Free Flexural Vibration of Composite Beam by Spectral Element Method

Manas Ranjan Pradhan

Department of Civil Engineering
National Institute of Technology
Rourkela – 769 008, India

Free Flexural Vibration of Composite Beam by Spectral Element Method

Dissertation submitted in

May 2015

to the Department of

Civil Engineering

of

National Institute of Technology Rourkela

in partial fulfillment of the requirements

for the degree of

Master of Technology in

Structural Engineering

by

Manas Ranjan Pradhan

(Roll No. 213CE2070)

under the supervision of

Prof. Manoranjan Barik

Department of Civil Engineering

National Institute of Technology

Rourkela – 769 008, India

Dedicated to

My beloved Parents & Mentor Barik sir...

Department of Civil Engineering
National Institute of Technology Rourkela
Rourkela-769 008 , Odisha , India. www.nitrkl.ac.in

Dr. Manoranjan Barik
Associate Professor

May , 2015

Certificate

This is to certify that the work in the thesis entitled *Free Flexural Vibration of Composite Beam by Spectral Element Method* by *Manas Ranjan Pradhan*, bearing Roll Number 213CE2070, is a record of an original research work carried out by him under my supervision and guidance in partial fulfillment of the requirements for the award of the degree of *Master of Technology* in *Structural Engineering, Department of Civil Engineering*. Neither this thesis nor any part of it has been submitted for any degree or academic award elsewhere.

Manoranjan Barik

Acknowledgement

First I thank God for the blessings on me to pursue this program successfully in spite of all challenges faced. My wholehearted gratitude to my beloved guide, *Prof. Manoranjan Barik, (Dept. of Civil Engineering, NIT, Rourkela)*, for his imparallel guidance, precious time, valuable suggestions and constructive criticisms that he imparted throughout the work. The encouragement and suggestions by him was always there during my research as well as thesis writing. I did my level best to obey every order and suggestion of my guide. His support made me grab this improved final product.

My special thanks to Head of the Department, *Prof. S.K. Sahu*, for the consistent support in our academic activities. I express my sincere thanks to all the faculty members and staffs of Civil Engineering Department, who supported throughout my research work. I am obliged to express my sincere thanks to *Prof. K. C. Biswal* for his continuous encouragement and invaluable advice.

My warmly thanks to all my friends for their positive criticism, loving cooperation, excellent advice and consistent support during the preparation of this thesis. Special thanks to Biraja Prasad Mishra(PhD scholar) and my colleague Pranab Kumar Ojah for their valuable support and advice in crucial time.

Most profound regards to my mother *Smt.Sukanti Pradhan* and my father *Shri Dwitiyanand Pradhan*, for their exceptional love and support. I feel short of words to express my gratitude towards my beloved parents for their sacrifice on so many things for my betterment.

Manas Ranjan Pradhan
Structural Engineering
Roll No. 213CE2070

Abstract

The composite beam may be of the forms of uniform, non-uniform or tapered. Free vibration of uniform composite beam has been studied by researchers with different boundary conditions using various methods of analysis such as First Order Shear Deformation Theory(FSDT), Wittrick Williams algorithm, Trigonometric Shear Deformation Technique, Runge Kutta Nystran Method, Higher Order Beam Theory, Dynamic Finite Element Method(DFEM), Lagrange Multiplier Method etc., besides the conventional analytical methods and finite element methods. Rarely any literature is available on free vibration on composite stepped beam. The one employed by Lam and Sathiyamoorthy [1] is based on Runge-Kutta-Nystran Method. They studied the behaviour of uniform composite beam and validated the result by Runge-Kutta-Nystran Method and extended it to the stepped composite beam. For the free vibration of composite tapered beam only few papers are available where methods like FSDT, higher order finite element method, complementary function method have been used. In this present work the Spectral Element Method (SEM) has been used for the free vibrations of composite uniform, stepped and tapered Timoshenko beam. The results obtained for all the above beams by Spectral Element Method (SEM) are more promising in accuracy compared to the other methods even for higher modes and with lesser degrees of freedom.

***Keywords*:** composite uniform beam, composite stepped beam, composite tapered beam, Spectral Element Method (SEM), FSDT, Dynamic Finite Element Methods (DFEM).

Contents

List of Figures

List of Tables

Chapter 1

Introduction

The composite materials have many advantages over others due to their high strength to weight ratio (specific strength), design flexibility so that it gives the designer free hand to create any shape or configuration at a very low cost, long term corrosion resistance from severe chemical and temperature exposures, long life span and low maintenance requirements. So they are widely being used in helicopter blades, aircraft wings, propeller and turbine blades, axles of vehicles, robot arms, satellite antennas etc. Non uniform composite beams like stepped beam and tapered beam are used for better strength and weight distribution with good architectural and functional requirements. The dynamic analysis of structures is very important in engineering field. By appropriate structural coupling (obtained by perfect lay up and fiber orientation) and material tailoring the dynamic characteristics of composite laminated structures can be enhanced. The dynamic analysis of structures can be carried out mainly by the following methods:

1. Finite element Method (FEM)
2. Dynamic Stiffness Method (DSM)
3. Spectral Analysis Method (SAM)
4. Spectral Element Method (SEM)

1.1 Finite Element Method (FEM)

In finite element formulation frequency independent shape functions are used but by using these shape functions, frequency waves at higher modes can't be captured as associated wavelengths are much shorter. To improve the solution accuracy mesh refining can be done but this approach make the system size extremely large and not suitable from computational aspect.

1.2 Dynamic Stiffness Method (DSM)

In DSM the main purpose is to calculate exact dynamic stiffness matrix in frequency domain so the time domain governing equations are transformed into frequency domain to get the wave solutions. By using wave solutions, shape functions are obtained to finally formulate dynamic stiffness matrix. In DSM exact frequency domain solutions are obtained from governing equations so in other words DSM is called as exact solution method. The accuracy of DSM is totally dependent on the accuracy of governing equations. In DSM no meshings are required so degree of freedom (DOFs) decreases so also the computational cost, time for improving accuracy by reducing round-off errors. With minimum number of DOFs, DSM provides infinite eigenvalues from dynamic stiffness matrix.

1.3 Spectral Analysis Method (SAM)

In SAM wave mode superpositions of different frequencies by first fourier transformation (FFT) and inverse first fourier transformation (IFFT) are done to find out solutions of governing differential equations.

1.4 Spectral Element Method (SEM)

The SEM is the combination of key apects of FEM, DSM and SAM. In short the key aspects are:

- FEM: discretization and assembling of finite elements
- DSM: formulation of dynamic stiffness matrix with less number of DOFs
- SAM: superposition of wave modes by FFT theory

Just like conventional FEM, the SEM also is an element method. So when any external force exists or geometrical or material discontinuities exist, mesh refining is done in SEM like FEM. Some advantages of SEM are:

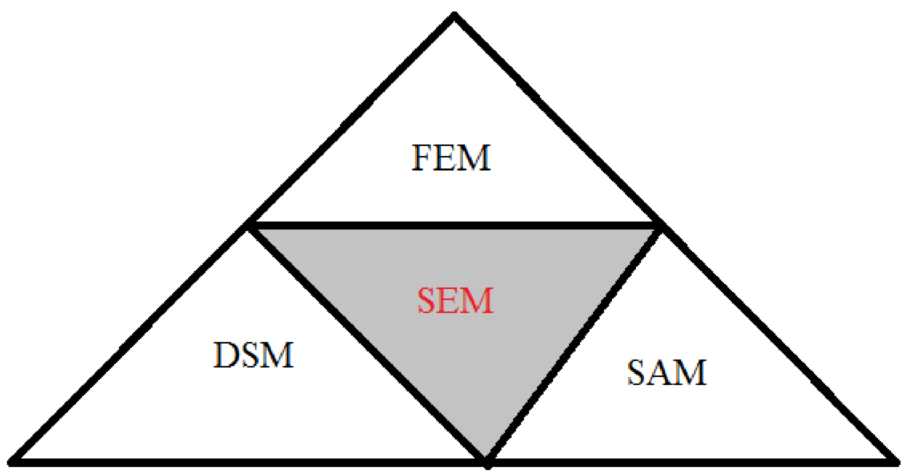


Figure 1.1: Key features of SEM

- highly accurate
- smallness of problem size and degree of freedoms(DOFs)
- low computational cost
- effective to deal with digitalized data
- effective to deal with frequency domain problems

1.5 Objectives

The primary objectives of this research work are summarized as follows:

1. To study the free flexural vibration of uniform and non uniform (stepped and tapered) composite laminated Timoshenko Beam using the Spectral Element Method (SEM).

2. To compare the natural frequencies found using SEM with those found by other methods in the published results.

Chapter 2

Literature Review

2.1 Uniform Laminated Composite Beam

There are many research papers available on uniform composite beam considering various boundary conditions. Teoh and Huang [2] presented an analytical method that takes account of rotary inertia, fiber orientation and shear deformation in free vibration analysis of fiber reinforced composite beam. Chandrasekhar et al. [3] used First Order Shear Deformation Theory (FSDT) to predict exact free vibration of symmetrical composite beam. They also showed how free vibration is affected by anisotropy of materials, shear deformation and various end conditions. Abramovich [4] studied the free vibrational analysis of composite beam but ignored coupling effect of shear deformation and rotary inertia. Krishnaswamy et al. [5] used a series solution embedded with Langrange Multipliers to get anlytical solution of layered composite beam. They used Langrange Multipliers due to ease in picking of displacement functions as displacement functions don't require to satisfy boundary conditions. End conditions which are not satisfied by assumed series declared as constraint. Abramovich and Livshits [6] used FSDT theory for free vibrational analysis of non symmetrical laminated composite beam. They took longitudinal deformation with shear deformation and rotary inertia for analysis of beam. Eisenberger et al. [7] used exact shape functions to get exact dynamic stiffness matrix to ultimately get exact natural frequencies. Hassan et al. [8] showed that change in fiber orientation gives the designer another angle of making structure more stiffened. They also cited that without changing the material and geometrical properties, desired natural frequencies can be found by just changing the orientation.

Teboub and Hajela [9] used FSDT to analysis free vibration of both symmetric and non-symmetric composite beam considering effect of beam geometry, poisson's effect, material anisotropy, boundary conditions. The more accurate governing equations are used for accurate results using symbolic computation of Maple software. Banerjee and Williams [10] formulated a dynamic stiffness matrix (DSM) for Timoshenko beam. In this case they accounted coupling between bending and torsional deformation. Lam and Sathiyamoothy [1] used Runge-Kutta-Nystran numerical method to get non dimensional frequencies of composite laminated beam i.e symmetrical beam, non symmetrical beam, angle ply laminate and cross ply laminate. Banerjee [11] analyzed a composite Timoshenko beam taking account of axial force, shear rigidity and rotation to develop a dynamic exact stiffeness matrix to get modal frequencies using Wittrick-Williams algorithm. Shi and Lam [12] used Shear Deformation Theory based on third order to get required stiffness and mass matrices for composite beam. The coupled mass matrix of axial displacement components and high order mass matrix showed their significant effect on modal frequencies at higher modes in flexure. Bassiouni et al. [13] analyzed a finite element model to get natural frequenices and modal shapes of composite laminated beam considering effect of shear deformation and fiber orientation. They got that change in fiber orientation at core of the beam which carried very less effect on modal frequency but frequencies got increased as orientation value got incremented at the envelope of the beam. Chen et al. [14] used elasticity theory to develop a new method to analyze free vibration of laminated beam considering single-ply, cross-ply, angle-ply, multi-ply, unsymmetric and symmetric beam called state space quadrature (SSQD) technique. Jun et al. [15] formulated a dynamically exact stiffness matrix based on exact solution of governing equation.

