c$$

Where,

$\alpha_s$ = A dimensionless factor of the support domain

$d_c$ = Characteristic length that relates to the nodal spacing near the point at $x$

### 2.5.2. Determination of Characteristic length or local nodal spacing

For our 1D problem, Averaged nodal spacing method is utilized

$$d_c = \frac{D_s}{(n\,D_s - 1)}$$

Where,

$D_s$ = An estimate of $d_s$ (There are no specifications but reasonably good estimate is sufficient).

$n_{D_s}$ = Number of nodes that are covered by a domain with known size $D_s$.

***CHAPTER 3***

# MESHLESS LOCAL PETROV GALERKIN FORMULATION

### 3.1. Introduction

A simply supported beam of length (L), subjected to uniformly distributed load (w) is shown in the fig.02.

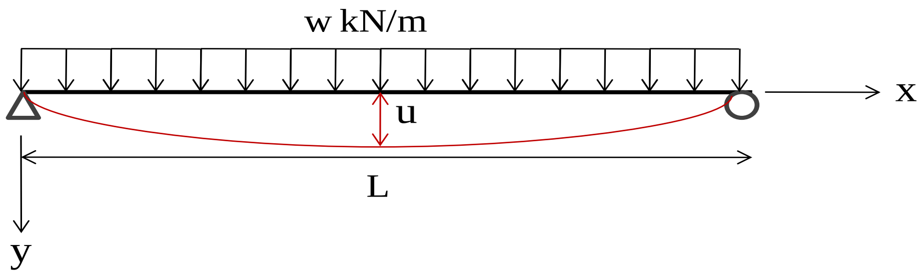



Fig. 02.Schematics of simply supported beam of length (L)

The governing differential equation for beam bending is a 4th order differential equation. Eq. (1.) is solved using the MLPG method with necessary boundary conditions as specified in Eqn. (2.).

$$EI\frac{d^4u}{dx^4} = f \qquad \text{in global domain } \Omega \tag{1.}$$

Where, u is transverse displacement, EI is bending stiffness and f is distributed load over the beam.

$$u(x) = \overline{u(x)} \text{ on } \Gamma_u \text{ and } \frac{\partial u(x)}{\partial x} = \overline{\theta(x)} \text{on} \Gamma_\theta \tag{2.}$$

Where, $\Gamma_\theta$ and $\Gamma_u$ are the boundary regions where displacement, slope are specified. The moment and shear force are related to the displacement through the Eq. (3.)

$$M = EI\frac{d^2u}{dx^2} \text{ and } V = -EI\frac{d^3u}{dx^3} \tag{3.}$$

**3.2. Moving Least Squares Approximation**

To achieve a non mesh type interpolation, a meshfree method uses a local interpolation or approximation to symbolize the trial function, with values (or the fictious nodal values) of the unknown variable at some arbitrarily located nodes. The moving least squares interpolation is one such accepted scheme which does not need any mesh information. Additionally, the required efficiency of the approximation function can be easily attained by the moving least squares interpolation technique. Due to these reasons the moving least squares may be a good technique for approximating the unknown variables in boundary value problems. In Fig. 03, the local sub domains, nodes in support domain, the influence domain of the moving least squares approximation for the test function at a node is defined. A support domain $\Omega_x$, in the locality of a point x, indicates the influence domain of moving least squares approximation for the test function at node x. To approximate the distribution of u in $\Omega_x$ over arbitrarily positioned nodes $\{x_i\}, i = 1,2\ 3, \ldots. n$, the moving least squares approximant $u^h(x)$ of $u, \forall x \in \Omega_x$, may be defined as

$$u^h(x) = \sum_{j=1}^{m} P_j(x)a_j(x) = p^T(x).a(x), \quad \forall x \in \Omega \tag{4.}$$

Where, $P_j(x), j = 1, 2, 3, \ldots, m$ are whole monomial basis of order $m$. The coefficient vector $a(x)$ is found out by minimizing a weighted discrete $L_2$ norm of the error function $J(x)$, as defined as

$$J(x) = \sum_{I=1}^{n} v_I(x)\,[p^T(x_I).\,a(x) - u_I]^2 \quad (5.)$$

Where $v_I(x)$ = weight function coupled with node i, with $v_I(x) > 0, \forall\, x$ in the support of $v_I(x)$. The stationary condition of $J(x)$ in Eq. (5.) with regard to $a(x)$ leads to the following linear relation among $a(x)$ and $u_i$.

$$[A]\{a\} = [B]\{u_I\} \quad (6.)$$

Where [A] and [B] are the matrices attained from both the trial and test functions. Solving for $a(x)$ from Eq. (6.) and substituting it into Eq. (4.) gives a relation, which may be written as the form of an interpolation function as that utilized in the FEM, is attained as

$$u^h(x) = \sum_{i}^{n}\sum_{j}^{m} P_x(x)(A^{-1}(x)B(x))_{ji}\,u_i = \sum_{i=1}^{n} \phi_i(x).\,u_i \quad (7.)$$

Where, the shape function $\phi_i(x)$ is defined as

$$\phi_I(x) = \sum_{j=1}^{m} p_j(x)[A^{-1}(x).\,B(x)]_{jI} = P^T A^{-1} B \quad (8.)$$

$$u^h(x) = \sum_{I=1}^{n} \phi_I(x) u_I \text{ and } \frac{du^h(x)}{dx} = \sum_{I=1}^{n} \frac{d\phi_I(x)}{dx} u_I \quad (9.)$$

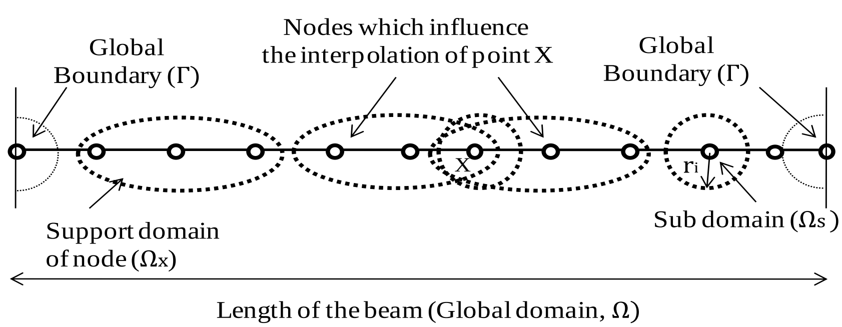



Fig. 03. Schematics of the MLPG method

The efficiency of the shape functions is determined by that of the basis function. In this work we have utilized the dissimilar weight functions with compacted support domain. The weight function equivalent to node i may be written as

$$v_i(x) = \begin{cases} \begin{cases} \frac{2}{3} - 4r^2 + 4r^3 & for \quad r \le \frac{1}{2} \\ \frac{4}{3} - 4r + 4r^2 - \frac{4}{3}r^3 & for \quad \frac{1}{2} \le r \le 1 \\ 0 & for \, r > 1 \end{cases} \\ \begin{cases} 1 - 6r^2 + 8r^3 - 3r^4 & for \quad r \le 1 \\ 0 & for \quad r > 1 \end{cases} \\ \begin{cases} 1 - 10r^2 + 20r^3 - 15r^4 + 4r^5 & for \quad r \le 1 \\ 0 & for \, r > 1 \end{cases} \\ \begin{cases} 1 - \frac{47}{10}r^2 + 12r^4 - 10r^5 + \frac{1}{2}r^6 + \frac{6}{5}r^7 & for \quad r \le 1 \\ 0 & for \quad r > 1 \end{cases} \end{cases} \tag{10.}$$

Where, $r = \frac{d_i}{R_i}$ where $d_i = |x - x_i|$ is the space between node $x_i$ and x and $R_i$ is the size of the support for weight function $v_i$. The span of the backing of the weight capacity $v_i$ associated with hub i ought to be picked such that $R_i$ is sufficiently substantial to have an adequate number of hubs secured in the area of impact of each example point ($n \ge m$) to guarantee the normality of [A]. To build a very much characterized shape work, the quantity of hubs (n) affecting the concerned point must be more noteworthy than monomial premise of request m. An essential condition for a very much characterized shape capacity is that atleast m weight capacities are non zero (i.e., $n \ge m$) for every example point it can be seen that each weight function has different characteristics when it comes to parameters like maximum value, steepness, $C^n$ continuity it offers, the nature of derivatives, number of integration cells etc. All these influence the result of MLPG. Due consideration must be provided to these and then the weight function chosen.

For beams at least C1 continuity must be available, the weight function be non-negative and the first and second order partial derivatives be nonsingular (X. L .Chen, 2003). When it comes to basis, the essential requirement is that it should have an order that would provide continuous and smooth derivatives upto the order required.

In the current work, quadratic, cubic and linear bases are utilized. The bases are usually chosen to be ‘complete’ polynomials. Pascal’s triangle is utilized to determine the monomial terms to be included in the basis. The requirements for basis are similar to those in FEM. The order of basis affects the minimum number of nodes that must be included in the domain.

Linear- $1 + x$

Quadratic- $1 + x + x^2$

Cubic- $1 + x + x^2 + x^3$

The influence domain utilized in the current EFGM formulation is rectangular. The mesh is also a regular one. The benefit of using a regular mesh is that implementation of influence domain size becomes very easy and usually the regular mesh has been reported to be more accurate and easy to handle (Belinha et al, 2006). Further the rectangular domain suits the current problem of rectangular plates and laminates. The rectangular domain’s size is defined using a parameter ‘d’ which is the ratio between the domain size in a direction by the mesh size in that direction.

A typical 1D moving least squares weight function and shape function is given in figure 04 and 05. The consistency of MLS shape function mainly depends on the order of the chosen monomial. If the complete order of the complete monomial is $\boldsymbol{k}$, the moving least squares shape function will posses $\boldsymbol{C^k}$ consistency. The interpolation and approximation of moving least squares and Finite element shape functions are described in Fig. 06.

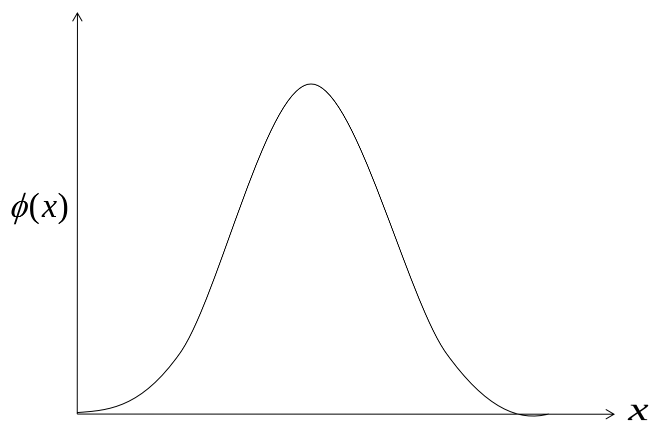



Fig. 04. Moving Least Squares 1D Shape function

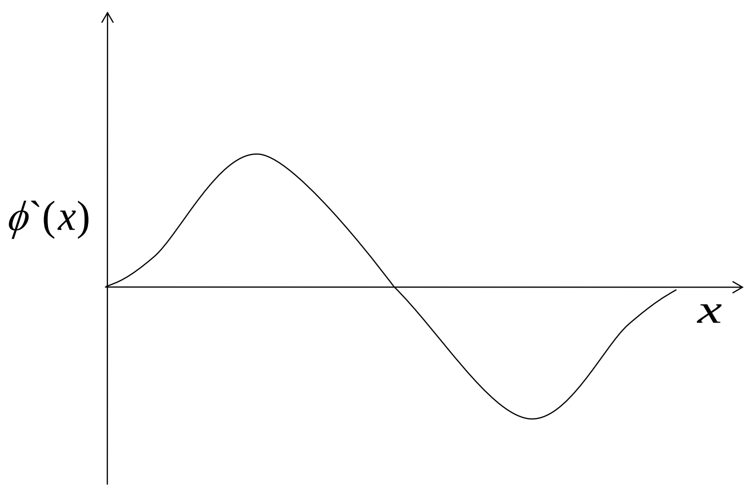
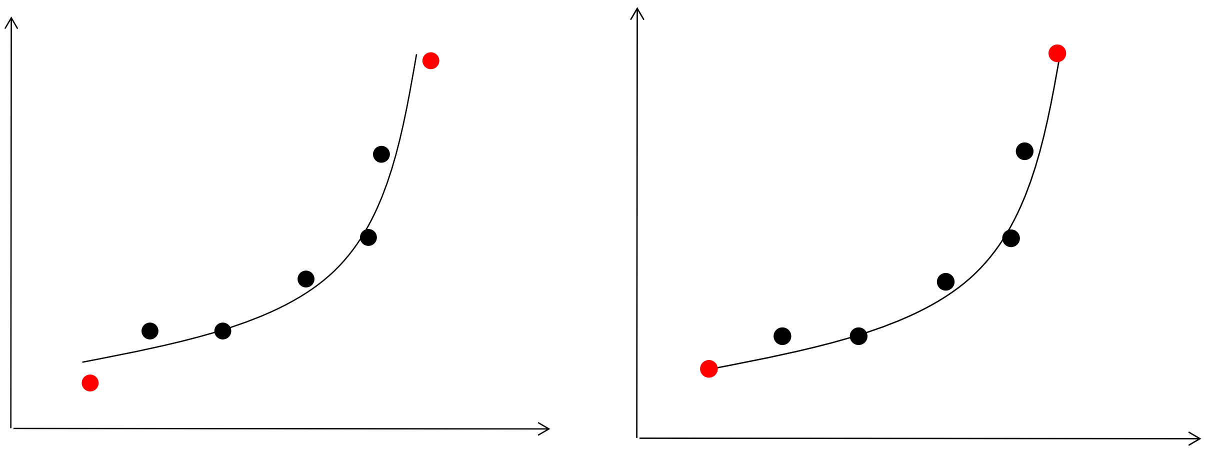



Fig. 05. Derivative of Moving Least Squares Shape function



(a). MLS approximation (b). FEM interpolation

Fig. 06. Interpolation and Approximation

### 3.3. Local Symmetric Weak Form

The MLPG method is utilized to work out the fourth order ordinary differential equation Eq. (1.). A generalized local weak form of the differential Eq. (1.) after imposing boundary conditions over the local sub domain $\Omega_s$, may be written as

$$\int_{\Omega_s}\left(EI\,\frac{d^4u}{dx^4}-f\right)v\,dx+\ \alpha_u\,[(u-\bar{u})v\,]_{\partial\Omega_s\cap\Gamma_u}=0,\ \text{for all v} \tag{11.}$$