The governing equations are derived from Hamilton's principle to analyze free vibration and buckling analysis of laminated composite beam. They also studied the effect of axial forces and boundary conditions on natural mode shapes and frequencies.

Jun et al. [16] used the dynamic finite element technique to study natural frequency of composite beam based on FSDT. They took account of poisson's effect, axial -bending-torsional coupling, shear deformation and rotary inertia. They also studied the effect of material anisotropy, boundary condition, shear deformation and rotary inertia on natural frequencies of the composite beam. Jun and Hongxing [17] used free differential governing beam equations to get dynamic stiffened matrix of composite beam based on Trigonometric Shear Deformation Theory. They used Wittrick-Williams theory to get natural frequencies and mode shapes of the laminated beam.

2.2 Stepped Laminated Composite Beam

Subramanian and Balasubramanian [18] showed that due to stepping in uniform beam, the dynamic behaviour of structure altered as stepping down stiffen the structure. They also showed that stepping up weaken the structure if the beam ends are not held down. They also explained about dynamic stiffness reduction and procedure of avoiding resonance by achieving the natural frequencies as needed by introducing step sections. Dong et al. [19] used Timoshenko beam theory to calculate the natural frequency of cantilever composite stepped beam with surface bonded piezoelectric material taking shear deformation and rotary inertia effect. They found out the effect of step location on natural frequencies of cantilever beam. Rarely any literature is available on free vibration on composite stepped beam. Lam and Sathiyamoorthy [1] studied the behaviour of uniform composite beam and validated the result by Runge-Kutta-Nystran Method and extended it to the stepped composite beam.

2.3 Tapered Laminated Composite Beam

Taber and Viano [20] calculated resonant frequency and mode shapes for tapered composite Timoshenko cantilever beam using transfer matrix method. In bending vibration anlysis shear deformation and flexural rigidity parameters were considered. Rao and Ganesan [21] used higher order shear deformation theory (HSDT) based finite element method to study the harmonic response of laminated tapered composite beam. They also showed that with the increase in non uniformity parameter (β) the natural frequencies go on decreasing for all boundary cases. Ganesan and Zabihollah [22] studied the free vibration of undamped composite tapered beam using higher-order finite element formulation. They considered effect like non uniformity parameter, tapered profile, boundary conditions, laminate configuration and tapered profile. Çalım [23] used Timoshenko beam theory to get governing equations for deriving dynamic stiffness matrix of non uniform composite beam. He considered the possible effect of orientation angle and non uniform parameter(β) on frequency. He also showed that due to rigidity clamped-clamped end condition has highest frequency value than clamped-free end case.

Chapter 3

Laminated Composite Timoshenko Beam

3.1 Spectral Element Methodology

As the spectral element method (SEM) can't be applied to time variant system, so governing differential equations obtained from time domain is transformed into frequency domain by the use of Discrete Fourier Transformation (DFT) or First Fourier Transformation (FFT). Then the obtained differential equations based on frequency domain are solved exactly to get exact wave solutions which are used to formulate frequency dependent dynamic shape function [24]. Then the spectral element matrix which is frequency dependent is derived from the dynamic shape function. Spectral element matrices (Spectral Dynamic Stiffness Matrix) are assembled to obtain global spectral element matrix and boundary conditions are imposed to get reduced global stiffness matrix. Finally the natural frequencies are obtained when the determinant of that matrix is equated to zero and solved. Overall the spectral element method can be summerised as follows:

- Time domain governing equation is converted into frequency domain equation then formulation of dynamic shape function and spectral element matrix calculation.

- Assembling of spectral element matrices and to get reduced global matix by applying boundary conditions.
- Determination of frequency by equating determinant of reduced matrix to zero and solving.

3.2 Laminated Composite Structural Mechanics

For the orthotropic materials the three dimensional strain-stress relationship is

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \tag{3.1}$$

where S_{ij} is the compliance matrix element determined from nine independent engineering constants as:

$$\begin{aligned} S_{11} = \frac{1}{E_1}, \quad S_{22} = \frac{1}{E_2}, \quad S_{33} = \frac{1}{E_3} \\ S_{44} = \frac{1}{G_{23}}, \quad S_{55} = \frac{1}{G_{31}}, \quad S_{66} = \frac{1}{G_{12}} \\ S_{12} = -\frac{\nu_{21}}{E_2}, \quad S_{13} = -\frac{\nu_{31}}{E_3}, \quad S_{23} = -\frac{\nu_{32}}{E_3} \end{aligned} \tag{3.2}$$

where E_{ij},G_{ij} and ν_{ij} are the Young modulus of elasticity, shear rigidity and poisson's ratio respectively and the subscripts denote principal material coordinates such that j= direction and i=plane perpendicular to that direction
For an orthotropic lamina σ_3=0, so from Eq.(3.1) we can obtain

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} \tag{3.3}$$

$$\begin{Bmatrix} \epsilon_4 \\ \epsilon_5 \end{Bmatrix} = \begin{bmatrix} S_{44} & 0 \\ 0 & S_{55} \end{bmatrix} \begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix} \tag{3.4}$$

In other way, we can have stress-strain relationship from the reduced stiffnesses Q_{ij} as:

$$\begin{gathered} Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \qquad Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2} = \frac{E_2}{1-\nu_{12}\nu_{21}} \\ Q_{12} = -\frac{S_{12}}{S_{11}S_{22} - S_{12}^2} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} \\ Q_{44} = \frac{1}{S_{44}} = G_{23}, \qquad Q_{55} = \frac{1}{S_{55}} = G_{31}, \qquad Q_{66} = \frac{1}{S_{66}} = G_{12} \end{gathered} \tag{3.5}$$

The stress-strain relationship with respect to global coordinate (x,y,z) is

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{11} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \tag{3.6}$$

$$\begin{Bmatrix} \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{44} & \overline{Q}_{45} \\ \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{xx} \\ \gamma_{yy} \end{Bmatrix} \tag{3.7}$$

where $\overline{Q}_{ij}$ are the reduced transform stiffnesses. γ_{xy}= shear strain in x direction and y is the plane perpendicular to that direction. γ_{xy} =2ϵ_{xy}, γ_{yz} =2ϵ_{yz}, γ_{zx} =2ϵ_{zx}. The relationship between reduced stiffness matrix and reduced transformed stiffness matrix is as follows:

$$\begin{Bmatrix} \overline{Q}_{11} \\ \overline{Q}_{22} \\ \overline{Q}_{12} \\ \overline{Q}_{16} \\ \overline{Q}_{26} \\ \overline{Q}_{66} \end{Bmatrix} = \begin{bmatrix} m^4 & n^4 & 2m^2n^2 & 4m^2n^2 \\ n^4 & m^4 & 2m^2n^2 & 4m^2n^2 \\ m^2n^2 & m^2n^2 & m^4+n^4 & -4m^2n^2 \\ m^3n & -mn^3 & mn^3-m^3n & 2(mn^3-m^3n) \\ mn^3 & -m^3n & m^3n-mn^3 & 2(m^3n-mn^3) \\ m^2n^2 & m^2n^2 & -2m^2n^2 & (m^2-n^2)^2 \end{bmatrix} \begin{Bmatrix} Q_{11} \\ Q_{22} \\ Q_{12} \\ Q_{66} \end{Bmatrix} \tag{3.8}$$

$$\begin{aligned} \overline{Q}_{44} &= Q_{44}m^2 + Q_{55}n^2 \\ \overline{Q}_{55} &= Q_{55}m^2 + Q_{44}n^2 \\ \overline{Q}_{45} &= (Q_{55} - Q_{44})mn \end{aligned} \tag{3.9}$$

Here $m = \cos\phi$ and $n = \sin\phi$, ϕ is the angle between material coordinate axes (1, 2, 3) and global coordinate axes (x,y,z) about z direction. Now we consider a composite laminate of kth layer having width b, thickness h and κ is the shear correction factor. The extensional stiffness matrix, coupling stiffness matrix and

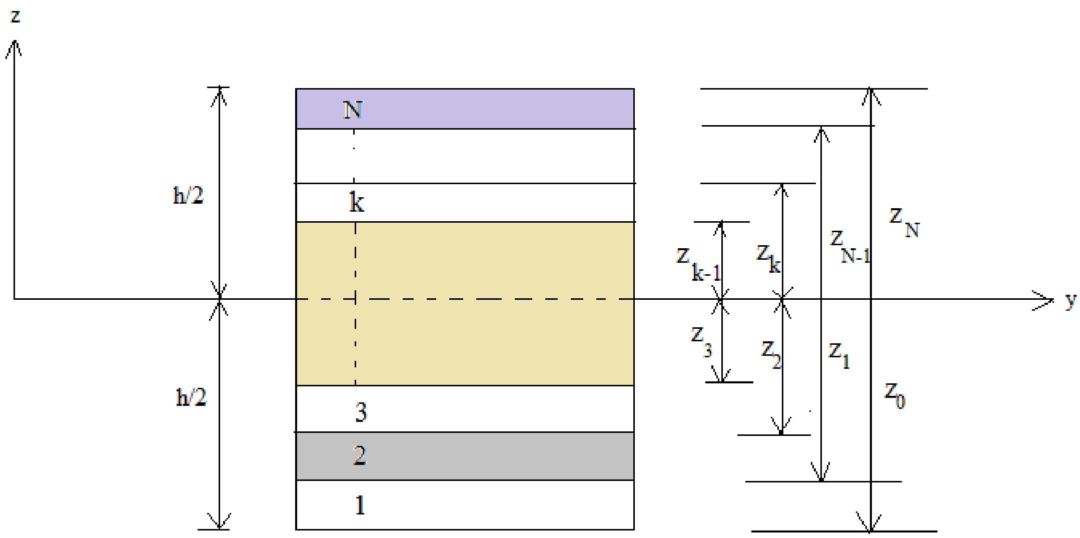


Figure 3.1: Geometry of an N-layered laminate

the bending stiffness matrix can be written respectively as:

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{N} \overline{Q}_{ij}^{(k)}(z_k - z_{k-1}) \\ B_{ij} &= \frac{1}{2}\sum_{k=1}^{N} \overline{Q}_{ij}^{(k)}(z_k^2 - z_{k-1}^2) \\ D_{ij} &= \frac{1}{3}\sum_{k=1}^{N} \overline{Q}_{ij}^{(k)}(z_k^3 - z_{k-1}^3) \end{aligned} \tag{3.10}$$

Some other parameters related to composite laminated beam are EA=extensional rigidity= bA_{11}, EI=flexural rigidity= bD_{11}, κGA=shear rigidity= $b\kappa A_{55}$, and

K_A=coupling rigidity= bB_{11}.

$$
\begin{aligned}
\rho A &= b\sum_{k=1}^{N}\rho^{(k)}(z_k - z_{k-1}) \\
\rho R &= \frac{b}{2}\sum_{k=1}^{N}\rho^{(k)}(z_k^2 - z_{k-1}^2) \\
\rho I &= \frac{b}{3}\sum_{k=1}^{N}\rho^{(k)}(z_k^3 - z_{k-1}^3)
\end{aligned}
\tag{3.11}
$$

where ρA= mass per unit length of beam

ρR= first order mass moment of inertia of beam

ρI= second order mass moment of inertia of beam

NB: For symmetric laminated composite beam coupling stiffness matrix B and ρR becomes zero due to mid plane symmetry.