Where, u is trial (basis) function; v is test function; $\alpha_u$ is the penalty factor to impose essential boundary conditions and $\Gamma_u$ is the boundary over which the essential boundary conditions are specified. The penalty factor utilized is generally a very large number. Using integration by parts the following expression is attained:

$$\int_{\Omega_s} EI\frac{d^2u}{dx^2}\frac{d^2v}{dx^2}ds-\int_{\Omega_s} fv\,dx+\ \alpha_u\,[(u-\bar{u})v]_{\partial\Omega_s\cap\Gamma_u}-\left[\bar{n}\,EI\frac{d^2u}{dx^2}\frac{dv}{dx}\right]_{\partial\Omega_s\cap\Gamma_\theta}+\left[\bar{n}EI\frac{d^3u}{dx^3}\,v\right]_{\partial\Omega_s\cap\Gamma_u} = 0 \tag{12.}$$

In which $\partial\Omega_s$ is boundary of the sub domain $\Omega_s$ and $\bar{n}$ is the outward unit normal to the boundary $\partial\Omega_s$. It is taken as $\bar{n} = 1$ if boundary $\partial\Omega_s$ is on the right side of $\Omega s$, and $\bar{n} = -1$ if it is on the left side of $\Omega s$. After rearranging the terms of Eq. (12.) the concise form of the LSWF attained is:

$$K^{(node)}u + K^{(boundary)}u - f^{(node)} - f^{(boundary)} = 0 \tag{13.}$$

Where,

$$\mathbf{K}_{\mathbf{ij}}^{\mathbf{node}} = \int_{\Omega s} EI \frac{d^2u}{dx^2}\frac{d^2v}{dx^2}dx \qquad \mathbf{K}_{\mathbf{ij}}^{\mathbf{boundary}} = [\alpha\, u\, v]_{\Gamma su} - \left[\bar{n}\, EI \frac{d^2u}{dx^2}\frac{dv}{dx}\right]_{\partial\Omega_s \cap \Gamma_\theta} + \left[\bar{n} EI \frac{d^3u}{dx^3}\, v\right]_{\partial\Omega_s \cap \Gamma_u}$$

$$\mathbf{f}_{\mathbf{i}}^{\mathbf{node}} = \int_{\Omega s} fv\, dx \qquad \mathbf{f}_{\mathbf{i}}^{\mathbf{boundary}} = [\alpha\bar{u}v]_{\Gamma su}$$

Where, v is the weight function and $u^h(x) = \sum_{i=1}^{n} \phi_i(x).u_i$ , $\phi_i(x)$ is the shape function attained from the moving least squares approximation. It is to be noted that the resulting displacements are fictious after that we need to convert them to true displacements and slopes using Eq. (9.).

### 3.4. Numerical Integration

The above eqn. 13 need integration over the local sub domain (quadrature domain) and on the boundary that intersects with sub domain. The integration has carried out using numerical quadrature schemes or gauss quadrature schemes. In practice each quadrature domain often needs to be further divided into cells, and the gauss quadrature technique is utilized to evaluate the integration for each cell. Therefore there will be a number of issues involved in the process such as number of cells and the number of Gauss points to be utilized.

In general there are certain difficulties in attaining the exact (to machine accuracy) numerical integration meshfree methods. Insufficient integration may cause deterioration in the numerical solution. Particularly in MLPG the difficulty is more because of the complexity of the integrand that results from Petrov Galerkin formulation. The shape functions derived using moving least squares scheme have a complex feature, they have different forms in each small integration region and the derivatives of the shape functions are extremely complex compared to normal shape functions. Other complicacy is that the overlapping of the interpolation domains makes the integrand in the overlapping domain very complex. To improve the numerical integration the quadrature domain need more and more divisions, as small as possible partitions. In each small partition, more Gauss quadrature points should be utilized.

**3.5. Procedure for enforcement of essential boundary condition**

Mainly there are three different procedures available in the literature they are Lagrange interpolation scheme, method of direct interpolation and penalty methods. In the current project penalty method is utilized for the enforcement of necessary boundary conditions. Moving least squares approximations produce shape functions that do not have the kronecker delta property, i.e. $\Phi_I^H(x_J) \neq \delta_{IJ}$. This leads to $u^h(x_J) = \sum_I^n \Phi_I^H(x_I)u_I \neq u_j$ which imply that one cannot impose the essential boundary conditions in the same way as in conventional FEM. It involves the choice of penalty factor $\alpha$. Erroneous displacements will sometimes occur If $\alpha$ is chosen improperly. The use of penalty method produces equation systems of the same dimensions that conventional FEM produces with the same number of nodes.

# CHAPTER 4

# THEORETICAL FORMULATION

### 4.1. Computation of deflection for unstrengthened beam using IS 456:2000 Code:

Fig.02. shows the problem geometry, loading and boundary conditions. The deflection profile of the unstrengthened RC beam can be attained using the Eq. (14.).

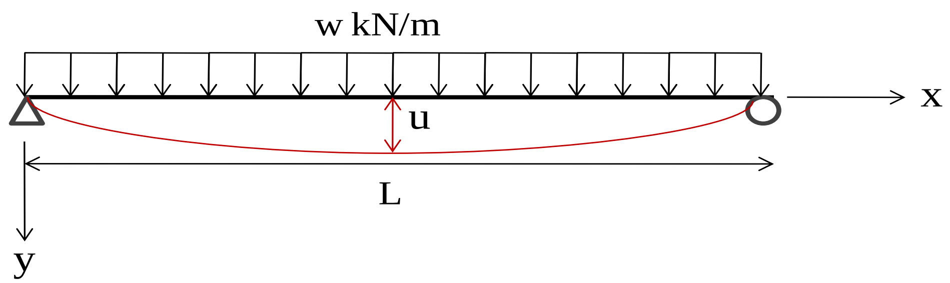



Fig.02 (Repeated). Schematics of simply supported beam of length (L)

$$y = \frac{wx}{24E_c I_{eff}}(L^3 - 2Lx^2 + x^3) \tag{14.}$$

$$y_{max} = \frac{5wl^4}{384E_c I_{eff}} \quad \text{at } x = 0.5L \tag{15.}$$

### 4.2. Effective flexural rigidity ($E_c I_{eff}$)

For the purpose of evaluating short term deflections in reinforced cement concrete flexural members, equations based on elastic theory are made utilized. A significant parameter that needs to be considered in these computation is the flexural rigidity $EI$, which is the product of young's modulus of elasticity of cement concrete $E = E_c$ , and the moment of inertia, $I$, of the cross section. The Modulus of elasticity of cement concrete depends on factors such as cement concrete quality, age, particular stress level and duration of the applied load. However for the short term loading upto service level, the code expression for the short term static young's modulus is satisfactory $E_c = 5000\sqrt{f_{ck}}$.

The moment of inertia, $I$, need to be considered in the deflection computations is influenced by the amount of reinforcement as well as the degree of flexural cracking, which in turn depends on the functional bending moment and the modulus of rupture $f_{cr}$ of cement concrete. Various empirical expressions for the "effective moment of inertia" $I_{eff}$, have been proposed and incorporated in different codes. Some of these formulations are based on assumed transition moment-curvature relations, whereas the others are based on assumed transition of stresses/strains in the region between cracks (and involve stress strain relations and equilibrium of forces).

The expression given in the Indian code (IS 456:2000, C1. C-2.1) is based on earlier version of British code, which assumes an idealized trilinear moment- curvature relation as given in Eq. (16.). Fig. 04. Shows a typical T shaped cross section of an RCC beam with flange width (bf), flange depth (Df), effective depth (d), overall depth (D) web width (bw) and depth of neutral axis from the topmost fiber (Xu).

$$I_{eff} = \frac{I_r}{1.2 - \frac{M_r}{M}\frac{z}{d}\left(1 - \frac{x}{d}\right)\frac{b_w}{b}} \quad (16.)$$

$$I_r \leq I_{eff} \leq I_{gr} \quad (17.)$$

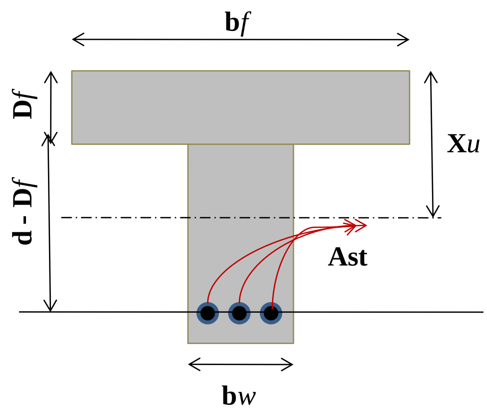



Fig.04. Schematics of typical T shaped RCC Cross sections

The cracking moment ( $M_r$) and the neutral axis in Eq. (16.) has been computed from Eqs. (18.)-(19.)

$$M_r = \frac{(0.7\sqrt{f_{ck}})(I_{Gr})}{\left(\frac{D}{2}\right)} \text{ and} \tag{18.}$$

$$b_f d_f\left(x_u - \frac{d_f}{2}\right) + \frac{b_w(x_u - d_f)^2}{2} = m\, A_{st}(d - x_u) \tag{19.}$$

Where, $I_{Gr}$ is the gross moment of inertia of the cross section about its bending axis, $m$ is the modular ratio its value is taken as $E_s/E_c$. If we made bf and $b_w$ equal then the above computational procedure is also suitable for rectangular cross section. In case of doubly reinforced cross sections and extra parameter denoting area of compression reinforcement must be included while computing neutral axis depth and young`s modulus of the composite section.

# CHAPTER 5

# RESULTS AND DISCUSSIONS

## 5.1. Introduction

In the present study, the Petrov Galerkin system is used in every nearby sub domain. In the ordinary Galerkin strategy, the trial and test functions are browsed the same function space; then again, in the Petrov Galerkin system, the trial and test functions are looked over changed function spaces. Also in the Petrov Galerkin method the measurement of nearby sub domain and support domain of the nodal shape function for the test and trial function are divergent. In the event that we made alike it prompts typical Galerkin approximation process.

In the present study, the support domain of the node is considered as proportional to the nodal dispersing increased by a scaling variable. A standard designation of nodes is used to discretize the worldwide issue domain, with vertical removal as the unidentified nodal field variable. On the off chance that the conveyance is not standard, size of the test function will be the minimum separation between any two progressive nodes. Likewise key scaling element is expected to land at a proper size of the support domain. For such a case, the computation of the test function turns out to be hard, on the grounds that an all around characterized shape function obliges a more prominent number of nodes than the polynomial evaluation. In this way, the typical disseminations of nodes are utilized to accomplish the computational precision. Thus, the rate of union of the arrangement relies on upon the nodal appropriations and the measurement of support domain, which are ascertained through the scaling variable. The aggregate number of nodes in the worldwide domain is chosen in such a path, to the point that the amalgamation of all such neighborhood sub domains covers the worldwide domain however much as could be expected, with least convergence of sub domains. The higher the quantity of nodes included the worldwide domain the more mix focuses are obliged to achieve the computational exactness comparing to a lesser nodal separation. This sort of intricacy may bring about the trouble of the shape functions in the MLPG technique. It is less demanding in the processing of the MLPG system if the nodal dispersing is same.

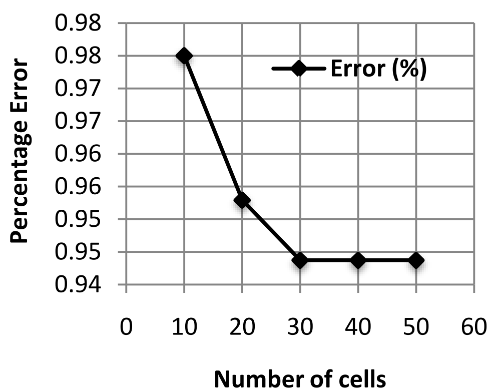
In processing the span of the support domain is taken as $3.5\delta x$, with $\delta x$ being the separation between two continuous nodes. For numerical reconciliation, ten point gauss quadrature is used. A meeting study is done to discover the important number of nodes and the quantity of mix cells needed in accomplishing a littlest sum blunder in the figured displacement

**5.2. Determination of Optimum Number of Nodes and Integration Cells:**

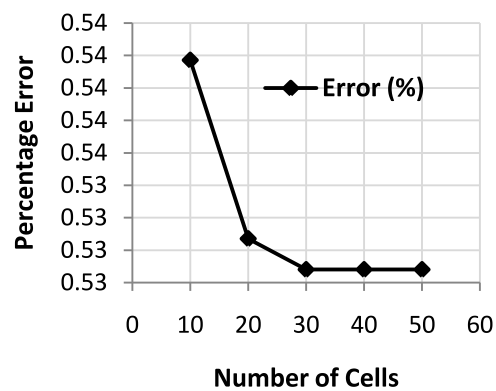
This example is studied to find the optimal number of nodes and integration cells in the sub domain. A simple isotropic beam of Young's modulus; moment of inertia; length of the beam (L) and the uniformly distributed load over the beam is taken as unity. The choice of total number of nodes and the total number of integration cells in each local sub domain is very important. Innecessary choice leads to erroneous displacements. Fig. 07 shows the level of percentage error in the computation of mid span displacement with integration cells in the sub domain varying from 10 to 50 and each graph is made for constant number of nodes i.e. Fig. 07(a) 31 nodes, Fig. 07(b) 41 nodes and so on Fig. 07(f) 81 nodes. The error norm is evaluated as follow:

$$\| e \| = \left| \frac{(U_1 - U_2)\, 100}{U_1} \right| \tag{20.}$$

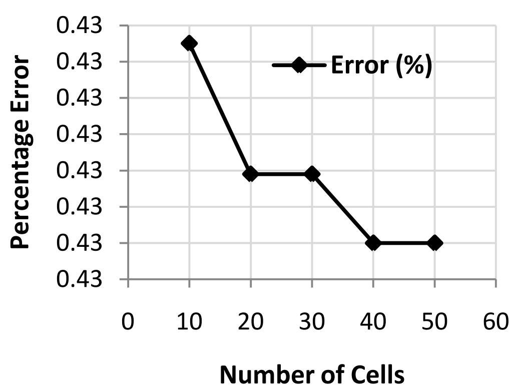
Where,$U_1$ is analytical displacement at centre of the span which is 0.0130 m and $U_2$is MLPG displacement.At the selection of 81 nodes and 50 integration cells in the sub domain, the computed displacement is converged with the analytical displacement with percentage error of 0.07%. On the basis of the preceding observations, the following examples are studied and compared with analytical displacements.