3.3 Spectral Element Matrix for Laminated Composite Beam (Uniform Section)

3.3.1 Symmetrical Ply Oriented Beam

The spectral element matrix for symmetric composite beam can be derived by considering Timoshenko beam (T-beam) theory for the laminated composite beam which has two degrees of freedom per node i.e. transverse and rotational.
The governing equations concerned with free vibration of symmetric composite beam are given as [24])

$$\begin{aligned} &EI\theta^{''} + \kappa GA(w^{'} - \theta) - \rho I\ddot{\theta} = 0 \\ &\kappa GA(w^{''} - \theta^{'}) - \rho A\ddot{w} = 0 \end{aligned} \tag{3.12}$$

where $\theta(x,t)$= slope due to bending,
$w(x,t)$=transverse displacement,
EI=Flexural rigidity,
κ=shear correction factor which depends upon shape of the cross-section,
κGA=Shear rigidity,
ρI= second order mass moment of inertia,
ρR= first order mass moment of inertia

$$\begin{aligned} &M_t(x,t) = EI\theta^{'}(x,t) \\ &Q_t(x,t) = \kappa GA[w^{'}(x,t) - \theta(x,t)] \end{aligned} \tag{3.13}$$

where $M_t(x,t)$=internal bending moment, $Q_t(x,t)$=transverse shear force.
Let the solution to Eq.(3.12) in spectral form be

$$w(x,t) = \frac{1}{N}\sum_{n=0}^{N-1} W_n(x;\omega_n)e^{i\omega_n t} \tag{3.14}$$

$$\theta(x,t) = \frac{1}{N}\sum_{n=0}^{N-1} \Theta_n(x;\omega_n)e^{i\omega_n t} \tag{3.15}$$

Substituting Eq.(3.14) & (3.15) into Eq.(3.12) gives an eigenvalue problem

$$\kappa GA(W^{''} - \Theta^{'}) + \rho A\omega^2 W = 0 \tag{3.16}$$

$$EI\Theta^{''} - \kappa GA(W^{'} - \Theta^{'}) + \rho I\omega^2\Theta = 0 \tag{3.17}$$

Let the general solution to Eq.(3.16) & (3.17) be

$$W(x) = ae^{-ik(\omega)x} \tag{3.18}$$

$$\Theta(x) = \beta ae^{-ik(\omega)x} \tag{3.19}$$

Substituting Eq.(3.18) & (3.19) into Eq.(3.16) & (3.17) yields an eigenvalue problem as

$$\begin{bmatrix} \kappa GAk^{''} - \rho A\omega^2 & -ik\kappa GA \\ ik\kappa GA & EIk^2 + \kappa GA - \rho A\omega^2 \end{bmatrix} \begin{Bmatrix} 1 \\ \beta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.20}$$

Eq.(3.20) gives a dispersion relation as:

$$k^4 - \eta k_F^4 k^2 - k_F^4(1 - \eta_1 k_G^4) = 0 \tag{3.21}$$

where

$$k_F = \sqrt{\omega}\left(\frac{\rho A}{EI}\right), k_G = \sqrt{\omega}\left(\frac{\rho A}{\kappa EI}\right) \tag{3.22}$$

$$\eta = \eta_1 + \eta_2, \quad \eta_1 = \frac{\rho I}{\rho A}, \quad \eta_2 = \frac{EI}{\kappa GA} \tag{3.23}$$

Solving Eq.(3.21) gives four roots as

$$\begin{aligned} k_1 = -k_2 = \frac{1}{\sqrt{2}} k_F \sqrt{\eta k_F^2 + \sqrt{\eta^2 k_F^4 + 4(1 - \eta_1 k_G^4)}} = k_t \\ k_3 = -k_4 = \frac{1}{\sqrt{2}} k_F \sqrt{\eta k_F^2 - \sqrt{\eta^2 k_F^4 + 4(1 - \eta_1 k_G^4)}} = k_e \end{aligned} \tag{3.24}$$

From the first row of Eq.(3.20) we can obtain the wavemode ratio as

$$\beta_p(\omega) = \frac{1}{ik_p}(k_p^2 - k_G^2) = -ir_p(\omega) \ (p = 1, 2, 3, 4) \tag{3.25}$$

$$\text{where} \quad r_p(\omega) = \frac{1}{k_p}(k_p^2 - k_G^4) \tag{3.26}$$

By using the four wavenumbers given by Eq.(3.24) the general solution of Eq.(3.16) & (3.17) can be obtained as

$$W(x) = a_1 e^{-ik_t x} + a_2 e^{ik_t x} + a_3 e^{-ik_e x} + a_4 e^{ik_e x} = \mathbf{e_w}(x;\omega)\mathbf{a}$$
$$\Theta(x) = \beta_1 a_1 e^{-ik_t x} + \beta_2 a_2 e^{ik_t x} + \beta_3 a_3 e^{-ik_e x} + \beta_4 a_4 e^{ik_e x} = \mathbf{e}_\theta(x;\omega)\mathbf{a} \tag{3.27}$$

$$\text{where} \quad \mathbf{a} = \begin{Bmatrix} a_1 & a_2 & a_3 & a_4 \end{Bmatrix}^T \tag{3.28}$$

$$\mathbf{e}_w(x;\omega) = [e^{-ik_t x} \;\; e^{ik_t x} \;\; e^{-ik_e x} \;\; e^{ik_e x}]$$
$$\mathbf{e}_\theta(x;\omega) = \mathbf{e}_w(x;\omega)\mathbf{B}(\omega) \tag{3.29}$$
$$\mathbf{B}(\omega) = diag[\beta_p(\omega)]$$

The spectral nodal displacements and slopes of the beam element of length L are related to the displacement field by

$$\mathbf{d} = \begin{Bmatrix} W_1 \\ \Theta_1 \\ W_2 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} W(0) \\ \Theta(0) \\ W(L) \\ \Theta(L) \end{Bmatrix} \tag{3.30}$$

Substituting Eq.(3.27) into right hand side of Eq.(3.30) gives

$$\mathbf{d} = \begin{bmatrix} e_\omega(0;\omega) \\ e_\theta(0;\omega) \\ e_\omega(L;\omega) \\ e_\theta(L;\omega) \end{bmatrix} \mathbf{a} = \mathbf{H}_T(\omega)\mathbf{a} \tag{3.31}$$

where

$$\mathbf{H}_T(\omega) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -ir_t & ir_t & -ir_e & ir_e \\ e_t & e_t^{-1} & e_e & e_e^{-1} \\ -ir_t e_t & ir_t e_t^{-1} & -ir_e e_e & ir_e e_e^{-1} \end{bmatrix} \tag{3.32}$$

the values of matrix elements are as follows :

$$e_t = e^{-ik_t L} \;\; , \; e_e = e^{-ik_e L} \;\; , \quad r_t = \frac{1}{k_t}(k_t^2 - k_G^4) \;\; , \; r_e = \frac{1}{k_e}(k_e^2 - k_G^4) \tag{3.33}$$

From Eq.(3.31) we have

$$\mathbf{a} = \mathbf{H}_T^{-1}(\omega)\mathbf{d} \tag{3.34}$$

Substituting the value of '**a**' from Eq.(3.34) into Eq.(3.27), the general solution can be expressed as

$$\begin{aligned} W(x) &= \mathbf{e}_w(x;\omega)\mathbf{H}_T^{-1}\mathbf{d} \\ \Theta(x) &= \mathbf{e}_\theta(x;\omega)\mathbf{H}_T^{-1}\mathbf{d} \end{aligned} \tag{3.35}$$

From Eq.(3.13) the spectral components of the transverse shear force and bending moment can be related to $W(x)$ and $\Theta(x)$ as

$$Q = \kappa GA(W^{'} - \Theta) \quad , \; M = EI\Theta^{'} \tag{3.36}$$

The spectral nodal transverse shear forces and bending moments defined for the beam element correspond to the forces and the moments as given below (Fig. 2).

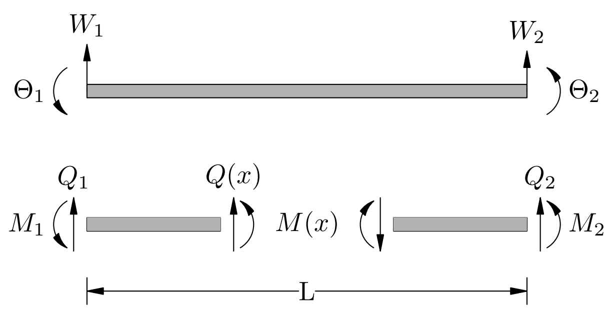


Figure 3.2: Timoshenko spectral beam element with nodal forces and displacements

The spectral nodal force vector can be written as:

$$\mathbf{f}_c(\omega) = \begin{Bmatrix} Q_1 \\ M_1 \\ Q_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} -Q(0) \\ -M(0) \\ +Q(L) \\ +M(L) \end{Bmatrix} \tag{3.37}$$

Substituting Eq.(3.35) into Eq.(3.36) and its results into right-hand side of Eq.(3.37), we have

$$\mathbf{f}_c = \begin{Bmatrix} -EIW'''(0) \\ -EIW''(0) \\ EIW'''(L) \\ EIW''(L) \end{Bmatrix} = \begin{Bmatrix} -\kappa GA\left\{e'_\omega(0;\omega) - e_\theta(0;\omega)\right\} \\ -EIe'_\theta(0;\omega) \\ \kappa GA\left\{e'_\omega(L;\omega) - e_\theta(L;\omega)\right\} \\ EIe'_\theta(L;\omega) \end{Bmatrix} \mathbf{H}_T^{-1}\mathbf{d} = \mathbf{S}_T(\omega)\mathbf{d} \quad (3.38)$$

where $\mathbf{S}_T(\omega)$ is spectral element (dynamic stiffness) matrix for the beam element given by

$$\mathbf{S}_T(\omega) = \begin{Bmatrix} -\kappa GA\left\{e'_\omega(0;\omega) - e_\theta(0;\omega)\right\} \\ -EIe'_\theta(0;\omega) \\ \kappa GA\left\{e'_\omega(L;\omega) - e_\theta(L;\omega)\right\} \\ EIe'_\theta(L;\omega) \end{Bmatrix} \mathbf{H}_T^{-1} \quad (3.39)$$

3.3.2 Non Symmetrical Ply Oriented Beam

The spectral element matrix for non-symmetric composite Timoshenko beam can be derived by considering extended Timoshenko beam (ET-beam) theory for the laminated composite beam which has three degrees of freedom per node i.e. axial displacement, transverse displacement and rotation. The spectral nodal degrees

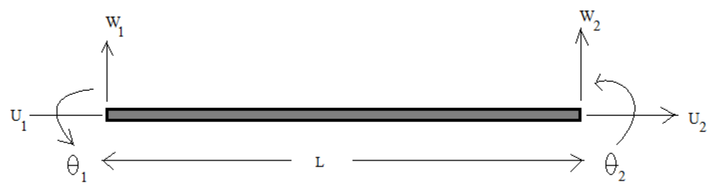


Figure 3.3: Spectral nodal degrees of freedom

of freedom(DOFs) vector can be written as

$$d = \begin{Bmatrix} U_1 \\ W_1 \\ \Theta_1 \\ U_2 \\ W_2 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} U(0) \\ W(0) \\ \Theta(0) \\ U(L) \\ W(L) \\ \Theta(L) \end{Bmatrix} \tag{3.40}$$

The governing equations concerned with free vibration of non-symmetric composite beams are derived from Hamilton's principle.

$$\int_0^t (\delta T - \delta U + \delta W) dt = 0$$

where T = kinetic energy, U = strain energy and δW = virtual work done by external forces and moment on composite laminated beam. So the governing equations obtained by Hamilton's Principle in spectral form are [24]

$$\begin{aligned} \kappa G A_{eq}(W'' - \Theta') + P W'' + \omega^2 \rho A_{eq} W = 0 \\ E A_{eq} U'' - K_{Aeq}\Theta'' + \omega^2 \rho A_{eq} U - \omega^2 \rho R_{eq}\Theta = 0 \\ E I_{eq}\Theta'' + \kappa G A_{eq}(W' - \Theta) - K_{Aeq} U'' - \omega^2 \rho R_{eq} U + \omega^2 \rho I_{eq}\Theta = 0 \end{aligned} \tag{3.41}$$

where EA_{eq}=extensional rigidity= bA_{11}

EI_{eq}=flexural rigidity= bD_{11}

κGA_{eq}=shear rigidity= $b\kappa A_{55}$

K_{Aeq}=coupling rigidity= bB_{11}

ρA_{eq}= mass per unit length of beam

ρR_{eq}= first order mass moment of inertia of beam

ρI_{eq}= second order mass moment of inertia of beam

P= constant axial tensile force which acts through the center of mass of cross section of beam.

$$\begin{aligned}
\rho A_{eq} &= b\sum_{k=1}^{N}\rho^{(k)}(z_k - z_{k-1}) \\
\rho R_{eq} &= \frac{b}{2}\sum_{k=1}^{N}\rho^{(k)}(z_k^2 - z_{k-1}^2) \\
\rho I_{eq} &= \frac{b}{3}\sum_{k=1}^{N}\rho^{(k)}(z_k^3 - z_{k-1}^3)
\end{aligned} \tag{3.42}$$

The subscript "*eq*" is used to denote the apparent structural rigidities and mass inertia properties for a composite laminated beam.