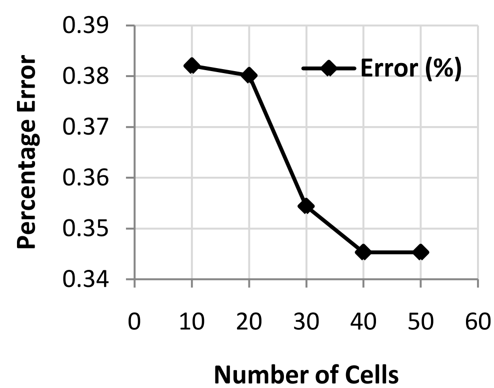

(a) Percentage error vs. number of integration cells in sub domain



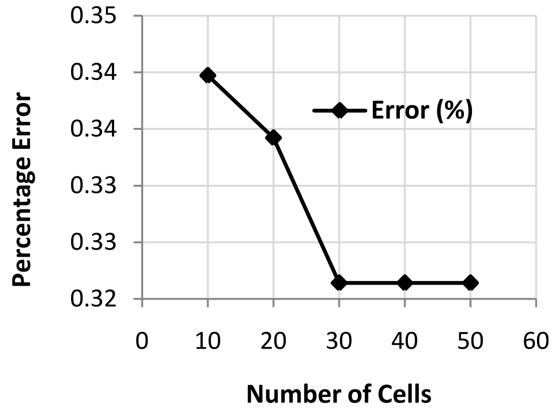

(b) Percentage error vs. number of integration cells in sub domain



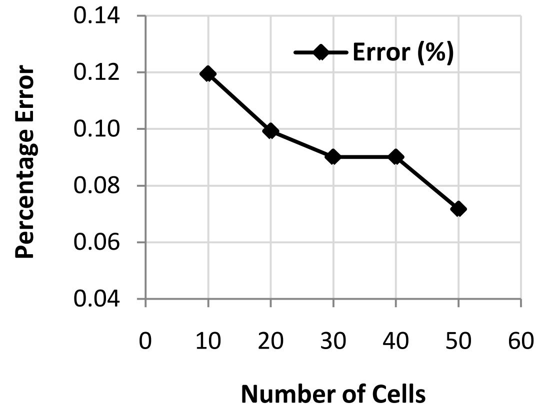

(c) Percentage error vs. number of integration cells in sub domain




(d) Percentage error vs. number of integration cells in sub domain




(e) Percentage error vs. number of integration cells in sub domain

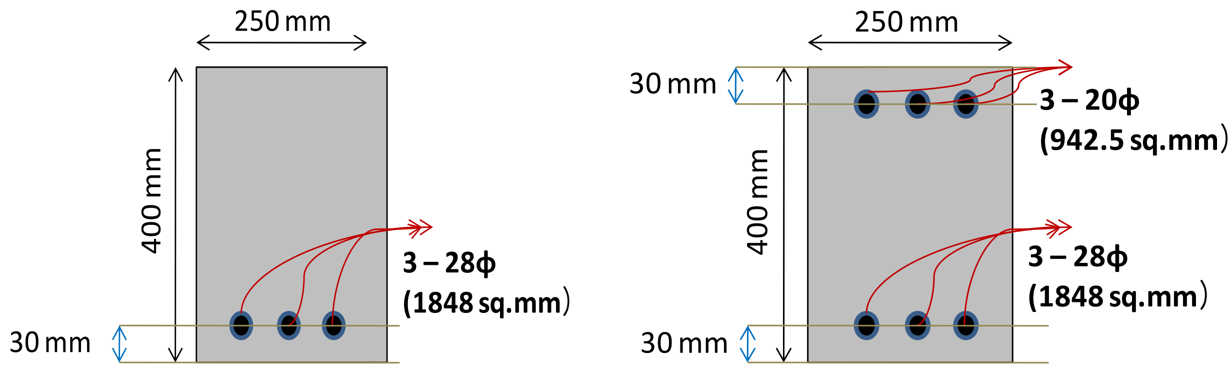



(f) Percentage error vs. number of integration cells in sub domain

Fig. 07. Percentage error norm disparity corresponding to the disparity in number of nodes and integration cells in the sub domain

**5.3. Unstrengthened Singly and Doubly Reinforced Rectangular Cross Section:**

Evaluation of mid span displacement in simply supported beam with singly reinforced and doubly reinforced rectangular cross sections is addressed in this example. The length of the beam is 1m and the uniformly distributed load is 10 kN/m. cross section details of the beam taken for analysis are shown in Fig. 08. As illustrated in example 1, a total of 81 nodes in the global domain and 50 integration cells in each sub domain are considered for the analysis. It is observed that the deflection decreases with increase in the stiffness which in turn due to young`s modulus i.e. the central deflection of beam with doubly reinforced cross section is less than with the singly reinforced cross section. The computed MLPG, IS 456:200 displacements and corresponding error are presented in Table. 02. The absolute error norm is evaluated using the Eq. (20.)




(a) Singly reinforced section (b) Doubly reinforced section

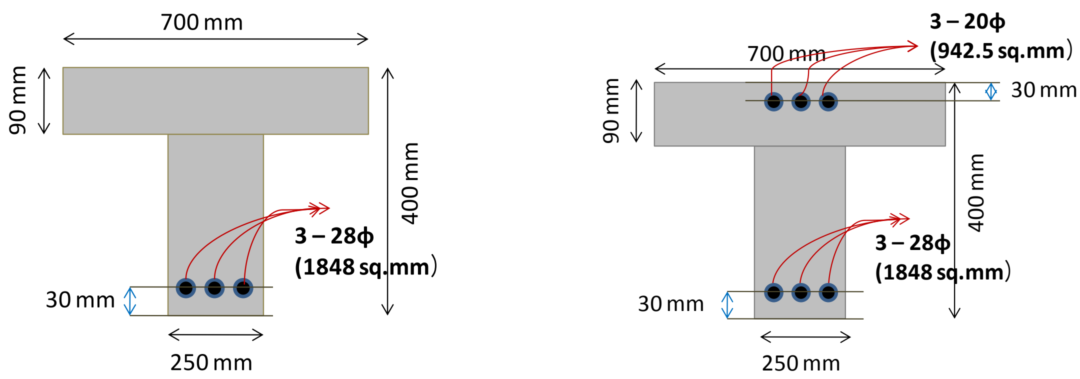
Fig. 08.Detailing of rectangular RCC cross sections

Table. 02: IS 456:2000 and MLPG Displacements at 0.5L in a unstrengthened simply supported beam with singly and doubly reinforced rectangular cross section

| | IS 456:2000 displacement (mm) | MLPG displacement (mm) | Error (%) |
|---|---|---|---|
| Singly reinforced rectangular cross section | 5.4470 | 5.4519 | 0.08 |
| Doubly reinforced rectangular cross section | 4.3673 | 4.3713 | 0.07 |

### 5.4. Unstrengthened Singly and Doubly Reinforced T Shaped Cross Section:

Evaluation displacement at 0.5L in simply supported beam with singly reinforced and doubly reinforced T shaped cross sections is addressed in this example. The length of the beam is 1m and loaded with 10 kN/m uniformly distributed load. Fig. 09 shows the cross section details of the beam taken for analysis. As illustrated in example 1, a total of 81 nodes in the global domain and 50 integration cells in each sub domain are considered for the analysis. It is observed that the deflection decreases with increase in stiffness. The computed MLPG, analytical displacements and corresponding error are presented in Table. 03. The absolute error norm is evaluated using the Eq. (20.).




(a) Singly reinforced section (b) Doubly reinforced section

Fig. 09.Detailing of T shaped RCC cross sections

Table. 03: IS 456:2000 and MLPG Displacements at 0.5L in a unstrengthened simply supported beam with singly and doubly reinforced T shaped cross section

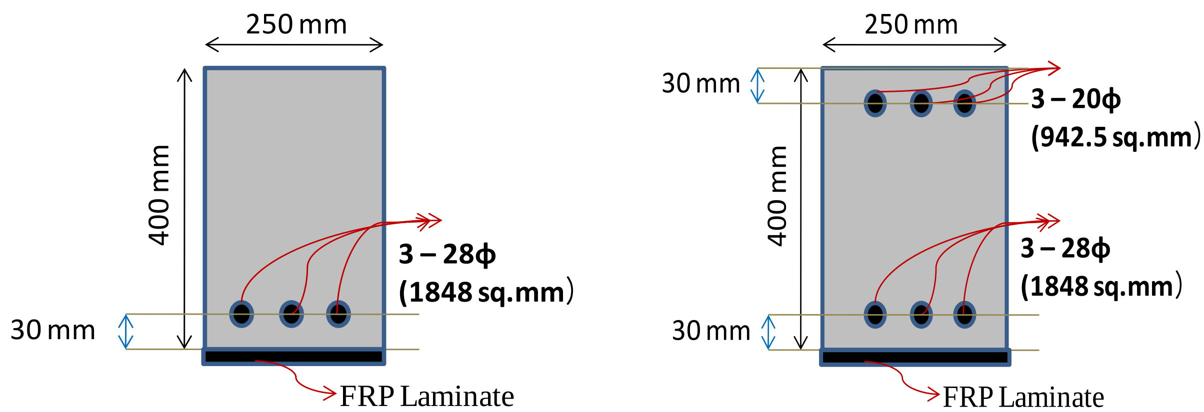
| | IS 456:2000 displacement (mm) | MLPG displacement (mm) | Percentage error (%) |
|---|---|---|---|
| Singly reinforced rectangular cross section | 4.0845 | 4.0882 | 0.09 |
| Doubly reinforced rectangular cross section | 4.0674 | 4.0711 | 0.08 |

**5.5. FRP Strengthened Singly and Doubly Reinforced Rectangular Cross Sections:**

On the basis of the comparative studies performed in the above examples 2 and 3, it is ascertained that the MLPG codes gives satisfactory displacements. The developed codes are then utilized to compute the displacements in FRP strengthened beams. In this example, the effect of strengthening with FRP laminates on displacement of a beam with singly and doubly reinforced rectangular cross section is studied. The beam is strengthened with high strength carbon fibre reinforced polymer (CFRP), basalt fibre reinforced polymer (BFRP) and glass fibre reinforced polymer (GFRP) laminates as shown in Fig. 10. Longitudinal young`s modulus of the laminates are given in table 04. It is observed that the beam deflection had decreased considerably as the stiffness is increased. The young`s modulus is computed by idealizing the section as cracked transformed section. A total of 81 nodes and 50 integration cells are utilized in the sub domain. The computed MLPG displacements are presented in Table 4. The graphical disparity displacement along the length of beam for different FRP laminates is presented in Fig. 12 and Fig. 13.

Table 04. Elastic properties of FRP laminates

| S. No | FRP Laminate | Longitudinal young's modulus (MPa) |
|---|---|---|
| 1. | Glass fibre epoxy (GFRP) | 7207.4 |
| 2. | Basalt fibre epoxy (BFRP) | 13130 |
| 3. | High strength carbon fibre epoxy (CFRP) | 139000 |




(a) Singly reinforced section strengthened with FRP laminate

(b) Doubly reinforced section strengthened with FRP laminate

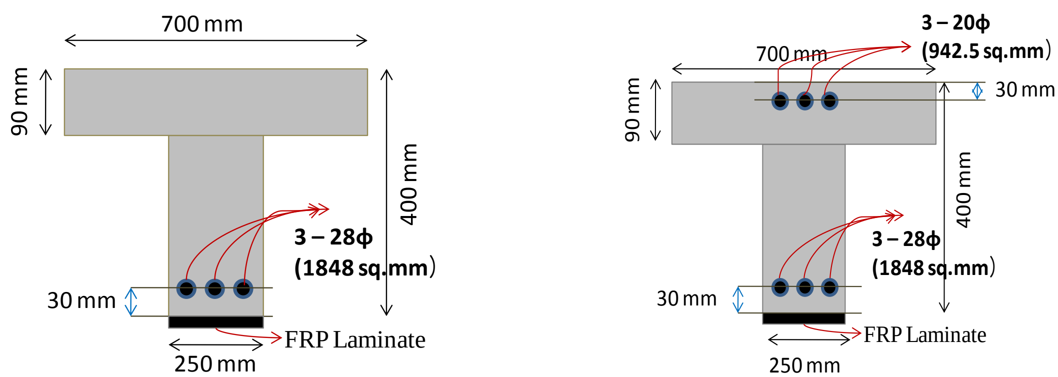
Fig. 10.Detailing of rectangular RCC cross sections strengthened with FRP laminates

Table. 05: Displacements attained from MLPG at 0.5L in a simply supported beam with singly and doubly reinforced rectangular cross section strengthened with FRP laminates

| | Singly Reinforced (mm) | Doubly Reinforced (mm) |
|---|---|---|
| Unstrengthened beam | 5.4519 | 4.3713 |
| GFRP | 4.2510 | 3.9740 |
| BFRP | 3.8463 | 3.5818 |
| CFRP | 1.7862 | 1.5923 |

**5.6. FRP Strengthened Singly and Doubly Reinforced T- Shaped Cross Sections:**

The effect of strengthening with FRP laminates on displacement of a beam with singly and doubly reinforced T shaped cross section is studied in this example. The beam is strengthened with high strength CFRP, BFRP and GFRP laminates as shown in Fig. 11. Material properties of the laminates are given in table 3. It is observed that the beam deflection had decreased considerably as the stiffness is increased. The young`s modulus is computed by idealizing the section as cracked transformed section. A total of 81 nodes and 50 integration cells are utilized in the sub domain. The computed MLPG displacements are presented in Table. 06. The graphical disparity displacement along the length of beam for different FRP laminates is presented in Fig. 14 and Fig. 15.




(a) Singly reinforced section strengthened with FRP laminate

(b) Doubly reinforced section strengthened with FRP laminate

Fig. 11.Detailing of T shaped RCC cross sections strengthened with FRP laminates

Table. 06: Displacements attained from MLPG at 0.5L in a simply supported beam with singly and doubly reinforced T shaped cross section strengthened with FRP laminates

| | Singly Reinforced (mm) | Doubly Reinforced (mm) |
|---|---|---|
| Unstrengthened beam | 4.0882 | 4.0711 |
| GFRP | 3.7229 | 3.5846 |
| BFRP | 3.4767 | 3.3429 |
| CFRP | 1.7063 | 1.6048 |

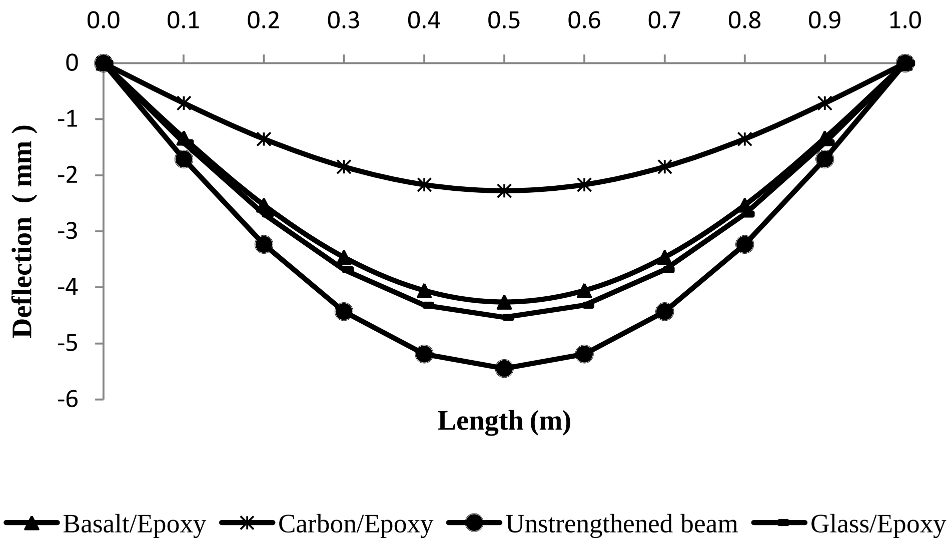



Fig. 12. Graph showing disparity of displacement along the length of singly reinforced rectangular cross section