To find out the spectral element matrix of non symmetric composite Timonshenko beam, the general solution of Eq.(3.41) can be expressed as

$$\begin{aligned}
W(x) &= ae^{-ikx} \\
\Theta(x) &= \alpha a e^{-ikx} \\
U(x) &= \beta a e^{-ikx}
\end{aligned} \tag{3.43}$$

Substituting values from Eq.(3.43) into Eq.(3.41) we get in matrix form :

$$\begin{bmatrix} X_{11} & X_{12} & 0 \\ -X_{12} & X_{22} & X_{23} \\ 0 & X_{23} & X_{33} \end{bmatrix} \begin{Bmatrix} 1 \\ \alpha \\ \beta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.44}$$

where

$$\begin{aligned}
X_{11} =& -(\kappa GA_{eq}+P)k^2+\omega^2\rho A_{eq} \\
X_{22} =& -EI_{eq}k^2+\omega^2\rho I_{eq}-\kappa GA_{eq} \\
X_{12} =& i\kappa GA_{eq}k \\
X_{33} =& -EA_{eq}k^2+\omega^2\rho A_{eq} \\
X_{23} =& X_{32}=K_{Aeq}k^2-\omega^2\rho R_{eq}
\end{aligned} \tag{3.45}$$

The determinant of Eq.(3.44) gives a dispersive relation as

$$\mathbf{c_1k_n^6+c_2k_n^4+c_3k_n^3+c_4=0} \tag{3.46}$$

where

$$\begin{aligned}
c_1 =& \quad -(EA_{eq}EI_{eq}-K_{Aeq}^2)(\kappa GA_{eq}+P) \\
c_2 =& \quad \omega^2[EA_{eq}(EI_{eq}\rho A_{eq}+\kappa GA_{eq}\rho I_{eq}+P\rho I_{eq}) \\
& \quad -\rho A_{eq}K_{Aeq}^2+(P+\kappa GA_{eq})(EI_{eq}\rho A_{eq} \\
& \quad -2\rho R_{eq}K_{Aeq})]-EA_{eq}\kappa GA_{eq}P \\
c_3 =& \quad P\kappa GA_{eq}\rho A_{eq}\omega^2+EA_{eq}\kappa GA_{eq}\rho I_{eq}\omega^2 \\
& \quad +\omega^4[-EI_{eq}\rho A_{eq}^2-EA_{eq}\rho A_{eq}\rho I_{eq} \\
& \quad -\kappa GA_{eq}\rho A_{eq}\rho I_{eq}+\kappa GA_{eq}\rho R_{eq}^2 \\
& \quad +2\kappa K_{Aeq}\rho A_{eq}\rho R_{eq}+P(\rho R_{eq}^2-\rho A_{eq}\rho I_{eq})] \\
c_4 =& \quad (\rho A_{eq}^2\rho I_{eq}-\rho A_{eq}\rho R_{eq}^2)\omega^6-\kappa GA_{eq}\rho A_{eq}^2\omega^4
\end{aligned}$$

Now we get six roots from Eq.(3.46) as k_p for p=1,2...6 known as six wave numbers

$$\begin{aligned}
k_{1,2} =& \pm\sqrt{B_3+B_4-\frac{1}{3}(c_2/c_1)} \\
k_{3,4} =& \pm\sqrt{-\frac{1}{2}(B_3+B_4)-\frac{1}{3}(c_2/c_1)+i(\sqrt{3}/2)(B_3-B_4)} \\
k_{5,6} =& \pm\sqrt{-\frac{1}{2}(B_3+B_4)-\frac{1}{3}(c_2/c_1)-i(\sqrt{3}/2)(B_3-B_4)}
\end{aligned} \tag{3.47}$$

$$\text{where}\quad \begin{aligned} B_1 =& \frac{1}{9}[3(c_3/c_1) - (c_2/c_1)^2] \\ B_2 =& \frac{1}{54}[9(c_2c_3/c_1^2) - 27(c_4/c_1) - 2(c_2/c_1)^3] \\ B_3 =& \sqrt[3]{B_2 + \sqrt{B_1^3 + B_2^2}} \\ B_4 =& \sqrt[3]{B_2 - \sqrt{B_1^3 + B_2^2}} \end{aligned} \tag{3.48}$$

To obtain the values of α_p and β_p (p=1, 2...6) from Eq.(3.45) the wavenumber values k_p from Eq.(3.47) are substituted in Eq.(3.45). So we get

$$\begin{aligned} \alpha_p =& i\frac{\omega^2\rho A_{eq} - \kappa G A_{eq}k_p^2 - Pk_p^2}{\kappa G A_{eq}k_p} \\ \beta_p =& (\frac{\omega^2\rho R_{eq} - K_{Aeq}k_p^2}{\omega^2\rho A_{eq} - EA_{eq}k_p^2})\alpha_p \end{aligned} \tag{3.49}$$

The general solutions of the governing Eq.(3.41) by using k_p from Eq.(3.47) can be written as:

$$\begin{aligned} W(x) =& \sum_{p=1}^{6} a_p e^{-ik_px} \\ \Theta(x) =& \sum_{p=1}^{6} \alpha_p a_p e^{-ik_px} \\ U(x) =& \sum_{p=1}^{6} \beta_p a_p e^{-ik_px} \end{aligned} \tag{3.50}$$

In other way the assumed general solutions can be expressed as

$$W(x) = e(x;\omega)a, \quad \Theta(x) = e(x;\omega)A(\omega)a, \quad U(x) = e(x;\omega)B(\omega)a \tag{3.51}$$

$$\text{where}\quad e(x;\omega) = [e^{-ik_1x} \quad e^{-ik_2x} \quad e^{-ik_3x} \quad e^{-ik_4x} \quad e^{-ik_5x} \quad e^{-ik_6x}]$$

$$A(\omega) = diag[\alpha_p(\omega)], \quad B(\omega) = diag[\beta_p(\omega)], \quad a = \left\{a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6\right\}^T$$

By substituting Eq.(3.51) to Eq.(3.40) we have,

$$\mathbf{d} = \mathbf{H}(\omega)\mathbf{a}$$

$$\text{where} \quad H(\omega) = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \beta_1 e_1 & \beta_2 e_2 & \beta_3 e_3 & \beta_4 e_4 & \beta_5 e_5 & \beta_6 e_6 \\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \alpha_1 e_1 & \alpha_2 e_2 & \alpha_3 e_3 & \alpha_4 e_4 & \alpha_5 e_5 & \alpha_6 e_6 \end{bmatrix} \tag{3.52}$$

$$e_p = e^{-ik_pL}(p = 1, 2...6)$$

Finally the spectral element matrix for a non symmetrical composite beam can be expressed as:

$$\mathbf{S}(\omega) = \mathbf{H^{-T}}(\omega)\mathbf{D}(\omega)\mathbf{H^{-1}}(\omega) \tag{3.53}$$

$$\begin{aligned} \text{where} \quad D(\omega) = \quad & \kappa G A_{eq}[-KEK + i(A^T EK) + A^T EA] - EI_{eq}A^T KEKA \\ & -EA_{eq}B^T KEKB + K_{Aeq}(A^T KEKB + B^T KEKA) - PKEK \\ & -\omega^2[\rho A_{eq}(B^T EB + E) + \rho I_{eq}A^T EA - \rho R_{eq}(A^T EB + B^T EA)] \end{aligned}$$

$$E(\omega) = [E_{rs}\omega](r, s = 1, 2, ...6)$$

$$E_(\omega) = \begin{cases} \dfrac{i}{k_r + k_s}[e^{-i(k_r+k_s)L} - 1] & \text{if } k_r + k_s \neq 0 \\ L & \text{if } k_r + k_s = 0 \end{cases}$$

3.4 Spectral Element Matrix for Laminated Composite Beam (Non-Uniform Section)

For composite beam of symmetrical and non-symmetrical orientation having non uniform section i.e. stepped and tapered beam, the spectral element matrix formation is same as that of uniform one with the usual orientation except few changes which are implemented as follows:

- In the composite stepped beam, three types of geometries are considered.
  - constant width with variation in thickness.
  - constant thickness in variation in width.
  - variation in width and thickness.

  The spectral element matrices for each section of stepped beam are determined and assembled to find the global spectral element matrix and then boundary conditions are applied. The frequencies are obtained by equating the determinant of the reduced global spectral element matrix to zero. The process is same as for uniform beam, because each section (single, stepped, doubled) behaves like uniform beam.

- A tapered composite beam is considered keeping the width of the beam as constant and varying the thickness of the beam profile. The variation of the thickness is as per the equation given by

$$h(x) = h_0(1 - \beta\frac{x}{L})$$

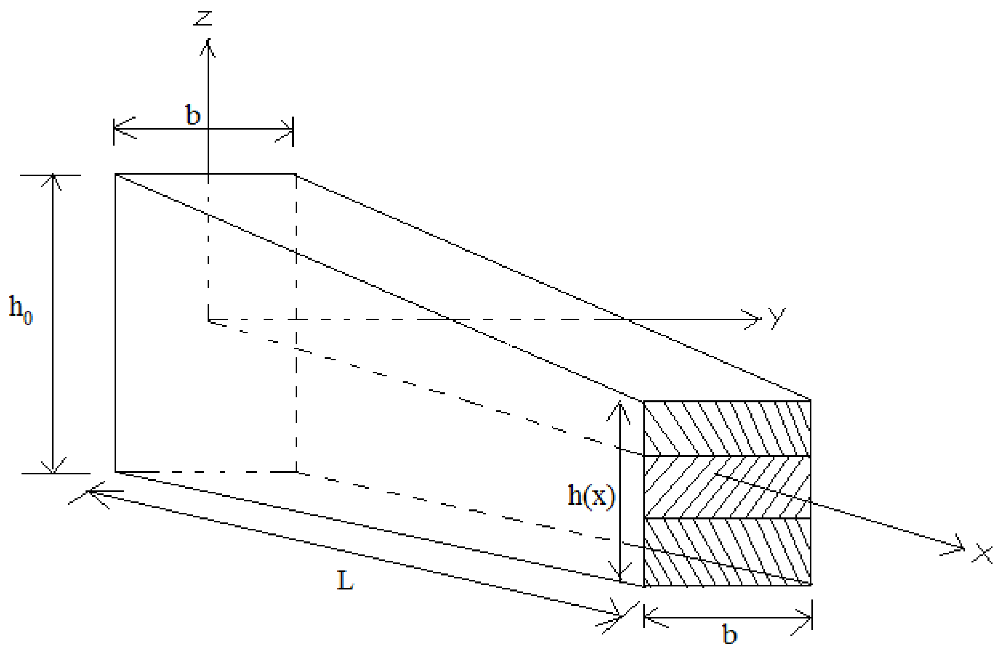
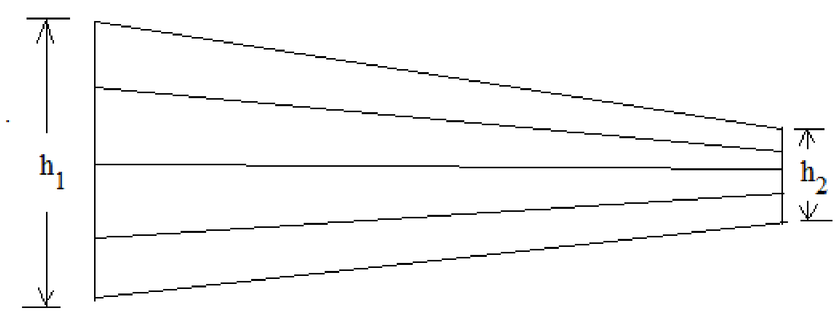


Figure 3.4: Geometry and coordinate system of tapered composite beam

where $h(x)$= thickness of composite tapered beam at any cross section $\beta =$ non uniformity parameter $= 1 - \dfrac{h_2}{h_1}$, $\quad h_1 > h_2$



h_1 = maximum thickness of tapered section,

h_2= minimum thickness of tapered section

Here the non-uniformity parameter β is taken as 0.25, 0.5, 0.75 .

If β is taken as 0 then the beam will become a uniform one.

3.5 Globalisation of Dynamic Stiffness Matrix

The assembling of global system equation in SEM is same as the procedure adopted in conventional FEM. The two points which are the backbone of assembling theory as:

- Between the local spectral nodal degrees of freedom (DOFs) and global spectral nodal degrees of freedom the inter-element continuity conditions exists.
- Between local force coordinate and global force coordinate equilibrium condition exists.

After assembling the spectral element matrix of each element the global spectral element matrix (dynamic stiffness matrix) is obtained. After that associated boundary conditions are imposed to obtain the reduced global equation in the form of:

$$\mathbf{S_g}(\omega)\mathbf{d_g} = \mathbf{f_g}$$

where $\mathbf{S_g}$, $\mathbf{f_g}$ and $\mathbf{d_g}$ respectively represents exact global spectral element matrix, global spectral nodal force vector and global spectral nodal degrees of freedom vector.

3.6 Solution Methodology

The required frequency or eigenvalues are obtained by making the determinant of $S_g(\omega)$ equal to zero at $\omega = \omega_i$ for i=(1,2,3.........∞). In other words the frequency for which dynamic stiffness matrix becomes singular that becomes the one of the natural frequencies.

Hence,
$$\left|\mathbf{S_g}(\omega_\mathbf{i})\right| = \mathbf{0} \tag{3.54}$$

where ω_i refers to i th natural frequency of the system. The eigen frequencies values are obtained by varying ω in small steps starting from ω_1 and computing the determinant of $S_g(\omega)$. If the determinant value becomes zero then that becomes a natural frequency value. The process is repeated for next incremented value of ω .

Chapter 4

Results & Discussion

4.1 Uniform Laminated Composite Beam

The following notations are being used for various boundary conditions

- C: clamped
- S: simply supported
- F: free

4.1.1 Symmetric Laminated Composite Beam

4.1.1.1 [0°/0°/0°/0°] Ply Oriented Beam

An orthotropic graphite epoxy beam [0°] having material properties $E_1 = 144.84\,GPa$, $E_2 = 9.65\,GPa$, $G_{12} = G_{13} = 4.14\,GPa$, $G_{23} = 3.45\,GPa$, $\nu_{12} = 0.3$, $\rho = 1389.79\,kg/m^3$, $b = 0.0254\,m$ is considered. The first five natural frequencies of thin beam (h=0.00635 m, L/h=120) and thick beam (h=0.0254 m, L/h=15) are computed using spectral element method (SEM) for various boundary conditions considering two number of spectral elements and the results by SEM are presented in Table 4.1 and 4.2 and are compared with those by other methods of reference [12] and [3]. The shear correction value (κ) is taken as 5/6.

Table 4.1: Natural frequency (kHz) of thick graphite epoxy composite beams (h=0.0254 m, L/h=15)

Mode No.	BOUNDARY CONDITIONS					
	S-S			C-C	C-S	C-F
	SEM	Ref. [12]	Ref. [3]	SEM	SEM	SEM
1	0.755	0.753	0.755	1.378	1.06	0.279
2	2.548	2.537	2.548	3.077	2.829	1.471
3	4.716	4.68	4.716	5.066	4.897	3.428
4	6.961	6.868	6.96	7.17	7.068	5.589
5	9.195	9.011	9.194	9.322	9.259	7.823

Table 4.2: Natural frequency (kHz) of thin graphite epoxy composite beams (h=0.00635 m, L/h=120)

Mode No.	BOUNDARY CONDITIONS					
	S-S			C-C	C-S	C-F
	SEM	Ref. [12]	Ref. [3]	SEM	SEM	SEM
1	0.0507	0.051	0.051	0.114	0.079	0.018
2	0.2021	0.202	0.203	0.313	0.2548	0.1129
3	0.452	0.451	0.457	0.608	0.5275	0.3143
4	0.7968	0.794	0.812	0.994	0.8932	0.611
5	1.232	1.232	1.269	1.464	1.346	1.000

4.1.1.2 [0°/90°/90°/0°] Ply Oriented Beam

A [0°/90°/90°/0°] symmetric composite beam with material and geometrical properties $E_1 = 144.84\,GPa,\ E_2 = 9.65\,GPa,\ G_{12} = G_{13} = 4.14\,GPa,\ G_{23} = 3.45\,GPa,\ \nu_{12} = 0.3,\ \rho = 1389\,kg/m^3,\ b = 0.01\,m,\ h = 0.01\,m,\ L = 0.15\,m$ is considered for first ten non dimensional natural frequency parameters for the beam and the results are compared with the exact solutions from other methods. The results obtained by SEM given in table 4.3 considering two elements of the beam exactly match with those in [1] and [3]. Shear correction factor (κ)=5/6. The non dimensional frequency ($\overline{\omega}$) is given by $\overline{\omega} = \omega L^2.\sqrt{\rho/(E_1 h^2)}$.

Table 4.3: Non dimensional frequencies ($\overline{\omega} = \omega L^2 \sqrt{\rho/(E_1 h^2)}$) of $[0^\circ/90^\circ/90^\circ/0^\circ]$ composite beam

Mode No.	BOUNDARY CONDITIONS											
	C-F			C-C			S-S			C-S		
	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]
1	0.924	0.924	0.925	4.594	4.594	4.636	2.502	2.502	2.502	3.525	3.524	3.54
2	4.893	4.893	4.923	10.292	10.291	10.493	8.4819	8.481	8.483	9.443	9.442	9.538
3	11.441	11.44	11.582	16.968	16.966	17.423	15.757	15.756	15.789	16.386	16.384	16.636
4	18.699	18.697	19.02	24.045	24.041	24.839	23.312	23.309	23.44	23.688	23.685	24.157
5	26.216	26.212	26.796	31.293	31.287	32.475	30.844	30.839	31.217	31.071	31.066	31.844
6	33.722			38.57			38.288			38.431		
7	41.169			45.832			45.647			45.74		
8	48.533			53.059			52.935			52.998		
9	55.832			60.253			60.167			60.21		
10	63.067			67.415			67.354			67.384		

4.1.1.3 ANGLE PLY BEAMS $[\theta/-\theta/-\theta/\theta]$ with (θ =0°, 15°, 30°, 45°, 60°, 75°, 90°)

Angle ply composite beams of $[\theta/-\theta/-\theta/\theta]$ with (θ =0°, 15°, 30°, 45°, 60°, 75°, 90°) having material properties same as used for Table 4.3 and geometrical properties are taken as length (L)=150 mm, width (b)=height (h)=10 mm. Two beam elements are taken and non dimensional frequencies ($\overline{\omega} = \omega L^2\sqrt{\rho/(E_1h^2)}$) are calculated upto fifth mode. The fundamental frequencies obtained by SEM have very good agreement with that of other methods of [1] and [3] for each ply orientation as can be seen from Table 4.4 - 4.10. Other modal frequency values (from 2nd to 5th mode) are also determined for all boundary conditions. As the ply oriented angle increases, the fundamental frequency decreases for all the boundary conditions.

Table 4.4: Non dimensional frequencies ($\overline{\omega} = \omega L^2\sqrt{\rho/(E_1 h^2)}$) of [0°/0°/0°/0°] composite beam

Mode No.	BOUNDARY CONDITIONS											
	C-C			C-S			S-S			C-F		
	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]
1	4.849	4.849	4.907	3.7307	3.731	3.751	2.656	2.656	2.657	0.982	0.982	0.983
2	10.826			9.9536			8.962			5.174		
3	17.819			17.2277			16.589			12.059		
4	25.22			24.8625			24.484			19.661		
5	32.792			32.5712			32.344			27.517		

Table 4.5: Non dimensional frequencies ($\overline{\omega} = \omega L^2\sqrt{\rho/(E_1 h^2)}$) of [15°/-15°/-15°/15°] composite beam

Mode No.	BOUNDARY CONDITIONS											
	C-C			C-S			S-S			C-F		
	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]
1	4.664	4.664	4.713	3.559	3.559	3.577	2.5105	2.515	2.511	0.9249	0.925	0.926
2	10.5238			9.6137			8.589			4.946		
3	17.414			16.774			16.081			11.649		
4	24.741			24.34			23.916			19.1411		
5	32.263			32.01			31.751			26.9327		

Table 4.6: Non dimensional frequencies ($\overline{\omega} = \omega L^2 \sqrt{\rho/(E_1 h^2)}$) of [30°/-30°/-30°/30°] composite beam

Mode No.	BOUNDARY CONDITIONS											
	C-C			C-S			S-S			C-F		
	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]
1	4.098	4.098	4.127	3.057	3.057	3.067	2.1032	2.103	2.103	0.7677	0.768	0.768
2	9.5613			8.5612			7.47			4.2725		
3	16.1143			15.3255			14.4701			10.3854		
4	23.1916			22.6435			22.0591			17.4891		
5	30.54			30.1705			29.7859			25.0406		

Table 4.7: Non dimensional frequencies ($\overline{\omega} = \omega L^2 \sqrt{\rho/(E_1 h^2)}$) of [45°/-45°/-45°/45°] composite beam

Mode No.	BOUNDARY CONDITIONS											
	C-C			C-S			S-S			C-F		
	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]
1	3.1844	3.184	3.196	2.3032	2.303	2.307	1.5367	1.537	1.537	0.555	0.555	0.555
2	7.8377			6.7944			5.7214			3.2427		
3	13.6962			12.7071			11.6567			8.2679		
4	20.2334			19.4156			18.5438			14.5177		
5	27.1767			26.5437			25.8761			21.4646		