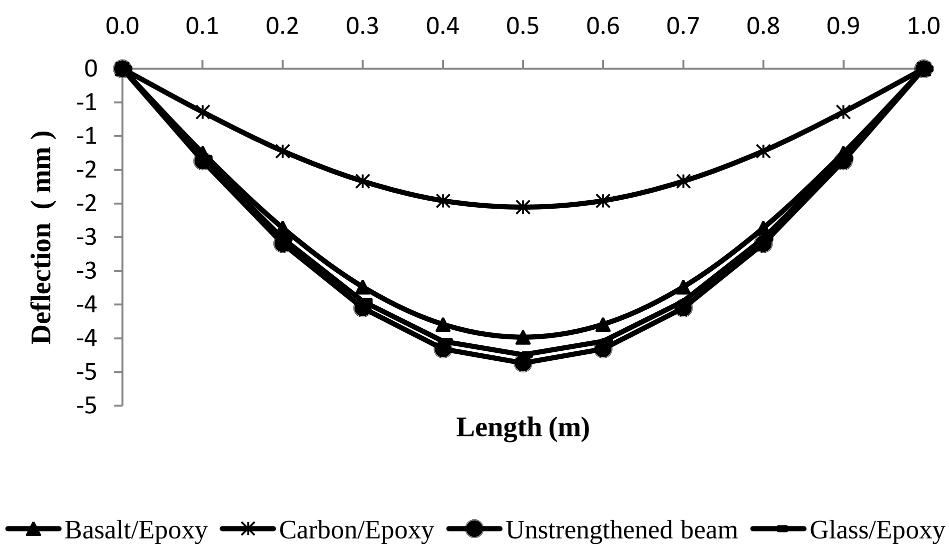



Fig. 13. Graph showing disparity of displacement along the length of Doubly reinforced rectangular cross section

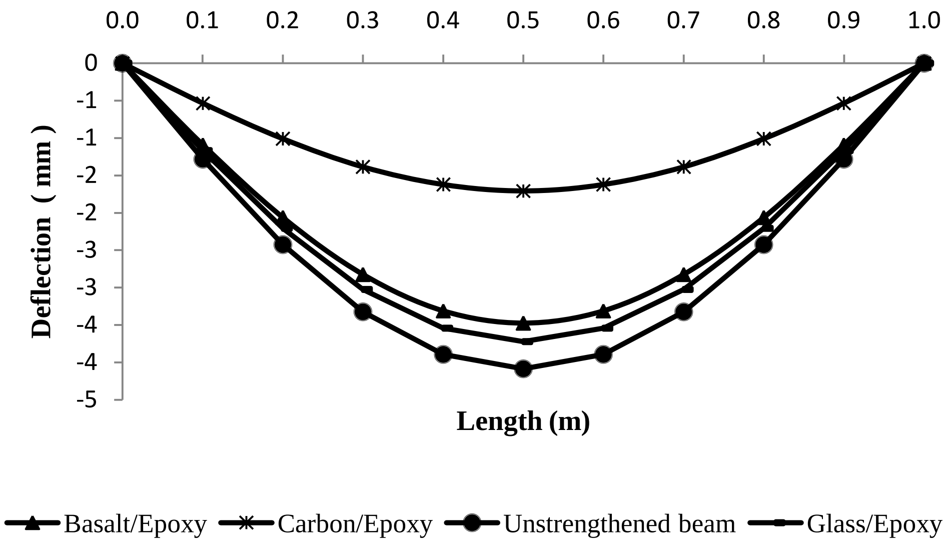



Fig. 14. Graph showing disparity of displacement along the length of Doubly reinforced T Shaped cross section

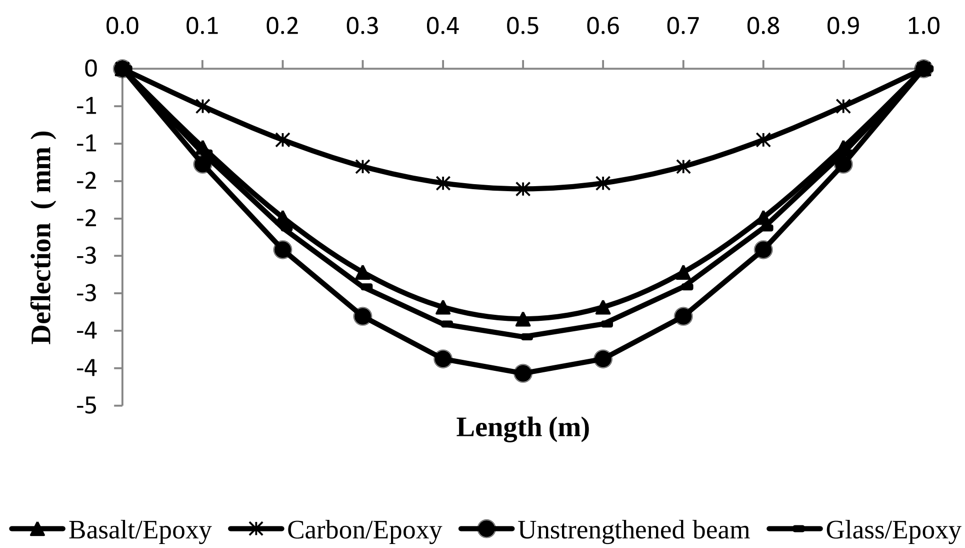



Fig. 15. Graph showing disparity of displacement along the length of Doubly reinforced T Shaped cross section

## *CHAPTER 6*

# CONCLUSION

### 6.1. Conclusion

The validity and correctness of the present numerical simulation schemes are verified by comparing the MLPG displacements with analytical displacements, and good accord is found. The displacement is greatly influenced by the young`s modulus of the cross section, length of the beam, load over the beam, effect of number of nodes in the global domain, scaling parameter for shape function evaluation and the integration cells in sub domain appears to be significant, from the parametric study it is observed that at 81 nodes and 50 integration cells with scaling parameter as 3.5 times nodal spacing for support domain the percentage error is almost negligible.

In case of unstrengthened rectangular and T shaped RCC beams the percentage error between IS 456:2000 and MLPG displacements is within the range of 0.07% to 0.09%, hence the displacements are acceptable. Later the effect of strengthening on the displacement is studied with high strength carbon fibre reinforced polymer (CFRP), basalt fibre reinforced polymer (BFRP) and glass fibre reinforced polymer (GFRP) laminates are utilized for the study. The additional FRP laminate shifts the neutral axis towards the tension side and enhances the stiffness of the beam. Hence the displacement is reduced due to increased stiffness.

On the basis of the studies performed in first three examples. It is found out that the current MLPG codes gives acceptable displacements. The developed codes are then extended to compute the displacements in FRP strengthened beams, computed displacements are presented. Hence the attained displacements are acceptable. Unlike FEM the MLPG requires no structured mesh, since only a scattered set of nodal points required in the domain of interest.

There is no need for connectivity between the nodes, since mesh generation of complex structures is more time consuming and costly effort than solving discrete set of equations, the current meshless method MLPG provides an alternative to the finite element method for solving fourth order differential equations for beam bending.

### 6.2. Future Scope of Research

1. In this project only determinate simply supported beam is considered. The formulation can also be extended to indeterminate continuous beams.