Table 4.8: Non dimensional frequencies ($\overline{\omega} = \omega L^2 \sqrt{\rho/(E_1 h^2)}$) of [60°/-60°/-60°/60°] composite beam

Mode No.	BOUNDARY CONDITIONS											
	C-C			C-S			S-S			C-F		
	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]
1	2.198	2.198	2.202	1.551	1.551	1.552	1.0122	1.012	1.012	0.363	0.363	0.363
2	5.694			4.783			3.9035			2.1965		
3	10.399			9.363			8.3155			5.8428		
4	15.936			14.903			13.833			10.7234		
5	22.046			21.092			20.096			16.488		

Table 4.9: Non dimensional frequencies ($\overline{\omega} = \omega L^2 \sqrt{\rho/(E_1 h^2)}$) of [75°/-75°/-75°/75°] composite beam

Mode No.	BOUNDARY CONDITIONS											
	C-C			C-S			S-S			C-F		
	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]
1	1.681	1.682	1.683	1.175	1.175	1.176	0.7609	0.761	0.761	0.272	0.272	0.272
2	4.449			3.687			2.9716			1.667		
3	8.304			7.365			6.444			4.511		
4	12.991			11.968			10.938			8.447		
5	18.309			17.273			16.2151			13.2519		

Table 4.10: Non dimensional frequencies ($\overline{\omega} = \omega L^2\sqrt{\rho/(E_1h^2)}$) of [90°/-90°/-90°/90°] composite beam

Mode No.	BOUNDARY CONDITIONS											
	C-C			C-S			S-S			C-F		
	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]	SEM	Ref. [3]	Ref. [1]
1	1.6193	1.62	1.621	1.1308	1.131	1.132	0.7318	0.732	0.732	0.2617	0.262	0.262
2	4.294			3.554			2.8612			1.605		
3	8.0325			7.1142			6.2154			4.3497		
4	12.5934			11.5836			10.5698			8.1599		
5	17.783			16.7515			15.7			12.8267		

4.1.2 Non-symmetric Laminated Composite Beam

4.1.2.1 [0°/90°] Ply Oriented Beam

A two layered unsymmetrical composite beam having ply orientation [0°/90°] is considered for the analysis. The beam has material properties as $E_1 = 144.84\,GPa,\ E_2 = 9.65\,GPa,\ G_{12} = G_{13} = 4.14\,GPa,\ G_{23} = 3.45\,GPa,\ \nu_{12} = 0.3,\ \rho = 1389\,kg/m^3$. Width (b) =thickness (h)=10 mm. The length to thickness ratio (L/h) is taken as 15. There is a very good agreement between non dimensional frequencies as compared with [1] and [5] in Table 4.11. The frequency values for other boundary conditions (simply-simply and clamped-free) are also presented which are not there in [1] and [5]. Here two numbers of spectral elements are considered for the analysis.

Table 4.11: Non dimensional frequency ($\overline{\omega} = \omega L^2\sqrt{\rho/(E_1h^2)}$) of unsymmetrical composite graphite epoxy beams [0°/90°]

Mode No.	BOUNDARY CONDITIONS							
	C-C			C-S			S-S	C-F
	SEM	Ref. [1]	Ref. [5]	SEM	Ref. [1]	Ref. [5]	SEM	SEM
1	2.819	2.853	2.84	2.017	2.176	2.414	1.332	0.4801
2	7.076	7.246	7.283	6.0533	6.255	6.518	5.028	2.8417
3	12.5601	13.016	12.613	11.512	11.876	11.873	10.419	7.3583
4	18.783	19.692	19.087	17.849	18.466	18.218	16.856	13.1105
5	25.475	26.977	25.518	24.697	25.69	24.911	23.875	17.1852

4.1.2.2 [0°/45°/0°/45°] Ply Oriented Beam

An unsymmetric [0°/45°/0°/45°] ply oriented graphite epoxy composite Timoshenko beam having material and geometrical properties same as used in Table 4.11 is considered. The shear correction factor is taken as $\kappa = \frac{5}{6}$. Two number of beam elements are used to compare non dimensional frequency of clamped clamped (CC) and clamped-simply support (CS) as shown in Table 4.12 and it is found that results obtained by SEM are in close agreement with that of

Ref. [5]. Other boundary conditions(Simply supported, clamped-free) results by SEM are also presented.

Table 4.12: Non dimensional frequency ($\overline{\omega} = \omega L^2 \sqrt{\rho/(E_1 h^2)}$) of unsymmetrical composite graphite epoxy beams [0°/45°/0°/45°]

Mode No.	BOUNDARY CONDITIONS					
	C-C		C-S		S-S	C-F
	SEM	Ref. [5]	SEM	Ref. [5]	SEM	SEM
1	4.14	3.86	3.1	2.96	2.14	0.78
2	9.64	9.48	8.65	8.23	7.57	4.33
3	16.22	15.54	15.45	14.76	14.61	10.49
4	23.31	22.8	22.78	21.95	22.22	17.62
5	30.67	29.77	30.31	29.37	29.94	19.08

4.1.2.3 [0°/90°/0°/90°] Ply Oriented Composite Beam

CASE-I

In this composite unsymmetric beam the material properties are same as used in Table 4.11 but the geometrical properties are considered as width (b)=0.0254 m and thickness (h)=0.0254 m. The length to width ratio (L/h)= 15 . Density (ρ)=1389.79 kg/m^3. For the evaluation of natural frequencies four boundary conditions (clamped-clamped, clamped-free, simply-simply, clamped-free) with two spectral elements are employed. Shear correction factor κ =5/6. The present results obtained by SEM are in close agreement with that of [16] and [24] as shown in Table 4.13.

CASE-II

For another case with the same material properties that are used in Table 4.13 but having geometrical properties as: width (b)=thickness (h)=10 mm, length (L)=150 mm is considered for calculation of frequency in kHz and in non dimensional form ($\overline{\omega} = \omega L^2 \sqrt{\rho/(E_1 h^2)}$). Density ($\rho$)=1389 kg/m^3.The two

Table 4.13: Natural frequency (Hz) of unsymmetrical composite cross ply graphite epoxy beams [0°/90°/0°/90°]

Mode No.	BOUNDARY CONDITIONS										
	C-C			S-S			C-S			C-F	
	SEM	Ref. [16]	Ref. [24]	SEM	Ref. [16]	Ref. [24]	SEM	Ref. [16]	Ref. [24]	SEM	Ref. [24]
1	1054.97	1054.4	1055	528.139	558.9	528	777.42	784.9	777	191.96	192
2	2510.06	2509.2	2510	1911.20	1907.7	1911	2220.53	2223.9	2221	1089.13	1089
3	4282.46	4281.5		3773.61	3787.2		4037.76	4038.6		2694.94	
4	6216.62	6215.8		5841.92	5824.8		6035.61	6031.5		4599.49	
5	8240.54	8239.9		7978.58	7980.5		8112.62	8104.8		4908.99	

number of element are taken and results are given in Table 4.15 and Table 4.14. The results found by clamped-clamped (CC), clamped-simply (CS) and simply-simply (SS) boundary conditions agree well with that of [5] and [12]. The non dimensional frequencies of clamped-free (CF) boundary condition is also presented. By comparing natural frequencies from both the case-I and case-II as given in Table 4.13 and Table 4.15 it can be concluded that natural frequency increases with decrease in geometrical properties having same material properties.

Table 4.14: Non dimensional frequency of unsymmetrical composite cross ply graphite epoxy beams [0°/90°/0°/90°]

Mode No.	BOUNDARY CONDITIONS							
	C-C			C-S		S-S		C-F
	SEM	Ref. [5]	Ref. [12]	SEM	Ref. [5]	SEM	Ref. [12]	SEM
1	3.71	3.736	3.6994	2.734	2.879	1.8576	1.9619	0.675
2	8.829	9.187	8.8119	7.8105	7.988	6.7225	6.6566	3.8309
3	15.063	15.102	15.0012	14.2025	14.343	13.273	13.122	9.479
4	21.866	22.19	21.6318	21.229	21.353	20.5485	19.94	16.1784
5	28.985	29.002	28.3575	28.535	28.605	28.0641	27.0	17.267

Table 4.15: Natural frequency (kHz) of unsymmetrical composite cross ply graphite epoxy beams [0°/90°/0°/90°]

Mode No.	BOUNDARY CONDITIONS			
	C-C	C-S	S-S	C-F
	SEM	SEM	SEM	SEM
1	2.680	1.975	1.341	0.487
2	6.377	5.641	4.855	2.767
3	10.88	10.258	9.587	6.847
4	15.79	15.33	14.842	11.686
5	20.93	20.61	20.271	12.472

4.2 Non-uniform Laminated Composite Beam

4.2.1 Stepped Laminated Composite Beam

4.2.1.1 Single Stepped Laminated Beam-I

The material properties considered for free vibration of stepped composite beam are : E_1=144.8 GPa E_2=9.65 GPa G_{12}=G_{13}=4.14 GPa G_{23}=3.45 GPa ν_{12}=0.3 ρ=1389 kg/m^3

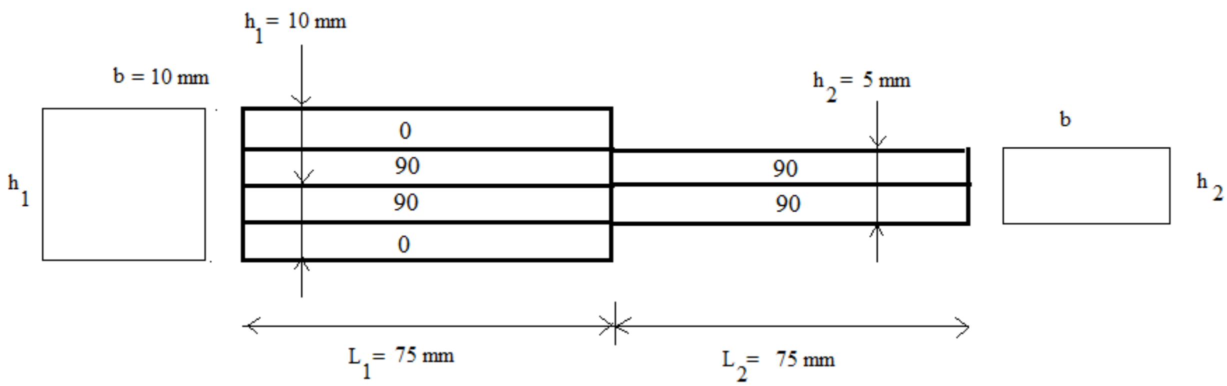


Figure 4.1: Geometry and ply orientation details of composite stepped beam- I

The Fig.4.1 shows the geometry of the beam where the width (b) for both the spans are taken as 10 mm having ply orientation of span 1 and span 2 as [0°/90°/90°/0°] and [90°/90°] respectively. The thickness of the two spans are 10 mm and 5 mm respectively and are of lengths 75 mm each. Four number of beam elements have been considered and the natural frequency parameter ($\overline{\omega} = \omega L^2 \sqrt{\rho/(E_1 h_1^2)}$) as shown in Table 4.16 are obtained for different boundary conditions and compared with that of Lam and Sathiyamoorthy [1].

Table 4.16: Natural frequency parameter of two span laminated stepped composite beam

Mode No.	BOUNDARY CONDITIONS							
	C-F		S-S		C-C		C-S	
	SEM(4)	Ref. [1]	SEM(4)	Ref. [1]	SEM(4)	Ref. [1]	SEM(4)	Ref. [1]
1	0.606	0.644	0.5569	0.538	2.4513	2.316	1.98	1.924
2	2.408	2.28	3.308	3.69	4.8706	5.042	4.00	4.046
3	4.905	5.205	8.43	8.713	10.1375	10.21	8.863	9.55
4	10.2847	10.563	12.394	13.791	13.926	14.99	13.192	13.927

4.2.1.2 Single Stepped Laminated Beam-II

A second beam with different ply orientation of span 1 and span 2 which are [90°/0°/0°/90°] and [0°/0°] respectively with length of span 75 mm each is considered. The width (b) of both span is taken as 20 mm. Four number of spectral elements are considered for analysis. The material properties of graphite-epoxy are E_1=144.8 GPa, E_2=9.65 GPa, G_{12}=G_{13}=4.14 GPa, G_{23}=3.45 GPa, ν_{12}=0.3, ρ=1389 kg/m^3.

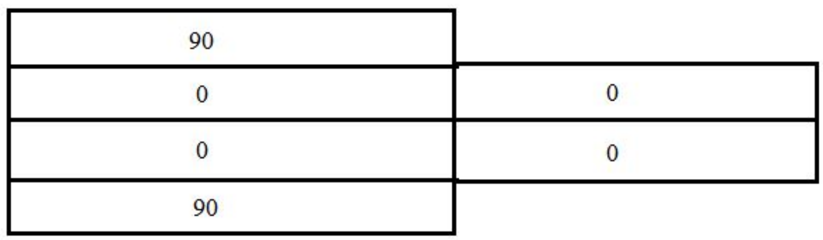


Figure 4.2: Ply orientation details of composite stepped beam-II

CASE-I

Three different thickness ratio (h_1/h_2) has been taken as 1.3, 2 and 4 where h_1 is 20 mm. The results are obtained for various boundary conditions and are presented in Table 4.17. The results are obtained by SEM.

Table 4.17: Natural frequency (kHz) of composite stepped beam for various boundary conditions

Ratio h_1/h_2	Mode no	Boundary Condition CC	CS	SS	CF
1.3	1	3.587	2.897	2.00	0.706
	2	8.034	7.604	6.922	3.818
	3	12.758	12.558	12.025	9.413
	4	18.30	18.172	17.913	14.336
	5	23.29	23.232	23.035	20.203
2.0	1	3.239	2.618	1.727	0.828
	2	7.515	6.803	6.145	3.574
	3	12.153	11.742	11.185	8.376
	4	17.635	17.328	17.049	13.207
	5	22.543	22.414	22.170	19.370
4.0	1	2.457	2.053	0.935	0.938
	2	6.541	5.471	5.011	2.570
	3	10.189	9.645	8.961	6.827
	4	15.669	14.775	14.543	10.586
	5	20.559	20.234	19.831	16.861

CASE-II

In this case E_1 has been taken as 200 GPa and all other material properties are same as used in Table 4.16. The thicknesses are h_1=20 mm and h_2=10 mm. Width of beam is taken as 20 mm for both the span. Results which are shown in Table 4.18 are obtained by SEM.

Table 4.18: Natural frequency in kHz of stepped composite beam

Mode No.	BOUNDARY CONDITIONS							
	C-C		C-S		S-S		C-F	
	case-I	case-II	case-I	case-II	case-I	case-II	case-I	case-II
1	3.239	3.493	2.618	2.886	1.727	1.947	0.828	0.923
2	7.515	7.913	6.803	7.285	6.145	6.693	3.5748	3.910
3	12.153	12.638	11.742	12.312	11.185	11.847	8.376	9.014
4	17.635	18.213	17.328	17.989	17.049	17.778	13.207	13.837
5	22.543	23.102	22.414	23.012	22.170	22.832	19.370	20.132

CASE-III

Here five different types of ply orientation (PO) are considered for frequency analysis for two span stepped beam. The geometrical properties are width (b)=20 mm, thicknesses h_1 and h_2 are 20 and 10 mm respectively. The material properties are E_1=144.8 GPa, E_2=9.65 GPa, G_{12}=G_{13}=4.14 GPa, G_{23}=3.45 GPa, ν_{12}=0.3 and ρ=1389 kg/m^3 . The natural frequency values for clamped-clamped and simply supported cases are presented for all orientations.

- PO1: span1-[90°/0°/0°/90°], span2- [0°/0°]
- PO2: span1-[0°/0°/0°/0°], span2- [0°/0°]
- PO3: span1-[90°/90°/90°/90°], span2- [90°/90°]
- PO4: span1-[90°/45°/45°/90°], span2- [45°/45°]
- PO5: span1-[45°/90°/90°/45°], span2- [90°/90°]

Table 4.19: Natural frequency (kHz) of stepped composite beam for various ply angles with C-C & S-S Boundary condition

Boundary Condition	Mode No	Ply Orientation				
		PO1	PO2	PO3	PO4	PO5
C-C	1	3.239	4.036	1.707	1.963	2.922
	2	7.515	8.757	4.637	5.224	5.537
	3	12.153	13.610	7.805	8.675	9.332
	4	17.635	19.076	12.241	13.392	13.555
	5	22.543	24.712	16.394	17.884	19
S-S	1	1.727	2.107	0.723	0.835	0.779
	2	6.145	7.789	3.282	3.785	4.125
	3	11.185	13.207	6.349	7.228	8.258
	4	17.049	18.762	10.834	12.078	12.404
	5	22.170	24.501	15.148	16.799	18.146

It can be seen from Table 4.19 that 0° ply oriented lamina gives maximum natural frequency where 90° orientation gives minimum frequency values. So as per frequency values obtained, from the above Table it may observed that the frequency values for the ply orientation can be arranged as: PO2 > PO1 > PO5 > PO4 > PO3.

CASE-IV

In this case all properties are same as case-III. The total length L_t of two span stepped beam is 150 mm. The length of the span one L_1 is varied as $0.25 \times L_t$,$0.5 \times L_t$ and $0.75 \times L_t$. The natural frequency values are obtained for clamped clamped and simply supported cases considering two no of beam elements.

Table 4.20: Natural frequency in kHz of stepped composite beam for various L_1

Mode No.	Span Length L_1					
	L_1=0.25 $\times$ L_t		L_1=0.5 $\times$ L_t		L_1=0.75 $\times$ L_t	
	C-C	S-S	C-C	S-S	C-C	S-S
1	3.704	1.883	3.239	1.727	2.935	1.644
2	7.802	6.153	7.515	6.145	6.949	5.786
3	12.379	11.421	12.153	11.185	12.099	11.232
4	17.889	17.397	17.635	17.049	17.301	16.651
5	23.451	23.171	22.543	22.170	22.115	21.646

4.2.2 Double Stepped Laminated Beam

The material properties are same as that is used in Table 4.16. The width of three span stepped beam is 30 mm. The ply orientation of span 1,2 and 3 are [90°/0°/90°/90°/0°/90°], [0°/90°/90°/0°] and [90°/90°] respectively. Six number of spectral elements are used for free vibration analysis and results are presented in Table 4.21.

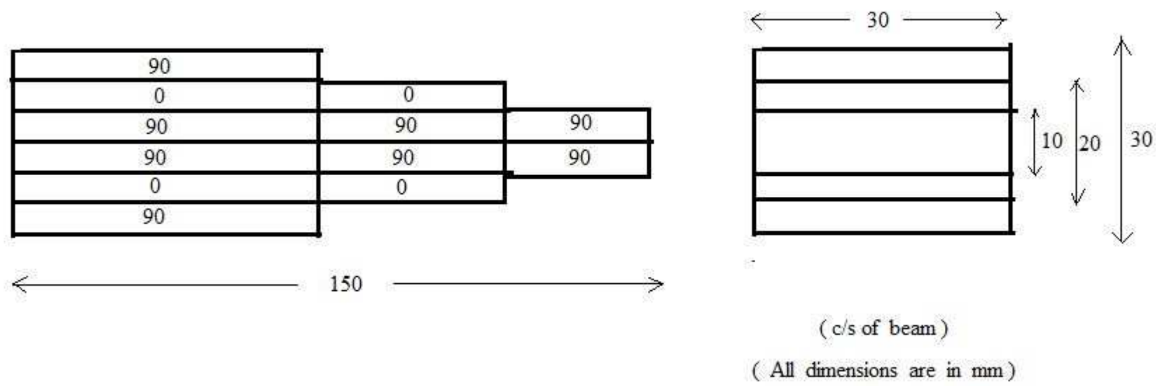


Figure 4.3: Dimensions and ply orientation of double stepped beam

Table 4.21: Free vibration analysis in kHz of double stepped laminated beam

Mode No.	BOUNDARY CONDITIONS			
	C-C	C-S	S-S	C-F
	SEM	SEM	SEM	SEM
1	2.895	2.426	1.061	1.419
2	7.446	6.719	6.321	3.092
3	11.135	9.844	9.706	7.488
4	16.584	16.095	15.967	11.591
5	21.409	20.670	20.668	16.954

4.2.3 Tapered Laminated Composite Beam

4.2.3.1 [0°/0°/0°/0°] Tapered Composite Beam

A tapered composite beam [0°/0°/0°/0°] having properties: $E_1 = 144.8\,GPa$, $E_2 = 9.65\,GPa$, $G_{12} = G_{13} = 4.14\,GPa$, $G_{23} = 3.45\,GPa$, $\nu_{12} = 0.3$, $\rho = 1389.23\,kg/m^3$, $b = 0.0254\,m$ is taken.The variation of thickness of tapered beam is given by:

$$h(x) = h_0(1 - \beta\frac{x}{L})$$

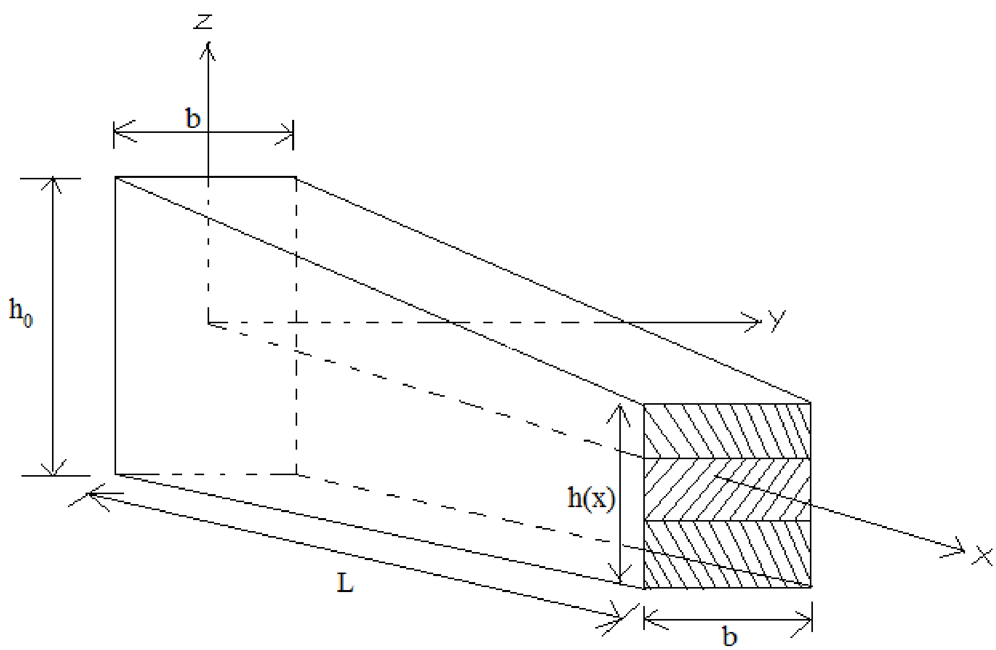
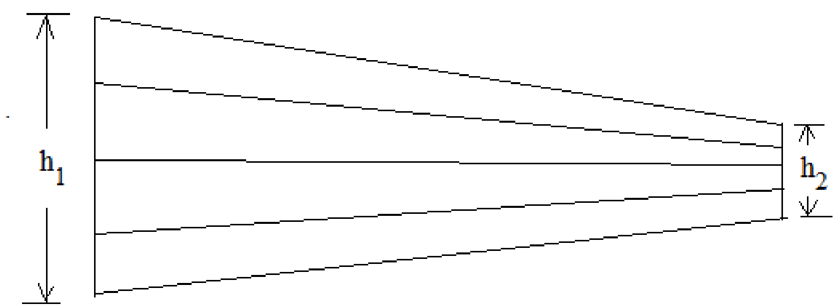