2. The dynamic and buckling analysis of beams can be conducted as an extension to the current research work.

3. The stiffness and force matrices generated by using the above formulation can be utilized to analyse the frame structures by taking orientation of the matrices into consideration.

## REFERENCES


1 Nayroles, B., Touzot. G & Villon. P (1992). Generalizing the finite element method: diffuse approximation and diffuse elements. Computational Mechanics, 10 (5), 307-318.

2 Belytschko, T, Lu, Y. Y &Gu, L. (1994). Element free Galerkin methods. International Journal for Numerical Methods in Engineering, 37(1), 229-256.

3 Duarte, C. A & Oden, J. T (1996). Hp clouds - an hp meshless method. Numerical Methods for Partial Differential Equations, 12(6), 673-705.

4 Liu, W. K., Jun. S & Zhang, Y. F (1995). Reproducing kernel particle methods. International Journal for Numerical Methods in Engineering, 20(2), 1081-1106.

5 Melenk, J. M &Babuska. I (1996). The partition of unity finite element method basic theory and applications. Computer Methods in Applied Mechanics and Engineering, 139(2), 289-314.

6 Atluri, S. N., & Zhu, T. (1998). A new meshless local Petrov-Galerkin (MLPG) approach in computational mechanics. Computational Mechanics, 22(2), 117-127.

7 Lin, H. &Atluri, S. N. (2000). Meshless Local Petrov-Galerkin (MLPG) method for convection- diffusion problems. Computer Modeling in Engineering and Sciences, 1(2), 45-60.

8 Liu, G. R &Gu, Y. T (2000). Meshless Local Petrov-Galerkin (MLPG) method in combination with finite element and boundary element approaches. Computational Mechanics, 26(6), 536-546.

9 Ching, H. K &Batra, R. C (2001). Determination of crack tip fields in linear

elastostatics by the Meshless Local Petrov-Galerkin (MLPG) method. Computer Modeling in Engineering and Sciences, 2(2), 273-289.

10 Gu, Y. T & Liu, G. R (2001a). A Meshless Local Petrov-Galerkin (MLPG) method for free and forced vibration analyses for solids. Computational Mechanics, 27(3), 188-198.

11 Qian. L. F., Batra. R. C & Chen. L. M (2003a). Elastostatic deformations of a thick plate by using a higher-order shear and normal deformable plate theory and two Meshless Local Petrov-Galerkin (MLPG) Methods. Computer Modeling in Engineering and Sciences, 4(1), 161-175.

12 Qian, L. F, Batra, R. C & Chen, L. M (2003b). Free and forced vibrations of thick rectangular plates by using higher-order shear and normal deformable plate theory and Meshless Local Petrov-Galerkin (MLPG) Method. Computer Modeling in Engineering and Sciences, 4(5), 519-534.

13 H. G. Kim & S. N. Atluri. (2000). Random placement of secondary nodes, and error control, in the meshless local Petrov-Galerkin (MLPG) method. Computer Modeling in Engineering and Sciences, 1(3), 11–32.

14 J. Ren& K. M. Liew (2002). Mesh-free method revisited: two new approaches for the treatment of essential boundary conditions. International Journal of Computational Engineering Science, 3(2), 219–233.

15 S. N. Atluri& S. Shen (2002). The meshless local Petrov-Galerkin (MLPG) method: a simple & less-costly alternative to the finite element and boundary element methods. Computer Modeling in Engineering and Sciences, 3(1), 11–51.

16 S. Fernandez-Mendez & A. Huerta (2004). Imposing essential boundary conditions in

mesh-free methods. Computer Methods Applied Mechanics and Engineering, 193(12)–(14), 1257–1273.

17 Atluri, S. N., &Shen, S. P. (2005). Simulation of a 4th order ode: illustration of various primal & mixed MLPG methods. Computer Modeling in Engineering & Sciences, 7(3), 241–268.

18 Zuo, C.W &Nie. Y.F (2005). The selection about the radius of influence in MLS. Chinese J. Engineering. Math, 5(1), 833-838.

19 S. Ikuno, K. Takakura& A. Kamitani (2007). Influence of method for imposing essential boundary condition on meshless Galerkin/ Petrov-Galerkin approach. IEEE Transactions Magazine, 43(4), 1501–1504.

20 A. R. Fonseca., S. A. Viana., E. J. Silva., & R. C. Mesquita. (2008). Imposing boundary conditions in the meshless local Petrov-Galerkin method. IET Sci. Meas. Technology, 6(2), 387–394.

21 Sladek. J., Sladek. V., Solek. P &Atluri, S. N (2008). Modeling of Intelligent Material Systems by the MLPG. Computer Modeling in Engineering & Sciences, 34(2008), No. 3, 273-300.

22 R. Trobec., M. Šterk& B. Robic . Computational complexity and parallelization of the meshless local Petrov–Galerkin method. Computers and Structures, 87(2009), No. 1, 81–90.

23 A. Abdollahifar., M. R. Nami., & A. R. Shafiei. A new MLPG method for elastostatic problems. Engineering Analysis with Boundary Elements, 36(2012), No. 3, 451–457.

24 J. Sladek., P. Stanak., Z. D. Han., V. Sladek& S.N. Atluri.. Applications of the MLPG Method in Engineering &Sciences: A Review. Computer Modeling in Engineering &

Sciences, 92(2005), No. 5, 423-475.

25 David V. Hutton, Fundamentals Of Finite Element Analysis, Tata McGraw Hill (2004), New York USA.

26 G.R. Liu, Mesh Free Methods Moving Beyond The Finite Element Method, CRC Press (2003), Boca Raton, Florida

27 Standard, Indian. "IS-456. 2000. Plain and Reinforced Cement concrete-Code of Practice, Bureau of Indian Standards, Manak Bhawan, 9 (2000).

28 U. Andreaus, R. C. Batra, and M. Porfii, Vibrations of cracked Euler-Bernoulli beams using meshless local Petrov Galerkin (MLPG) method, Computer Modeling in Engineering and Sciences, vol. 9 (2005), no. 2, pp. 111–131.

29 H. G. Kim and S. N. Atluri, Random placement of secondary nodes, and error control, in the meshless local Petrov-Galerkin (MLPG) method, Computer Modeling in Engineering and Sciences, vol. 1(2000), no. 3, pp. 11–32.

30 R. Vaghefi G. H. Baradaran, and H. Koohkan, Three dimensional static analysis of thick functionally graded plates by using meshless local Petrov-Galerkin (MLPG) method, Engineering Analysis with Boundary Elements, vol. 34 (2010), no. 6, pp. 564–573.

31 S. D. Daxini, J. M. Prajapati, A review on recent contribution of meshfree methods to structure and fracture mechanics applications, The Scientific World Journal, Volume 2014, Article ID 247172, 13 pages, 2014.

32 D. L. Logan, A first course in Finite Element Method, THOMSON, 2007

L. Kaiyuan, L. Shuyao, and L. Guangyao, A simple and less costly meshless local Petrov-Galerkin (MLPG) method for the dynamic fracture problem, Engineering

Analysis with Boundary Elements, vol. 30 (2006), no. 1, pp. 72–76.

33 D. A. Hu, S. Y. Long, X. Han, and G. Y. Li, A meshless local Petrov-Galerkin method for large deformation contact analysis of elastomers, Engineering Analysis with Boundary Elements, vol. 31 (2007), no. 7, pp. 657–666.

34 T. Belytschko, Y. Krongauz, D. Organ, M. Fleming, and P. Krysl, Meshless methods: an overview and recent developments, Computer Methods in Applied Mechanics and Engineering, vol. 139 (1996), no. 1–4, pp. 3–47.