Figure 4.4: Geometry and coordinate system of tapered composite beam

where $h(x)$= thickness of composite tapered beam at any cross section, $h_0 = 0.0254\,m$, length $L = 0.381\,m$, β = non uniformity parameter $= 1 - \dfrac{h_2}{h_1}$, $h_1 > h_2$,



h_1 = maximum thickness of tapered section, h_2= minimum thickness of tapered section. Here non uniformity parameter is taken as β=0, 0.25, 0.5, 0.75. The β=0 value represents a uniform beam. The fundamental natural frequency of tapered beam for various boundary conditions by SEM for various tapered profile (β=0, 0.25, 0.5, 0.75) are presented (shown in Table 4.22 to 4.24) and are found to be in close agreement with that of [23]. In this problem 4, 8 and 12 number of beam elements are taken for convergence study. Except the fundamental one, results of all other modes are evaluated using 12 number of beam elements. In this problem the width (b) of beam remains constant and thickness varies according to non uniformity parameter (β) and span considered (x). From the result it can be concluded that for the C-C and S-S boundary conditions, as the non uniformity parameter increases fundamental frequency decreases and opposite in C-F boundary conditions. Among the fundamental natural frequencies of each boundary condition the C-C has the highest and C-F has the lowest value because of rigidities of support.

4.2.3.2 [90°/-90°/-90°/90°] Composite Tapered Beam

An angle ply [90°/-90°/-90°/90°] composite tapered beam having properties same as used in Table 4.22 is taken for evaluation. The fundamental natural frequency of tapered beam (Hz) for clamped-clamped boundary conditions (C-C) by SEM for various tapered profile (β=0, 0.25, 0.5, 0.75) are determined (shown in Table 4.25) and compared with [23]. In this problem 4, 8 and 12 number of beam elements are taken to have the convergence study.

Table 4.22: Natural frequencies (Hz) of [0°/0°/0°/0°] tapered composite beam for clamped-free (CF) boundary condition

Mode No.	Non-Uniformity Parameter															
	β=0				β=0.25				β=0.5				β=0.75			
	Ref. [23]	SEM(4)	SEM(8)	SEM(12)	Ref. [23]	SEM(4)	SEM(8)	SEM(12)	Ref. [23]	SEM(4)	SEM(8)	SEM(12)	Ref. [23]	SEM(4)	SEM(8)	SEM(12)
1	278.43	279.19	279.19	279.19	288.52	286.26	288.55	288.9	304.04	297.4	302.9	304.05	332.79	316.8	329.4	331.8
2				1471.02				1397				1304.2				1188.7
3				3428.3				3216				2943.6				2575.3
4				5589.54				5305				4907.2				4312.8
5				7823.01				7511				7042.7				6278.1

Table 4.23: Natural frequencies (Hz) of [0°/0°/0°/0°] tapered composite beam for clamped-clamped (CC) boundary condition

Mode No.	Non-Uniformity Parameter															
	β=0				β=0.25				β=0.5				β=0.75			
	Ref. [23]	SEM(4)	SEM(8)	SEM(12)	Ref. [23]	SEM(4)	SEM(8)	SEM(12)	Ref. [23]	SEM(4)	SEM(8)	SEM(12)	Ref. [23]	SEM(4)	SEM(8)	SEM(12)
1	1376.46	1378.61	1378.61	1378.61	1267.57	1271.64	1270.5	1270.2	1125.86	1137.3	1131.3	1129.6	928.81	957.57	943.3	937.1
2				3077.82				2908.9				2667.6				2294.7
3				5066.03				4852.8				4536.7				4009.7
4				7170.14				6934.5				6575.1				5940
5				9322.94				9081.05				8701.2				7999.3

Table 4.24: Natural frequencies (Hz) of $[0°/0°/0°/0°]$ tapered composite beam for simply-simply (SS) boundary condition

Mode No.	Non-Uniformity Parameter															
	β=0				β=0.25				β=0.5				β=0.75			
	Ref. [23]	SEM(4)	SEM(8)	SEM(12)	Ref. [23]	SEM(4)	SEM(8)	SEM(12)	Ref. [23]	SEM(4)	SEM(8)	SEM(12)	Ref. [23]	SEM(4)	SEM(8)	SEM(12)
1	753.22	755.12	755.12	755.12	665.36	666.36	666.96	667.07	560.88	558.18	561.4	561.9	427.28	413.44	424.6	426.79
2				2548.03				2330.07				2059.4				1698.4
3				4716.5				4430.1				9033.7				3438.4
4				6960.8				6658.38				6204.8				5450
5				9195.41				3902.19				8435.04				7590.7

Table 4.25: Natural frequencies (Hz) of [90°/-90°/-90°/90°] tapered composite beam for clamped-clamped (C-C) boundary condition

Mode No.	Non-Uniformity Parameter															
	β=0				β=0.25				β=0.5				β=0.75			
	Ref. [23]	SEM(4)	SEM(8)	SEM(12)	Ref. [23]	SEM(4)	SEM(8)	SEM(12)	Ref. [23]	SEM(4)	SEM(8)	SEM(12)	Ref. [23]	SEM(4)	SEM(8)	SEM(12)
1	459.16	460.37	460.37	460.37	402.86	404.7	404.25	404.12	340.14	344.7	342.4	341.7	265.95	275.7	270	268.86
2				1220.8				1080.6				920.35				727.49
3				2283.6				2040.46				1753.7				1397.03
4				3580.3				3230.22				2804.37				2255.4
5				5055.7				4604				4037.82				3281.11

Chapter 5

CONCLUSION

The following conclusions can be derived from the present work :

1. In this work laminated composite Timoshenko beams are analyzed by the spectral element method (SEM). During the analysis two number of elements are considered for uniform composite beam, whereas more number of elements maximum upto 12 numbers are employed as section changes to stepped and tapered ones.

2. There is very good agreement with the results of other investigators and SEM is found to be very efficient.

3. In the free vibration of symmetrical composite beam it is found that with the increase in slenderness ratio, the natural frequency values decrease.

4. For angle ply laminated beam the frequency parameter decreases with increase in ply angle.

5. The free vibration of non symmetrical beam with same material properties shows that frequency is inversely proportional to the beam span L.

6. For the two-span stepped composite beam as the thickness ratio (h_1/h_2) increases, the natural frequency decreases. It is also concluded that as the

material property (E_1) value increases, the natural frequency increases. It is observed that the natural frequency values get maximum with all lamina having 0° orientation and minimum when it is 90°. The natural frequency of two span laminated stepped beam varies inversely with the thicker portion beam length.

7. For the tapered composite beam as the non uniformity parameters increase (from 0 to 0.75) frequency values tend to decrease.

Bibliography

[1] KY Lam and TS Sathiyamoorthy. Low-velocity impact response for laminated stepped beam. *Composite structures*, 35(4):343–355, 1996.

[2] LS Teoh and C-C Huang. The vibration of beams of fibre reinforced material. *Journal of Sound and Vibration*, 51(4):467–473, 1977.

[3] K Chandrashekhara, K Krishnamurthy, and S Roy. Free vibration of composite beams including rotary inertia and shear deformation. *Composite Structures*, 14(4):269–279, 1990.

[4] Haim Abramovich. Shear deformation and rotary inertia effects of vibrating composite beams. *Composite Structures*, 20(3):165–173, 1992.

[5] S Krishnaswamy, K Chandrashekhara, and WZB Wu. Analytical solutions to vibration of generally layered composite beams. *Journal of Sound and Vibration*, 159(1):85–99, 1992.

[6] Haim Abramovich and A Livshits. Free vibrations of non-symmetric cross-ply laminated composite beams. *Journal of sound and vibration*, 176(5):597–612, 1994.

[7] Moshe Eisenberger, Haim Abramovich, and Oleg Shulepov. Dynamic stiffness analysis of laminated beams using a first order shear deformation theory. *Composite Structures*, 31(4):265–271, 1995.

[8] Galal A Hassan, Mohammed A Fahmy, and Ibrahim Goda Mohammed. Effects of fiber orientation and laminate stacking sequence on out-of-plane and in-plane bending natural frequencies of laminated composite beams.

[9] Y Teboub and P Hajela. Free vibration of generally layered composite beams using symbolic computations. *Composite structures*, 33(3):123–134, 1995.

[10] JR Bannerjee and FW Williams. Exact dynamic stiffness matrix for composite timoshenko beams with applications. *Journal of Sound and Vibration*, 194(4):573–585, 1996.

[11] JR Banerjee. Free vibration of axially loaded composite timoshenko beams using the dynamic stiffness matrix method. *Computers & structures*, 69(2):197–208, 1998.

[12] G Shi and KY Lam. Finite element vibration analysis of composite beams based on higher-order beam theory. *Journal of Sound and Vibration*, 219(4):707–721, 1999.

[13] AS Bassiouni, RM Gad-Elrab, and TH Elmahdy. Dynamic analysis for laminated composite beams. *composite structures*, 44(2):81–87, 1999.

[14] WQ Chen, CF Lv, and ZG Bian. Free vibration analysis of generally laminated beams via state-space-based differential quadrature. *Composite Structures*, 63(3):417–425, 2004.

[15] Jun Li, Hongxing Hua, and Rongying Shen. Dynamic stiffness analysis for free vibrations of axially loaded laminated composite beams. *Composite Structures*, 84(1):87–98, 2008.

[16] Li Jun, Hua Hongxing, and Shen Rongying. Dynamic finite element method for generally laminated composite beams. *International Journal of Mechanical Sciences*, 50(3):466–480, 2008.

[17] Li Jun and Hua Hongxing. Dynamic stiffness analysis of laminated composite beams using trigonometric shear deformation theory. *Composite Structures*, 89(3):433–442, 2009.

[18] G Subramanian and TS Balasubramanian. Beneficial effects of steps on the free vibration characteristics of beams. *Journal of Sound and vibration*, 118(3):555–560, 1987.

[19] Xing-Jian Dong, Guang Meng, Hong-Guang Li, and Lin Ye. Vibration analysis of a stepped laminated composite timoshenko beam. *Mechanics Research Communications*, 32(5):572–581, 2005.

[20] LA Taber and DC Viano. Comparison of analytical and experimental results for free vibration of non-uniform composite beams. *Journal of Sound and Vibration*, 83(2):219–228, 1982.

[21] S Ramalingerswara Rao and N Ganesan. Dynamic response of tapered composite beams using higher order shear deformation theory. *Journal of sound and vibration*, 187(5):737–756, 1995.

[22] Rajamohan Ganesan and Abolghassem Zabihollah. Vibration analysis of tapered composite beams using a higher-order finite element. part i: Formulation. *Composite structures*, 77(3):306–318, 2007.

[23] Faruk Fırat Çalım. Free and forced vibrations of non-uniform composite beams. *Composite Structures*, 88(3):413–423, 2009.

[24] Usik Lee. *Spectral element method in structural dynamics*. John Wiley & Sons, 2009.